Abstract. In this paper, we study \(f\)-biharmonic curves as the critical points of the \(f\)-bienergy functional \(E_f(\psi) = \int_M f |\tau(\psi)|^2 \, \vartheta_g\), on a Lorentzian para-Sasakian manifold \(M\). We give necessary and sufficient conditions for a curve such that has a timelike principal normal vector on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be an \(f\)-biharmonic curve. Moreover, we introduce proper \(f\)-biharmonic curves on the Lorentzian sphere \(S^4_1\).

Keywords: \(f\)-biharmonic curves; \(f\)-bienergy functional; para-Sasakian manifold; Lorentzian sphere.

1. Introduction

Harmonic maps \(\psi: (M, g) \rightarrow (N, h)\) between Riemannian manifolds are the critical points of the energy functional defined by

\[
E(\psi) = \frac{1}{2} \int_\Omega |d\psi|^2 \vartheta_g,
\]

for every compact domain \(\Omega \subset M\). The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

\[
\tau(\psi) = \text{trace} \nabla d\psi,
\]

where \(\tau(\psi)\) is called the tension field of the map \(\psi\).

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J.H. Sampson [7]. Biharmonic maps
between Riemannian manifolds $\psi : (M, g) \rightarrow (N, h)$ are the critical points of the bienergy functional
\begin{equation}
E_2(\psi) = \frac{1}{2} \int_{\Omega} |\tau(\psi)|^2 \, \vartheta_g,
\end{equation}
for any compact domain $\Omega \subset M$.

In [3], G.Y. Jiang derived the first and the second variation formulas for the bienergy, showing that the Euler-Lagrange equation associated to $E_2$ is
\begin{equation}
\tau_2(\psi) = -J^\psi(\tau(\psi)) = -\triangle^\psi(\Psi) - \text{trace} R^N(d\psi, \tau(\psi))d\psi,
\end{equation}
where $J^\psi$ is the Jacobi operator of $\psi$. The equation $\tau_2(\psi) = 0$ is called biharmonic equation. Clearly, any harmonic maps is always a biharmonic map. A biharmonic map that is not harmonic is called a proper biharmonic map.

For some recent geometric study of biharmonic maps see [14, 17, 18, 19, 24] and the references therein. Also for some recent progress on biharmonic submanifolds see [1, 2, 16, 20, 21] and for biharmonic conformal immersions and submersions see [15, 25, 27].

The concept of $f-$biharmonic maps were initiated by W.J. Lu [23]. A smooth map $\psi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is called an $f-$biharmonic map if it is a critical point of the $f-$bienergy functional defined by
\begin{equation}
E_{2,f}(\psi) = \frac{1}{2} \int_{\Omega} f |\tau(\psi)|^2 \, \vartheta_g,
\end{equation}
for every compact domain $\Omega \subset M$.

The Euler-Lagrange equation gives the $f-$biharmonic map equation [23]
\begin{equation}
\tau_{2,f} = f\tau_2(\psi) + (\triangle f)\tau(\psi) + 2\nabla^\psi_{g,\nabla f(\psi)}\tau(\psi) = 0,
\end{equation}
where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of $\psi$, respectively. Therefore, we have the following relationship among these types of maps [26]:
\begin{equation}
(1.5) \quad \text{Harmonic maps} \subset \text{Biharmonic maps} \subset f-\text{Biharmonic maps}.
\end{equation}
From now on we will call an $f-$biharmonic map, which is neither harmonic nor biharmonic, a proper $f-$biharmonic map (see also [28]).

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [9]. He also introduced the notion of Lorentzian para-Sasakian manifold. In [4], I. Mihai and R. Rosca defined the same notion independently and there after many authors [5, 11, 22] studied Lorentzian para-Sasakian manifolds.

Moreover, in [17] some geometric result for spacelike and timelike curves in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be proper biharmonic were given. Motivated by this work, we introduced $f-$biharmonic curves on Lorentzian para-Sasakian manifold and Lorentzian sphere $S^4_1$. 
2. Preliminaries

2.1. $f$–Biharmonic Maps

$f$–Biharmonic maps are critical points of the $f$–bienergy functional for maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\psi) = \frac{1}{2} \int_{\Omega} f | \tau(\psi) |^2 \, \vartheta_g,$$

where $\Omega$ is a compact domain of $M$.

The following Theorem was proved in [23]:

**Theorem 2.1.** A map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is an $f$–biharmonic map if and only if

$$\tau_{2,f} = f \tau_2(\psi) + (\Delta f) \tau(\psi) + 2 \nabla_{\text{grad} f} \tau(\psi) = 0,$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of $\psi$, respectively. $\tau_{2,f}(\psi)$ is called the $f$–bitension field of map $\psi$.

