

## ON $(p, q)$ -STANCU-SZÁSZ-BETA OPERATORS AND THEIR APPROXIMATION PROPERTIES

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**Abstract.** In the present paper, we have introduced the generalized form of  $(p, q)$ -analogue of the Szász-Beta operators with Stancu type parameters. We have studied the local approximation properties of these operators and obtained the convergence rate and weighted approximation.

**Keywords:** Szász-Beta operators; Stancu type parameters; weighted approximation.

### 1. Introduction and preliminaries

In the last two decades, the applications of  $q$ -calculus emerged as a new area in the field of approximation theory. The development of  $q$ -calculus has led to the discovery of various modifications of Bernstein polynomials involving  $q$ -integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [11] introduced the first  $q$ -analogue of the classical Bernstein operators and investigated its approximating and shape preserving properties. Another  $q$ -generalization of the classical Bernstein polynomial is due to Phillips [20]. Several generalization of well known positive linear operators based on  $q$ -integers were introduced and their approximation properties have been studied by several researchers.

Recently, Mursaleen *et al* introduced the use of  $(p, q)$ -calculus in approximation theory and constructed the  $(p, q)$ -analogue of Bernstein operators [13] and  $(p, q)$ -analogue of Bernstein-Stancu operators [15]. Most recently, the  $(p, q)$ -analogue of

some more operators have been studied in [1]- [3], [5], [12], [14], [16], [17], [18] and [19].

The  $(p, q)$ -integer was introduced to generalize or unify several forms of  $q$ -oscillator algebras well known in the Physics literature related to the representation theory of single parameter quantum algebras. The  $(p, q)$ -integer is defined by

$$(1.1) [n]_{p,q} = p^{n-1} + qp^{n-2} + \dots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p - q} & (p \neq q \neq 1) \\ \frac{1 - q^n}{1 - q} & (p = 1) \\ n & (p = q = 1) \end{cases}$$

The  $(p, q)$ -binomial expansion is

$$(ax + by)_{p,q}^n := \sum_{k=0}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)_{p,q}^n := (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

$$(1 - x)_{p,q}^n := (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x).$$

The  $(p, q)$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}.$$

The definite integral of a function  $f$  is defined by

$$\int_0^a f(t) d_{p,q}t = (q - p)a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}}a\right) \frac{p^k}{q^{k+1}}, \quad \text{if } \left| \frac{p}{q} \right| < 1,$$

$$\int_0^a f(t) d_{p,q}t = (p - q)a \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right) \frac{q^k}{p^{k+1}}, \quad \text{if } \left| \frac{q}{p} \right| < 1.$$

There are two  $(p, q)$ -analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

which satisfy the equality  $e_{p,q}(x)E_{p,q}(-x) = 1$ . For  $p = 1$ ,  $e_{p,q}(x)$  and  $E_{p,q}(x)$  reduce to  $q$ -exponential functions.

For  $m, n \in \mathbb{N}$ , the  $(p, q)$ -Beta and the  $(p, q)$ -Gamma functions are defined by

$$B_{p,q}(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} d_{p,q}x,$$

and

$$\Gamma_{p,q}(n) = \int_0^{\infty} p^{\frac{n(n-1)}{2}} E_{p,q}(-qx) d_{p,q}x, \quad \Gamma_{p,q}(n+1) = [n]_{p,q}!$$

respectively. The two functions are connected through

$$B_{p,q}(m, n) = q^{\frac{2-m(m-1)}{2}} p^{-\frac{m(m+1)}{2}} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

For  $p = 1$ , all the notions of the  $(p, q)$ -calculus reduce to those of  $q$ -calculus.

Based on  $(p, q)$ -calculus, very recently Acar [1] defined the  $(p, q)$  analogue of Szász operators as

$$(1.2) \quad S_{n,p,q}(f; x) = \sum_{k=0}^n s_{n,k}^{p,q}(x) f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right)$$

for  $x \in [0, \infty), 0 < q < p \leq 1$ , where

$$s_{n,k}^{p,q}(x) = \frac{q^{\frac{k(k-1)}{2}}}{E_{p,q}([n]_{p,q}x)} \frac{([n]_{p,q}x)^k}{[k]_{p,q}!}.$$

