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CONVERGENCE OF S-ITERATIVE METHOD TO A SOLUTION OF FREDHOLM INTEGRAL EQUATION AND DATA DEPENDENCY

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Abstract. The convergence of normal S-iterative method to solution of a nonlinear Fredholm integral equation with modified argument is established. The corresponding data dependence result has also been proved. An example in support of the established results is included in our analysis.
Key words: Fredholm equation, data dependency, Fixed-point theorem.

1. Introduction and Preliminaries

The past few decades have witnessed substantial developments in the field of integral equations and their applications have arisen in many areas, ranging from economics to engineering. Now it is an unquestionable fact that the theory of iterative approximation of fixed points plays a significant role in recent progress of integral equations and their applications. In this context, fixed point iterative methods for solving integral equations have already gained a splendid boost over the past few years (see, for example [1],[2],[4],[5],[7],[8],[16],[17],[19],[20]).

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In 2011, Sahu [23] introduced a normal S-iterative method as follows:

$$\left\{\begin{array}{l} x_0 \in X, \\ x_{n+1} = Ty_n, \\ y_n = (1-\xi_n)\, x_n + \xi_n T x_n, \quad n \in \mathbb{N} \end{array}\right. \tag{1.1}$$

where X is an ambient space, T is a self-map of X and $\{\xi_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ satisfying certain control condition(s).

It has been shown both analytically and numerically in [23] and [12] that the iterative method (1.1) converges faster than Picard [22], Mann [21], and Ishikawa [10] iterative processes in the sense of Berinde [3] for the class of contraction mappings.

This iterative method, due to its simplicity and fastness, has attracted the attention of many researchers and has been examined in various settings (see [9],[11],[13], [14],[15],[18],[24]).

In this paper, inspired by the above mentioned achievements of normal S-iterative method (1.1), we will use it to show that normal S-iterative method (1.1) converges strongly to the solution of the following integral equation which has been considered in [6]:

$$x(t) = \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))\, ds + f(t), \quad t \in [a,b], \tag{1.2}$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$, $h : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ and $f,\ x : [a,b] \to \mathbb{R}$.

Also we give a data dependence result for the solution of integral equation (1.2) with the help of normal S-iterative method (1.1).

We need the following pair of known results:

Theorem 1.1. [6] *Assume that the following conditions are satisfied:*

(A_1) $K \in C([a,b] \times [a,b])$;

(A_2) $h \in C\left([a,b] \times \mathbb{R}^3\right)$;

(A_3) $f,\ x \in C[a,b]$;

(A_4) there exist constants $\alpha,\ \beta,\ \gamma > 0$ such that

$$|h(s,u_1,u_2,u_3) - h(s,v_1,v_2,v_3)| \le \alpha |u_1 - v_1| + \beta |u_2 - v_2| + \gamma |u_3 - v_3|,$$

for all $s \in [a,b]$, $u_i,\ v_i \in \mathbb{R}$, $i = 1,2,3$;

(A_5) $M_K(\alpha+\beta+\gamma)(b-a) < 1$,

where M_K denotes a positive constant such that for all $t,s \in [a,b]$

$$|K(t,s)| \le M_K.$$

Then the equation (1.2) *has a unique solution $x^* \in C[a,b]$, which can be obtained by the successive approximations method starting with any element $x_0 \in C[a,b]$. Moreover, if x_n is the n-th successive approximation, then one has:*

$$|x_n - x^*| \le \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^n}{1 - M_K(\alpha+\beta+\gamma)(b-a)} \cdot |x_0 - x_1|.$$

Lemma 1.1. [25] *Let* $\{\beta_n\}_{n=0}^{\infty}$ *be a sequence of non negative numbers for which one assumes there exists* $n_0 \in \mathbb{N}$ *(set of natural numbers), such that for all* $n \geq n_0$

$$\beta_{n+1} \leq (1-\mu_n)\,\beta_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$, *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. *Then the following inequality holds:*

$$0 \leq \lim\sup_{n\to\infty} \beta_n \leq \lim\sup_{n\to\infty} \gamma_n.$$

2. Main Results

Theorem 2.1. *Assume that all the conditions* $(A_1)-(A_5)$ *in Theorem* 1.1 *are fulfilled. Let* $\{\xi_n\}_{n=0}^{\infty}$ *be a real sequence in* $[0,1]$ *satisfying* $\sum\limits_{n=0}^{\infty} \xi_n = \infty$. *Then equation* (1.2) *has a unique solution* $x^* \in C[a,b]$ *and normal S-iterative method* (1.1) *converges to* x^* *with the following estimate:*

$$\|x_{n+1} - x^*\| \leq \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^{n+1}}{e^{(1-M_K(\alpha+\beta+\gamma)(b-a))\sum_{k=0}^{n}\xi_k}} \|x_0 - x^*\|.$$

