A SURVEY ON THE AUTOMORPHISM GROUPS OF THE COMMUTING GRAPHS AND POWER GRAPHS

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Abstract. Let $G$ be a finite group. The power graph $P(G)$ of a group $G$ is the graph whose vertex set is the set of group elements where two elements are adjacent if one is a power of the other. The commuting graph $\Delta(G)$ of a group $G$, is the graph whose vertices are the group elements, two of them are joined if they commute. When the vertex set is $G \setminus Z(G)$, this graph is denoted by $\Gamma(G)$. Since the results based on the automorphism groups of these kinds of graphs are so sporadic, in this paper, we give a survey of all results on the automorphism groups of power graphs and commuting graphs obtained in the literature.

Keywords. Finite group; graph; vertex set; commuting graph; automorphism groups.

1. Introduction

There are many connections between graphs and groups. Generating graphs from semigroups and groups has a long history. In 1964, Bosak [6] studied a certain graph over semigroups. In [13], Zelinka studied the intersection graphs of nontrivial subgroups of finite Abelian groups. The well-known study of a directed graphs defined on the elements of a group is the Cayley digraph [7, 22, 40]. The investigation of graphs like these is very important, because they have valuable and numerous applications presented, for example, in the books [27], [28] and [29]. The directed power graph of a group was introduced by Kelarev and Quinn [24]. The definition was formulated so that it applied to semigroups as well. Accordingly, the power graphs of semigroups were first considered in [25], [23] and [26]. It is also explained in the survey [2] that the definition given in [24] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [24] (see [2] for more detailed explanations). All of these papers used only the brief term ’power graph’, even though they covered both directed and undirected power graphs. Kelarve and Quinn [23] defined another interesting classes of directed graphs, namely,
the divisibility graphs of semigroups. Let S be a semigroup, the divisibility graph, \( \text{Div}(S) \), of a semigroup S is a directed graph with vertex set S and there is an arc from u to v if and only if \( u \neq v \) and \( u \mid v \), i.e., the ideal generated by v contains u. On the other hand, the power graph, \( \overrightarrow{P}(S) \), of a semigroup S is a directed graph in which the set of vertices is again S and for \( a, b \in S \) there is an arc from a to b if and only if \( a \neq b \) and \( b = a^m \) for some positive integer m.

The undirected power graph \( P(S) \) was also considered by Chakrabarty, Ghosh and Sen in [11]. Recall that \( P(S) \) has vertex set S and two vertices \( a, b \in S \) are adjacent if and only if \( a \neq b \) and \( \langle a \rangle \subseteq \langle b \rangle \) or \( \langle b \rangle \subseteq \langle a \rangle \) (which is equivalent to saying \( a \neq b \) and \( a^m = b \) or \( b^m = a \) for some positive integer m). As a consequence, they proved that \( P(G) \) is connected for any finite group G and \( P(G) \) is complete if and only if G is a cyclic group of order 1 or \( p^m \) [11].
The undirected power graphs became the main focus of study in [11] and in the subsequent papers by P. J. Cameron et al. [8, 9], which introduced the use of the brief term ‘power graph’ in the second meaning of an undirected power graph. For a group $G$, the digraph $P(G)$ was considered in [37] as the main subject of study. The interested readers can be consulted [2, 32, 1] for more information about the power graphs. In this paper, we are also interested in the well-known commuting graphs and their automorphism groups. Let $G$ be a non-abelian group and let $Z(G)$ be the center of $G$. Associate a graph $\Gamma(G)$ with $G$ as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma(G)$ and join two distinct vertices $x$ and $y$, whenever $xy = yx$. The complement of the $\Gamma(G)$ is said to be the noncommuting graph. The noncommuting graph was first considered by Paul Erdős, when he posed the following problem in 1975 [36]: Let $G$ be a group whose noncommuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of the noncommuting graph of $G$? B. H. Neumann [36] answered positively Erdős’ question. We refer the readers to [3, 4, 14, 35, 31] for more details about the noncommuting graph. In [1], authors related the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices $x$ and $y$ are adjacent if they generate a cyclic group. They called this graph as the enhanced power graph of $G$. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We consider the commuting graph with vertex set $G$ and denoted it by $\Delta(G)$. 

Figure 2. The undirected power graph of the dihedral group $D_8$. 

Figure 3. The commuting graph $\Delta(D_8)$. 

