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A NEW CHARACTERIZATION OF CURVES IN MINKOWSKI 4-SPACE $\mathbb{E}_1^4$

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Abstract. In this study, we attend to the curves whose position vectors are written as a linear combination of their Serret-Frenet vectors in Minkowski 4-space $\mathbb{E}_1^4$. We characterize such curves with regard to their curvatures. Further, we get certain consequences of T-constant and N-constant types of curves in $\mathbb{E}_1^4$.
Keywords: Constant ratio curves, T-constant curves, N-constant curves, Minkowski space.

1. Introduction

The term rectifying curves is presented by B.Y. Chen in [7]. Afterwards, Chen and Dillen gave the connection between these curves and centrodes that have a place in mechanics and kinematics as well as in differential geometry [10]. The rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$ were investigated in [12, 16, 17]. For a regular curve $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ given with the arclength parameter, the hyperplanes spanned by $\{T, N_1, N_3\}$ and $\{T, N_2, N_3\}$ are known as the first osculating hyperplane and the second osculating hyperplane, respectively. If x lies on its first (second) osculating hyperplane, then $x(s)$ is called as an osculating curve of first (second) kind. In [1], the authors considered the rectifying curves in Minkowski 4-space $\mathbb{E}_1^4$. They characterized the rectifying curves with the equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \upsilon(s)N_3(s)$$

for given differentiable functions $\lambda(s), \mu(s)$ and $\upsilon(s)$. Actually, these curves are osculating curves of a second kind. The rectifying curves in $\mathbb{E}_1^4$ are studied by the authors in [18, 19].

The notion of constant ratio curves in Minkowski spaces is given by B. Y. Chen in [9]. In the same paper, the author gave the necessary and sufficient conditions, $x^T = 0$ or the ratio $\left\|x^T\right\| : \|x\|$ is constant, for curves to become constant ratio.

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Moreover, in [8], the same author introduces *T-constant* and *N-constant* types of curves. If the norm of the tangential component (normal component) is constant, the curve is called as *T-constant* (*N-constant*). Also, if this norm is equal to zero, then the curve is a *T-constant* (*N-constant*) curve of first kind, otherwise second kind [15]. Recently, the authors have studied the mentioned curves in some spaces in [2, 3, 4, 5, 6, 15, 20, 21, 22, 28, 29, 30, 31].

In this study, we deal with spacelike curves with spacelike principal normal in $\mathbb{E}_1^4$ with respect to the their Frenet frame $\{T, N_1, N_2, N_3\}$. Since $\{T, N_1, N_2, N_3\}$ is an orthonormal basis in $\mathbb{E}_1^4$, we write the position vector of the curve as

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{1.1}$$

for some differentiable functions $m_i(s)$, $i = 0, 1, 2, 3$. We classify osculating curves of the first and the second kind with regard to their curvature functions $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$. We give W-curves in $\mathbb{E}_1^4$. Furthermore, we get certain consequences of these types curves to become ccr-curves. We consider T-constant and N-constant curves in $\mathbb{E}_1^4$.

2. Basic Consepts

Minkowski 4-space is 4-dimensional pseudo-Euclidean space defined by the Lorentzian inner product

$$\langle v, w\rangle_{\mathbb{L}} = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4,$$

where v_i, w_i, i=1,2,3,4 are the components of the vectors v and w. Any arbitrary vector v is called timelike, lightlike or spacelike if the Lorentzian inner product $\langle v, v\rangle_{\mathbb{L}}$ is negative definite, zero or positive definite, respectively. Then, the length of the vector $v \in \mathbb{E}_1^4$ is calculated by

$$\|v\| = \sqrt{|\langle v, v\rangle_{\mathbb{L}}|}.$$

The sets

$$\mathbb{S}_1^3(r^2) = \left\{v \in \mathbb{E}_1^4 : \langle v, v\rangle_{\mathbb{L}} = r^2\right\}$$

and

$$\mathbb{H}_0^3(-r^2) = \left\{v \in \mathbb{E}_1^4 : \langle v, v\rangle_{\mathbb{L}} = -r^2\right\}$$

are called pseudo-Riemannian and pseudo-Hyperbolic spaces in $\mathbb{E}_1^4$ for positive number r, respectively [11].

