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$f-$BIHARMONIC CURVES WITH TIMELIKE NORMAL VECTOR ON LORENTZIAN SPHERE

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Abstract. In this paper, we study $f-$biharmonic curves as the critical points of the $f-$bienergy functional $E_2(\psi) = \int_M f \mid \tau(\psi)^2 \mid \vartheta_g$, on a Lorentzian para-Sasakian manifold M. We give necessary and sufficient conditions for a curve such that has a timelike principal normal vector on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be an $f-$biharmonic curve. Moreover, we introduce proper $f-$biharmonic curves on the Lorentzian sphere S_1^4.
Keywords: $f-$biharmonic curves; $f-$bienergy functional; para-Sasakian manifold; Lorentzian sphere.

1. Introduction

Harmonic maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds are the critical points of the energy functional defined by

$$E(\psi) = \frac{1}{2} \int_\Omega \mid d\psi \mid^2 \vartheta_g, \tag{1.1}$$

for every compact domain $\Omega \subset M$. The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

$$\tau(\psi) = trace \nabla d\psi, \tag{1.2}$$

where $\tau(\psi)$ is called the tension field of the map ψ.

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J.H. Sampson [7]. Biharmonic maps

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between Riemannian manifolds $\psi : (M, g) \to (N, h)$ are the critical points of the bienergy functional

$$E_2(\psi) = \frac{1}{2}\int_\Omega \mid \tau(\psi) \mid^2 \vartheta_g, \tag{1.3}$$

for any compact domain $\Omega \subset M$.

In [3], G.Y. Jiang derived the first and the second variation formulas for the bienergy, showing that the Euler-Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(\psi) &= -J^\psi(\tau(\psi)) \\ &= -\triangle\tau(\Psi) - traceR^N(d\psi, \tau(\psi))d\psi, \end{aligned}$$

where J^ψ is the Jacobi operator of ψ. The equation $\tau_2(\psi) = 0$ is called biharmonic equation. Clearly, any harmonic maps is always a biharmonic map. A biharmonic map that is not harmonic is called a proper biharmonic map.

For some recent geometric study of biharmonic maps see [14, 17, 18, 19, 24] and the references therein. Also for some recent progress on biharmonic submanifolds see [1, 2, 16, 20, 21] and for biharmonic conformal immersions and submersions see [15, 25, 27].

The concept of $f-$biharmonic maps were initiated by W.J. Lu [23]. A smooth map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is called an $f-$biharmonic map if it is a critical point of the $f-$bienergy functional defined by

$$E_{2,f}(\psi) = \frac{1}{2}\int_\Omega f \mid \tau(\psi) \mid^2 \vartheta_g, \tag{1.4}$$

for every compact domain $\Omega \subset M$.

The Euler-Lagrange equation gives the $f-$biharmonic map equation [23]

$$\begin{aligned} \tau_{2,f} &= f\tau_2(\psi) + (\triangle f)\tau(\psi) + 2\nabla^\psi_{gradf}\tau(\psi) \\ &= 0, \end{aligned}$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of ψ, respectively. Therefore, we have the following relationship among these types of maps [26]:

$$Harmonic\ \ maps \subset Biharmonic\ \ maps \subset f - Biharmonic\ \ maps. \tag{1.5}$$

From now on we will call an $f-$biharmonic map, which is neither harmonic nor biharmonic, a proper $f-$biharmonic map (see also [28]).

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [9]. He also introduced the notion of Lorentzian para-Sasakian manifold. In [4], I. Mihai and R. Rosca defined the same notion independently and there after many authors [5, 11, 22] studied Lorentzian para-Sasakian manifolds.

Moreover, in [17] some geometric result for spacelike and timelike curves in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be proper biharmonic were given. Motivated by this work, we introduced $f-$biharmonic curves on Lorentzian para-Sasakian manifold and Lorentzian sphere S^4_1.

2. Preliminaries

2.1. $f-$Biharmonic Maps

$f-$Biharmonic maps are critical points of the $f-$bienergy functional for maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\psi) = \frac{1}{2}\int_{\Omega} f \mid \tau(\psi) \mid^2 \vartheta_g, \tag{2.1}$$

where Ω is a compact domain of M.

