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(CLR)-PROPERTY ON QUASI-PARTIAL METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Abstract. In this paper, we introduce the concept of common limit range ((CLR)−property) in the framework of quasi-partial metric spaces. By using this concept, some fixed point theorems involving two pairs of contraction mappings are proved without using the completeness condition of the whole space. Our results extend some results in literature, such as Nazir and Abbas [8] and Vetro et al. [11].
Keywords: quasi-partial metric spaces; (CLR)−property; contraction mappings.

1. Introduction

The connotation of partial metric spaces (PMS for short) was defined by Matthews in [9]. He amended metric spaces via setting self-distances to be not always identical to zero. Additionally, he relocated the Banach contraction principle in the setting of (PMS). Since then, there has been extensive research into fixed point results related to partial metric spaces (see [2, 3, 4, 7]). By dropping the symmetry condition, in 2013 Karapinar et al. [6] defined the notation of quasi-partial metric spaces (QPMS for short) and established some fixed point results in these spaces.

Let us first present some definitions and consequences which we need in the sequel.

Definition 1.1. [6] *The function $\sigma : X \times X \to [0, \infty)$ is a quasi-partial metric if the following conditions are satisfied for all $\gamma, \omega, \delta \in X$:*
(1) If $0 \le \sigma(\gamma,\gamma) = \sigma(\gamma,\omega) = \sigma(\omega,\omega) \Rightarrow \gamma = \omega$;
(2) $\sigma(\gamma,\omega) \ge \sigma(\gamma,\gamma)$;
(3) $\sigma(\omega,\gamma) \ge \sigma(\gamma,\gamma)$;
(4) $\sigma(\gamma,\delta) \le \sigma(\gamma,\omega) + \sigma(\omega,\delta) - \sigma(\omega,\omega)$.
The couple (X, σ) is known as a (QPMS).

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For each partial metric p on X, the function $d_p : X \times X \to [0, \infty)$ defined by

$$d_p(\gamma, \omega) = 2p(\gamma, \omega) - p(\gamma, \gamma) - p(\omega, \omega), \tag{1.1}$$

is a metric on X. Similarly, if (X, σ) is a (QPMS), then the function $d_\sigma : X \times X \to [0, \infty)$ defined by

$$d_\sigma(\gamma, \omega) = \sigma(\gamma, \omega) + \sigma(\omega, \gamma) - \sigma(\gamma, \gamma) - \sigma(\omega, \omega), \tag{1.2}$$

is also a metric on X.

Definition 1.2. [6] *Let (X, σ) be a quasi-partial metric space.*
1. A sequence $\{x_n\}$ is called convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \sigma(x_n, x) = \lim_{n\to\infty} \sigma(x, x_n) = \lim_{n\to\infty} \sigma(x_n, x_n) = \sigma(x, x)$;
2. A sequence $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ and $\lim_{n,m\to\infty} \sigma(x_m, x_n)$ exist and are finite;
3. (X, σ) is called complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some $x \in X$. Further, $\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n,m\to\infty} \sigma(x_m, x_n) = \sigma(x, x)$.

In 1996, Jungck [5] introduced the concept of weakly compatible mappings (w-compatible for short).

Definition 1.3. [5] *Let X be a nonempty set. Given $S, H : X \to X$. The mappings H and S are w-compatible if and only if $SH\mu = HS\mu$ for $\mu \in C(S, H)$, where $C(S, H) = \{u,\ fu = gu\}$.*

Definition 1.4. [1] *Let S and H be two self-mappings on a metric space (X, d). The mappings S and H fulfill the $(E.A)$-property if there exists a sequence $\{a_n\}$ in X such that*

$$\lim_{n\to\infty} Ha_n = \lim_{n\to\infty} Sa_n = \mu$$

for $\mu \in X$.

Note that the $(E.A)$-property exchanges the completeness condition of the space with closedness of the range. The connotation of (CLR)-property was defined by Sintunavarat and Kumam in [10]. Its significance is that one does no longer refer to the closeness condition of the range of subspaces.

Definition 1.5. [10] *Let (X, d) be a metric space and S, H be two self-mappings on X. These maps satisfy the (CLR_S)-property, if there exists a sequence $\{a_n\}$ in X so that*

$$\lim_{n\to\infty} Ha_n = \lim_{n\to\infty} Sa_n = \mu,$$

where $\mu \in S(X)$.