A curve $x = x(s) : I \to \mathbb{E}_1^4$ is timelike (lightlike (null), spacelike) if all tangent vectors $x'(s)$ are timelike (lightlike (null), spacelike). If $\|x'(s)\| = 1$, x is a unit speed curve [25].

The light cone $\mathcal{LC}$ of $\mathbb{E}_1^4$ is defined as

$$\mathcal{LC} = \left\{v \in \mathbb{E}_1^4,\ \ \langle v, v\rangle_{\mathbb{L}} = 0\right\}.$$

Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and $\{T, N_1, N_2, N_3\}$ be the Frenet frame of x in $\mathbb{E}_1^4$. Then, the Frenet formulas are

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \varepsilon\kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s) - \varepsilon\kappa_3(s)N_3(s), \\ N_3'(s) &= -\varepsilon\kappa_3(s)N_2(s), \end{aligned} \tag{2.1}$$

where $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ are the first, the second, and the third curvatures of the curve x and

$$\varepsilon = \langle N_2(s), N_2(s)\rangle_L = -\langle N_3(s), N_3(s)\rangle_L = \pm 1$$

[26].

Screw lines or helices, called as *W-curves* by F. Klein and S. Lie [23], are the curves with constant curvatures, and they are mentioned in [13, 14]. Moreover, a regular curve is a ccr-curve, constant curvature ratios, if its curvature's ratios are constants [24, 27].

3. Characterization of Spacelike Curves in $\mathbb{E}_1^4$

Now, we shall consider curves given with the equality (1.1) in $\mathbb{E}_1^4$. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal, and $\kappa_1(s) \neq 0$, $\kappa_2(s)$ and $\kappa_3(s)$ be the curvatures of x. Differentiating (1.1) according to s and using (2.1), we get

$$\begin{aligned} x'(s) = \; & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & +(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & +(m_2'(s) + \varepsilon\kappa_2(s)m_1(s) - \varepsilon\kappa_3(s)m_3(s))N_2(s) \\ & +(m_3'(s) - \varepsilon\kappa_3(s)m_2(s))N_3, \end{aligned}$$

which follows

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \varepsilon\kappa_2 m_1 - \varepsilon\kappa_3 m_3 &= 0, \\ m_3' - \varepsilon\kappa_3 m_2 &= 0. \end{aligned} \tag{3.1}$$

The following theorem determines the W-curves in $\mathbb{E}_1^4$.

Theorem 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike* *principal normal. If x is a W-curve in $\mathbb{E}_1^4$, then*

$$
\begin{aligned}
m_0(s) &= -\frac{2\kappa_1}{\sqrt{-2\lambda+2\mu}}\left\{c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} - c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s}\right\} \\
&\quad -\frac{2\kappa_1}{\sqrt{2\lambda+2\mu}}\left\{c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} - c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}\right\}, \\
m_1(s) &= \frac{-1}{\kappa_1} + c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} + c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s} \\
&\quad + c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} + c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}, \\
m_2(s) &= \frac{1}{\kappa_2}\left\{\begin{array}{l}
-c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s}\left(\frac{-\lambda+\mu+2\kappa_1^2}{\sqrt{-2\lambda+2\mu}}\right) \\
+c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s}\left(\frac{-\lambda+\mu+2\kappa_1^2}{\sqrt{-2\lambda+2\mu}}\right) \\
-c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s}\left(\frac{\lambda+\mu+2\kappa_1^2}{\sqrt{2\lambda+2\mu}}\right) \\
+c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}\left(\frac{\lambda+\mu+2\kappa_1^2}{\sqrt{2\lambda+2\mu}}\right)
\end{array}\right., \\
m_3(s) &= \varepsilon\kappa_3 \int m_2(s)ds.
\end{aligned}
$$

Here, c_i $(1 \le i \le 4)$ are integral constants and

$$
\begin{aligned}
\lambda &= \sqrt{\kappa_1^4 + 2\kappa_1^2\kappa_3^2 + 2\varepsilon\kappa_1^2\kappa_2^2 + \kappa_3^4 - 2\varepsilon\kappa_2^2\kappa_3^2 + \kappa_2^4}, \\
\mu &= -\kappa_1^2 + \kappa_3^2 - \varepsilon\kappa_2^2
\end{aligned}
$$

are real constants.

