ON CAPABLE GROUPS OF ORDER $p^4$ *

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Abstract. A group $H$ is said to be capable, if there exists another group $G$ such that $\frac{G}{Z(G)} \cong H$, where $Z(G)$ denotes the center of $G$. In a recent paper [5], the authors considered the problem of capability of five non-abelian $p$–groups of order $p^4$ into account. In this paper, we try to solve the problem of capability by considering three other groups of order $p^4$. It is proved that the group

$$H_6 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = xyz, yz = zy \rangle$$

is not capable. Moreover, if $p > 3$ is a prime number and $d \not\equiv 0, 1 \pmod{p}$ then the following groups are not capable:

$$H_1^7 = \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^3, yx = x^4y, zx = xyz, zy = yz \rangle,$$

$$H_2^7 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{p+1}yz, zy = x^pyz \rangle,$$

$$H_1^8 = \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^{-3}, yx = x^4y, zx = xyz, zy = yz \rangle,$$

$$H_2^8 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{dp+1}yz, zy = x^{dp}yz \rangle.$$

Keywords: Capable group; $p$–group; non-abelian $p$–groups; center.

1. Introduction

A group $H$ is said to be capable if there exists another group $G$ such that $\frac{G}{Z(G)} \cong H$, or equivalently $H$ can be represented as the inner automorphism group of a given group $G$. The capability of groups was first studied by Baer [1] who was asked the question “which conditions a group $H$ must fulfill in order to be the group of inner automorphisms of a group $G$?”. In the mentioned paper, he determined all capable groups which are direct products of cyclic groups. Since the time that

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Hall and Senior published their inovating work [3], such groups are called capable.
It is well-known that the classification of capable groups is the first step towards
the classification of prime power order groups [4]. The following theorem of Baer is
well-known in the context of capable groups.

**Theorem 1.1.** Let $A$ be a finite abelian group written as $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$
such that each integer $n_{i+1}$ is divisible by $n_i$. Then $A$ is capable if and only if $k \geq 2$
and $n_{k-1} = n_k$.

Burnside [2] was classified all $p-$groups of order $p^4$ which $p$ is an odd prime
number. This classification is expressed in the following theorem:

**Theorem 1.2.** Suppose $p$ is an odd prime number and $d \neq 0, 1 \pmod{p}$. Then
there are fifteen different groups of order $p^4$ up to isomorphisms. Five of those are
abelian and the non-abelian groups are in the list below.

\[
\begin{align*}
H_1 &= \langle x, y \mid x^3 = y^p = 1, xyx^{-1} = x^{p+1} \rangle, \\
H_2 &= \langle x, y, z \mid x^p = y^p = z^p = 1, xy = yz, xz = zx, xyz^{-1} = xz^p \rangle, \\
H_3 &= \langle x, y \mid x^2 = y^p = 1, xyx^{-1} = x^{p+1} \rangle, \\
H_4 &= \langle x, y, z \mid x^p = y^p = z^p = 1, xy = yz, xz =zx, xyz^{-1} = xz^p \rangle, \\
H_5 &= \langle x, y, z \mid x^2 = y^p = z^p = 1, xy = yz = zx, xz = yz^{-1} = xy \rangle, \\
H_6 &= \langle x, y, z \mid x^p = y^p = z^p = 1, xyx^{-1} = xz^p, yz^{-1} = xy, yz = zy \rangle, \\
H_7^1 &= \langle x, y, z \mid x^3 = y^3 = 1, [y, z] = 1, x^3, y^{-1} xy = x^4, z^{-1} xz = xy^{-1} \rangle, \\
H_7^2 &= \langle x, y, z \mid x^2 = y^p = z^p = 1, xyx^{-1} = z^{p+1}, xz^{-1} = xz^p, yz^{-1} = xz^p \rangle, \\
H_8^1 &= \langle x, y, z \mid x^3 = y^3 = 1, [y, z] = 1, x^3, y^{-1} xy = x^4, z^{-1} xz = xy^{-1} \rangle, \\
H_8^2 &= \langle x, y, z \mid x^2 = y^p = z^p = 1, xyx^{-1} = z^{p+1}, xz^{-1} = xz^p, yz^{-1} = xz^p \rangle, \\
H_9 &= \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, tz^{-1} = xz \rangle, \\
H_{10}^1 &= \langle x, y, z \mid x^3 = y^3 = 1, xy = yz, z^{-1} xz = xy, yz^{-1} = xz^{-1} \rangle, \\
H_{10}^2 &= \langle x, y, z \mid x^3 = y^3 = 1, xy = yz, z^{-1} xz = xy, yz^{-1} = xz^{-1} \rangle, \\
H_{10}^3 &= \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [t, y] x^{-1} = [t, z] y^{-1} = 1 \rangle \quad p > 3.
\end{align*}
\]