Gupta and Noor [9] proposed Szász-Beta operators and obtained some direct results in simultaneous approximation. Gupta and Aral [8] extended the studies and they proposed the  $q$ -analogue of Szász-Beta operators. Later on Aral and Gupta [4] introduced the  $(p, q)$ -analogue of the Szász-Beta operators as follows

$$(1.3) \quad D_n^{(p,q)}(f; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} f(p^{k+1}qt) d_{p,q}t$$

where  $s_{n,k}^{p,q}(x)$  is defined in (1.2). In this paper, we have generalized this operator (1.3) with Stancu type parameters. Assuming that  $0 \leq \alpha \leq \beta$ , for  $x \in [0, \infty)$ ,  $0 < q < p \leq 1$ , we define

$$D_{n,p,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} f\left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta}\right) d_{p,q}t. \quad (1.4)$$

## 2. Auxiliary results

**Lemma 2.1.** For  $x \in [0, \infty)$ ,  $0 < q < p \leq 1$ , we have

$$\begin{aligned} (i) D_n^{p,q}(1;x) &= 1, \\ (ii) D_n^{p,q}(t;x) &= x, \\ (iii) D_n^{p,q}(t^2;x) &= \frac{[2]_{p,q} qx}{p[n-1]_{p,q}} + \frac{p[n]_{p,q} x^2}{[n-1]_{p,q}}, \\ (iv) D_n^{p,q}(t^3;x) &= \frac{p^3 [n]_{p,q}^2}{q^6 [n-1]_{p,q} [n-2]_{p,q}} x^3 \\ &+ \left( \frac{(p[2]_{p,q} + p^2)[n]_{p,q}}{p^2 q^6 ([n-1]_{p,q} [n-2]_{p,q})} + \frac{(p^2 q + 2pq^2)[n]_{p,q}}{q^6 ([n-1]_{p,q} [n-2]_{p,q})} \right) x^2 \\ &+ \left( \frac{[2]_{p,q}}{p^3 q^5 ([n-1]_{p,q} [n-2]_{p,q})} + \frac{(p[2]_{p,q} + p^2)}{p^3 q^5 ([n-1]_{p,q} [n-2]_{p,q})} \right) x, \\ (v) D_n^{p,q}(t^4;x) &= \frac{p^6 [n]_{p,q}^3}{q^{12} [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^4 \\ &+ \frac{[n]_{p,q}^2 (p^5 + 3p^3 q^2 + 2p^3 q + 2p^2 q^3 + pq^4 + q^3)}{q^{11} [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^3 \\ &+ \frac{[n]_{p,q}}{p^5 q^9 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \left( p^8 + 3p^7 q + 5p^6 q^2 \right. \\ &\left. + 5p^5 q^3 + 2p^4 q^4 + p^4 q^2 + p^3 q^4 + 2p^3 q^3 + 2p^2 q^4 + pq^5 \right) x^2 \\ &+ \frac{(p^6 + 2p^5 q + p^4 q^2 + p^3 q^3 + p^3 q^2 + p^3 q + 2p^2 q^4 + 2p^2 q^2 + 2pq^5 + pq^3 + q^6)}{p^6 q^6 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x. \end{aligned}$$

**Lemma 2.2.** Let  $e_r(t) = t^r$ ,  $r \in \mathbb{N} \cup \{0\}$ . For  $x \in [0, \infty)$ ,  $0 < q < p \leq 1$ ,  $0 \leq \alpha \leq \beta$ , we have

$$\begin{aligned} (i) D_{n,p,q}^{(\alpha,\beta)}(e_0;x) &= 1, \\ (ii) D_{n,p,q}^{(\alpha,\beta)}(e_1;x) &= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta}, \end{aligned}$$