Proof. We consider the Banach space $B = (C[a,b], \|\cdot\|_C)$, where $\|\cdot\|_C$ is the Chebyshev's norm on $C[a,b]$, defined by $\|\cdot\|_C = \{\sup|x(t)| : t \in [a,b]\}$. Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by Normal-S iteration method (1.1) for the operator $T : B \to B$ defined by

$$T(x(t)) = \int_a^b K(t,s)\cdot h(s, x(s), x(a), x(b))\,ds + f(t),\ t \in [a,b]. \tag{2.1}$$

We will show that $x_n \to x^*$ as $n \to \infty$.

From (1.1), (2.1), and assumptions $(A_1)-(A_4)$, we have that

$$\begin{aligned}
|x_{n+1}(t) - x^*(t)| &= |T(y_n(t)) - T(x^*(t))| \\
&= \left| \int_a^b K(t,s)\cdot \left[\begin{array}{c} h(s, y_n(s), y_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{array} \right] ds \right| \\
&\leq \int_a^b |K(t,s)|\cdot \left| \begin{array}{c} h(s, y_n(s), y_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{array} \right| ds \\
&\leq M_K \int_a^b \left[\begin{array}{c} \alpha|y_n(s) - x^*(s)| + \beta|y_n(a) - x^*(a)| \\ +\gamma|y_n(b) - x^*(b)| \end{array} \right] ds,
\end{aligned}$$

$$\begin{aligned}
|y_n(t) - x^*(t)| &\leq (1-\xi_n)|x_n(t) - x^*(t)| + \xi_n |T(x_n)(t) - T(x^*)(t)| \\
&= (1-\xi_n)|x_n(t) - x^*(t)| \\
&\quad + \xi_n \left| \int_a^b K(t,s) \cdot \begin{bmatrix} h(s, x_n(s), x_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{bmatrix} ds \right| \\
&\leq (1-\xi_n)|x_n(t) - x^*(t)| \\
&\quad + \xi_n M_K \int_a^b \begin{bmatrix} \alpha |x_n(s) - x^*(s)| + \beta |x_n(a) - x^*(a)| \\ +\gamma |x_n(b) - x^*(b)| \end{bmatrix} ds
\end{aligned}$$

Now, by taking supremum in the above inequalities, we get

$$\|x_{n+1} - x^*\| \leq M_K(\alpha+\beta+\gamma)(b-a)\|y_n - x^*\|, \tag{2.2}$$

and

$$\|y_n - x^*\| \leq [1 - \xi_n(1 - M_K(\alpha+\beta+\gamma)(b-a))]\|x_n - x^*\|, \tag{2.3}$$

respectively.

Combining (2.2) with (2.3), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq M_K(\alpha+\beta+\gamma)(b-a)[1 - \xi_n(1 - M_K(\alpha+\beta+\gamma)(b-a))]\|x_n - x^*\|.
\end{aligned} \tag{2.4}$$

Thus, by induction, we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\| [M_K(\alpha+\beta+\gamma)(b-a)]^{n+1} \\
&\quad \times \prod_{k=0}^{n} [1 - \xi_k(1 - M_K(\alpha+\beta+\gamma)(b-a))].
\end{aligned} \tag{2.5}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}$, the assumption (A_5) yields

$$\xi_k(1 - M_K(\alpha+\beta+\gamma)(b-a)) < 1. \tag{2.6}$$

From the classical analysis, we know that $1 - x \leq e^{-x}$ for all $x \in [0,1]$. Hence by utilizing this fact with (2.6) in (2.5), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\| [M_K(\alpha+\beta+\gamma)(b-a)]^{n+1} \\
&\quad \times e^{-(1 - M_K(\alpha+\beta+\gamma)(b-a)) \sum_{k=0}^{n} \xi_k},
\end{aligned} \tag{2.7}$$

which yields $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

We now prove a closeness of solutions of integral equation (1.2) with the help of the normal-S iterative method (1.1).

We consider the following equation:

$$\widetilde{T}(\widetilde{x}(t)) = \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{x}(s), \widetilde{x}(a), \widetilde{x}(b))\, ds + g(t), t \in [a,b], \tag{2.8}$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$, $\widetilde{h} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$.