Zainal et al. [5] examined the capability of five groups out of ten non-abelian
groups of order $p^4$ and proved that among first five groups the previous theorem,
only the group number 3 is capable. We record this result in the following theorem:

**Theorem 1.3.** (See [5]) The groups $H_i, 1 \leq i \leq 5,$ is capable if and only if $i = 3.$


2. Main Results

Our aim in this section is to prove the groups numbers 6, 7 and 8 in Theorem 1.2
are not capable.

**Theorem 2.1.** The group $H_6$ is not capable.
Proof. By definition of $H_6$ and some calculations we have the following equations,

\begin{align}
y^j x^i &= x^{ijp^i} y^j, \\
z^k x^i &= x^{\frac{(n-1)}{2} k_i} y_i^k z^k.
\end{align}

We put $i = p$ and $j = k = 1$ in Equations 2.1 and 2.2. Since $p$ is odd and $x^{p^2} = y^p = 1, y x^p = x^p y$ and $z x^p = x^p z$. Thus $\langle x^p \rangle \subseteq Z(H_6)$ and $|Z(H_6)| = p$ or $p^2$. Suppose $|Z(H_6)| = p^2$. Then for every $h \in H_6 \setminus Z(H_6)$, $Z(H_6) \langle CH_6(h) \rangle \leq H_6$ and so $|CH_6(h)| = p^3$. This proves that the conjugacy class $hH_6$ has size $p$. Choose $j, k$ with this condition that $0 \leq j, k \leq p - 1$. Since $x$ is not central and by Equations 2.1 and 2.2,

\begin{align}
y^j x^i &= x^{ip^2 + 1} \quad \text{and} \quad z^k x^i &= x^{ip^2 + 1} y^i z^k,
\end{align}

we find that $|xH_6| > p$ which is not possible. Therefore $|Z(H_6)| = p$ and $Z(H_6) = \langle x^p \rangle$.

If $H_6$ is capable then there exists a non-abelian group $G$ with center $Z$ such that $H_6 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

\[
G = \left\langle aZ, bZ, cZ \mid (aZ)^p = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \right. \\
\left. (cZ)(aZ) = (aZ)(bZ)(cZ), (bZ)(cZ) = (cZ)(bZ) \right\rangle.
\]

By definition, $a^p, b^p, c^p \in Z$ and by Equation 2.1 one can see the following equation:

\[
(2.3) \quad ba^p = a^p b.
\]

By Equation 2.2 and some calculations, we have:

\[
(2.4) \quad (aZcZ)^n = (aZ)^{t_n}(aZ)^{n}(bZ)^{\frac{n(n-1)}{2}}(cZ)^n
\]

in which $t_n = \frac{n(n-1)(n-2)}{6}$. By substituting $n = p$ in Equation 2.4, we obtain the following equality:

\[
(2.5) \quad (aZcZ)^p = (aZ)^{t_p}(aZ)^p.
\]

We now consider two cases that $p = 3$ or $p > 3$.

1. $p > 3$. Then $p | t_p$ and so by Equation 2.5 and this fact that $a^p \in Z$,

\[
(ac)^p Z = (acZ)^p = (aZcZ)^p = (aZ)^{t_p}(aZ)^p = (aZ)^p = a^p Z.
\]

Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^p c$. Finally, we apply Equation 2.3 to conclude that $a^p \in Z$ which is a contradiction.
2. \( p = 3 \). Then \( t_p = 1 \) and by Equation 2.5, \((ac)^3Z = (aZcZ)^3 = (aZ)^3(aZ)^3 = (aZ)^6 = a^6Z\). Hence there exists \( z \in Z \) such that \((ac)^3 = a^6z\) and so \(ca^6 = a^6c\). By these equations and and Equation 2.3, we conclude that \(a^6 \in Z\) which is our final contradiction.