$$\begin{aligned}
 (iii) D_{n,p,q}^{(\alpha,\beta)}(e_2; x) &= \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q} + \beta)^2} x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q} + 2\alpha p[n-1]_{p,q})}{p([n]_{p,q} + \beta)^2 [n-1]_{p,q}} x \\
 &\quad + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}, \\
 (iv) D_{n,p,q}^{(\alpha,\beta)}(e_3; x) &= \frac{p^3[n]_{p,q}^5}{q^6([n]_{p,q} + \beta)^3 [n-1]_{p,q} [n-2]_{p,q}} x^3 \\
 &\quad + \frac{[n]_{p,q}^3([n]_{p,q}(p^3q + 2p^2q^2 + 2p + q) + 3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q} + \beta)^3 [n-1]_{p,q} [n-2]_{p,q}} x^2 \\
 &\quad + \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^3} \left( \frac{[n]_{p,q}^2([2]_{p,q} + p[2]_{p,q} + p^2)}{q^5p^3[n-1]_{p,q} [n-2]_{p,q}} + \frac{3q\alpha[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}} + 3\alpha^3 \right) x \\
 &\quad + \frac{\alpha^3}{([n]_{p,q} + \beta)^3}, \\
 (v) D_{n,p,q}^{(\alpha,\beta)}(e_4; x) &= \frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^4 \\
 &\quad + \frac{[n]_{p,q}^5([n]_{p,q}(p^5 + 3p^3q^2 + 2p^3q + 2p^2q^3 + pq^4 + q^3) + 4\alpha p^3q^5[n-3]_{p,q})}{q^{11}([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^3 \\
 &\quad + \frac{[n]_{p,q}^3}{p^5q^9([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \left( [n]_{p,q}^2(p^8 + 3p^7q + 5p^6q^2 \right. \\
 &\quad + 5p^5q^3 + 2p^4q^4 + p^4q^2 + p^3q^4 + 2p^3q^3 + 2p^2q^4 + pq^5) \\
 &\quad + (4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4 + 2p^6q^5 + 2p^5q^3 + p^4q^4)) \\
 &\quad \left. + (6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q}) \right) x^2 \\
 &\quad + \frac{[n]_{p,q}}{p^6q^6([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \left( [n]_{p,q}^3(p^6 + 2p^5q + p^4q^2 \right. \\
 &\quad + p^3q^3 + p^3q^2 + p^3q + 2p^2q^4 + 2p^2q^2 + 2pq^5 + pq^3 + q^6) + 4\alpha[n]_{p,q}^2[n-3]_{p,q} \\
 &\quad (2p^5q + p^4q^2 + p^4q + p^3q^2) + 6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7 + p^5q^8) \\
 &\quad \left. + \alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6 \right) x + \frac{\alpha^4}{([n]_{p,q} + \beta)^4}.
 \end{aligned}$$

*Proof.* Using Lemma 2.1, we can easily say,  $(i) D_{n,p,q}^{(\alpha,\beta)}(e_0; x) = 1$ . Moreover

$$\begin{aligned}
 (ii) D_{n,p,q}^{(\alpha,\beta)}(e_1; x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left( \frac{[n]_{p,q}p^{k+1}qt + \alpha}{[n]_{p,q} + \beta} \right) d_{p,q}t \\
 &= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1}q}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^k}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 &\quad + \frac{\alpha}{[n]_{p,q} + \beta} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t
 \end{aligned}$$

$$\begin{aligned}
&= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} D_n^{p,q}(e_1; x) + \frac{\alpha}{[n]_{p,q} + \beta} D_n^{p,q}(e_0; x) \\
&= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta}. \\
\\
(iii) D_{n,p,q}^{(\alpha,\beta)}(e_2; x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left( \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} \right)^2 d_{p,q}t \\
&= \frac{[n]_{p,q}^2}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2} q^2}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{2\alpha[n]_{p,q}}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^k}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&+ \frac{[n]_{p,q}^2}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_2; x) + \frac{2\alpha[n]_{p,q}}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_1; x) \\
&\quad + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_0; x) \\
&= \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q} + \beta)^2} x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q} + 2\alpha p[n-1]_{p,q})}{p([n]_{p,q} + \beta)^2 [n-1]_{p,q}} x \\
&\quad + \frac{\alpha^2}{([n]_{p,q} + \beta)^2}. \\
\\
(iv) D_{n,p,q}^{(\alpha,\beta)}(e_3; x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left( \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} \right)^3 d_{p,q}t \\
&= \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{3k+3} q^3}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{3\alpha[n]_{p,q}^2}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2} q^2}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{3\alpha^2[n]_{p,q}}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^k}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&\quad + \frac{\alpha^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
&+ \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_3; x) + \frac{3\alpha[n]_{p,q}^2}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_2; x)
\end{aligned}$$