Proof. Assume $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a unit speed spacelike curve with spacelike principal normal. From (3.1), we get the differential equation

$$m_1^{(iv)} + (\kappa_1^2 + \varepsilon\kappa_2^2 - \kappa_3^2)m_1'' - \kappa_1^2\kappa_3^2 m_1 - \kappa_1\kappa_3^2 = 0,$$

which has a solution

$$
\begin{aligned}
m_1(s) &= \frac{-1}{\kappa_1} + c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} + c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s} \\
&\quad + c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} + c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}.
\end{aligned}
$$

Thus, the theorem is proved. □

3.1. Osculating Curve of First Kind in $\mathbb{E}_1^4$

Definition 3.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal. If x lies in the hyperplane spanned by $\{T, N_1, N_3\}$, then x is called an osculating curve of first kind in $\mathbb{E}_1^4$.

In [19], authors consider the osculating curves of first kind in $\mathbb{E}_1^4$. It means that the differentiable function $m_2(s)$ vanishes identically. Thus, from (3.1), the system

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 &= 0, \\ \kappa_2 m_1 - \kappa_3 m_3 &= 0, \\ m_3' &= 0 \end{aligned}$$

is obtained. Therefore,

$$\begin{aligned} m_0 &= \frac{-cH_2'}{\kappa_1}, \\ m_1 &= cH_2, \\ m_3 &= c, \end{aligned}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$, $c \in \mathbb{R}$. Thus, one can write x as in the following

$$x(s) = c\left\{ \frac{-H_2'}{\kappa_1}(s)T(s) + H_2(s)N_1(s) + N_3(s) \right\}.$$

In [19], authors give the Lemma 3.1.

Lemma 3.1. *[19] Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal. The necessary and sufficient condition for x to correspond an osculating curve of first kind is*

$$\left(\frac{cH_2'}{\kappa_1} \right)' + c\kappa_1 H_2 + 1 = 0, \tag{3.2}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$, $c \in \mathbb{R}$.

Corollary 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and corresponds to an osculating curve of the first kind in $\mathbb{E}_1^4$. If x is a ccr-curve, then*

$$H_2 = -\frac{1}{c\kappa_1},$$

where $c = m_3$ is a real constant.

We give a classification assuming only one of the curvature functions is non-constant as follows:

Assume $\kappa_1(s)$ =constant> 0, $\kappa_2(s)$ =constant$\neq 0$ and $\kappa_3(s)$ is a non-constant function. From (3.2), we obtain the differential equation

$$c\kappa_3''(s) + c\kappa_1^2\kappa_3(s) + \kappa_1\kappa_2 = 0,$$

which has a solution

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1 \cos(\kappa_1 s) + c_2 \sin(\kappa_1 s).$$

Similarly, assume that $\kappa_1(s) = constant > 0$, $\kappa_3(s) = constant \neq 0$ and $\kappa_2(s)$ is a non-constant function. Then, (3.2) implies the differential equation

$$\frac{c\kappa_3}{\kappa_1}\left(\frac{1}{\kappa_2(s)}\right)'' + \frac{c\kappa_1\kappa_3}{\kappa_2(s)} + 1 = 0, \tag{3.3}$$

with solution

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{-c_1\kappa_3\cos(\kappa_1 s) + c_2\kappa_3\sin(\kappa_1 s) - 1}.$$

Summing up these calculations, we give the Theorem 3.2.