Now, we define the following normal-S iterative methods associated with T in (2.1) and $\widetilde{T}$ in (2.8), respectively:

$$\begin{cases} x_0 \in C[a,b], \\ x_{n+1} = \int_a^b K(t,s) \cdot h(s, y_n(s), y_n(a), y_n(b))\, ds + f(t), \\ y_n = (1-\xi_n) x_n \\ \qquad + \xi_n \int_a^b K(t,s) \cdot h(s, x_n(s), x_n(a), x_n(b))\, ds + f(t), t \in [a,b], n \in \mathbb{N}, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \widetilde{x}_0 \in C[a,b], \\ \widetilde{x}_{n+1} = \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{y}_n(s), \widetilde{y}_n(a), \widetilde{y}_n(b))\, ds + g(t), \\ \widetilde{y}_n = (1-\xi_n) \widetilde{x}_n \\ \qquad + \xi_n \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{x}_n(s), \widetilde{x}_n(a), \widetilde{x}_n(b))\, ds + g(t), t \in [a,b], n \in \mathbb{N}, \end{cases} \tag{2.10}$$

where $\{\xi_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$, $K : [a,b] \times [a,b] \to \mathbb{R}$, h, $\widetilde{h} : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ and f,$g : [a,b] \to \mathbb{R}$.

Theorem 2.2. *Consider the sequences* $\{x_n\}_{n=0}$ *and* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *generated by* (2.9) *and* (2.10)*, respectively, with the real sequence* $\{\xi_n\}_{n=0}^{\infty}$*in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_n$ *for all* $n \in \mathbb{N}$*. Assume that:*

(i) all the conditions of Theorem 2.1 hold and x^* *and* $\widetilde{x}^*$ *are solutions of equations* (2.1) *and* (2.8)*, respectively;*

(ii) there exist non negative constants ε_1 *and* ε_2 *such that* $\left| h(s,u,v,w) - \widetilde{h}(s,u,v,w) \right| \leq \varepsilon_1$ *and* $|f(t) - g(t)| \leq \varepsilon_2$*, for all* $t, s \in [a,b]$*,* $u, v, w \in \mathbb{R}$*.*

If the sequence $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *converge to* $\widetilde{x}^*$*, then we have*

$$\|x^* - \widetilde{x}^*\| \leq \frac{3\left[M_K(b-a)\varepsilon_1 + \varepsilon_2\right]}{1 - M_K(\alpha+\beta+\gamma)(b-a)}. \tag{2.11}$$

Proof. Using (1.1), (2.1), (2.8)-(2.10), and assumptions (A_1)-(A_4) and (ii), we obtain

$$\begin{aligned} |x_{n+1}(t) - \widetilde{x}_{n+1}(t)| &= \left| T(y_n)(t) - \widetilde{T}(\widetilde{y}_n)(t) \right| \\ &= \left| \int_a^b K(t,s) \cdot h(s, y_n(s), y_n(a), y_n(b))\, ds + f(t) \right. \end{aligned}$$

$$
\begin{aligned}
&-\int_a^b K(t,s)\cdot \widetilde{h}\left(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)\right)ds-g(t)\Bigg| \\
\leq\ & \left|\int_a^b K(t,s)\cdot\left[\begin{array}{c} h\left(s,y_n(s),y_n(a),y_n(b)\right) \\ -\widetilde{h}\left(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)\right)\end{array}\right]ds\right| \\
& +|f(t)-g(t)| \\
\leq\ & M_K\int_a^b\left(\begin{array}{c}\left|\begin{array}{c} h\left(s,y_n(s),y_n(a),y_n(b)\right) \\ -h\left(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)\right)\end{array}\right| \\ +\left|\begin{array}{c} h\left(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)\right) \\ -\widetilde{h}\left(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)\right)\end{array}\right|\end{array}\right)ds \\
& +|f(t)-g(t)| \\
\leq\ & M_K\int_a^b\left(\begin{array}{c}\alpha\left|y_n(s)-\widetilde{y}_n(s)\right| \\ +\beta\left|y_n(a)-\widetilde{y}_n(a)\right|+\gamma\left|y_n(b)-\widetilde{y}_n(b)\right|+\varepsilon_1\end{array}\right)ds \\
& +\varepsilon_2 \\
\leq\ & M_K\int_a^b\left(\begin{array}{c}\alpha\left|y_n(s)-\widetilde{y}_n(s)\right| \\ +\beta\left|y_n(a)-\widetilde{y}_n(a)\right|+\gamma\left|y_n(b)-\widetilde{y}_n(b)\right|\end{array}\right)ds \\
& +M_K\int_a^b \varepsilon_1 ds+\varepsilon_2,
\end{aligned}
$$