$$\begin{aligned}
 & + \frac{3\alpha^2 [n]_{p,q}}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_1; x) + \frac{\alpha^3}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_0; x) \\
 & = \frac{p^3 [n]_{p,q}^5}{q^6 ([n]_{p,q} + \beta)^3 [n-1]_{p,q} [n-2]_{p,q}} x^3 \\
 & + \frac{[n]_{p,q}^3 ([n]_{p,q} (p^3 q + 2p^2 q^2 + 2p + q) + 3p^2 q^6 \alpha [n-2]_{p,q})}{pq^6 ([n]_{p,q} + \beta)^3 [n-1]_{p,q} [n-2]_{p,q}} x^2 \\
 & + \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^3} \left( \frac{[n]_{p,q}^2 ([2]_{p,q} + p[2]_{p,q} + p^2)}{q^5 p^3 [n-1]_{p,q} [n-2]_{p,q}} + \frac{3q\alpha [2]_{p,q} [n]_{p,q}}{p[n-1]_{p,q}} + 3\alpha^3 \right) x \\
 & + \frac{\alpha^3}{([n]_{p,q} + \beta)^3}. \\
 (v) D_{n,p,q}^{(\alpha,\beta)}(e_4; x) & = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left( \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} \right)^4 d_{p,q}t \\
 & = \frac{[n]_{p,q}^4}{([n]_{p,q} + \beta)^4} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{4k+4} q^4}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+3}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 & = \frac{4\alpha [n]_{p,q}^3}{([n]_{p,q} + \beta)^4} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{3k+3} q^3}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 & + \frac{6\alpha^2 [n]_{p,q}^2}{([n]_{p,q} + \beta)^4} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2} q^2}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 & + \frac{4\alpha^3 [n]_{p,q}}{([n]_{p,q} + \beta)^4} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^k}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 & + \frac{\alpha^4}{([n]_{p,q} + \beta)^4} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\
 & + \frac{[n]_{p,q}^4}{([n]_{p,q} + \beta)^4} D_n^{p,q}(e_4; x) + \frac{4\alpha [n]_{p,q}^3}{([n]_{p,q} + \beta)^4} D_n^{p,q}(e_3; x) + \frac{6\alpha^2 [n]_{p,q}^2}{([n]_{p,q} + \beta)^4} D_n^{p,q}(e_2; x) \\
 & + \frac{4\alpha^3 [n]_{p,q}}{([n]_{p,q} + \beta)^4} D_n^{p,q}(e_1; x) + \frac{\alpha^4}{([n]_{p,q} + \beta)^4} D_n^{p,q}(e_0; x) \\
 & = \frac{p^6 [n]_{p,q}^7}{q^{12} ([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^4 \\
 & + \frac{[n]_{p,q}^5 ([n]_{p,q} (p^5 + 3p^3 q^2 + 2p^3 q + 2p^2 q^3 + pq^4 + q^3) + 4\alpha p^3 q^5 [n-3]_{p,q})}{q^{11} ([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^3 \\
 & + \frac{[n]_{p,q}^3}{p^5 q^9 ([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \left( [n]_{p,q}^2 (p^8 + 3p^7 q + 5p^6 q^2 \right. \\
 & \left. + 5p^5 q^3 + 2p^4 q^4 + p^4 q^2 + p^3 q^4 + 2p^3 q^3 + 2p^2 q^4 + pq^5) \right)
 \end{aligned}$$

$$\begin{aligned}
& + (4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4 + 2p^6q^5 + 2p^5q^3 + p^4q^4) \\
& + (6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q}))x^2 \\
& + \frac{[n]_{p,q}}{p^6q^6([n]_{p,q} + \beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} \left( [n]_{p,q}^3(p^6 + 2p^5q + p^4q^2 \right. \\
& + p^3q^3 + p^3q^2 + p^3q + 2p^2q^4 + 2p^2q^2 + 2pq^5 + pq^3 + q^6) + 4\alpha[n]_{p,q}^2[n-3]_{p,q} \\
& (2p^5q + p^4q^2 + p^4q + p^3q^2) + 6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7 + p^5q^8) \\
& \left. + \alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6 \right) x + \frac{\alpha^4}{([n]_{p,q} + \beta)^4}.
\end{aligned}$$

□

We readily obtain the following lemma.

