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TAUBERIAN CONDITIONS FOR q-CESÀRO INTEGRABILITY

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Abstract. Given a q-integrable function f on $[0,\infty)$, we define $s(x)=\int_0^x f(t)d_qt$ and $\sigma(s(x))=\frac{1}{x}\int_0^x s(t)d_qt$ for $x>0$. It is known that if $\lim_{x\to\infty} s(x)$ exists and is equal to A, then $\lim_{x\to\infty}\sigma(s(x))=A$. But the converse of this implication is not true in general. Our goal is to obtain Tauberian conditions imposed on the general control modulo of $s(x)$ under which the converse implication holds. These conditions generalize some previously obtained Tauberian conditions.
Keywords: q-integrable function; Tauberian conditions; q-derivative; q-integrals; quantum calculus.

1. Introduction

The first formulae of what we now call quantum calculus or q-calculus were introduced by Euler in the 18th century. Many notable results were obtained in the 19th century. In the early 20th century, Jackson defined the notions of q-derivative [9] and definite q-integral [10]. Also, he was the first to develop q-calculus in a systematic way. Following Jackson's papers, q-calculus has received an increasing attention of many researchers due to its vast applications in mathematics and physics.

We wiill now give some concepts of the q-calculus necessary for the understanding of this work. We follow the terminology and notations from the book of Kac and Cheung [11]. In what follows, q is a real number satisfying $0<q<1$.

The q-derivative $D_qf(x)$ of an arbitrary function $f(x)$ is defined by

$$D_qf(x)=\frac{f(x)-f(qx)}{x-qx},\ \text{ if } x\neq 0,$$

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where $D_q f(0) = f'(0)$ provided $f'(0)$ exists. If $f(x)$ is differentiable, then $D_q f(x)$ tends to $f'(x)$ as q tends to 1.
Notice that the q-derivative satisfies the following q-analogue of Leibniz rule

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x).$$

The q-integrals from 0 to a and from 0 to ∞ are given by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^{\infty} f(x)d_qx = (1-q)a\sum_{n=-\infty}^{\infty} f(q^n)q^n$$

provided the sums converge absolutely. On a general interval $[a,b]$, the q-integral is defined by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

The q-integral and the q-derivative are related by the fundamental theorem of quantum calculus as follows:

If $F(x)$ is an anti q-derivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, then

$$\int_a^b f(x)d_qx = F(b) - F(a), \quad 0 \le a < b \le \infty.$$

In addition, we have

$$D_q\left(\int_0^x f(t)d_qt\right) = f(x).$$

A function $f(x)$ is said to be q-integrable on $\mathbb{R}_+ := [0,\infty)$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely. We denote the set of all functions that are q-integrable on $\mathbb{R}_+$ by $L_q^1(\mathbb{R}_{q,+})$, where

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$

One may consult the recent books [2, 1] for further results and several applications of q-calculus.

Throughout this paper we assume that $f(x)$ is q-integrable on $\mathbb{R}_+$ and $s(x) = \int_0^x f(t)d_qt$. The symbol $s(x) = o(1)$ means that $\lim_{x\to\infty} s(x) = 0$. The q-Cesàro mean of $s(x)$ are defined by

$$\sigma(x) = \sigma(s(x)) = \frac{1}{x}\int_0^x s(t)d_qt.$$

The integral $\int_0^\infty f(t)d_qt$ is said to be q-Cesàro integrable (or $(C_q,1)$ integrable) to a finite A, in symbols: $s(x)\to A(C_q,1)$, if

$$\lim_{x\to\infty}\sigma(x)=A. \tag{1.1}$$

If the q-integral

$$\int_0^\infty f(t)d_qt=A \tag{1.2}$$

exists, then the limit (1.1) also exists [6]. That is, q-Cesàro integrability method is regular. The converse is not necessarily true (see [15], Example 1). Adding some suitable condition to (1.1), which is called a Tauberian condition, may imply (1.2). Any theorem which states that the convergence of the q-integral follows from its q-Cesàro integrability and some Tauberian condition is called a Tauberian theorem.