Currently, Nazir and Abbas [8] established some fixed point results via the (E.A)-property in the class of (PMS). However, we see that the circumstance $p(t,t)=0$ in [4, Definition 1.7] is superfluous. In our current work, we shall give the definition of (CLR)-property (for two pairs of self-mappings) on (QPMS). Additionally, by using this concept, we employ a different method compared with that in the proof of [4, Theorem 2.1] in order to prove our main results in the class of (QPMS). Some illustrated examples are also given.

2. Main results

First, let $\psi:[0,\infty)\to[0,\infty)$ be a function such that
(a) ψ is nondecreasing and continuous;
(b) $\psi(\mu)=0 \Leftrightarrow \mu=0$.
Denote $\mathcal{F}$ (resp. $\mathcal{G}$) the set of functions verifying the conditions (a) and (b) (resp. (b) and (c): ψ is lower-semicontinuous).

Now, we introduce the concept of (CLR)-property first for a single pair and after for a double pair of self-mappings on a (QPMS).

Definition 2.1. *Let (X,σ) be a (QPMS). The pair of self-mappings (f,S) on X satisfies the (CLR_S)-property, if there exists $\{x_n\}\subset X$ such that*

$$\lim_{n\to\infty}\sigma(fx_n,w)=\lim_{n\to\infty}\sigma(w,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,w)=\lim_{n\to\infty}\sigma(w,Sx_n)=\sigma(w,w),\ w\in SX.$$

Example 2.1. *Let $X=(0,\infty)$ and $\sigma(x,y)=|x-y|+x$ for all $x,y\in X$. Clearly, (X,σ) is a (QPMS). Let (f,S) be a pair of self-mappings on X such that $fx=\frac{3x+2}{2}$ and $Sx=2x$. Choose $\{x_n\}=\{\frac{2n+1}{n}\}$. We have*

$$\lim_{n\to\infty}\sigma(fx_n,4)=\lim_{n\to\infty}\sigma(4,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,4)=\lim_{n\to\infty}\sigma(4,Sx_n)=\sigma(4,4)=S2=4.$$

Hence the pair (f,S) satisfies the (CLR_S)-property.

Definition 2.2. *Let (X,σ) be a (QPMS). The pairs of self-mappings (f,S) and (g,H) on X satisfy the (CLR_{SH})-property, if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\begin{aligned}\lim_{n\to\infty}\sigma(fx_n,w) &= \lim_{n\to\infty}\sigma(w,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,w)=\lim_{n\to\infty}\sigma(w,Sx_n)\\ &= \lim_{n\to\infty}\sigma(w,gy_n)=\lim_{n\to\infty}\sigma(gy_n,w)\\ &= \lim_{n\to\infty}\sigma(Hy_n,w)=\lim_{n\to\infty}\sigma(w,Hy_n)=\sigma(w,w),\ \ w\in SX\cap HX.\end{aligned}$$

We illustrate Definition 2.2 by the following example.

Example 2.2. *Let $X = (0,2)$ be equipped with the quasi-partial metric $\sigma(x,y) = |x-y| + x$ for all $x, y \in X$. Let (f,S) and (g,H) be two pairs of self-mappings on X defined as*

$$fx = \begin{cases} 1 & ; x \in (0,1] \\ \frac{4}{3} & ; x \in (1,2) \end{cases} \qquad gx = \begin{cases} 1 & ; x \in (0,1] \\ \frac{3}{2} & ; x \in (1,2) \end{cases}$$

$$Sx = \begin{cases} x^2 & ; x \in (0,1] \\ x-1 & ; x \in (1,2) \end{cases} \qquad Hx = \begin{cases} x & ; x \in (0,1] \\ 2-x & ; x \in (1,2). \end{cases}$$

Consider $\{x_n\} = \{1 - \frac{1}{n}\}$ and $\{y_n\} = \{\frac{5n^2-4}{5n^2+2}\}$. We have

$$\lim_{n\to\infty} \sigma(fx_n, 1) = \lim_{n\to\infty} \sigma(1, fx_n) = \lim_{n\to\infty} \sigma(Sx_n, 1) = \lim_{n\to\infty} \sigma(1, Sx_n) = \sigma(1,1) = S1 = 1.$$

Moreover,

$$\lim_{n\to\infty} \sigma(gy_n, 1) = \lim_{n\to\infty} \sigma(1, gy_n) = \lim_{n\to\infty} \sigma(Hy_n, 1) = \lim_{n\to\infty} \sigma(1, Hy_n) = \sigma(1,1) = H1 = 1.$$

Hence the two pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property.