The following Theorem was proved in [23]:

Theorem 2.1. *A map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is an $f-$biharmonic map if and only if*

$$\tau_{2,f} = f\tau_2(\psi) + (\triangle f)\tau(\psi) + 2\nabla^{\psi}_{gradf}\tau(\psi) = 0, \tag{2.2}$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of ψ, respectively. $\tau_{2,\,f}\,(\psi)$ is called the $f-$bitension field of map ψ.

A special case of $f-$biharmonic maps is $f-$biharmonic curves. We have the following.

Lemma 2.1. *[26] An arclength parametrized curve $\gamma : (a, b) \to (N^m, g)$ is an $f-$biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if*

$$f(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' - R^N(\gamma', \nabla^N_{\gamma'}\gamma')\gamma') + 2f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' + f''\nabla^N_{\gamma'}\gamma' = 0. \tag{2.3}$$

2.2. Lorentzian almost paracontact manifolds

Let M be an n-dimensional differentiable manifold with a Lorentzian metric g, i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in M$, the tensor

$$g_p : T_pM \times T_pM \to R,$$

is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM is the tangent space of M at the point p. Then (M, g) is called a Lorentzian manifold. A non-zero vector $X_p \in T_pM$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$ or $g_p(X_p, X_p) < 0$, respectively.

Let M be an n-dimensional differentiable manifold equipped with a structure (φ, ξ, η), where φ is a $(1, 1)$-tensor field, ξ is a vector field, η is a 1-form on M such that [9]

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.4}$$

$$\eta(\xi) = -1. \tag{2.5}$$

The above equations imply that

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad rank(\varphi) = n - 1.$$

Then M admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$$

and M is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). Then we get

$$g(X, \xi) = \eta(X). \tag{2.6}$$

The manifold M endowed with a Lorentzian almost paracontact structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold [9, 10]. In equations (2.4) and (2.5) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by I. Sato [6].

A Lorentzian almost paracontact manifold equipped with the structure (φ, ξ, η, g) is called a Lorentzian para-Sasakian manifold [9] if

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{2.7}$$

The conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)W &= R(X,Y)W - \frac{1}{n-2}\left\{ \begin{array}{c} S(Y,W)X - S(X,W)Y \\ +g(Y,W)QX - g(X,W)QY \end{array} \right\} \\ &\quad + \frac{r}{(n-1)(n-2)}\left\{ g(Y,W)X - g(X,W)Y \right\}, \end{aligned}$$

where $S(X, Y) = g(QX, Y)$. The Lorentzian para-Sasakian manifold is called conformally flat if conformal curvature tensor vanishes i.e., $C = 0$.

The quasi-conformal curvature tensor $\hat{C}$ is defined by

$$\begin{aligned} \hat{C}(X,Y)W &= aR(X,Y)W - b\left\{ \begin{array}{c} S(Y,W)X - S(X,W)Y \\ +g(Y,W)QX - g(X,W)QY \end{array} \right\} \\ &\quad - \frac{r}{n}\left(\frac{a}{(n-1)} + 2b \right)\left\{ g(Y,W)X - g(X,W)Y \right\}, \end{aligned}$$

where a, b constants such that $ab \neq 0$. Similarly the Lorentzian para-Sasakian manifold is called quasi-conformally flat if $\hat{C} = 0$.

We know that a conformally flat and quasi-conformally flat Lorentzian para-Sasakian manifold M^n $(n > 3)$ is of constant curvature 1 and also a Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation $R(X, Y)\cdot C = 0$ holds on M [12]. For a conformally symmetric Riemannian manifold [13], we get $\nabla C = 0$. Thus for a conformally symmetric space the relation $R(X, Y)\cdot$

$C = 0$ satisfies. Hence a conformally symmetric Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere [12].

Therefore, for a conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M, we have [12]

$$R(X,Y)W = g(Y,W)X - g(X,W)Y, \tag{2.8}$$

for any vector fields $X, Y, W \in TM$.