2. Preliminaries and background information

An action of a group $G$ on a set $X$ is the choice, for each $g \in G$ of a permutation $\pi_g : X \to X$ such that the following two conditions hold:

1. $\pi_e$ is the identity: $\pi_e(x) = x$ for each $x \in X$,
2. for every $g_1, g_2$ in $G$, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

For example, any group $G$ acts on itself $(X = G)$ by left multiplication functions. A group action of $G$ on $X$ is said to be faithful if different elements of $G$ act on $X$ in different ways: when $g_1 \neq g_2$ in $G$, there is an $x \in X$ such that $g_1 x \neq g_2 x$. For any graph $\Gamma$, we denote the sets of the vertices and the edges of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $v \in V(\Gamma)$ and $V_1(\Gamma) \subseteq V(\Gamma)$, then $N(v)$ is the set of neighbours of $v$ and $(V_1(\Gamma))$ is the subgraph of $\Gamma$ induced by $V_1(\Gamma)$. The closed neighbourhood of a vertex $x$, denoted by $N[x]$, is the set of its neighbours and itself. The complement of $\Gamma$ is the graph $\bar{\Gamma}$ on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in $\Gamma$. For two graphs with disjoint vertex sets $V_1$ and $V_2$ their union is the graph $\Gamma$ in which $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$. Define $nH$ to be the union of $n$ disjoint copies of $G$. The automorphism group of a graph $\Gamma$ is that set of all permutations on $V(\Gamma)$ that fix as a set the edges $E(\Gamma)$. The set of all automorphisms of a graph $\Gamma$ forms a permutation group, $\text{Aut}(\Gamma)$, acting on the object set $V(\Gamma)$. See [10] for the terminology and main results of permutation group theory. Let $A$ and $B$ be permutation groups acting on object sets $X$ and $Y$, respectively. Define $B \wr A = \{(a, f) \mid a \in A, f : X \to B\}$, $(a, f)(x, y) = (ax, b_2 y)$ where $f(x) = b_x$. $B \wr A$ is said to be wreath product. It acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \ldots, b_n$ (where $n = |X|$) in $B$, there is a unique permutation in $A \wr B$ written $(a; b_1, \ldots, b_n)$, and $(a; b_1, \ldots, b_n)(x_1, y_1, \ldots, y_n) = (ax_1, b_2 y_1, \ldots, b_n y_n)$. Suppose $S_n$ denotes the symmetric group on $\{1, 2, \ldots, n\}$, $\varphi$ is the Euler’s totient function. In what follows, we describe some important results relating the automorphism groups of a graph which are crucial in this paper. Frucht [18] described if $\Gamma$ is a connected graph, then $\text{Aut}(n \Gamma) \cong (\text{Aut}(\Gamma)) \wr S_n$, if no component of $\Gamma_1$ is isomorphic with a component of $\Gamma_2$, then $\text{Aut}(\Gamma_1 \cup \Gamma_2) \cong \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2)$ and applying the last two theorems we have the result: Let $\Gamma = n_1 \Gamma_1 \cup n_2 \Gamma_2 \cup \cdots \cup n_r \Gamma_r$, where $n_i$ is the number of components of $\Gamma$ isomorphic to $\Gamma_i$, then

$$\text{Aut}(\Gamma) \cong ((\text{Aut}(\Gamma_1)) \wr S_{n_1}) \times ((\text{Aut}(\Gamma_2)) \wr S_{n_2}) \times \cdots \times ((\text{Aut}(\Gamma_r)) \wr S_{n_r}).$$

An operation $\cdot$ on the set $S$ is associative if it satisfies the following associative law: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$. A semigroup is a set $S$ equipped with an associative binary operation $\cdot$. The set of the orders of all elements of $G$ is denoted by $\pi_e(G)$ and is said to be the spectrum of $G$. For $n \in \mathbb{N}$, the cyclic group of order $n$ can be defined as the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of residues modulo $n$, the set $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ is the cyclic group generated by $g$ in $G$. For a prime $p$, a group $G$ is said to be an elementary abelian $p$-group if $G$ is finite, abelian and
every nontrivial element of $G$ has order $p$. A group $G$ is an AC-group, whenever the centralizers of non-central elements are abelian. The dihedral group $D_{2n}$ is an example of an AC-group. The group $G$ is said to be an EPPO-group, if all elements of $G$ have prime power order.

3. Automorphism groups of power graphs

The first result about the automorphism groups of power graphs was obtained by P. Cameron in [8], where he explained that when the automorphism group and its graph are equal. P. Cameron proved the only finite group $G$ for which $\text{Aut}(G) = \text{Aut}(P(G))$ is the Klein group $Z_2 \times Z_2$.

In 2013, Doostabadi, Erfanian and Jafarzadeh asserted that the full automorphism group of the power graph of the cyclic group $Z_n$ is isomorphic to the direct product of some symmetry groups.