**Lemma 2.3.** For  $x \in [0, \infty)$ ,  $0 < q < p \leq 1$ ,  $0 \leq \alpha \leq \beta$ , we have

$$\begin{aligned}
(i) D_{n,p,q}^{\alpha,\beta}((t-x); x) &= \left( \frac{[n]_{p,q}}{([n]_{p,q} + \beta)} - 1 \right) x + \frac{\alpha}{([n]_{p,q} + \beta)}, \\
(ii) D_{n,p,q}^{\alpha,\beta}((t-x)^2; x) &= \left( \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q} + \beta)^2} - \frac{2[n]_{p,q}}{([n]_{p,q} + \beta)} + 1 \right) x^2 \\
&+ \left( \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^2} \left( \frac{2[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}} + 2\alpha \right) - \frac{2\alpha}{([n]_{p,q} + \beta)} \right) x \\
&+ \frac{\alpha^2}{([n]_{p,q} + \beta)^2}, \\
(iii) D_{n,p,q}^{\alpha,\beta}((t-x)^4; x) &= \left( \frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q} + \beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} \right. \\
&- \frac{4p^3[n]_{p,q}^5}{q^6([n]_{p,q} + \beta)^3[n-1]_{p,q}[n-2]_{p,q}} \\
&+ \left. \frac{6p[n]_{p,q}^3}{([n]_{p,q} + \beta)^2[n-1]_{p,q}} - \frac{4[n]_{p,q}}{([n]_{p,q} + \beta)} + 1 \right) x^4 \\
&+ \left( \frac{[n]_{p,q}^5}{q^{11}([n]_{p,q} + \beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} \right. \\
&\left. \left[ [n]_{p,q}(p^5 + 3p^3q^2 + 2p^3q + 2p^2q^3 + pq^4 + q^3) + 4\alpha p^3q^5[n-3]_{p,q} \right] \right. \\
&- \frac{4[n]_{p,q}^3([n]_{p,q}(p^3q + 2p^2q^2 + 2p + q) + 3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q} + \beta)^3[n-1]_{p,q}[n-2]_{p,q}} \\
&\left. + \frac{6[n]_{p,q}([n]_{p,q}(pq + q^2) + 2\alpha p[n-1]_{p,q})}{p([n]_{p,q} + \beta)^2[n-1]_{p,q}} - \frac{4\alpha}{([n]_{p,q} + \beta)} \right) x^3
\end{aligned}$$



$$\begin{aligned}
 & + \left( \frac{[n]_{p,q}^3}{p^5 q^9 ([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \right. \\
 & \left[ [n]_{p,q}^2 (p^8 + 3p^7 q + 5p^6 q^2 + 5p^5 q^3 + 2p^4 q^4 + p^4 q^2 + p^3 q^4 + 2p^3 q^3 \right. \\
 & \left. + 2p^2 q^4 + p q^5) + 4\alpha [n]_{p,q} [n-3]_{p,q} (p^7 q^4 + 2p^6 q^5 + 2p^5 q^3 + p^4 q^4) \right. \\
 & \left. + 6\alpha^2 p^6 q^9 [n-2]_{p,q} [n-3]_{p,q} \right] - \frac{4[n]_{p,q}}{p^3 q^5 ([n]_{p,q} + \beta)^3 [n-1]_{p,q} [n-2]_{p,q}} \\
 & \left[ [n]_{p,q}^2 (2p^2 + pq + p + q) + 3\alpha [n-2]_{p,q} (p^3 q^6 + p^2 q^7) \right. \\
 & \left. + 3\alpha^2 p^3 q^5 [n-1]_{p,q} [n-2]_{p,q} \right] + \frac{6\alpha^2}{([n]_{p,q} + \beta)^2} \Big) x^2 \\
 & + \left( \frac{[n]_{p,q}}{p^6 q^6 ([n]_{p,q} + \beta)^4 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} \right. \\
 & \left[ [n]_{p,q}^3 (p^6 + 2p^5 q + p^4 q^2 + p^3 q^3 + p^3 q^2 + p^3 q + 2p^2 q^4 + 2p^2 q^2 + 2p q^5 \right. \\
 & \left. + p q^3 + q^6) + 4\alpha [n]_{p,q}^2 [n-3]_{p,q} (2p^5 q + p^4 q^2 + p^4 q + p^3 q^2) \right. \\
 & \left. + 6\alpha^2 [n]_{p,q} [n-2]_{p,q} [n-3]_{p,q} (p^6 q^7 + p^5 q^8) \right. \\
 & \left. + \alpha^3 p^6 q^6 [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q} \right] - \frac{4\alpha^3}{([n]_{p,q} + \beta)^3} \Big) x \\
 & + \frac{\alpha^4}{([n]_{p,q} + \beta)^4}.
 \end{aligned}$$