A special case of $f$–biharmonic maps is $f$–biharmonic curves. We have the following.

**Lemma 2.1.** [26] An arclength parametrized curve $\gamma : (a, b) \to (N^m, g)$ is an $f$–biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if

$$f(\nabla_{\gamma'}^N \nabla_{\gamma'}^N \gamma' - R^N(\gamma', \nabla_{\gamma'}^N \gamma') \gamma') + 2f' \nabla_{\gamma'}^N \nabla_{\gamma'}^N \gamma' + f'' \nabla_{\gamma'}^N \gamma' = 0.$$

2.2. Lorentzian almost paracontact manifolds

Let $M$ be an $n$-dimensional differentiable manifold with a Lorentzian metric $g$, i.e., $g$ is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p \in M$, the tensor

$$g_p : T_p M \times T_p M \to R,$$

is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where $T_p M$ is the tangent space of $M$ at the point $p$. Then $(M, g)$ is called a Lorentzian manifold. A non-zero vector $X_p \in T_p M$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$ or $g_p(X_p, X_p) < 0$, respectively.

Let $M$ be an $n$-dimensional differentiable manifold equipped with a structure $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1,1)$-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form on $M$ such that [9]

$$\varphi^2 X = X + \eta(X)\xi,$$
The above equations imply that
\[ \eta \circ \varphi = 0, \quad \varphi \xi = 0, \quad \text{rank}(\varphi) = n - 1. \]

Then \( M \) admits a Lorentzian metric \( g \), such that
\[ g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \]
and \( M \) is said to admit a Lorentzian almost paracontact structure \( (\varphi, \xi, \eta, g) \). Then we get
\[ g(X, \xi) = \eta(X). \]

The manifold \( M \) endowed with a Lorentzian almost paracontact structure \( (\varphi, \xi, \eta, g) \) is called a Lorentzian almost paracontact manifold \([9, 10]\). In equations (2.4) and (2.5) if we replace \( \xi \) by \( -\xi \), we obtain an almost paracontact structure on \( M \) defined by I. Sato \([6]\).

A Lorentzian almost paracontact manifold equipped with the structure \( (\varphi, \xi, \eta, g) \) is called a Lorentzian para-Sasakian manifold \([9]\) if
\[ C(X, Y)W = R(X, Y)W - \frac{1}{n-2} \left\{ \begin{array}{c}
S(Y, W)X - S(X, W)Y \\
+g(Y, W)QX - g(X, W)QY
\end{array} \right\} 
+ \frac{r}{(n-1)(n-2)} \{ g(Y, W)X - g(X, W)Y \}, \]
where \( S(X, Y) = g(QX, Y) \). The Lorentzian para-Sasakian manifold is called conformally flat if conformal curvature tensor vanishes i.e., \( C = 0 \).

The quasi-conformal curvature tensor \( \hat{C} \) is defined by
\[ \hat{C}(X, Y)W = aR(X, Y)W - b \left\{ \begin{array}{c}
S(Y, W)X - S(X, W)Y \\
+g(Y, W)QX - g(X, W)QY
\end{array} \right\} 
- \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \{ g(Y, W)X - g(X, W)Y \}, \]
where \( a, b \) constants such that \( ab \neq 0 \). Similarly the Lorentzian para-Sasakian manifold is called quasi-conformally flat if \( \hat{C} = 0 \).

We know that a conformally flat and quasi-conformally flat Lorentzian para-Sasakian manifold \( M^n (n > 3) \) is of constant curvature 1 and also a Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation \( R(X, Y) \cdot C = 0 \) holds on \( M \) \([12]\). For a conformally symmetric Riemannian manifold \([13]\), we get \( \nabla C = 0 \). Thus for a conformally symmetric space the relation \( R(X, Y) \cdot C = 0 \).
$f$–Biharmonic Curves with Timelike Normal Vector on Lorentzian Sphere

$C = 0$ satisfies. Hence a conformally symmetric Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere [12].

Therefore, for a conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold $M$, we have [12]

\begin{equation}
R(X, Y)W = g(Y, W)X - g(X, W)Y,
\end{equation}

for any vector fields $X, Y, W \in TM$.

3. $f$–Biharmonic Curves in Lorentzian Para-Sasakian Manifolds

For a Lorentzian para-Sasakian manifold $M$, an arbitrary curve $\gamma : I \to M$, $\gamma = \gamma(s)$ is called spacelike, timelike or lightlike (null), if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike (null), respectively. In this section, we give some conditions for a curve having timelike normal vector on a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold $M$ to be an $f$–biharmonic curve.