Therefore, the group \(H_6\) is not capable. \(\square\)

**Theorem 2.2.** The group \(H_7^1\) is not capable.

**Proof.** By definition of \(H_7^1\) and some tedious calculations, one can see that

\[\begin{align*}
y^j x^i &= x^{3ij + i} y^j \\
z^k x^i &= x^{4kij + i} y^j z^k
\end{align*}\]

We put \(i = 3\) and \(j = k = 1\) in Equations 2.6 and 2.7. Since \(x^3 = y^3 = 1\), \(y^ix^3 = x^3y\) and \(zz^3 = x^3z\) and so \((x^3) \in Z(H_7^1)\). On the other hand, \(|H_7^1| = 3^4\) and hence \(|Z(H_7^1)| = 3\) or \(|Z(H_7^1)| = 9\). Suppose \(|Z(H_7^1)| = 9\). Then for every \(h \in H_7^1 \setminus Z(H_7^1)\), \(Z(H_7^1)/C_{H_7^1}(h)\leq H_7^1\) which implies that \(|C_{H_7^1}(h)| = 3^3\) or equivalently \(|H_7^1| = 3^3\). Note that \(x \in H_7^1 \setminus Z(H_7^1)\). Choose \(j, k\) such that \(0 \leq j, k \leq 2\). By Equations 2.6 and 2.7, \(y^j x y^{-j} = x^{3j+1}\) and \(z^k x z^{-k} = x y^k\) which shows that \(|x H_7^1| > 3\). This contradiction implies that \(|Z(H_7^1)| = 3\) and \(Z(H_7^1) = (x^3)\). If \(H_7^1\) is capable, there is a non-abelian group \(G\) with center \(Z\) such that \(H_7^1 \cong \frac{G}{Z}\). Since \(G\) is not centerless, there are elements \(a, b, c \in G \setminus Z\) such that

\[\frac{G}{Z} = \left\langle aZ, bZ, cZ \mid (aZ)^3 = (bZ)^3 = (cZ)^3 = (aZ)(bZ)(cZ) = (aZ)^4(bZ), (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ)\right\rangle.
\]

Obviously \(a^9, b^3, c^9 \in Z\) and by Equation 2.6,

\[\begin{align*}
(aZbZ)^n &= (aZ)^{3(n-1)}(bZ)^n\nonumber\end{align*}\]

In above equation, we put \(n = 3\). Since \(a^9, b^3 \in Z\), \((ab)^3Z = (abZ)^3 = (aZbZ)^3 = (aZ)^3(bZ)^3 = (aZ)^4 = a^3Z\) and so there exists \(z \in Z\) such that \((ab)^3 = a^3z\). Therefore,

\[\begin{align*}
ba^3 &= a^3b
\end{align*}\]

On the other hand, \(a^3Z = c^3Z\) and so there exists \(z_1 \in Z\) such that

\[\begin{align*}
a^3 &= c^3z_1
\end{align*}\]

Put \(k = 1\) and \(i = 3\) in Equation 2.7. Since \(o(aZ) = 9\) and \(o(bZ) = 3\),

\[\begin{align*}
ca^3Z &= (cZ)(aZ)^3 \\
&= (aZ)^9(aZ)^3(bZ)^3(cZ) \\
&= (aZ)^3(cZ) \\
&= a^3cZ.
\end{align*}\]
Thus there exists $z_2 \in Z$ such that
\[
(2.10) \quad ca^3 = a^3 c z_2.
\]
Now by inserting the Equation 2.9 in 2.10, $cc^3 z_1 = c^3 z_1 c z_2$ which shows that $z_2 = 1$. Apply again Equation 2.10 to conclude that $ca^3 = a^3 c$. Now by Equation 2.8 $a^3 \in Z$ and hence $(aZ)^3 = Z$ which is our final contradiction. \(\blacksquare\)