$$
\begin{aligned}
\left|y_n(t)-\widetilde{y}_n(t)\right| \leq\ & (1-\xi_n)\left|x_n(t)-\widetilde{x}_n(t)\right|+\xi_n\left|T(x_n)(t)-\widetilde{T}(\widetilde{x}_n)(t)\right| \\
\leq\ & (1-\xi_n)\left|x_n(t)-\widetilde{x}_n(t)\right| \\
& +\xi_n M_K\int_a^b\left(\begin{array}{c}\left|\begin{array}{c} h\left(s,x_n(s),x_n(a),x_n(b)\right) \\ -h\left(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)\right)\end{array}\right| \\ +\left|\begin{array}{c} h\left(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)\right) \\ -\widetilde{h}\left(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)\right)\end{array}\right|\end{array}\right)ds \\
& +\xi_n\left|f(t)-g(t)\right| \\
\leq\ & (1-\xi_n)\left|x_n(t)-\widetilde{x}_n(t)\right| \\
& +\xi_n M_K\int_a^b\left(\begin{array}{c}\alpha\left|x_n(s)-\widetilde{x}_n(s)\right| \\ +\beta\left|x_n(a)-\widetilde{x}_n(a)\right|+\gamma\left|x_n(b)-\widetilde{x}_n(b)\right|+\varepsilon_1\end{array}\right)ds \\
& +\xi_n\varepsilon_2.
\end{aligned}
$$

Now, by taking supremum in the above inequalities, we get

$$
\begin{aligned}
\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| \leq\ & M_K(\alpha+\beta+\gamma)(b-a)\left\|y_n-\widetilde{y}_n\right\| \\
& +M_K(b-a)\varepsilon_1+\varepsilon_2,
\end{aligned} \tag{2.12}
$$

and

$$\|y_n - \widetilde{y}_n\| \leq [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| + \xi_n M_K (b - a) \varepsilon_1 + \xi_n \varepsilon_2, \tag{2.13}$$

respectively.

Combining (2.12) with (2.13) and using assumptions (A_5) and $\frac{1}{2} \leq \xi_n$ for all $n \in \mathbb{N}$ in the resulting inequality, we get

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_{n+1}\| &\leq [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| \\ &\quad + \xi_n M_K (b - a) \varepsilon_1 + \xi_n \varepsilon_2 + 2\xi_n M_K (b - a) \varepsilon_1 + 2\xi_n \varepsilon_2 \\ &= [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| \\ &\quad + \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a)) \\ &\quad \times \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}. \end{aligned} \tag{2.14}$$

Denote by

$$\begin{aligned} \beta_n &= \|x_n - \widetilde{x}_n\|, \\ \mu_n &= \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a)) \in (0, 1), \\ \gamma_n &= \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)} \geq 0. \end{aligned}$$

The assumption $\frac{1}{2} \leq \xi_n$ for all $n \in \mathbb{N}$ implies $\sum_{n=0}^{\infty} \xi_n = \infty$. Now it can be easily seen that (2.14) satisfies all the conditions of Lemma 1.1. Hence it follows by its conclusion that

$$0 \leq \limsup_{n \to \infty} \|x_n - \widetilde{x}_n\| \leq \limsup_{n \to \infty} \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}.$$

By (i), we have that $\lim_{n \to \infty} x_n = x^*$. Using this fact and the assumption $\lim_{n \to \infty} \widetilde{x}_n = \widetilde{x}^*$, we get

$$\|x^* - \widetilde{x}^*\| \leq \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}.$$

□

Remark 2.1. The result given in Theorem 2.2 relate the solutions of equations (2.1) and (2.8) in the sense that if f is close to g and h is close to $\widetilde{h}$, then not only the solutions of equations (2.1) and (2.8) are close to each other, but also depend continuously on the functions involved therein. Further, if $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, then the solution x^* of equation (2.1) tends the solution $\widetilde{x}^*$ of the equation (2.8).