**Conjecture 3.1.** [16] For every positive integer $n$,

$$\text{Aut}(P(Z_n)) \cong S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1, n\}} S_{\varphi(d)}$$

where $D(n)$ is the set of positive divisors of $n$, and $\varphi$ is the Euler’s totient function.

In fact, if $n$ is a prime power, then $P(Z_n)$ is a complete graph by [11] which implies that $\text{Aut}(P(Z_n)) \cong S_n$. Hence, the conjecture does not hold if $n = p^m$ for any prime $p$ and integer $m > 2$. In [17], proved that this conjecture holds for the remaining case. Feng, Ma and Wang [17], describe the full automorphism group of the power (di)graph of an arbitrary finite group. As an application, this conjecture is valid if $n$ is not a prime power. Denote by $C(G)$ the set of all cyclic subgroups of $G$. For $C \in C(G)$, let $[C]$ denote the set of all generators of $C$. Write

$$C(G) = \{C_1, \ldots, C_k\} \text{ and } [C_i] = \{[C_i]_1, \ldots, [C_i]_{s_i}\}.$$  

Define $P(G)$ as the set of permutations $\sigma$ on $C(G)$ preserving order, inclusion and noninclusion, i.e., $[C_i^\sigma] = [C_i]$ for each $i \in \{1, \ldots, k\}$ and $C_i \subseteq C_j$ if and only if $C_i^\sigma \subseteq C_j^\sigma$. Note that $P(G)$ is a permutation group on $C(G)$. This group induces the faithful action on the set $G$:

$$(3.1) \quad G \times P(G) \rightarrow G, \quad ([C_i]_j, \sigma) \mapsto [C_i^\sigma]_j.$$  

For $\Omega \subseteq G$, let $S_{\Omega}$ denote the symmetric group on $\Omega$. Since $G$ is the disjoint union of $[C_1], \ldots, [C_k]$, we get the faithful group action on the set $G$:

$$(3.2) \quad G \times \prod_{i=1}^{k} S_{[C_i]} \rightarrow G, \quad ([C_i]_j, (\xi_1, \ldots, \xi_k)) \mapsto ([C_i]_j)^{\xi_i}.$$  

By using the above-mentioned symbols we have:
Theorem 3.1. [17] Let $G$ be a finite group. Then

$$Aut(\overrightarrow{P}(G)) = \left(\prod_{i=1}^{k} S_{C_i}\right) \times P(G),$$

where $P(G)$ and $\prod_{i=1}^{k} S_{C_i}$ act on $G$ as in (3.1) and (3.2), respectively.

In the power graph $P(G)$, for $x, y \in G$, define $x \equiv y$ if $N[x] = N[y]$. Observe that $\equiv$ is an equivalence relation. Let $\bar{x}$ denote the equivalence class containing $x$. Write $U(G) = \{\bar{x} | x \in G\} = \{\bar{u}_1, \ldots, \bar{u}_l\}$.

Since $G$ is the disjoint union of $u_1, \ldots, u_l$, the following is a faithful group action on the set $G$:

(3.3) \quad G \times \prod_{i=1}^{l} S_{\bar{u}_i} \rightarrow G, \quad (x, (\tau_1, \tau_2, \ldots, \tau_l)) \mapsto x^{\tau_i}, \text{ where } x \in \bar{u}_i.

Similar to the last theorem, for the automorphism groups of undirected power graphs we have:

Theorem 3.2. [17] Let $G$ be a finite group. Then

$$Aut(P(G)) = \left(\prod_{i=1}^{l} S_{\bar{u}_i}\right) \times P(G),$$

where $P(G)$ and $\prod_{i=1}^{l} S_{\bar{u}_i}$ act on $G$ as in (3.1) and (3.3), respectively.

By combining Theorems 3.1 and 3.2, the authors in [17], obtained that $Aut(P(G)) = Aut(\overrightarrow{P}(G))$ if and only if $x = [x]$ for each $x \in G$. Indeed, this result demonstrates relationship between power graphs and directed power graphs.

A graph $\Gamma$ is said to be a subgraph of another graph $\Delta$ (or $\Delta$ is a supergraph of $\Gamma$), if $V(\Gamma) \subset V(\Delta)$ and $E(\Gamma) \subset E(\Delta)$. Hamzeh and Ashrafi [19] defined the main supergraph $S(G)$ of $P(G)$ with the vertex set $G$ and two elements $x, y \in G$ are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$ and proved that there is not a group $G$, such that $Aut(S(G)) = Aut(G)$. In what follows, $\Omega_{a_i}(G) = |\{y \mid o(y) = a_i\}|$. Authors in [19] also define the graph $\Delta$ with vertex set $V(\delta) = \pi_e(G)$ and two vertices $a_i$ and $a_j$ are adjacent if and only if $a_i \mid a_j$ or $a_j \mid a_i$.