The difference between $s(x)$ and its q-Cesàro mean is given by the identity [6]

$$s(x)-\sigma(x)=qv(x), \tag{1.3}$$

where $v(x)=\frac{1}{x}\int_0^x tf(t)d_qt$. The identity (1.3) will be used in the various steps of proofs.

For each integer, $m\geq 0$, $\sigma_m(x)$ and $v_m(x)$ are defined by

$$\sigma_m(x)=\begin{cases}\dfrac{1}{x}\displaystyle\int_0^x\sigma_{m-1}(t)d_qt & ,m\geq 1\\ s(x) & ,m=0\end{cases}$$

and

$$v_m(x)=\begin{cases}\dfrac{1}{x}\displaystyle\int_0^x v_{m-1}(t)d_qt & ,m\geq 1\\ v(x) & ,m=0\end{cases}$$

The relationship between $\sigma_m(x)$ and $v_m(x)$ can be easily obtained by (1.3) as follows:

$$\sigma_m(x)-\sigma_{m+1}(x)=qv_m(x). \tag{1.4}$$

The classical control modulo of $s(x)=\int_0^x f(t)d_qt$ is denoted by

$$\omega_0(x)=xD_q(s(x))=xf(x),$$

and the general control modulo of integer order $m\geq 1$ of $s(x)$ is defined by

$$\omega_{m-1}(x)-\sigma(\omega_{m-1}(x))=q\omega_m(x).$$

Note that the concepts of classical and general control modulo were first introduced by Çanak and Totur [3] for the integrals in standard calculus.

A function $f(x)$ is said to satisfy the property (P) (see [7]), if for all $\epsilon>0$ there exists $K>0$ such that

$$|f(x)-f(qx)|<\epsilon$$

for all $x > K$.

Recently, Fitouhi and Brahim [7], Çanak et al. [6] and Totur et al. [15] have determined Tauberian conditions using this property. Moreover, Çanak et al. [6] showed that if $s(x)$ satisfies the property $(\mathcal{P})$, its q-Cesàro mean $\sigma(x)$ then also satisfies the property $(\mathcal{P})$.

Slowly oscillating real-valued functions were introduced by Schmidt [14]. A function $f(x)$ is said to be slowly oscillating, if for every $\varepsilon > 0$ there exists $K > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x > y > K$ and $x/y \to 1$. Slow oscillation condition were used in a number of Tauberian theorems for the Cesàro integrability [4, 5], logarithmic integrability [12, 16] and weighted mean integrability [13, 17] in standard calculus. Consider that, as q tends to 1, the property $(\mathcal{P})$ corresponds to slow oscillation of a function.

The following theorems are the q-analogues of classical Tauberian theorems due to the Hardy [8] and Schmidt [14], respectively.

Theorem 1.1. ([7]) *If $s(x)$ is q-Cesàro integrable to A and*

$$\omega_0(x) = o(1), \tag{1.5}$$

then $\int_0^\infty f(t)d_qt = A$.

Theorem 1.2. ([6],[7]) *If $s(x)$ is q-Cesàro integrable to A and satisfies the property $(\mathcal{P})$, then $\int_0^\infty f(t)d_qt = A$.*

The purpose of this study is to generalize the above theorems by imposing Tauberian conditions on the general control modulo of integer order $m \geq 1$.

2. Main Results

In this paper, we shall prove the following Tauberian theorems.

Theorem 2.1. *If $s(x)$ is q-Cesàro integrable to A and*

$$\omega_m(x) = o(1) \tag{2.1}$$

for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

Remark 2.1. It follows from the definition of the general control modulo that condition (1.5) implies the condition (2.1).

Theorem 2.2. *If $s(x)$ is q-Cesàro integrable to A and $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$ for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.*

Remark 2.2. Let the function $s(x)$ satisfy the property $(\mathcal{P})$, then so does the function $\sigma(\omega_m(x))$ for any non-negative integer m.

Remark 2.3. For the case $m = 0$ in Theorem 2.2, we observe that $v(x)$ satisfies the property $(\mathcal{P})$ which means that it is a Tauberian condition for the q-Cesàro integrability [6].

3. Auxiliary Results

In this section we state and prove some lemmas which are needed for the brevity of proofs of our main results.