The following lemma is crucial in order to prove our main result (Theorem 2.1).

Lemma 2.1. *Let (X,σ) be a (QPMS). Suppose that the self-mappings $f, g, S, H : X \to X$ are such that*
(i) $fX \subseteq HX$ (or $gX \subseteq SX$);
(ii) the pair (f,S) satisfies the (CLR_S)-property (or (g,H) satisfies the (CLR_H)-property);
(iii) HX (or SX) is closed;
(iv) $\{gy_n\}$ (or $\{fy_n\}$) is bounded for every sequence $\{y_n\}$ in X;
(v) there exist $\beta \in \mathcal{F}$ and $\alpha \in \mathcal{G}$ such that

$$\beta(\sigma(fa, gb)) \leq \beta(\Lambda(a,b)) - \alpha(\Lambda(a,b)), \tag{2.1}$$

where $\Lambda(a,b) = \max\{\sigma(Sa,Hb), \sigma(fa,Sa), \sigma(Hb,gb), \sigma(fa,Hb), \sigma(Sa,gb)\}$. Then the pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property.

Proof. From Condition (ii), if (f,S) satisfies the (CLR_S)-property, then there exists $\{x_n\} \subset X$, so that

(2.2)
$$\lim_{n\to\infty} \sigma(fx_n, w) = \lim_{n\to\infty} \sigma(w, fx_n) = \lim_{n\to\infty} \sigma(Sx_n, w) = \lim_{n\to\infty} \sigma(w, Sx_n) = \sigma(w,w); \quad w \in SX.$$

Since $fX \subseteq HX$, there exists $\{y_n\}$ such that

$$fx_n = Hy_n. \tag{2.3}$$

Due to (2.2) and (2.3), we write $\lim_{n\to\infty} \sigma(Hy_n, w) = \sigma(w,w)$, so from the closedness condition of HX, we have

$$w \in SX \cap HX.$$

Now, we want to prove that $gy_n \to w$ as $n \to \infty$. We have

$$\begin{aligned}\sigma(fx_n, gy_n) &\le \sigma(fx_n, Sx_n) + \sigma(Sx_n, gy_n) - \sigma(Sx_n, Sx_n)\\ &\le \sigma(fx_n, w) + \sigma(w, Sx_n) - \sigma(w, w) + \sigma(Sx_n, gy_n) - \sigma(Sx_n, Sx_n).\end{aligned}$$

By (2.2), $\lim_{n\to\infty} \sigma(Sx_n, Sx_n) = \sigma(w, w)$. We also get

$$\limsup_{n\to\infty} \sigma(fx_n, gy_n) - \limsup_{n\to\infty} \sigma(Sx_n, gy_n) \le 0. \tag{2.4}$$

Again, by (2.2), $\lim_{n\to\infty} \sigma(fx_n, fx_n) = \sigma(w, w)$, so similarly,

$$\limsup_{n\to\infty} \sigma(Sx_n, gy_n) - \limsup_{n\to\infty} \sigma(fx_n, gy_n) \le 0. \tag{2.5}$$

As $\{gy_n\}$ is bounded, $\limsup_{n\to\infty} \sigma(fx_n, gy_n)$ and $\limsup_{n\to\infty} \sigma(Sx_n, gy_n)$ are finite numbers. Using (2.4) and (2.5), there exists $\delta \ge 0$ such that one writes

$$\limsup_{n\to\infty} \sigma(Sx_n, gy_n) = \limsup_{n\to\infty} \sigma(fx_n, gy_n) = \delta. \tag{2.6}$$

So there are subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that

$$\lim_{k\to\infty} \sigma(Sx_{n_k}, gy_{n_k}) = \lim_{k\to\infty} \sigma(fx_{n_k}, gy_{n_k}) = \delta. \tag{2.7}$$