Theorem 3.3. [19] Let $G$ be a finite group with spectrum $\pi_e(G) = \{a_1, \ldots, a_k\}$ and choose a representative set $\{t_1, t_2, \ldots, t_k\}$, where for each $i$, $1 \leq i \leq k, ti \in K_{\Omega_{a_i}}(G)$. Then,

1. If deg($t_i$)'s are distinct then $Aut(S(G)) = S_{\Omega_{a_1}}(G) \times \cdots \times S_{\Omega_{a_k}}(G)$.
2. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), any two distinct vertices of \( K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G) \) are adjacent and \( N_{\Delta}[a_{i_1}] = \cdots = N_{\Delta}[a_{i_r}] \) then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{i_1}}}(G) \cdots S_{\Omega_{a_{i_r}}}(G) \).

3. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), all vertices of \( K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G) \) are adjacent and \( N_{\Delta}[a_{i_1}] \)'s are distinct then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G) \).

4. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), \( N_{\Delta}[a_{i_1}] = \cdots = N_{\Delta}[a_{i_r}] \) and for each two \( m, n, 1 \leq m, n \leq r \), \( K_{\Omega_{a_{im}}}(G) \) and \( K_{\Omega_{a_{in}}}(G) \) are disjoint then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G) \).

5. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), \( N_{\Delta}[a_{i_1}] \)'s are distinct and for each \( m, n, 1 \leq m, n \leq r \), \( K_{\Omega_{a_{im}}}(G) \) and \( K_{\Omega_{a_{in}}}(G) \) are disjoint then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G) \).

6. \( \text{Aut}(S(G)) = A_1 \times \cdots \times A_q \), where \( A_i, 1 \leq i \leq q \), are subgroups appeared in Cases (2–5).

In [20], Theorem 2.8, it is proved that if \( G \) is an EPPO-group of order \( p_1^{n_1} \cdots p_k^{n_k} \) and \( V_i = \{ 1 \neq g \in G \mid o(g)p_i^{n_i} \} \) then \( S(G) = K_1 + (\bigcup_{i=1}^{k} K_{V_i}) \). The authors applied the structure of \( S(G) \) to determine its automorphism.

**Theorem 3.4.** [19] Let \( G \) be a finite group and \( e_1, \cdots, e_t \) are distinct values of \( |V_1|, \cdots, |V_k| \). Define \( B_i = \{|V_j| \mid |V_j| = e_i\} \). Then,

\[
\text{Aut}(S(G)) = (S_{|V_1| \backslash B_1}) \times \cdots \times (S_{|V_k| \backslash B_k}).
\]

Suppose \( G \) is a finite group and \( C(G) = \{C_1, \cdots, C_k\} \) is the set of all cyclic subgroups of \( G \). Define \( L_G \) to be the graph with vertex set \( C(G) \) in which two cyclic subgroups \( C_i \) and \( C_j \) are adjacent if one is contained in the other or there is a cyclic subgroup \( C_k \) such that \( C_i \subseteq C_k \) and \( C_j \subseteq C_k \). It is clear that the subgraphs of \( L(G) \) induced by a cyclic subgroup are complete. So, \( P(G) = W_G[K_{b_1}, K_{b_2}, \cdots, K_{b_k}] \) with \( b_i = \varphi(|C_i|) \).

**Theorem 3.5.** [19] Let \( G \) be a finite group with \( C(G) = \{C_1, \cdots, C_k\} \) and choose a representative set \( \{t_1, t_2, \cdots, t_k\} \), where for each \( i, 1 \leq i \leq k, ti \in K_{b_i} \). Then,

1. If \( \text{deg}(t_i) \)'s are distinct then \( \text{Aut}(P(G)) = S_{b_1} \times \cdots \times S_{b_k} \).

2. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), any two distinct vertices of \( K_{b_{i_1}}, \cdots, K_{b_{i_r}} \) are adjacent and \( N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}] \) then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times \cdots \times S_{b_{i_r}} \).
3. If \( \deg(t_{i_1}) = \cdots = \deg(t_{i_r}) \), all vertices of \( K_{b_{i_1}}, \cdots, K_{b_{i_r}} \) are adjacent and \( N_{WG}[C_{i_1}] \)'s are distinct then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times \cdots \times S_{b_{i_r}} \).