**Theorem 3.1.** Let $\gamma : I \to M$ be a curve parametrized by arclength and $M$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Assume that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along $\gamma$ such that principal normal vector $N$ is timelike. Then $\gamma$ is a proper $f$–biharmonic curve if and only if one of the following cases happens:

i) The first curvature $\kappa_1$ of the $\gamma$ solves the following ordinary differential equation,

\begin{equation}
3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 - 4\kappa_1^2,
\end{equation}

with $f = t_1\kappa_1^{-2}$ and $\kappa_2 = 0$.

ii) The first curvature $\kappa_1$ of the $\gamma$ solves the following ordinary differential equation,

\begin{equation}
3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 + 4\kappa_1^4 t_3^2 - 4\kappa_1^2,
\end{equation}

with $f = t_1\kappa_1^{-2}$, $\kappa_2 \neq 0$, $\kappa_3 = 0$, $\frac{\kappa_1}{\kappa_1} = t_3$.

**Proof.** Let $\gamma$ be a curve parametrized by arclength on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold $M$ and let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along $\gamma$ such that principal normal vector $N$ is timelike.

In this case for this curve, the Frenet frame equations are given by [8]

\begin{equation}
\begin{bmatrix}
\nabla_T T \\
\nabla_T N \\
\nabla_T B_1 \\
\nabla_T B_2
\end{bmatrix} = \begin{bmatrix}
0 & \kappa_1 & 0 & 0 \\
\kappa_1 & 0 & \kappa_2 & 0 \\
0 & \kappa_2 & 0 & \kappa_3 \\
0 & 0 & -\kappa_3 & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}
\end{equation}
where $T$, $N$, $B_1$, $B_2$ are mutually orthogonal vectors and $\kappa_1$, $\kappa_2$ and $\kappa_3$ are respectively the first, the second and the third curvature of the $\gamma$.

In view of the Frenet formulas given in (3.3) and equation (2.8), we obtain

$$\nabla_T T = \kappa_1 N,$$

$$\nabla_T \nabla_T T = \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1,$$

$$\nabla_T \nabla_T \nabla_T T = (3 \kappa_1 \kappa_1') T + (\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2) N$$

$$+ (2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2) B_1 + (\kappa_1 \kappa_2 \kappa_3) B_2,$$

and

$$R(T, \nabla_T T) T = -\kappa_1 N,$$

where $\kappa_1$, $\kappa_2$ and $\kappa_3$ are the first, the second and the third curvature of the $\gamma$, respectively.

Considering Theorem 2.1 and equation (2.3), we get

$$\tau_{2, f} = f \left[ (3 \kappa_1 \kappa_1') T + (\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 N) \right]$$

$$+ (2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2') B_1 + (\kappa_1 \kappa_2 \kappa_3) B_2$$

$$+ 2f' \left[ \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1 \right] + f'' \left[ \kappa_1 N \right]$$

$$= 0.$$

Comparing the coefficients of above equation, we obtain that $\gamma$ is an $f$–biharmonic curve if and only if

$$3 \kappa_1 \kappa_1' + 2 \kappa_1^2 \frac{f'}{f} = 0,$$

(3.4)

$$\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 + 2 \kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0,$$

(3.5)

$$2 \kappa_1' \kappa_2 + \kappa_1 \kappa_2' + 2 \kappa_1 \kappa_2 \frac{f'}{f} = 0,$$

(3.6)

$$\kappa_1 \kappa_2 \kappa_3 = 0.$$

(3.7)

Let $\kappa_1$ be a non zero constant. Then from (3.4) we get $f$ is constant. So $\gamma$ is biharmonic. Let $\kappa_2$ be a non zero constant. From (3.4) and (3.6) one can easily see that $f$ is constant and $\gamma$ is biharmonic.
By using (3.4) - (3.7), if $\kappa_2 = 0$, then $f$–biharmonic curve equation reduces to

\begin{equation}
3\kappa_1\kappa_1' + 2\kappa_1^2 \frac{f'}{f} = 0,
\end{equation}

(3.8)

\begin{equation}
\kappa_1'' + \kappa_1^3 + \kappa_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0.
\end{equation}

(3.9)

Integrating the equation (3.8) we get $f = t_1 \kappa_1^{-\frac{3}{2}}$ and using this result in (3.9), we arrive at (i).