**Theorem 2.3.** The group $H_2^z$ is not capable.

**Proof.** By presentation of $H_2^z$ and some tedious calculations one can see that
\[
(2.11) \quad y_j^i x_i = x_i^{ijp+i} y_j^i,
\]
\[
(2.12) \quad z_k^i x_i = x_i^{(i-1)p+ik} y_i^j z_k^i,
\]
\[
\quad z_k^i y_i^j = x_i^{jkp} y_i^j z_k^i.
\]
By substituting $i = p$ and $j = k = 1$ in Equations 2.11 and 2.12 we have $yx^p = x^p y$ and $zx^p = x^p z$. Hence $(x^p) \leq Z(H_2^z)$ and arguments similar to the proof of Theorem 2.1 show that $Z(H_2^z) = (x^p)$. If $H_2^z$ is capable, there is a non-abelian group $G$ with center $Z$ such that and $H_2^z \cong \frac{Q}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that
\[
\frac{G}{Z} = \left\{ aZ, bZ, cZ \mid (aZ)^p = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), 
\quad (cZ)(aZ) = (aZ)^{p+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^p(bZ)(cZ) \right\}.
\]
Thus $a^p, b^p, c^p \in Z$. Now by Equation 2.11 and a similar argument as Theorem 2.1,
\[
(2.13) \quad ba^p = a^pb.
\]
Apply Equation 2.12 to conclude that
\[
(aZcZ)^n = (aZ)^{k_n p}(aZ)^n (bZ)^{\frac{n(n-1)}{6}} (cZ)^n
\]
in which $k_n = \frac{n(n-1)(2n-1)}{6}$. Next we assume that $n = p$. Since $b^p, c^p$ are central,
\[
\quad (ac)^p Z = (acZ)^p = (aZcZ)^p = (aZ)^{k_p p}(aZ)^p (bZ)^{\frac{p(p-1)}{2}} (cZ)^p = (aZ)^{(k_p+1)p} = a^{(k_p+1)p} Z.
\]
Hence there exists $z \in Z$ such that
\[
(2.14) \quad (ac)^p = a^{(k_p+1)p} z.
\]
It is clear that $p \mid 6k_p$. Since $p > 3$, $p \mid k_p$ and so $p \nmid k_p + 1$. Since $(ac)^p (ac) = (ac)(ac)^p$, Equation 2.14 implies that $ca^{(k_p+1)p} = a^{(k_p+1)p} c$ and by Equation 2.13, $a^{(k_p+1)p} \in Z$. So, $(aZ)^{(k_p+1)p} \in Z$. But $o(aZ) = p^2$ and hence $p^2 \mid (k_p + 1)p$ which implies that $p \mid k_p + 1$. This contradiction completes the proof. \(\blacksquare\)
Theorem 2.4. The group $H_{18}^1$ is not capable.

Proof. By presentation of $H_{18}^1$ we have:

\begin{align}
    y^j x^i &= x^{3ij + j} y^j, \\
    z^k x^i &= x^{3k(n-1) + j} y^j z^k.
\end{align}

Again substitute $i = 3$ and $j = k = 1$ in Equations 2.15 and 2.16. Since $x^3 = y^3 = 1$, $yx^3 = x^3y$ and $xz^3 = x^3z$. Thus $(x^3) \leq Z(H_{18}^1)$. Similar to the proof of Theorem 2.2, $Z(H_{18}^1) = \langle x^3 \rangle$. If $H_{18}^1$ is capable, there is a non-abelian group $G$ with center $Z$ such that $H_{18}^1 \cong G / Z$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

\[
    G / Z = \left\langle aZ, bZ, cZ \mid (aZ)^9 = (bZ)^3 = 1, (cZ)^3 = (aZ)^{-3}, (bZ)(aZ) = (aZ)^4(bZ), (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ) \right\}.
\]

Obviously, $a^9, b^3, c^9 \in Z$ and by Equation 2.15,

\[
    (aZbZ)^n = (aZ)^{3(n-1)}(bZ)^n.
\]

Put $n = 3$. Since $a^9, b^3 \in Z$,

\[
    (ab)^3 Z = (abZ)^3 = (aZbZ)^3 = (aZ)^9(aZ)^3(bZ)^3 = (aZ)^3 = a^3 Z.
\]