Theorem 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4_1$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}^4_1$. Then x corresponds to an osculating curve of first kind if*

i) $\kappa_1(s) =$constant> 0, $\kappa_2(s) =$constant$\neq 0$ and

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1 \cos(\kappa_1 s) + c_2 \sin(\kappa_1 s),$$

ii) $\kappa_1(s) =$constant> 0, $\kappa_3(s) =$constant$\neq 0$ and

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{-c_1\kappa_3\cos(\kappa_1 s) + c_2\kappa_3\sin(\kappa_1 s) - 1},$$

where c, c_1 and $c_2 \in \mathbb{R}$.

3.2. Osculating Curve of the Second Kind in $\mathbb{E}^4_1$

Definition 3.2. Let $x : I \subset \mathbb{R} \to \mathbb{E}^4_1$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}^4_1$. If x lies in the hyperplane spanned by $\{T, N_2, N_3\}$, then x is an osculating curve of the second kind in $\mathbb{E}^4_1$.

In [1], the authors consider the spacelike osculating curve of the second kind in $\mathbb{E}^4_1$. Actually, they call them as rectifying curves in $\mathbb{E}^4_1$. In this case, the differentiable function $m_1(s)$ vanishes identically. Thus from (3.1), the equalities

$$\begin{aligned} m_0' &= 1, \\ \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' - \varepsilon\kappa_3 m_3 &= 0, \\ m_3' - \varepsilon\kappa_3 m_2 &= 0 \end{aligned} \tag{3.4}$$

hold. Therefore

$$\begin{aligned} m_0 &= s + b, \\ m_2 &= (s+b)H_1, \\ m_3 &= \frac{(s+b)H_1' + H_1}{\varepsilon\kappa_3}, \end{aligned} \tag{3.5}$$

where $b \in \mathbb{R}$ and $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ is the first harmonic curvature of x. Hence, x is

$$x(s) = (s+b)T(s) + (s+b)H_1 N_2(s) + \frac{(s+b)H_1' + H_1}{\varepsilon\kappa_3} N_3(s).$$

By the use of (3.4) and (3.5), we give the following results.

Theorem 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with a spacelike principal normal in $\mathbb{E}_1^4$. Then the necessary and sufficient condition for x to correspond an osculating curve of second kind is*

$$\left(\frac{(s+b)H_1' + H_1}{\varepsilon\kappa_3}\right)' - \varepsilon\kappa_3(s+b)H_1 = 0 \tag{3.6}$$

for $H_1(s) = \frac{\kappa_1}{\kappa_2}(s), b \in \mathbb{R}$.

Corollary 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and corresponds to an osculating curve of the second kind. If x is a ccr-curve, then*

$$\kappa_3(s) = \pm\frac{1}{\sqrt{c + s^2 + 2bs}}, \tag{3.7}$$

where $b, c \in \mathbb{R}$.

Proof. Let x be an osculating curve of second kind. If x is a ccr-curve, then the functions $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ are constants. Thus, by the use of (3.6), one can get

$$\kappa_3'(s) + (s+b)\kappa_3^3(s) = 0,$$

which has a solution (3.7). □

As a consequence of the differential equation (3.6), one can get the following solutions as in the previous section.

Corollary 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. Then, x is corresponds to an osculating curve of the second kind if*

i) $\kappa_1(s) =$ constant > 0, $\kappa_2(s) =$ constant $\neq 0$, and $\kappa_3(s) = \pm\frac{1}{\sqrt{c+s^2+2bs}}$,

ii) $\kappa_2(s) =$ constant $\neq 0$, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_1(s) = \frac{1}{s+b}\left(c_1 \sinh(\kappa_3 s) + c_2 \cosh(\kappa_3 s)\right),$$

iii) $\kappa_1(s) =$ constant > 0, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_2(s) = \frac{\kappa_3(s+b)}{c_1 \sinh(\kappa_3 s) - c_2 \cosh(\kappa_3 s)},$$

where $c_1, c_2, b \in \mathbb{R}$.