4. If \( \deg(t_{i_1}) = \cdots = \deg(t_{i_r}) \), \( N_{WG}[C_{i_1}] = \cdots = N_{WG}[C_{i_r}] \) and for each two \( m, n, 1 \leq m, n \leq r \), \( K_{b_{i_m}} \) and \( K_{b_{i_n}} \) are disjoint then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times S_r \).

5. If \( \deg(t_{i_1}) = \cdots = \deg(t_{i_r}) \), \( N_{WG}[C_{i_1}] \)'s are distinct and for each \( m, n, 1 \leq m, n \leq r \), \( K_{b_{i_m}} \) and \( K_{b_{i_n}} \) are disjoint then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times \cdots \times S_{b_{i_r}} \).  

6. \( \text{Aut}(P(G)) = A_1 \times \cdots \times A_q \), where \( A_i, 1 \leq i \leq q \), are subgroups appeared in Cases (2–5).

### 3.1. Examples

In this section, we present \( \text{Aut}(P(G)) \) and \( \text{Aut}(\overline{P}(G)) \) for some families of finite groups such as \( Z_n, Z_p^n, D_{2n}, Q_{4n}, U_{6n}, V_{8n} \) and so on. These results obtained in several papers in different ways. In [5], the authors used the graph structure from [30] and computed the automorphism groups of \( P(G) \) for the above groups. In [17], the authors by using Theorem 3.1 and Theorem 3.2, computed the automorphism groups of \( P(G) \) and \( \overline{P}(G) \) for these groups. In [19], authors obtained these results from Theorem 3.3.

**Example 3.1.** [17] If \( n \) be a positive integer then,

\[
\text{Aut}(\overline{P}(Z_n)) \cong \prod_{d \in D(n)} S_{\phi(d)},
\]

\[
\text{Aut}(P(Z_n)) \cong \begin{cases} 
S_n & \text{if } n \text{ is a prime power} \\
S_{\phi(n)+1} \times \prod_{d \in D(n) \setminus \{1, n\}} S_{\phi(d)} & \text{otherwise}
\end{cases},
\]

and if \( n \geq 2 \) then,

\[
\text{Aut}(P(Z^n_p)) = \text{Aut}(\overline{P}(Z^n_p)) \cong S_{p-1} \wr S_m,
\]

where \( m = \frac{n+1}{p-1} \) and \( Z^n_p \) denote the elementary abelian \( p \)-group.

In the [21, 15], the dihedral group \( D_{2n} \), semi-dihedral group \( SD_{2n} \), generalized quaternion group of \( Q_{4n} \), semidihedral groups \( SD_{8n} \) are defined by the following presentations:

\[
D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle,
\]

\[
SD_{2n} = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle,
\]

\[
Q_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle,
\]

\[
U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle,
\]

\[
V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.
\]

Now, we are ready to state next example.
Example 3.2. [17] For \( n \geq 3 \),
\[
\text{Aut}(\overline{P}(D_{2n})) \cong \prod_{d \in D(n)} S_{\varphi(d)} \times S_n,
\]
\[
\text{Aut}(P(D_{2n})) \cong \begin{cases} S_{n-1} \times S_n, & \text{n is a prime power} \\ S_n \times \prod_{d \in D(n)} S_{\varphi(d)} & \text{otherwise} \end{cases},
\]
and let \( n \geq 3 \) then,
\[
\text{Aut}(\overline{P}(Q_{4n})) \cong \prod_{d \in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n),
\]
\[
\text{Aut}(P(Q_{4n})) \cong \begin{cases} S_2 \times S_{2n-2} \times (S_2 \wr S_n), & \text{n is a power of 2} \\ \prod_{d \in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n) & \text{otherwise} \end{cases}.
\]