3. $f-$Biharmonic Curves in Lorentzian Para-Sasakian Manifolds

For a Lorentzian para-Sasakian manifold M, an arbitrary curve $\gamma : I \to M$, $\gamma = \gamma(s)$ is called spacelike, timelike or lightlike (null), if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike (null), respectively. In this section, we give some conditions for a curve having timelike normal vector on a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M to be an $f-$biharmonic curve.

Theorem 3.1. *Let $\gamma : I \to M$ be a curve parametrized by arclength and M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Asuume that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along γ such that principal normal vector N is timelike. Then γ is a proper $f-$biharmonic curve if and only if one of the following cases happens:*

i) The first curvature κ_1 of the γ solves the following ordinary differential equation,

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 - 4\kappa_1^2, \tag{3.1}$$

with $f = t_1\kappa_1^{-\frac{3}{2}}$ and $\kappa_2 = 0$.

ii) The first curvature κ_1 of the γ solves the following ordinary differential equation,

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 + 4\kappa_1^4 t_3^2 - 4\kappa_1^2, \tag{3.2}$$

with $f = t_1\kappa_1^{-\frac{3}{2}}, \kappa_2 \neq 0,\ \kappa_3 = 0,\ \frac{\kappa_2}{\kappa_1} = t_3$.

Proof. Let γ be a curve parametrized by arclength on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M and let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along γ such that principal normal vector N is timelike.

In this case for this curve, the Frenet frame equations are given by [8]

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{3.3}$$

where T, N, B_1, B_2 are mutually orthogonal vectors and κ_1, κ_2 and κ_3 are respectively the first, the second and the third curvature of the γ.

In view of the Frenet formulas given in (3.3) and equation (2.8), we obtain

$$\nabla_T T = \kappa_1 N,$$

$$\nabla_T \nabla_T T = \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= (3\kappa_1 \kappa_1')T + (\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2)N \\ &\quad + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')B_1 + (\kappa_1 \kappa_2 \kappa_3)B_2, \end{aligned}$$

and

$$R(T, \nabla_T T)T = -\kappa_1 N,$$

where κ_1, κ_2 and κ_3 are the first, the second and the third curvature of the γ, respectively.

Considering Theorem 2.1 and equation (2.3), we get

$$\begin{aligned} \tau_{2,f} &= f \begin{bmatrix} (3\kappa_1 \kappa_1')T + (\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 N) \\ +(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')B_1 + (\kappa_1 \kappa_2 \kappa_3)B_2 \end{bmatrix} \\ &\quad +2f' \left[\kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1\right] + f'' \left[\kappa_1 N\right] \\ &= 0. \end{aligned}$$

Comparing the coefficients of above equation, we obtain that γ is an $f-$biharmonic curve if and only if

$$3\kappa_1 \kappa_1' + 2\kappa_1^2 \frac{f'}{f} = 0, \tag{3.4}$$

$$\kappa_1'' + \kappa_1^3 + \kappa_1 \kappa_2^2 + \kappa_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0, \tag{3.5}$$

$$2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' + 2\kappa_1 \kappa_2 \frac{f'}{f} = 0, \tag{3.6}$$

$$\kappa_1 \kappa_2 \kappa_3 = 0. \tag{3.7}$$

Let κ_1 be a non zero constant. Then from (3.4) we get f is constant. So γ is biharmonic. Let κ_2 be a non zero constant. From (3.4) and (3.6) one can easily see that f is constant and γ is biharmonic.

By using (3.4) - (3.7), if $\kappa_2 = 0$, then $f-$biharmonic curve equation reduces to

$$3\kappa_1\kappa_1' + 2\kappa_1^2\frac{f'}{f} = 0, \tag{3.8}$$

$$\kappa_1'' + \kappa_1^3 + \kappa_1 + 2\kappa_1'\frac{f'}{f} + \kappa_1\frac{f''}{f} = 0. \tag{3.9}$$

Integrating the equation (3.8) we get $f = t_1\kappa_1^{-\frac{3}{2}}$ and using this result in (3.9), we arrive at (i).