Lemma 3.1. *For every integer $m \geq 1$,*

$$xD_q(\sigma_m(x)) = v_{m-1}(x). \tag{3.1}$$

Proof. Taking the q-derivative of $\sigma_m(x)$ gives

$$\begin{aligned} D_q(\sigma_m(x)) &= D_q\left(\frac{1}{x}\int_0^x \sigma_{m-1}(t)d_qt\right) \\ &= \frac{1}{qx}\sigma_{m-1}(x) - \frac{1}{qx^2}\int_0^x \sigma_{m-1}(t)d_qt \\ &= \frac{1}{qx}\left(\sigma_{m-1}(x) - \sigma_m(x)\right). \end{aligned}$$

Hence, applying the identity (1.3) to $\sigma_{m-1}(x)$, we get $D_q(\sigma_m(x)) = \dfrac{v_{m-1}(x)}{x}$, which completes the proof. □

Lemma 3.2. *For every integer $m \geq 1$,*

- **(i)** $xf(x) - v(x) = qxD_q(v(x))$
- **(ii)** $v_{m-1}(x) - v_m(x) = qxD_q(v_m(x))$.

Proof. (i) Taking the q-derivative and then multiplying both sides of identity (1.3) by x, we get

$$xD_q(s(x)) - xD_q(\sigma(x)) = qxD_q(v(x)).$$

It follows from Lemma 3.1 that

$$xf(x) - v(x) = qxD_q(v(x)).$$

(ii) Taking the $q-$derivative of both sides of (1.4), we have

$$D_q(\sigma_m(x)) - D_q(\sigma_{m+1}(x)) = qD_q(v_m(x)). \tag{3.2}$$

Then, multiplying (3.2) by x yields

$$xD_q(\sigma_m(x)) - xD_q(\sigma_{m+1}(x)) = qxD_q(v_m(x)).$$

Using Lemma 3.1, we prove that

$$v_{m-1}(x) - v_m(x) = qxD_q(v_m(x)).$$

□

Lemma 3.3. *For every integer $m \geq 1$,*

$$\sigma(xD_q(v_{m-1}(x))) = xD_q(v_m(x)). \tag{3.3}$$

Proof. Taking Cesàro means of both sides of the identity in Lemma 3.2 (ii), we find

$$\begin{aligned} \sigma(xD_q(v_{m-1}(x))) &= q^{-1}[\sigma(v_{m-2}(x)) - \sigma(v_{m-1}(x))] \\ &= q^{-1}\,(v_{m-1}(x) - v_m(x)) \\ &= xD_q(v_m(x)). \end{aligned}$$

□

For a function $f(x)$, we define

$$(xD_q)_m(f(x)) = (xD_q)_{m-1}(xD_q(f(x))) = xD_q((xD_q)_{m-1}(f(x))),$$

where $(xD_q)_0(f(x)) = f(x)$ and $(xD_q)_1(f(x)) = xD_q(f(x))$.

Lemma 3.4. *For every integer $m \geq 1$,*

$$\omega_m(x) = (xD_q)_m(v_{m-1}(x)). \tag{3.4}$$

Proof. We prove the assertion by using mathematical induction. From the definition of the general control modulo for $m = 1$ and Lemma 3.2 (i), we get

$$\omega_1(x) = q^{-1}(\omega_0(x) - \sigma(\omega_0(x))) = q^{-1}(xf(x) - v(x)) = xD_q(v(x)).$$

Assume the assertion holds for some positive integer $m = k$. That is, assume that

$$\omega_k(x) = (xD_q)_k(v_{k-1}(x)). \tag{3.5}$$

We show that the assertion is true for $m = k + 1$. That is,

$$\omega_{k+1}(x) = (xD_q)_{k+1}(v_k(x)).$$

By definition of the general control modulo for $m = k + 1$, we have

$$\omega_{k+1}(x) = q^{-1}(\omega_k(x) - \sigma(\omega_k(x))).$$

Considering Lemma 3.2 (ii) and Lemma 3.3 together with (3.5), we obtain

$$\begin{aligned} \omega_{k+1}(x) &= q^{-1}[(xD_q)_k(v_{k-1}(x)) - (xD_q)_k(v_k(x))] \\ &= q^{-1}(xD_q)_k(v_{k-1}(x) - v_k(x)) \\ &= (xD_q)_{k+1}(v_k(x)). \end{aligned}$$