Example 3.3. [5] If \( k \) is nonnegative integer and satisfies \( n = 3^k t \) for some positive integer \( t \) such that \( 3 \nmid t \) then,
\[
\text{Aut}(P(U_{6n})) \cong \begin{cases} \prod_{d|3n} S_{\varphi(d)} \times \prod_{d|2n,d|n} S_{\varphi(d)} \times (S_2 \wr S_3) & k = 0 \\ \prod_{d|2n,d|n} S_{\varphi(d)} \times (S_2 \wr S_3) \times \prod_{d|n,d \nmid t} S_{\varphi(d)} \times (S_2 \wr S_3) & k = 1 \\ \prod_{d|2n,d|n} S_{\varphi(d)} \times (S_2 \wr S_3) \times \prod_{d|n,d \nmid t} S_{\varphi(d)} \times (S_2 \wr S_3) & k \geq 2 \end{cases},
\]
if \( n = 2^k t \) for a nonnegative \( k \) and some positive odd integer \( t \) then,
\[
\text{Aut}(P(V_{8n})) \cong \begin{cases} S_{2n} \times S_2 \wr S_n \times \prod_{d|2n,d|n} S_{\varphi(d)} \times (S_2 \wr S_3) \times \prod_{d|2n} S_{\varphi(d)} & k = 0 \\ S_{2n} \times S_2 \wr S_n \times \prod_{d|2n,d|n} S_{\varphi(d)} \times (S_2 \wr S_3) \times \prod_{d|n,d \nmid t} S_{\varphi(d)} \times (S_2 \wr S_3) & k = 1 \\ S_{2n} \times S_2 \wr S_n \times \prod_{d|n,d \nmid t} S_{\varphi(d)} \times (S_2 \wr S_3) \times \prod_{d|2n,t,d|2t-1} S_{\varphi(d)} \times (S_2 \wr S_3) & t > 1, k \geq 1 \end{cases}.
\]
also,
\[
\text{Aut}(P(SD_{8n})) \cong \begin{cases} S_{3n-2} \times S_{2n} \times (S_2 \wr S_3), & \text{n is a power of 2} \\ \prod_{d|4n} S_{\varphi(d)} \times S_{2n} \times (S_2 \wr S_3) & \text{otherwise} \end{cases}.
\]
The smallest sporadic group is the first Mathieu group \( M_{11} \), it has order 7920. There are many presentations for the group \( M_{11} \), we give two of its known presentation, [39].
\[
M_{11} \cong \langle a, b, c | a^{11} = b^5 = c^4 - (ac)^3 = 1, b^4ab = a^4, c^3bc = b^2 \rangle,
\]
\[
\cong \langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = (ab)^5 = (bc)^3 = (bd)^3 = (abbd)^3 = 1 \rangle.
\]
The paper by Aron (1960) increased the interest to finite simple groups, as Janko in Australia found (1965) the first new sporadic group \( J_1 \) a century later after Mathieu’s. It turns out that \( J_1 \) had order 175560. A presentation for \( J_1 \) in terms of its standard generators is given below [12]:
\[
J_1 \cong \langle a, b | a^2 = b^3 = (ab)^7 = (ab(abab^{-1})^3)^5 = (ab(abab^{-1})^6(abab(ab^{-1})^2)^2 = 1 \rangle.
\]
The automorphism groups of \( M_{11} \) and \( J_1 \) are determined as follows:
Example 3.4. [5] Let $M_{11}$ be the first Mathieu group and $J_1$ be the first Janko group, then,

$$\text{Aut}(P(M_{11})) \cong (S_{10} \wr S_{44}) \times (S_4 \wr S_{396}) \times (S_2 \wr S_{55}) \times ((S_6 \wr S_3) \times (S_2 \wr S_4) \wr S_2) \wr S_{165},$$

$$\text{Aut}(P(J_1)) \cong (S_{10} \wr S_{506}) \times (S_6 \wr S_{4180}) \times (S_{18} \wr S_{154n})$$

$$\times ((S_2 \times S_8 \times S_4 \times (S_4 \wr S_3) \times (S_2 \wr S_3)) \wr S_2) \wr S_{1463}. $$

Moreover, in [30] the automorphism groups of $P(Z_{pq})$, $P(Z_{pqr})$ and $P(Z_{p^2q^2})$ are calculated as follows:

$$\text{Aut}(P(Z_{pq})) \cong S_{\varphi(pq)+1} \times S_{p-1} \times S_{q-1},$$

$$\text{Aut}(P(Z_{pqr})) \cong S_{\varphi(pqr)} \times S_{p-1} \times S_{q-1} \times S_{r-1} \times S_{\varphi(pq)} \times S_{\varphi(pr)} \times S_{\varphi(qr)},$$

$$\text{Aut}(P(Z_{p^2q^2})) \cong S_{\varphi(p^2q^2)+1} \times S_{p-1} \times S_{\varphi(p^2)} \times S_{q-1} \times S_{\varphi(q^2)} \times S_{\varphi(pq)} \times S_{\varphi(pq^2)} \times S_{\varphi(p^2q)}.$$

As we mentioned in above Theorem 3.4 is playing a main role in finding automorphism group of power graphs. In [19], the authors obtained the following results from Theorem 3.3.