Otherwise, by use of (3.4) - (3.7), if $\kappa_1 \neq \text{constant}$ and $\kappa_2 \neq \text{constant}$ $f$–biharmonic curve the equation is equivalent to

\begin{equation}
f^2 \kappa_1^3 = t_1^2,
\end{equation}

(3.10)

\begin{equation}
(f \kappa_1)'' = -f \kappa_1(\kappa_1^2 + \kappa_2^2 + 1),
\end{equation}

(3.11)

\begin{equation}
f^2 \kappa_1^2 \kappa_2 = t_2,
\end{equation}

(3.12)

\begin{equation}
\kappa_3 = 0.
\end{equation}

(3.13)

In view of (3.10), we find $f = t_1 \kappa_1^{-\frac{3}{2}}$ and using this result in (3.11), we get $\frac{\kappa_2}{\kappa_1} = t_3$. Finally substituting these equation in (3.11), we arrive at (ii). □

**Proposition 3.1.** Let $M$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold and $\gamma : I \to M$ be an $f$–biharmonic spacelike curve parametrized by arclength such that principal normal vector is timelike. If $\gamma$ has constant geodesic curvature then $\gamma$ is biharmonic.

4. $f$–Biharmonic Curves on Lorentzian Sphere $S_1^4$

Suppose that $M$ is a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Since $M$ is locally isometric to a Lorentzian unit sphere $S_1^4$, we give some characterizations for $f$–biharmonic curves in $S_1^4$. The Lorentzian unit sphere of radius 1 can be seen as the hyperquadric

\[ S_1^4 = \{ p \in \mathbb{R}_1^5 : <p, p> = 1 \}, \]

in a Minkowski space $\mathbb{R}_1^5$ with the metric

\[ < , > : -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2. \]
Let $\gamma : I \to S^4_1$ be a curve parametrized by arclength. For an arbitrary vector field $X$ along $\gamma$, we have

\[(4.1) \quad \nabla_T X = X' + <T, X > _\gamma,\]

where $\nabla$ is covariant derivative along $\gamma$ in $S^4_1$.

Since $S^4_1$ is a Lorentzian space form of the scalar curvature 1, we have


for all vector fields $X, Y, W$ in the tangent bundle of $S^4_1$, where $R$ is the curvature tensor of $S^4_1$.

Now, we give the following:

**Proposition 4.1.** Let $\gamma : I \to S^4_1$ be a non-geodesic $f$-biharmonic curve parametrized by arclength and \{T, N, B_1, B_2\} be a Frenet frame along $\gamma$ such that

\[g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(N, N) = -1.\]

Then, we have

\[(4.2) \quad \gamma^{(4)} - \left( \frac{\kappa''}{\kappa_1} + 2 \frac{\kappa'_1 f'}{\kappa_1} + \frac{f''}{f} \right) \gamma'' - \left( \frac{\kappa''}{\kappa_1} + 2 \frac{\kappa'_1 f'}{\kappa_1} + \frac{f''}{f} + 1 \right) \gamma = 0.\]

**Proof.** Using (3.5) and taking the covariant derivative of the second equation in (3.3), we get

\[\nabla^2_T N = \nabla_T (\kappa_1 T + \kappa_2 B_1) \]
\[= \kappa_1 \nabla_T T + \kappa_2 \nabla_T B_1 \]
\[= (\kappa_1^2 + \kappa_2^2) N + \kappa_2 \kappa_3 B_2.\]

Using (3.5) in (4.3), we have

\[(4.3) \quad \nabla^2_T N = - \left( \frac{\kappa''}{\kappa_1} + 2 \frac{\kappa'_1 f'}{\kappa_1} + \frac{f''}{f} + 1 \right) N.\]

On the other hand from (4.1), we arrive at

\[\nabla^2_T N = \nabla_T (N' + < T, N > _\gamma) \]
\[= N'' + < T, N > _\gamma \]
\[= N'' + < T, \nabla_T N - < N, T > _\gamma > _\gamma \]
\[= N'' + < T, \kappa_1 T + \kappa_2 B_1 > _\gamma \]
\[= N'' + \kappa_1 \gamma.\]

From (4.3) and (4.4), we obtain

\[(4.4) \quad \left( \frac{\kappa''}{\kappa_1} + 2 \frac{\kappa'_1 f'}{\kappa_1} + \frac{f''}{f} + 1 \right) N = N'' + \kappa_1 \gamma.\]
Also in view of (4.1), we have
\[ \nabla_T T = T' + \langle T, T \rangle \gamma = \gamma'' + \gamma, \]
which yields
\[ (4.5) \quad N = \frac{1}{\kappa_1} (\gamma'' + \gamma). \]
By use of (4.5) and (4.4), we obtain (4.2). □

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