Clearly, by (2.2),

$$\sigma(w, w) = \lim_{k\to\infty} \sigma(fx_{n_k}, Sx_{n_k}) = \lim_{k\to\infty} \sigma(Sx_{n_k}, fx_{n_k}). \tag{2.8}$$

Since $\sigma(fx_{n_k}, fx_{n_k}) \le \sigma(fx_{n_k}, Sx_{n_k})$, passing to the limit as $k \to \infty$, we obtain

$$\sigma(w, w) \le \delta. \tag{2.9}$$

We have

$$\begin{aligned}&\Lambda(fx_{n_k}, y_{n_k})\\ &= \max\{\sigma(Sx_{n_k}, Hy_{n_k}), \sigma(fx_{n_k}, Sx_{n_k}), \sigma(Hy_{n_k}, gy_{n_k}), \sigma(fx_{n_k}, Hy_{n_k}), \sigma(Sx_{n_k}, gy_{n_k})\}\\ &= \max\{\sigma(Sx_{n_k}, fx_{n_k}), \sigma(fx_{n_k}, Sx_{n_k}), \sigma(fx_{n_k}, gy_{n_k}), \sigma(fx_{n_k}, fx_{n_k}), \sigma(Sx_{n_k}, gy_{n_k})\}.\end{aligned}$$

Passing to the limit as $k \to \infty$, we get due to (2.9)

$$\lim_{k\to\infty} \Lambda(fx_{n_k}, y_{n_k}) = \max\{\sigma(w, w), \sigma(w, w), \delta, \sigma(w, w), \delta\} = \delta. \tag{2.10}$$

By using (2.1),

$$\beta(\sigma(fx_{n_k}, gy_{n_k})) \le \beta(\Lambda(x_{n_k}, y_{n_k})) - \alpha(\Lambda(x_{n_k}, y_{n_k})).$$

Taking the upper limit as $k \to \infty$ and using (2.8) and (2.10),

$$\beta(\delta) \le \beta(\delta) - \alpha(\delta),$$

i.e., $\alpha(\delta) = 0$, which yields that $\delta = 0$. Thus $\sigma(w, w) = \delta = 0$. So, by (2.6), we have

$$\lim_{n\to\infty} \sigma(fx_n, gy_n) = 0.$$

Consequently,

$$\lim_{k\to\infty} \sigma(gy_{n_k}, gy_{n_k}) = 0 = \sigma(w, w).$$

We obtained

$$\lim_{n\to\infty} \sigma(w, gy_n) = \lim_{n\to\infty} \sigma(gy_n, w) = \lim_{n\to\infty} \sigma(Hy_n, w) = \lim_{n\to\infty} \sigma(w, Hy_n) = \sigma(w, w).$$

So the pairs (f, S) and (g, H) satisfy the $(CLR_{SH})-$property. □

Now, we introduce and prove our main result by using the concept of (CLR)-property on the class of quasi-partial metric spaces.

Theorem 2.1. *Let f, g, H and S be self-mappings on a (QPMS) (X, σ) satisfying the condition (v) of Lemma 2.1. If the pairs (f, S) and (g, H) satisfy the (CLR_{SH})-property, then there exists $x \in X$ such that $fx = gx = Sx = Hx$. Furthermore, if (f, S) and (g, H) are w-compatible, then such x is the unique common fixed point of f, g, H and S.*

Proof. As (f, S) and (g, H) verify the (CLR_{SH})-property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}\lim_{n\to\infty} \sigma(fx_n, w) &= \lim_{n\to\infty} \sigma(w, fx_n) = \lim_{n\to\infty} \sigma(Sx_n, w) = \lim_{n\to\infty} \sigma(w, Sx_n)\\ &= \lim_{n\to\infty} \sigma(w, gy_n) = \lim_{n\to\infty} \sigma(gy_n, w)\\ &= \lim_{n\to\infty} \sigma(Hy_n, w) = \lim_{n\to\infty} \sigma(w, Hy_n) = \sigma(w, w);\ \ w \in SX \cap HX.\end{aligned}$$