Example 3.5. [19] If $n$ is odd, then

$$\text{Aut}(\mathcal{S}(D_{2n})) = \begin{cases} S_{n-1} \times S_n & n \text{ is a prime power} \\ S_n \times \prod_{d|n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

and if $n$ is even then

$$\text{Aut}(\mathcal{S}(D_{2n})) = \begin{cases} S_{2n} & n \text{ is a power of } 2 \\ S_{\varphi(n)+1} \times S_{n+1} \prod_{\{1, n, 2\} \neq d|n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

if $n$ is odd, then

$$\text{Aut}(\mathcal{S}(T_{4n})) = S_{2n} \times \prod_{d|2n} S_{\varphi(d)},$$

and if $n$ is even then

$$\text{Aut}(\mathcal{S}(T_{4n})) = \begin{cases} S_{4n} & n \text{ is a power of } 2 \\ S_{\varphi(2n)+1} \times S_{2n+2} \prod_{\{1, 2n, 4\} \neq d|2n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

for arbitrary $n$,

$$\text{Aut}(\mathcal{S}(SD_{4n})) = \begin{cases} S_{8n} & n \text{ is a power of } 2 \\ S_{\varphi(4n)+1} \times S_{2n+1} \times S_{2n+2} \prod_{\{1, 4n, 2, 4\} \neq d|4n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

if $n = 2^k$ then $\text{Aut}(\mathcal{S}(V_{8n})) \cong S_{8n}$, and if $n$ is an odd prime then $\text{Aut}(\mathcal{S}(V_{8n})) = S_{2n+3} \times S_{2n} \times S_{3\varphi(n)} \times \prod_{\{1, 2n, 2\} \neq d|2n} S_{\varphi(d)}$.
4. Automorphism groups of commuting graphs

The commuting graphs $\Delta(G)$ and $\Gamma(G)$ of a group $G$ are defined in the introduction. The following theorem established the relation between $\text{Aut}(G)$, $\text{Aut}(\Delta(G))$ and $\text{Aut}(\Gamma(G))$.

**Theorem 4.1.** [33] Let $G$ be a finite group, then

1. $\text{Aut}(G) = \text{Aut}(\Delta(G))$ if and only if $|G| = 1$.
2. $\text{Aut}(\Delta(G)) \cong \text{Aut}(\Gamma(G)) \times S_{Z(G)}$.

Mirzargar, Pach and Ashrafi studied the subgroups of $\text{Aut}(\Delta(G))$ in [33, 34]. The first subgroups are $\text{Aut}(\Gamma(G))$ and $\text{Aut}(G)$, then they added some automorphisms of graph to $\text{Aut}(G)$ and constructed bigger subgroups. Define two permutations $\Phi_{x,y}, \phi : G \rightarrow G$ as follows: $\Phi_{x,y}$ fixed each element $a \in G \setminus \{x, y\}$ and maps $x$ into $y$ and vice-versa; and, the permutation $\phi$ is defined by $x \rightarrow x^{-1}$ for each element $x \in G$. They also defined $\text{Aut}^*(G) = \langle \text{Aut}(G), \phi \rangle$ and considered to the equality of the subgroups and the main group.

**Theorem 4.2.** [33] $\text{Aut}^*(G) = \text{Aut}(\Delta(G))$ if and only if $G \cong S_3$.

Let the cosets $Z(G)x_1, Z(G)x_2, \ldots, Z(G)x_{m-1}$ of the group $G/Z(G)$ and define a new graph $\Delta^u(G)$ with $V(\Delta^u(G)) = \{x_0 = 1, x_1, \ldots, x_{m-1}\}$ and $E(\Delta^u(G)) = \{x_i x_j | x_i x_j = x_j x_i, 0 \leq i < j \leq m - 1\}$. Notice when $|Z(G)| = 1$ then $\Delta(G) \cong \Delta^u(G)$. It is clear that every two elements in one of these cosets commute. Hence we have a complete graph in any of these cosets. On the other hand, if there exists $x_i \in Z(G)x_i, x_j \in Z(G)x_j$ satisfying $x_i x_j = x_j x_i$, then for every $y_i \in Z(G)x_i, y_j \in Z(G)x_j$ we have $y_i y_j = y_j y_i$. Finally, the set of all $\phi \in \text{Aut}(\Delta(G))$ such that for $a, b \in G$ if $ab^{-1} \in Z(G)$, then $\phi(a)\phi(b)^{-1} \in Z(G)$ is denoted by $T$. These notations are applied in [33] to prove two following theorems.

**Theorem 4.3.** [33] Let $G$ be a group. Then,

1. $\text{Aut}(\Delta^u(G))$ is a subgroup of $\text{Aut}(\Delta(G))$. Moreover, $\text{Aut}(\Delta^u(G)) = \text{Aut}(\Delta(G))$ if and only if $|Z(G)| = 1$.
2. If $G$ is not centerless then $T$ is a subgroup of $\text{Aut}(\Delta(G))$, and $\text{Aut}(\Delta(G)) = T$ if and only if for each pair $a, b$ of elements of $G$ with $C_G(a) = C_G(b)$, we have $ab^{-1} \in Z(G)$.