### 3. Local approximation

In this section, we present local approximation theorem for operators  $D_{n,p,q}^{\alpha,\beta}$ . By  $C_B[0, \infty)$ , we denote the space of all real-valued continuous and bounded functions  $f$  defined on the interval  $[0, \infty)$ . The norm  $\| \cdot \|$  on the space  $C_B[0, \infty)$  is given by

$$\| f \| = \sup_{0 \leq x < \infty} | f(x) |.$$

Further, let us consider the following  $K$ -functional:

$$K_2(f, \delta) = \inf_{g \in W^2} \{ \| f - g \| + \delta \| g'' \| \}$$

where  $\delta > 0$  and  $W^2 = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \}$ . By Theorem 2.4 of [6], there exists an absolute constant  $C > 0$  such that

$$(3.1) \quad K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta})$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of  $f \in C_B[0, \infty)$ . The usual modulus of continuity of  $f \in C_B[0, \infty)$  is defined by

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|.$$

**Theorem 3.1.** *Let  $f \in C_B[0, \infty)$  and  $0 < q < p \leq 1$ ,  $0 \leq \alpha \leq \beta$ . Then for all  $n \in \mathbb{N}$ , there exists an absolute constant  $C > 0$  such that*

$$|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq C\omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{D_{n,p,q}^{\alpha,\beta}((t-x)^2; x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - x.$$

*Proof.* For  $x \in [0, \infty)$ , we consider the auxiliary operators  $\bar{D}_n^*$  defined by

$$\bar{D}_n^*(f; x) = D_{n,p,q}^{\alpha,\beta}(f; x) + f(x) - f\left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta}\right).$$

From Lemma 2.2 (i), (ii) and Lemma 2.3 (i), we observe that the operators  $\bar{D}_n^*(f; x)$  are linear and reproduce the linear functions. Hence

$$\begin{aligned} \bar{D}_n^*(1; x) &= D_{n,p,q}^{\alpha,\beta}(1; x) + 1 - 1 = 1, \\ \bar{D}_n^*(t; x) &= D_{n,p,q}^{\alpha,\beta}(t; x) + x - \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta}\right) = x, \\ \bar{D}_n^*((t-x); x) &= \bar{D}_n^*(t; x) - x\bar{D}_n^*(1; x) = 0. \end{aligned}$$

Let  $x \in [0, \infty)$  and  $g \in W^2$ . Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying  $\bar{D}_n^*$  to both sides of the above equation, we have

$$\begin{aligned} \bar{D}_n^*(g; x) - g(x) &= g'(x)\bar{D}_n^*((t-x); x) + \bar{D}_n^*\left(\int_x^t (t-u)g''(u)du; x\right) \\ &= D_{n,p,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du; x\right) \\ &\quad - \int_x^{\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - u\right)g''(u)du. \end{aligned}$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \int_x^t |t-u| |g''(u)| du \leq \|g''\| \int_x^t |t-u| du \leq (t-x)^2 \|g''\|$$

and

$$\begin{aligned} & \left| \int_x^{\frac{[n]_{p,q}x + \frac{\alpha}{[n]_{p,q} + \beta}}{[n]_{p,q} + \beta}} \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - u \right) g''(u) du \right| \\ & \leq \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - x \right)^2 \|g''\|. \end{aligned}$$

We conclude that

$$\begin{aligned} \left| \bar{D}_n^*(g; x) - g(x) \right| & \leq \left| D_{n,p,q}^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du; x \right) \right. \\ & \quad \left. - \int_x^{\frac{[n]_{p,q}x + \frac{\alpha}{[n]_{p,q} + \beta}}{[n]_{p,q} + \beta}} \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - u \right) g''(u) du \right| \\ & \leq \|g''\| D_{n,p,q}^{\alpha,\beta}((t-x)^2; x) + \|g''\| \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} - x \right)^2 \\ & = \|g''\| \delta_n^2(x). \end{aligned}$$

Now, taking into account boundedness of  $\bar{D}_n^*$ , we have

$$|\bar{D}_n^*(f; x)| \leq |D_{n,p,q}^{\alpha,\beta}(f; x)| + 2\|f\| \leq 3\|f\|.$$

Therefore

$$\begin{aligned} |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| & \leq |\bar{D}_n^*(f - g; x) - (f - g)(x)| + \left| f \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} \right) - f(x) \right| \\ & \quad + |\bar{D}_n^*(g; x) - g(x)| \\ & \leq |\bar{D}_n^*(f - g; x)| + |(f - g)(x)| + \left| f \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta}x + \frac{\alpha}{[n]_{p,q} + \beta} \right) - f(x) \right| \\ & \quad + |\bar{D}_n^*(g; x) - g(x)| \\ & \leq 4\|f - g\| + \omega(f, \alpha_n(x)) + \delta_n^2(x) \|g''\|. \end{aligned}$$