Since $w \in SX$, there exists $k \in X$ such that $Sk = w$. Now, we want to prove that $fk = Sk$. Suppose that $fk \neq Sk$. Obviously,

$$\lim_{n\to\infty} \sigma(Hy_n, gy_n) = \sigma(w, w), \tag{2.11}$$

and

$$\lim_{n\to\infty} \sigma(fk, Hy_n) = \lim_{n\to\infty} \sigma(fk, gy_n) = \sigma(fk, w). \tag{2.12}$$

From (2.1),

$$\beta(\sigma(fk, gy_n)) \leq \beta(\Lambda(k, y_n)) - \alpha(\Lambda((k, y_n)), \tag{2.13}$$

where

$$\Lambda(k, y_n) = \max\{\sigma(Sk, Hy_n), \sigma(fk, Sk), \sigma(Hy_n, gy_n), \sigma(fk, Hy_n), \sigma(Sk, gy_n)\}.$$

Taking the limit as $n \to \infty$ and using the equations (2.11) and (2.12), we get

$$(2.14)\quad \lim_{n\to\infty} \Lambda(k, y_n) = \max\{\sigma(w,w), \sigma(fk,w), \sigma(w,w), \sigma(fk,w), \sigma(w,w)\} = \sigma(fk, w).$$

Letting $n \to \infty$ in (2.13), by (2.12) and (2.14), we get

$$\beta(\sigma(fk,w)) \leq \beta(\sigma(fk,w)) - \alpha(\sigma(fk,w)).$$

So $\alpha(\sigma(fk,w)) = 0$, that is, $\sigma(fk,w) = 0$, i.e.,

$$(2.15)\qquad fk = Sk = w.$$

Since $w \in HX$, there exists $\nu \in X$ such that $H\nu = w$. As (2.11) and (2.12), we may write

$$(2.16)\qquad \lim_{n\to\infty} \sigma(fx_n, Sx_n) = \sigma(w,w),$$

and

$$(2.17)\qquad \lim_{n\to\infty} \sigma(Sy_n, g\nu) = \lim_{n\to\infty} \sigma(fx_n, g\nu) = \sigma(w, g\nu).$$

By (2.1),

$$\beta(\sigma(fx_n, g\nu)) \leq \beta(\Lambda(x_n,\nu)) - \alpha(\Lambda(x_n,\nu)),$$

where

$$\Lambda(x_n,\nu) = \max\{\sigma(Sx_n, H\nu), \sigma(fx_n, Sx_n), \sigma(H\nu, g\nu), \sigma(fx_n, H\nu), \sigma(Sx_n, g\nu)\}.$$

Due to (2.16) and (2.17),

$$(2.18)\quad \lim_{n\to\infty} \Lambda(x_n,\nu) = \max\{\sigma(w,w), \sigma(w,w), \sigma(w,g\nu), \sigma(w,w), \sigma(w,g\nu)\} = \sigma(w, g\nu).$$

By (2.17) and (2.18),

$$\beta(\sigma(w,g\nu)) \leq \beta(\sigma(w,g\nu)) - \alpha(\sigma(w,g\nu)).$$

This gives that $\alpha(\sigma(w,g\nu)) = 0$, hence $\sigma(w,g\nu)) = 0$. So $H\nu = g\nu = w$. The w-compatibility of (f,S) together with $fk = Sk$ implies that

$$fw = fSk = Sfk = Sw.$$

We shall prove that $fw = Sw = w$. We have

$$\beta(\sigma(fw,w)) = \beta(\sigma(fw,g\nu)) \leq \beta(\Lambda(w,\nu)) - \alpha(\Lambda(w,\nu)),$$

where

$$\begin{aligned}\Lambda(w,\nu) &= \max\{\sigma(Sw,H\nu),\sigma(fw,Sw),\sigma(H\nu,g\nu),\sigma(fw,H\nu),\sigma(Sw,g\nu)\}\\ &= \max\{\sigma(fw,w),\sigma(fw,fw),\sigma(w,w),\sigma(fw,w),\sigma(fw,w)\}\\ &= \sigma(fw,w).\end{aligned}$$