Example 2.1. Consider the following integral equation

$$x(t) = \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x(s)}{2} + \frac{x(0)+x(1)}{3}\right] ds + \frac{t+e^{-t}}{3}, t \in [0,1].$$

where $K \in C([0,1]\times[0,1])$, $K(t,s) = \frac{3t-2s}{5}$, $h \in C\left([0,1]\times\mathbb{R}^3\right)$, $h(s,u,v,w) = \frac{s-\sin u}{2} + \frac{v+w}{3}$, $f \in C[0,1]$, $f(t) = \frac{t+e^{-t}}{3}$, $x \in C[0,1]$ and its perturbed integral equation

$$\widetilde{x}(t) = \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin \widetilde{x}(s)}{2} + \frac{\widetilde{x}(0)+\widetilde{x}(1)}{3} - s + \frac{1}{7}\right] ds + \frac{t+2e^{-t}}{3}, t \in [0,1],$$

where $K \in C([0,1]\times[0,1])$, $K(t,s) = \frac{3t-2s}{5}$, $k \in C\left([0,1]\times\mathbb{R}^3\right)$, $k(s,u,v,w) = \frac{s-\sin u}{2} + \frac{v+w}{3} - s + \frac{1}{7}$, $g \in C[0,1]$, $g(t) = \frac{t+2e^{-t}}{3}$, $\widetilde{x} \in C[0,1]$.

Define the operator $T : C[0,1] \to C[0,1]$ by

$$T(x(t)) = \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x(s)}{2} + \frac{x(0)+x(1)}{3}\right] ds + \frac{t+e^{-t}}{3}, t \in [0,1].$$

We now show that the operator T is a contraction with contractivity factor $\frac{7}{10}$. Indeed,

$$\begin{aligned}
& |T(x_1(t)) - T(x_2(t))| \\
= & \left|\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x_1(s)}{2} + \frac{x_1(0)+x_1(1)}{3} - \frac{s-\sin x_2(s)}{2} - \frac{x_2(0)+x_2(1)}{3}\right] ds\right| \\
\leq & \left|\int_0^1 \left|\frac{3t-2s}{5}\right| \left|\frac{s-\sin x_1(s)}{2} + \frac{x_1(0)+x_1(1)}{3} - \frac{s-\sin x_2(s)}{2} - \frac{x_2(0)+x_2(1)}{3}\right| ds\right| \\
\leq & \left|\int_0^1 \left|\frac{3t-2s}{5}\right| \left[\frac{1}{2}|\sin x_1(s) - \sin x_2(s)| + \frac{1}{3}|x_1(0)-x_2(0)| + \frac{1}{3}|x_1(1)-x_2(1)|\right] ds\right|.
\end{aligned}$$

Now using the Chebyshev norm, we obtain

$$\begin{aligned}
\|Tx_1 - Tx_2\| & \leq \sup_{t,s\in[0,1]}\left|\frac{3t-2s}{5}\right|\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right)(1-0)\|x_1-x_2\| \\
& = \frac{7}{10}\|x_1-x_2\|.
\end{aligned}$$

One can easily show on the same lines as above that the mapping $\widetilde{T} : C[0,1] \to C[0,1]$ defined by

$$\widetilde{T}(\widetilde{x}(t)) = \int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin \widetilde{x}(s)}{2} + \frac{\widetilde{x}(0)+\widetilde{x}(1)}{3} - s + \frac{1}{7}\right] ds + \frac{t+2e^{-t}}{3}, t \in [0,1],$$

is also a contraction with contractivity factor $\frac{7}{10}$.

Since all the conditions of Theorem 2.1 are satisfied by the integral equations (2.1) and (2.8) so by its conclusion, normal S-iterative method (1.1) converges to unique solution x^* and $\widetilde{x}^*$, respectively in $C[0,1]$.

Now we have the following estimates:

$$|K(t,s)| = \left|\frac{3t-2s}{5}\right| \leq \frac{3}{5} = M_K, t, s \in [0,1],$$

$$|h(s,u,v,w) - k(s,u,v,w)| = \left|s - \frac{1}{7}\right| \leq \frac{1}{7} = \varepsilon_1, foralls \in [0,1], u, v, w \in \mathbb{R},$$

$$|f(t) - g(t)| = \left|\frac{t + e^{-t} - t - 2e^{-t}}{3}\right| = \frac{e^{-t}}{3} \leq \frac{1}{3} = \varepsilon_2, s \in [0,1].$$

In view of the above estimates, all the conditions of Theorem 2.2 are satisfied and hence from (2.11), we have

$$\|x^* - \widetilde{x}^*\| \leq \frac{88}{21}.$$

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