Therefore, we conclude that Lemma 3.4 is true for each integer $m \geq 1$. □

Lemma 3.5. *If $s(x)$ is q-Cesàro integrable to some finite number A, then for each non-negative integer m, $\sigma(\omega_m(x))$ is q-Cesàro integrable to 0.*

Proof. If $s(x) \to A(C_q, 1)$, then it is known that $\sigma(x) \to A(C_q, 1)$. Thus, it follows from the identity (1.3) that $v(x) = \sigma(\omega_0(x)) \to 0(C_q, 1)$. Replacing $s(x)$ with $v(x)$ in (1.3), we write

$$v(x) - v_1(x) = qxD_q(v_1(x)) = q\sigma(\omega_1(x)). \tag{3.6}$$

Then, (3.6) implies $\sigma(\omega_1(x)) \to 0(C_q, 1)$. Now, applying (1.3) to $xD_q(v_1(x))$, we get

$$xD_q(v_1(x)) - xD_q(v_2(x)) = q(xD_q)_2 v_2(x) = q\sigma(\omega_2(x)). \tag{3.7}$$

Hence from (3.7), $\sigma(\omega_2(x)) \to 0(C_q, 1)$. Continuing in the same manner, we obtain $\sigma(\omega_m(x)) \to 0(C_q, 1)$ for each non-negative integer m.

□

Lemma 3.6. *For every non-negative integer m and k,*

$$\sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x)). \tag{3.8}$$

Proof. Using Lemma 3.4 and Lemma 3.3 respectively, it follows

$$\begin{aligned} \sigma_k(\omega_m(x)) &= \sigma_k((xD_q)_m v_{m-1}(x)) \\ &= (xD_q)_{m+1}\sigma_{m+k}(x). \end{aligned} \tag{3.9}$$

On the other hand, taking Lemma 3.4 and Lemma 3.1 into account we find

$$\begin{aligned} \omega_m(\sigma_k(x)) &= (xD_q)_m v_{m-1}(\sigma_k(x)) \\ &= (xD_q)_{m+1}(\sigma_{m+k}(x)). \end{aligned} \tag{3.10}$$

Therefore, the proof is completed from the equality of (3.9) and (3.10). □

The following lemma shows a different representation of the difference $s(x) - \sigma(x)$.

Lemma 3.7. *For any function $s(x)$ defined on $(0, \infty)$, we have the identity*

$$s(x) - \sigma(x) = \frac{q}{1-q}(\sigma(x) - \sigma(qx)), \tag{3.11}$$

where $\sigma(qx) = \dfrac{1}{qx}\displaystyle\int_0^{qx} s(t)d_qt.$

Proof. By the definition of the q-integral, we may write

$$\begin{aligned} \int_0^{qx} s(t)d_qt &= (1-q)qx\sum_{n=0}^{\infty} s(xq^{n+1})q^n \\ &= (1-q)x\sum_{n=1}^{\infty} s(xq^n)q^n \\ &= (1-q)x\left(\sum_{n=0}^{\infty} s(xq^n)q^n - s(x)\right) \\ &= \int_0^x s(t)d_qt - (1-q)xs(x). \end{aligned}$$

Dividing the both sides of the last equality by qx, we get

$$\frac{q}{1-q}(\sigma(x)-\sigma(qx)) = s(x)-\sigma(x).$$

□

It is clear from Lemma 3.7 that, even if $\sigma(x)$ is convergent, $\sigma(x)$ and $\sigma(qx)$ do not tend to same value when $s(x)$ is not convergent.

4. Proofs

In this section, we give proofs of our main results.