**Theorem 4.4.** [33] Let $|Z(G)| \geq 2$, where $G$ be a nonabelian group. If $T = \text{Aut}(\Delta(G))$ then $G/Z(G)$ is an elementary abelian 2-group.
For a finite group $G$ define a labelled graph $\Delta^v(G)$ as follows. For $a, b \in G$ let $a \sim b$ if $C_G(a) = C_G(b)$. Clearly, $\sim$ is an equivalence relation, the equivalence class of $a \in G$ is $A(a) = \{ x | C_G(x) = C_G(a) \}$. Let us denote the equivalence classes by $A_1, \ldots, A_k$, these are the vertices of $\Delta^v(G)$. Two vertices $A_i$ and $A_j$ are connected if and only if $a_i a_j = a_j a_i$, for some $a_i \in A_i, a_j \in A_j$. At first, we note that if there exists $a_i \in A_i, a_j \in A_j$ satisfying $a_i a_j = a_j a_i$, then for every $b_i \in A_i, b_j \in A_j$ we have $a_i C_G(a_i) = C_G(b_i)$. So, $b_i \in C_G(a_i) = C_G(b_j)$ implies that $b_i b_j = b_j b_i$.

Each equivalence class is the union of some sets of the form $tZ(G)$, hence there exists a positive integers $c_i$ such that $|A_i| = c_i |Z(G)|$. Let $\alpha(A_i) = c_i$ be the label of the vertex $A_i$ in $\Delta^v(G)$. One can see $\phi : V(\Delta^v(G)) \to V(\Delta^v(G))$ is an automorphism of the labelled graph $\Delta^v(G)$ if $\phi$ is a bijection, it preserves the edges (and the non-edges) and it preserves the labels. The automorphism group formed by these automorphisms is denoted by $Aut(\Delta^v(G))$. Define $S_{A_i} = \{ f_{\sigma} | \sigma \in S_{A_i}, \forall x \in A_i, f_{\sigma}(x) = \sigma(x), \forall x \notin A_i, f_{\sigma}(x) = x \}, 1 \leq i \leq k$. Clearly, $S_{A_i}$ is a subgroup of $Aut(\Delta(G))$. The connection between $Aut(\Delta(G))$ and $Aut(\Delta^v(G))$ is described by the following theorem:

**Theorem 4.5.** [33] There is a subgroup $A$ of $Aut(\Delta(G))$ such that $A \cong Aut(\Delta^v(G))$ and $Aut(\Delta(G)) = (S_{A_1}, \ldots, S_{A_k}) \times A$.

In [38], Rocke proved that the following are equivalent:

1. $G$ has abelian centralizers;
2. If $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$;
3. If $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$;
4. If $U$ and $B$ are subgroups of $G$ and $Z(G) < C_G(U) \leq C_G(B) < G$ then $C_G(U) = C_G(B)$.

Therefore, the intersection of two proper element centralizers of an AC-group is the center of $G$. If $G$ is an AC-group, then $\Delta(G)$ is a union of some complete graphs with all vertices adjacent to the elements of $Z(G)$. So, $\Delta(G) = n_1(C_G(x_1) \setminus Z(G)) \cup n_2(C_G(x_2) \setminus Z(G)) \cup \cdots \cup n_r(C_G(x_r) \setminus Z(G))$ and also every element of $Z(G)$ is adjacent to all elements of $G$, such that for each $i, 1 \leq i \leq r$, we have $n_i$ isomorphic components with complete graph of size $|C_G(x_i) \setminus Z(G)|$. In [33], the authors proved that if $G$ is an AC-group with the above notations then,

$$Aut(\Delta(G)) \cong ((S_{C_G(x_1) \setminus |Z(G)|} \times S_{n_1}) \times ((S_{C_G(x_2) \setminus |Z(G)|} \times S_{n_2}) \times \cdots \times ((S_{C_G(x_r) \setminus |Z(G)|} \times S_{n_r}) \times S_{Z(G)}.$$

Finally, from [33], $|Aut(\Delta(G))|$ can not be a prime power or a square-free number. Moreover, $|Aut(\Delta(G))| = 1$ if and only if $G$ is trivial, $Aut(\Gamma(G))$ is abelian if and only if $G$ is a group of order 1 or 2. Also if $|G| > 2$ then $Aut(\Delta(G))$ is a nonabelian group.
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