Hence, taking the infimum on the right-hand side over all  $g \in W^2$ , we have the following result

$$|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq 4K_2(f, \delta_n^2(x)) + \omega(f, \alpha_n(x)).$$

In view of the property of  $K$ -functional, we get

$$|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq C\omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)).$$

This completes the proof of the theorem.  $\square$

#### 4. Approximation properties in weighted spaces

Let  $B_\rho[0, \infty)$  be the space of all real valued functions on  $[0, \infty)$  satisfying the condition  $|f(x)| \leq M_f \rho(x)$ , where  $M_f$  is a constant depending only on  $f$  and  $\rho(x)$  is a weight function.

Let  $C_\rho[0, \infty)$  be the space of all continuous functions in  $B_\rho[0, \infty)$  with the norm

$$\|f\|_\rho = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\rho(x)} \text{ and}$$

$$C_\rho^0 = \left\{ f \in C_\rho[0, \infty) : \lim_{x \rightarrow \infty} \frac{|f(x)|}{\rho(x)} < \infty \right\}.$$

In what follows, we assume the weight function as  $\rho(x) = 1 + x^2$ .

**Theorem 4.1.** *Let  $0 < q = q_n < p = p_n \leq 1$  such that  $q_n \rightarrow 1$ ,  $p_n \rightarrow 1$ , as  $n \rightarrow \infty$ . For each  $f \in C_\rho^0$ , we have*

$$\lim_{n \rightarrow \infty} \|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)\|_\rho = 0.$$

*Proof.* With elementary calculations, it can be easily followed that  $\lim_{n \rightarrow \infty} \|D_{n, p_n, q_n}^{\alpha, \beta}(e_i; \cdot) - e_i\|_\rho = 0$ , where  $e_i(x) = x^i$ ,  $i = 0, 1, 2$ . By weighted Korovkin theorem given in [7], we get the required result.  $\square$

Next we give the following theorem to approximate all functions in  $C_\rho^0$ . This type of result is discussed in [10] for locally integrable functions.

**Theorem 4.2.** *Let  $0 < q = q_n < p = p_n \leq 1$  such that  $q_n \rightarrow 1$ ,  $p_n \rightarrow 1$ ,  $q_n^n \rightarrow 1$ ,  $p_n^n \rightarrow 1$  as  $n \rightarrow \infty$ . For each  $f \in C_\rho^0$  and  $a > 0$ , we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+a}} = 0.$$

*Proof.* For any fixed  $x_0 > 0$ ,

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+a}} &\leq \sup_{x \leq x_0} \frac{|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+a}} + \sup_{x \geq x_0} \frac{|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+a}} \\ &\leq \|D_{n, p_n, q_n}^{\alpha, \beta}(f; x) - f(x)\|_{C[0, x_0]} \\ &\quad + \|f\|_\rho \sup_{x \geq x_0} \frac{|D_{n, p_n, q_n}^{\alpha, \beta}(1 + t^2; x)|}{(1 + x^2)^{1+a}} \\ &\quad + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x_0^2)^{1+a}} \end{aligned}$$

$$(4.1) \quad = I_1 + I_2 + I_3.$$

Since  $|f(x)| \leq \|f\|_\rho(1+x^2)$ , we have

$$I_3 = \sup_{x \geq x_0} \frac{|f(x)|}{(1+x^2)^{1+a}} \leq \sup_{x \geq x_0} \frac{\|f\|_\rho}{(1+x^2)^a} \leq \frac{\|f\|_\rho}{(1+x_0^2)^a}$$

Let  $\epsilon > 0$  be arbitrary. There exists  $n_1 \in \mathbb{N}$  such that

$$(4.2) \quad \begin{aligned} \|f\|_\rho \frac{|D_{n,p_n,q_n}^{\alpha,\beta}(1+t^2;x)|}{(1+x^2)^{1+a}} &< \frac{1}{(1+x^2)^{1+a}} \|f\|_\rho \left( (1+x^2) + \frac{\epsilon}{3\|f\|_\rho} \right), \quad \forall n \geq n_1 \\ &< \frac{\|f\|_\rho}{(1+x^2)^a} + \frac{\epsilon}{3} \quad \forall n \geq n_1. \end{aligned}$$

Hence

$$\|f\|_\rho \sup_{x \geq x_0} \frac{|D_{n,p_n,q_n}^{\alpha,\beta}(1+t^2;x)|}{(1+x^2)^{1+a}} < \frac{\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \quad \forall n \geq n_1.$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \quad \forall n \geq n_1.$$

Now, let us choose  $x_0$  to be so large that  $\frac{\|f\|_\rho}{(1+x^2)^a} < \frac{\epsilon}{6}$ .