Then

$$\beta(\sigma(fw,g\nu)) \le \beta(\sigma(fw,g\nu)) - \alpha(\sigma(fw,g\nu)).$$

This implies that $\alpha(\sigma(fw,w)) = 0$, that is, $\sigma(fw,w) = 0$, so $fw = w = Sw$. Again the w-compatibility condition of (g,H) and the fact that $g\nu = H\nu$ imply that $gw = gH\nu = Hg\nu = Hw$. Again, using (2.1),

$$\beta(\sigma(w,gw)) = \beta(\sigma(fk,gw)) \quad\le\quad \beta(\Lambda(k,w)) - \alpha(\Lambda(k,w)),$$

where

$$\begin{aligned}\Lambda(k,w) &= \max\{\sigma(Sk,Hw),\sigma(fk,Sk),\sigma(Hw,gw),\sigma(fk,Hk),\sigma(Sk,gk)\}\\ &= \max\{\sigma(w,gw),\sigma(w,w),\sigma(gw,gw),\sigma(w,gw),\sigma(w,gw)\}\\ &= \sigma(w,gw).\end{aligned}$$

Then

$$\beta(\sigma(w,gw)) = \beta(\sigma(fk,gw)) \le \beta(\sigma(w,gw)) - \alpha(\sigma(w,gw)),$$

hence, $\alpha(\sigma(w,gw)) = 0$. Thus $\sigma(w,gw) = 0$, so $w = gw = Hw$.

Finally, we shall show that w is unique. Consider that $\lambda = f\lambda = g\lambda = S\lambda = H\lambda$. From (2.1),

$$\beta(\sigma(w,\lambda)) = \beta(\sigma(fw,g\lambda)) \le \beta(\Lambda(w,\lambda)) - \alpha(\Lambda(w,w)).$$

Since

$$\begin{aligned}\Lambda(w,\lambda) &= \max\{\sigma(Sw,H\lambda),\sigma(fw,Sw),\sigma(H\lambda,g\lambda),\sigma(fw,H\lambda),\sigma(Sw,g\lambda)\}\\ &= \max\{\sigma(w,\lambda),\sigma(w,w),\sigma(\lambda,\lambda),\sigma(w,\lambda),\sigma(w,\lambda)\}\\ &= \sigma(w,\lambda),\end{aligned}$$

we get

$$\beta(\sigma(w,\lambda)) = \beta(\sigma(fw,g\lambda)) \le \beta(\sigma(w,\lambda)) - \alpha(\sigma(w,\lambda)).$$

Therefore, $\alpha(\sigma(w,\lambda)) = 0$, that is, $\sigma(w,\lambda) = 0$, hence $w = \lambda$. The proof is completed. $\square$

Example 2.3. *Take $A = [0,1]$. Consider the quasi-partial metric on A defined by*

$$\sigma(c,d) = |c-d| + c.$$

Given $f,g,H,S : A \to A$ as

$$f(d) = 0,\ \ g(d) = \frac{1}{8}d,\ \ S(d) = \frac{1}{2}d,\ \ H(d) = \frac{1}{3}d.$$

It is clear that $fA \subset HA$, $gA \subset SA$ and the pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property. Take $\beta(t) = 8t$ and $\alpha(t) = t$. We will prove that (2.1) holds. First,

$$\beta(\sigma(fc,gd)) = \beta(|fc-gd|+fc) = \beta(\frac{1}{8}d) = d. \tag{2.19}$$

Moreover,

$$\begin{aligned} \Lambda(c,d) &= \max\{\sigma(Sc,Hd),\sigma(fc,Sc),\sigma(Hd,gd),\sigma(fc,Hd),\sigma(Sc,gd)\} \\ &= \max\{\sigma(\frac{1}{2}c,\frac{1}{3}d),\sigma(0,\frac{1}{2}c),\sigma(\frac{1}{3}d,\frac{1}{8}d),\sigma(0,\frac{1}{3}d)\sigma(\frac{1}{2}c,\frac{1}{8}d)\} \\ &= \max\{|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c,\frac{1}{2}c,\frac{13}{24}d,\frac{1}{3}d,|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c\}. \end{aligned}$$

Case 1. Let $\Lambda(c,d) = \frac{13}{24}d$. We obtain

$$\beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) = \frac{13}{3}d - \frac{13}{24}d = \frac{91}{24}d > d = \beta(\sigma(fc,gd)). \tag{2.20}$$