4. T-Constant Curves in $\mathbb{E}_1^4$

Definition 4.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_t^n$. If the norm of the tangential component of x, i.e.$\left\|x^T\right\|$, is constant, then x is a *T-constant curve* [8]. Moreover, if this norm is equal to zero, i.e. $\left\|x^T\right\| = 0$, then the curve is a *T-constant* curve of the first kind, otherwise the second kind [15].

In view of (3.1), we give the results that determine T-constant curves in $\mathbb{E}_1^4$.

Theorem 4.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. The necessary and sufficient condition for x to become a T-constant curve of the first kind is*

$$\varepsilon H_2 R' + \left(\frac{\left(-\frac{R'}{\kappa_2}\right)'}{\varepsilon\kappa_3} - \frac{R}{H_2} \right)' = 0,$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x is a T-constant curve of the first kind. From (3.1), we get

$$m_1 = -\frac{1}{\kappa_1}, m_2 = \frac{m_1'}{\kappa_2}, m_3 = \frac{m_2' + \varepsilon\kappa_2 m_1}{\varepsilon\kappa_3}.$$

Further, substituting these values into $m_3' - \varepsilon\kappa_3 m_2 = 0$, we yield the expected result. □

Theorem 4.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with a spacelike principal normal in $\mathbb{E}_1^4$. The necessary and sufficient condition for x to become a T-constant curve of the second kind is*

$$\left(\frac{\left(\frac{-R'}{\kappa_2} + H_1 m_0\right)'}{\varepsilon\kappa_3} - \frac{R}{H_2} \right)' - \varepsilon H_2 \left(-R' + \kappa_1 m_0\right) = 0,$$

where $m_0 \in \mathbb{R}, H_1(s) = \frac{\kappa_1}{\kappa_2}(s), H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x is a T-constant curve of second kind. From (3.1), we get

$$m_1 = -\frac{1}{\kappa_1}, m_2 = \frac{m_1' + \kappa_1 m_0}{\kappa_2}, m_3 = \frac{m_2' + \varepsilon\kappa_2 m_1}{\varepsilon\kappa_3}.$$

Further, substituting these values into $m_3' - \varepsilon\kappa_3 m_2 = 0$, we get the result. □

Corollary 4.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed T-constant spacelike curve of second kind with spacelike principal normal in $\mathbb{E}_1^4$. If x is a W-curve of $\mathbb{E}_1^4$, then x has the parametrization of*

$$x(s) = \lambda T - RN_1 + H_1\lambda N_2 - \frac{R}{H_2}N_3,$$

where $R = \frac{1}{\kappa_1}, H_1 = \frac{\kappa_1}{\kappa_2}, H_2 = \frac{\kappa_3}{\kappa_2}, \lambda \in \mathbb{R}$ and c is an integral constant.

Theorem 4.3 gives a simple characterization of T-constant curves of second kind of $\mathbb{E}_1^4$.

Theorem 4.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed T-constant spacelike curve of second kind with spacelike principal normal in $\mathbb{E}_1^4$. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{2\lambda s + c}, \tag{4.1}$$

for some real constants $\lambda = m_0$ and c.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (1.1), we get $\rho\rho' = m_0$. If x is a T-constant curve of second kind, then by definition, the differentiable function $m_0(s)$ of x is constant. It is easy to show that this differential equation has a non-trivial solution (4.1). □

5. N-Constant Curves in $\mathbb{E}_1^4$

Definition 5.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_t^n$. If the norm of the normal component of x, i.e. $\|x^N\|$, is constant, then x is a *N-constant curve* [8]. Moreover, if this norm is equal to zero, i.e. $\|x^N\| = 0$, then the curve is a *N-constant* curve of the first kind, otherwise second kind [15].

Hence, for a N-constant curve x in $\mathbb{E}_1^4$

$$\|x^N(s)\|^2 = m_1^2(s) + \varepsilon m_2^2(s) - \varepsilon m_3^2(s)$$

becomes a constant function. Therefore, by differentiation

$$m_1m_1' + \varepsilon m_2m_2' - \varepsilon m_3m_3' = 0. \tag{5.1}$$

The following proposition gives a characterization of N-constant curves of the first kind.