4.1. Proof of Theorem 2.1

From the hypothesis we have

$$\omega_m(x) = xD_q\sigma(\omega_{m-1}(x)) = o(1), \tag{4.1}$$

for some integer $m \geq 1$. On the other hand, from Lemma 3.5, $\sigma(\omega_{m-1}(x)) \to 0(C_q, 1)$. Hence, applying Theorem 1.1 to $\sigma(\omega_{m-1}(x))$ we obtain

$$\sigma(\omega_{m-1}(x)) = o(1). \tag{4.2}$$

Considering (4.1) and (4.2) together with the identity

$$\omega_{m-1}(x) - \sigma(\omega_{m-1}(x)) = q\omega_m(x),$$

we get

$$\omega_{m-1}(x) = xD_q\sigma(\omega_{m-2}(x)) = o(1). \tag{4.3}$$

By Lemma 3.5, we also have $\sigma(\omega_{m-2}(x)) \to 0(C_q, 1)$. Now, applying Theorem 1.1 to $\sigma(\omega_{m-2}(x))$ we obtain

$$\sigma(\omega_{m-2}(x)) = o(1). \tag{4.4}$$

From (4.3), (4.4) and the identity

$$\omega_{m-2}(x) - \sigma(\omega_{m-2}(x)) = q\omega_{m-1}(x),$$

we find

$$\omega_{m-2}(x) = xD_q\sigma(\omega_{m-3}(x)) = o(1). \tag{4.5}$$

Taking (4.1), (4.3) and (4.5) into account and proceeding likewise, we observe that $\omega_0(x) = o(1)$. Therefore, the proof follows from Theorem 1.1. □

4.2. Proof of Theorem 2.2

Considering Lemma 3.7 we may construct the identity

$$\sigma(\omega_m(x)) - \sigma_2(\omega_m(x)) = \frac{q}{1-q}[\sigma_2(\omega_m(x)) - \sigma_2(\omega_m(qx))].$$

Since $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$, its $q-$Cesàro mean $\sigma_2(\omega_m(x))$ also satisfies the property $(\mathcal{P})$. Let $\epsilon > 0$ be given. Then, there exists $K > 0$ such that

$$-\epsilon < \sigma_2(\omega_m(x)) - \sigma(\omega_m(x)) < \epsilon \tag{4.6}$$

for every $x > K$. By (4.6), we write

$$\sigma(\omega_m(x)) - \epsilon < \sigma_2(\omega_m(x)) < \sigma(\omega_m(x)) + \epsilon. \tag{4.7}$$

Since $s(x) \to A(C_q, 1)$, we have by using Lemma 3.5 that $\lim_{x\to\infty} \sigma_2(\omega_m(x)) = 0$. Thus, it follows from (4.7)

$$-\epsilon < \liminf_{x\to\infty} \sigma(\omega_m(x)) < \limsup_{x\to\infty} \sigma(\omega_m(x)) < \epsilon,$$

which is equivalent to

$$\lim_{x\to\infty} \sigma(\omega_m(x)) = 0. \tag{4.8}$$

It yields from the equality

$$\begin{aligned} \sigma(\omega_m(x)) &= \sigma((xD_q)_m v_{m-1}(x)) \\ &= xD_q(xD_q)_{m-1} v_m(x) \\ &= xD_q \sigma_2(\omega_{m-1}(x)), \end{aligned}$$

that $xD_q\sigma_2(\omega_{m-1}(x)) = o(1)$. Also, by Lemma 3.5, $\sigma(\omega_{m-1}(x)) \to 0(C_q, 1)$. Further, regularity of q-Cesàro integrability implies $\sigma_2(\omega_{m-1}(x)) \to 0(C_q, 1)$. Then, if we apply Theorem 1.1 to $\sigma_2(\omega_{m-1}(x))$ we obtain

$$\lim_{x\to\infty} \sigma_2(\omega_{m-1}(x)) = 0. \tag{4.9}$$

From the definition of the general control modulo, it is easy to see

$$\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = q\sigma(\omega_m(x)). \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we reach

$$\lim_{x\to\infty} \sigma(\omega_{m-1}(x)) = 0. \tag{4.11}$$

Now, since

$$\begin{aligned} \sigma(\omega_{m-1}(x)) &= \sigma((xD_q)_{m-1} v_{m-2}(x)) \\ &= xD_q(xD_q)_{m-2} v_{m-1}(x) \\ &= xD_q \sigma_2(\omega_{m-2}(x)), \end{aligned}$$

we find $xD_q\sigma_2(\omega_{m-2}(x)) = o(1)$. Besides, we have $\sigma_2(\omega_{m-2}(x)) \to 0(C_q, 1)$ from Lemma 3.5 and the regularity of q-Cesàro integrability. Now, applying Theorem 1.1 to $\sigma_2(\omega_{m-2}(x))$ we get