Case 2. Let $\Lambda(c,d) = |\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c$. We have

$$\begin{aligned} \beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) &= 8(|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c) - (|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c) \\ &= 7(|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c) > 7(\frac{13}{24}d) > d = \beta(\sigma(fc,gd)). \end{aligned} \tag{2.21}$$

Case 3. Let $\Lambda(c,d) = |\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c$. We have

$$\begin{aligned} \beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) &= 8(|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c) - (|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c) \\ &= 7(|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c) > 7(\frac{13}{24}d) > d = \beta(\sigma(fc,gd)). \end{aligned} \tag{2.22}$$

From (2.20) to (2.21), the condition (2.1) holds. Here, 0 is the unique common fixed point, that is, $f0 = g0 = S0 = H0 = 0$.

Example 2.4. *Let $X = [0,7)$ and $\sigma(x,y) = |x-y|+x$ for all $x,y \in X$. (X,σ) is a (QPMS). Define (f,S) and (g,H) as two pairs of self-mappings on X, where*

$$f(x) = \begin{cases} 0 \ ; x \in \{0\} \cup [5,7) \\ 2 \ ; x \in (0,5), \end{cases} \qquad g(x) = \begin{cases} 0 \ ; x \in \{0\} \cup [5,7) \\ 4 \ ; x \in (0,5) \end{cases}$$

$$S(x) = \begin{cases} 0 \ ; x \in \{0\} \\ 5 \ ; x \in (0,5) \\ \frac{x+5}{2} \ ; x \in [5,7), \end{cases} \qquad H(x) = \begin{cases} 0 \ ; x \in \{0\} \\ 6 \ ; x \in (0,5) \\ x-5 \ ; x \in [5,7). \end{cases}$$

Also, define $\beta(t) = 8t$ and $\alpha(t) = \frac{t}{10}$. Choose $\{x_n\} = \{0\}$ and $\{y_n\} = \{5+\frac{1}{n}\}$. Then

$$\lim_{n\to\infty}\sigma(f(x_n),0) = \lim_{n\to\infty}\sigma(0,f(x_n)) = \lim_{n\to\infty}\sigma(S(x_n),0) = \lim_{n\to\infty}\sigma(0,S(x_n)) = \sigma(0,0) = S(0) = 0.$$

Also

$$\lim_{n\to\infty}\sigma(g(y_n),0)=\lim_{n\to\infty}\sigma(0,g(y_n))=\lim_{n\to\infty}\sigma(H(y_n),0)$$
$$=\lim_{n\to\infty}\sigma(0,H(y_n))=\sigma(0,0)=H(0)=0.$$

Hence the two pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property. Now, we will show that the contraction condition (2.1) holds. For this, we distinguish the following cases.

Case 1. $x,y\in\{0\}\cup[5,7)$. *Here, we have*

$$\beta(\sigma(fx,gy))=\beta(\sigma(0,0))=0\leq\beta(\Lambda(x,y))-\alpha(\Lambda(x,y)).$$

Case 2. $x\in\{0\}$ *and* $y\in(0,5)$. *We have*

$$\beta(\sigma(fx,gy))=\beta(\sigma(0,4))=32.$$

Also,

$$\begin{aligned}\Lambda(x,y)\quad&=\max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\&=\max\{\sigma(0,6),\sigma(0,0),\sigma(6,4),\sigma(0,6),\sigma(0,4)\}\\&=\max\{6,0,8,6,4\}=8.\end{aligned}$$

Hence,

$$\beta(\Lambda(x,y))-\alpha(\Lambda(x,y))=64-\frac{4}{5}>32=\beta(\sigma(fx,gy)).$$

Case 3. $x\in(0,5)$ *and* $y\in[5,7)$. *We have*

$$\beta(\sigma(fx,gy))=\beta(\sigma(2,0))=32.$$

Moreover,

$$\begin{aligned}\Lambda(x,y)\quad&=\max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\&=\max\{\sigma(5,y-5),\sigma(2,5),\sigma(y-5,0),\sigma(2,y-5),\sigma(5,0)\}\\&=\max\{|10-y|+5,5,2y-10,|7-y|+2,10\}=10.\end{aligned}$$