Otherwise, by use of (3.4) - (3.7), if $\kappa_1 \neq constant$ and $\kappa_2 \neq constant$ $f-$biharmonic curve the equation is equivalent to

$$f^2\kappa_1^3 = t_1^2, \tag{3.10}$$

$$(f\kappa_1)'' = -f\kappa_1(\kappa_1^2 + \kappa_2^2 + 1), \tag{3.11}$$

$$f^2\kappa_1^2\kappa_2 = t_2, \tag{3.12}$$

$$\kappa_3 = 0. \tag{3.13}$$

In view of (3.10), we find $f = t_1\kappa_1^{-\frac{3}{2}}$ and using this result in (3.11), we get $\frac{\kappa_2}{\kappa_1} = t_3$. Finally substituting these equation in (3.11), we arrive at (ii). □

Proposition 3.1. *Let M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold and $\gamma : I \to M$ be an $f-$biharmonic spacelike curve parametrized by arclength such that principal normal vector is timelike. If γ has constant geodesic curvature then γ is biharmonic.*

4. $f-$Biharmonic Curves on Lorentzian Sphere S_1^4

Suppose that M is a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Since M is locally isometric to a Lorentzian unit sphere S_1^4, we give some characterizations for $f-$biharmonic curves in S_1^4. The Lorentzian unit sphere of radius 1 can be seen as the hyperquadradic

$$S_1^4 = \{p \in \mathbb{R}_1^5 :< p, p >= 1\},$$

in a Minkowski space $\mathbb{R}_1^5$ with the metric

$$<\ ,\ >: -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2.$$

Let $\gamma : I \to S_1^4$ be a curve parametrized by arclength. For an arbitrary vector field X along γ, we have

$$\nabla_T X = X' + < T, X > \gamma, \tag{4.1}$$

where ∇ is covariant derivative along γ in S_1^4.

Since S_1^4 is a Lorentzian space form of the scalar curvature 1, we have

$$R(X,Y)W =< Y, W > X- < X, W > Y,$$

for all vector fields X, Y, W in the tangent bundle of S_1^4, where R is the curvature tensor of S_1^4.

Now, we give the following:

Proposition 4.1. *Let $\gamma : I \to S_1^4$ be a non-geodesic $f-$biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that*

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1, \ \ g(N,N) = -1.$$

Then, we have

$$\gamma^{(4)} - \left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f}\right)\gamma'' - \left(\kappa_1^2 + \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)\gamma = 0. \tag{4.2}$$

Proof. Using (3.5) and taking the covariant derivative of the second equation in (3.3), we get

$$\begin{aligned} \nabla_T^2 N &= \nabla_T(\kappa_1 T + \kappa_2 B_1) \\ &= \kappa_1 \nabla_T T + \kappa_2 \nabla_T B_1 \\ &= (\kappa_1^2 + \kappa_2^2)N + \kappa_2\kappa_3 B_2. \end{aligned}$$

Using (3.5) in (4.3), we have

$$\nabla_T^2 N = -\left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)N. \tag{4.3}$$

On the other hand from (4.1), we arrive at

$$\begin{aligned} \nabla_T^2 N &= \nabla_T(N' + < T, N > \gamma) \\ &= N'' + < T, N' > \gamma \\ &= N'' + < T, \nabla_T N- < N, T > \gamma > \gamma \\ &= N'' + < T, \kappa_1 T + \kappa_2 B_1 > \gamma \\ &= N'' + \kappa_1\gamma. \end{aligned}$$

From (4.3) and (4.4), we obtain

$$\left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)N = N'' + \kappa_1\gamma. \tag{4.4}$$

Also in view of (4.1), we have

$$\nabla_T T = T' + < T, T > \gamma = \gamma'' + \gamma,$$

which yields

$$N = \frac{1}{\kappa_1}(\gamma'' + \gamma). \tag{4.5}$$

By use of (4.5) and (4.4), we obtain (4.2). □

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