$$\lim_{x\to\infty} \sigma_2(\omega_{m-2}(x)) = 0. \tag{4.12}$$

Considering (4.11), (4.12) and the identity

$$\sigma(\omega_{m-2}(x)) - \sigma_2(\omega_{m-2}(x)) = q\sigma(\omega_{m-1}(x)), \tag{4.13}$$

we have

$$\lim_{x\to\infty} \sigma(\omega_{m-2}(x)) = 0. \tag{4.14}$$

In the light of (4.8), (4.11) and (4.14), continuing in the same fashion we conclude

$$\lim_{x\to\infty} \sigma(\omega_0(x)) = \lim_{x\to\infty} v(x) = 0.$$

Therefore, since $s(x) \to A(C_q, 1)$, we obtain via (1.3) that $\lim_{x\to\infty} s(x) = A$. □

5. Extensions

In this section, we will present the q-Hölder or (H_q, k) integrability method which is an obvious generalization of the q-Cesàro integrability. Later, we extend our main results to this method.

If

$$\lim_{x\to\infty} \sigma_k(x) = A,$$

then $\int_0^\infty f(t)d_qt$ is said to be integrable by the q-Hölder method of order $k \in \mathbb{N}_0$ (shortly, (H_q, k) integrable) to A, and this fact is denoted by $s(x) \to A(H_q, k)$. In particular, the method $(H_q, 0)$ indicates the convergence in the ordinary sense and the method $(H_q, 1)$ is equivalent to $(C_q, 1)$. The (H_q, k) methods are regular for any k and are compatible for all k. The power of the method increases with increasing k: The (H_q, k) integrability implies (H_q, k') integrability for any $k' > k$.

Theorem 5.1. *Let $s(x) \to A(H_q, k+1)$. If*

$$\omega_m(x) = o(1) \tag{5.1}$$

for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

Proof. By (5.1) and the regularity of the $(C_q, 1)$ method, we obtain $\sigma_k(\omega_m(x)) = o(1)$ for each integer $k \geq 0$. Then, from Lemma 3.6 it is clear that

$$\omega_m(\sigma_k(x)) = o(1) \quad \text{for each } k \in \mathbb{N}_0. \tag{5.2}$$

Besides, from the assumption since $\sigma_k(x) \to A(C_q, 1)$, Theorem 2.1 implies

$$\lim_{x\to\infty} \sigma_k(x) = A$$

which is also equivalent to $\sigma_{k-1}(x) \to A(C_q, 1)$. From (5.2), we know that $\omega_m(\sigma_{k-1}(x)) = o(1)$. Now, applying Theorem 2.1 to $\sigma_{k-1}(x)$ yields

$$\lim_{x\to\infty} \sigma_{k-1}(x) = A$$

which is also equivalent to $\sigma_{k-2}(x) \to A(C_q, 1)$. Repeating the same steps k-times we conclude

$$\lim_{x\to\infty} \sigma_0(x) = \int_0^\infty f(t) d_q t = A.$$

□

Theorem 5.2. *Let $s(x) \to A(H_q, k+1)$. If $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$ for some integer $m \geq 0$, then $\int_0^\infty f(t) d_q t = A$.*

Proof. If $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$, then so does $\sigma_k(\omega_m(x))$ for every non-negative integer k. From Lemma 3.6, since

$$\sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x))$$

we find that $\sigma(\omega_m(\sigma_k(x)))$ also satisfies $(\mathcal{P})$ for all $k \in \mathbb{N}_0$. Considering the hypothesis $\sigma_k(x) \to A(C_q, 1)$ and Theorem 2.2 we obtain

$$s(x) \to A(H_q, k)$$

which requires $\sigma_{k-1}(x) \to A(C_q, 1)$. Moreover, since $\sigma(\omega_m(\sigma_{k-1}(x)))$ satisfies $(\mathcal{P})$, we get

$$s(x) \to A(H_q, k-1)$$

which requires $\sigma_{k-2}(x) \to A(C_q, 1)$. Applying the same reasoning k-times we reach that

$$s(x) \to A(H_q, 0)$$

which means $\lim_{x\to\infty} s(x) = A$. □

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