Then

$$\beta(\Lambda(x,y))-\alpha(\Lambda(x,y))=79>32=\beta(\sigma(fx,gy)).$$

Case 4. $x\in(0,5)$ *and* $y=0$. *In this case,*

$$\beta(\sigma(fx,gy))=\beta(\sigma(2,0))=32.$$

Then

$$\begin{aligned}\Lambda(x,y)\quad&=\max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\&=\max\{\sigma(5,0),\sigma(2,5),\sigma(0,0),\sigma(2,0),\sigma(5,0)\}\\&=\max\{10,5,0,4,10\}=10,\end{aligned}$$

that is,

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 79 > 32 = \beta(\sigma(fx,gy)).$$

Case 5. $x, y \in (0,5)$. *Here,*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(2,4)) = 32.$$

Also,

$$\begin{aligned}\Lambda(x,y) &= \max\{\sigma(Sx,Hy), \sigma(fx,Sx), \sigma(Hy,gy), \sigma(fx,Hy), \sigma(Sx,gy)\}\\ &= \max\{\sigma(5,6), \sigma(2,5), \sigma(6,4), \sigma(2,6), \sigma(5,4)\}\\ &= \max\{6,5,8,6,6\} = 8.\end{aligned}$$

Then

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 64 - \frac{4}{5} > 32 = \beta(\sigma(fx,gy)).$$

Case 6. $x \in [5,7)$ *and* $y \in (0,5)$. *We have*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(0,4)) = 32.$$

Also,

$$\begin{aligned}\Lambda(x,y) &= \max\{\sigma(Sx,Hy), \sigma(fx,Sx), \sigma(Hy,gy), \sigma(fx,Hy), \sigma(Sx,gy)\}\\ &= \max\{\sigma(\frac{x+5}{2},6), \sigma(0,\frac{x+5}{2}), \sigma(6,4), \sigma(0,6), \sigma(\frac{x+5}{2},4)\}\\ &= \max\{6, \frac{x+5}{2}, 8, 6, |\frac{x+5}{2} - 4| + \frac{x+5}{2}\} = 8.\end{aligned}$$

Hence,

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 64 - \frac{4}{5} = \beta(\sigma(fx,gy)).$$

Therefore, all conditions of Theorem 2.1 are satisfied. So, the mappings f, g, H *and* S *have a common fixed point, which is 0.*

On the other hand, $fX = \{0,2\} \nsubseteq SX = \{0\} \cup [5,6)$ *and* $gX = \{0,4\} \nsubseteq HX = \{6\} \cup [0,2)$. *Note that the result of Nazir and Abbas [8] is not applicable because the hypothesis of containment among ranges of the mappings* f, g, S, H *in [[8], Theorem 2.1] does not hold here.*

Corollary 2.1. *Let* (X,σ) *be a (QPMS). Assume that* $f, S, g, H : X \to X$ *verify all conditions in Lemma 2.1. Suppose, in addition, that the pairs* (f,S) *and* (g,T) *are w-compatible. Then there exists a unique common fixed point of* f, g, H *and* S.

Proof. From Lemma 2.1, (f,S) and (g,H) share the (CLR_{SH})-property. All conditions of Theorem 2.1 are fulfilled. Then exists a unique $x \in X$ such that $fx = Sx = gx = Hx = x$. □

By taking $\beta(t) = \int_0^t \eta(s)ds$ in Lemma 2.1 and Theorem 2.1, where $\eta : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable summable mapping such that $\int_0^\epsilon \eta(t)dt > 0$ for $\epsilon > 0$, we state the following.

Corollary 2.2. *Let f, S, g and H be self-mappings on a (QPMS) (X, σ) such that*

$$\int_0^{\sigma(fx,gy))} \eta(s)ds \le \Lambda(x,y) - \alpha(\Lambda(x,y)), \tag{2.23}$$

where $\Lambda(x,y) = \int_0^{\max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}} \eta(s)ds$. Assume that (f,S) and (g,H) fulfill the (CLR_{SH})-property. Then $fx = Sx = gx = Hx$. Furthermore, if (f,S) and (g,T) are w-compatible, there exists only one point $x \in X$ so that $fx = Sx = gx = Hx = x$.

Remark 2.1. *Corollary 2.2 extends the paper by Vetro et al.* [11] *from metric spaces to (QPMS).*

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