Hence there exists $z \in Z$ such that $(ab)^3 = a^3 z$ and so

\[
    ba^3 = a^3 b.
\]

On the other hand, $c^3 Z = a^{-3} Z$ and so there exists $z_1 \in Z$ such that

\[
    a^3 = c^{-3} z_1.
\]

Since $o(aZ) = 9$ and $o(bZ) = 3$, by Equation 2.16 and substituting $k = 1$ and $i = 3$, we can see that

\[
    ca^3 Z = (caZ)(aZ)^3 = (aZ)^9(aZ)^3(bZ)^3(cZ) = (aZ)^3(cZ) = a^3 c Z
\]

and so there exists $z_2 \in Z$ such that

\[
    ca^3 = a^3 c z_2.
\]

We now insert Equation 2.18 in our last equation to deduce that $cc^{-3} z_1 = c^{-3} z_1 c z_2$. Thus $z_2 = 1$ and by Equation 2.19, $ca^3 = a^3 c$. Therefore, $a^3 Z$ and hence $9 = o(aZ) | 3$, which is impossible. This completes the proof. \qed

Theorem 2.5. The group $H_{18}^2$ is not capable.
Proof. By presentation of $H^2_0$ and some tedious calculations, we have

\begin{align}
(2.20) & \quad y^j x^i = x^{ip^j + i} y^j, \\
(2.21) & \quad z^k x^i = x^{\frac{d(n-1)}{nj} ip + k(n+1)} y^i z^k, \\
& \quad z^k y^j = x^{j kp^i + k} y^i z^k.
\end{align}

In Equations 2.20 and 2.21, we insert $i = p$ and $j = k = 1$. It is clear that $yx^p = x^p y$ and $zx^p = x^{p^2} z$ and so $\langle x^p \rangle \leq Z(H^2_0)$. Similar to Theorem 2.1, we can see that $Z(H^2_0) = \langle x^p \rangle$. If $H^2_0$ is capable, there is a non-abelian group $G$ with center $Z$ such that $H^2_0 \cong \frac{G}{Z}$. Since $G$ is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

\begin{equation}
G = \left\langle aZ, bZ, cZ \mid (aZ)^p = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), (cZ)(aZ) = (aZ)^{dp+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^{dp}(bZ)(cZ) \right\rangle,
\end{equation}

where $d \neq 0, 1 (\text{mod } p)$. It is obvious that $a^{p^2}, b^p, c^p \in Z$ and by Equations 2.20 and a similar argument used in the proof of the Theorem 2.1,

\begin{equation}
ba^p = a^pb.
\end{equation}

Moreover, by Equation 2.21,

\begin{equation}
(aZcZ)^n = (aZ)^{s_p}(aZ)^{t_p}(aZ)^n(bZ) \frac{n(n+1)}{d} (cZ)^n
\end{equation}

in which $s_n = \frac{n(n-1)(n+1)}{6}$ and $t_n = \frac{n(n-1)(n-2)}{6}$. It is easy to see that $p \mid s_n$ and $p \mid t_n$. Also by inserting $n = 1$ in Equation 2.23,

\begin{align*}
(ac)^p Z &= (acZ)^p = (aZcZ)^p \\
&= (aZ)^{s_p}(aZ)^{t_p}(aZ)^p(bZ) \frac{n(n+1)}{d} (cZ)^p \\
&= (aZ)^p = a^p Z.
\end{align*}

Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^pc$. This implies that $a^p \in Z$ and therefore $p^2 = o(aZ) \mid p$, which is our final contradiction. \qed

3. Concluding Remarks

In this paper the authors continued a recently published paper of Zainal et al. [5] in investigating finite $p$–groups of order $p^4$. It is proved that three non-abelian groups of this order are not capable. By results of [5] and our results to complete the classification of capable group of order $p^4$ it is enough to investigate the groups $H_9$ and $H_{10}$ in Theorem 1.2. Our calculations with computer algebra software GAP in working with small groups of order $p^4$ suggests the following conjecture:

Conjecture 3.1. The groups $H_9$ and $H_{10}$ are not capable.

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REFERENCES


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