Proposition 5.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed N-constant spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. If x is a N-constant curve of the first kind, then,*

i) x is congruent to a spacelike line which passes through the origin,

ii) x is a planar curve,

iii) x is an osculating curve of second kind,

iv) x lies in the hyperplane which is spanned by $\{T, N_1, N_2\}$.

Conversely, if $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$ with $\kappa_1 > 0$, and one of (i), (ii), (iii), (iv) holds, then x is a N-constant curve of the first kind.

Proof. Assume x is a N-constant curve of the first kind in $\mathbb{E}_1^4$. There are two possibilities; either $m_1 = m_2 = m_3 = 0$ or $m_1^2 + \varepsilon m_2^2 = \varepsilon m_3^2$. In the first case, $x(I)$ is congruent to a spacelike line which passes through the origin. Let $m_1^2 + \varepsilon m_2^2 = \varepsilon m_3^2$, then by the use of the equations (3.1), we get $\kappa_2 m_1 m_3 = 0$. If $\kappa_2 = 0$, x is a planar curve. If $m_1 = 0$, x is an osculating curve of second kind. Let $m_3 = 0$, then there are two possibilities; either $\kappa_3 = 0$ or $m_2 = 0$. If $m_2 = 0$, x is a planar curve. If $\kappa_3 = 0$, x lies in the hyperplane which is spanned by $\{T, N_1, N_2\}$. □

Further, for the N-constant curves of the second kind, we obtain the following result.

Theorem 5.1. *Let $x(s) \in \mathbb{E}_1^4$ be a spacelike curve with a spacelike principal normal given with the arclength function s and fully lies in $\mathbb{E}_1^4$. If x is a N-constant curve of the second kind, then x has a parametrization of*

$$x(s) = (s+c)T(s) + H_1(s+c)N_2(s) + \frac{H_1'(s+c) + H_1}{\varepsilon\kappa_3} N_3(s),$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s), c \in \mathbb{R}$.

Proof. Assume x is a N-constant curve of the second kind in $\mathbb{E}_1^4$. From the equalities (3.1) and (5.1), we get $m_1 = 0$, $m_0(s) = s + c$, $m_2(s) = \frac{\kappa_1}{\kappa_2}(s)m_0$ and $m_3(s) = \frac{m_2'(s)}{\varepsilon\kappa_3(s)}$ for some constant $c \in \mathbb{R}$. This completes the proof of the theorem. □

Remark 5.1. Every N-constant curve of the second kind is an osculating curve of second kind in $\mathbb{E}_1^4$.

Theorem 5.2 gives a simple characterization of N-constant curve of the second kind in $\mathbb{E}_1^4$.

Theorem 5.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a N-constant curve of second kind. Then, the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{s^2 + 2sc + 2b}, \quad (5.2)$$

for real constants b and c.

Example 5.1. Let us consider the W-curve $x(s) = \left(\sinh s, \cosh s, \sqrt{2}\sin s, -\sqrt{2}\cos s\right)$ in $\mathbb{E}_1^4$. The Frenet frame vectors and the curvatures of x are given as

$$\begin{aligned} T(s) &= \left(\cosh s, \sinh s, \sqrt{2}\cos s, \sqrt{2}\sin s\right), \\ N_1(s) &= \frac{1}{\sqrt{3}}\left(\sinh s, \cosh s, -\sqrt{2}\sin s, \sqrt{2}\cos s\right), \\ N_2(s) &= \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, \cos s, \sin s\right), \\ N_3(s) &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, \sin s, -\cos s\right) \end{aligned}$$

and

$$\kappa_1 = \sqrt{3}, \quad \kappa_2 = -\frac{2\sqrt{6}}{3}, \quad \kappa_3 = \frac{\sqrt{3}}{3},$$

respectively. We find the curvature functions as $m_0 = m_2 = 0$, $m_1 = -\frac{\sqrt{3}}{3}$ and $m_3 = \frac{2\sqrt{6}}{3}$, which shows that the curve x is a T-constant curve of the first kind and N-constant curve of the second kind.

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