Then,

$$(4.3) \quad I_2 + I_3 < \frac{2\epsilon}{3}, \quad \forall n \geq n_1.$$

$$(4.4) \quad I_1 = \|D_{n,p_n,q_n}^{\alpha,\beta}(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \quad \forall n \geq n_2.$$

Let  $n_0 = \max(n_1, n_2)$ . Then, combining (4.1)-(4.4), we get

$$\sup_{x \in [0,\infty)} \frac{|D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x)|}{(1+x^2)^{1+a}} < \epsilon, \quad \forall n \geq n_0.$$

This completes the proof.  $\square$

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$(4.5) \quad lip_M(\gamma) = \left\{ f \in C_B[0, \infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\gamma}{(t+x)^{\frac{\gamma}{2}}} \right\},$$

where  $M$  is a positive constant and  $0 < \gamma \leq 1$ .

**Theorem 4.3.** Let be  $f \in C_B[0, \infty)$ ,  $0 < q < p \leq 1$ ,  $0 \leq \alpha \leq \beta$ , then for any  $x \in (0, \infty)$ , the following inequality holds:

$$|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq M \left( \frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x} \right)^{\frac{\gamma}{2}},$$

where  $\varphi_{n,p,q}^{(\alpha,\beta)}(x) = D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x)$ .

*Proof.* First, we prove that the result is true for  $\gamma = 1$ . Then, for  $f \in \text{lip}_M(\gamma)$ , we obtain

$$\begin{aligned} |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| &\leq \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\ &\quad \times \left| f\left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right| d_{p,q}t \\ &\leq M \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\ &\quad \times \frac{\left| \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} - x \right|}{\sqrt{\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} + x}} d_{p,q}t. \end{aligned}$$

Using  $\sqrt{x} < \sqrt{\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} + x}$  and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| &\leq \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\ &\quad \times \left| \frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta} - x \right| d_{p,q}t \\ &= \frac{M}{\sqrt{x}} D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x) \leq M \sqrt{\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}}. \end{aligned}$$

Therefore, the result is true for  $\gamma = 1$ . We prove that the result is true for  $0 < \gamma \leq 1$ , applying Hölder's inequality with  $p = \frac{2}{\gamma}$ ,  $q = \frac{1}{2-\gamma}$ ,

$$\begin{aligned} |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| &\leq \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \\ &\quad \times \left| f\left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right| d_{p,q}t \\ &\leq \sum_{k=0}^{\infty} \left\{ s_{n,k}^{p,q}(x) \left( \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \right. \right. \end{aligned}$$

$$\begin{aligned} & \times \left| f\left(\frac{[n]_{p,q} p^{k+1} q t + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right|_{d_{p,q} t}^{\frac{2}{\gamma}} \Bigg\}^{\frac{\gamma}{2}} \\ & \times \left\{ \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q} t \right\}^{\frac{2-\gamma}{2}} \\ & \leq \left\{ \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \right. \\ & \quad \left. \times \left| f\left(\frac{[n]_{p,q} p^{k+1} q t + \alpha}{[n]_{p,q} + \beta}\right) - f(x) \right|_{d_{p,q} t}^{\frac{2}{\gamma}} \right\}^{\frac{\gamma}{2}}. \end{aligned}$$

Since  $f \in lip_M(\gamma)$ , we have

$$\begin{aligned} |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| & \leq \frac{M}{x^{\frac{\gamma}{2}}} \left\{ \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \right. \\ & \quad \left. \times \left( \frac{[n]_{p,q} p^{k+1} q t + \alpha}{[n]_{p,q} + \beta} - x \right)^2 d_{p,q} t \right\}^{\frac{\gamma}{2}} \\ & = \frac{M}{x^{\frac{\gamma}{2}}} \left( D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x) \right)^{\frac{\gamma}{2}} \leq M \left( \sqrt{\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}} \right)^{\gamma}. \end{aligned}$$

Therefore, the proof is completed.  $\square$

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