Abstract. The main objective of this article is to introduce the concepts of \( f \)-lacunary statistical convergence of order \( \alpha \) and strong \( f \)-lacunary summability of order \( \alpha \) of double sequences and give some inclusion relations between these concepts.

Keywords: \( f \)-lacunary statistical convergence; strong \( f \)-lacunary summability; sequence spaces.

1. Introduction

In 1951, Steinhaus [41] and Fast [19] introduced the concept of statistical convergence while later in 1959, Schoenberg [40] reintroduced it independently. Bhardwaj and Dhawan [4], Caserta et al. [5], Connor [6], Çakallı [11], Çınar et al. [12], Çolak [13], Et et al. ([15],[17]), Fridy [21], Işık [27], Salat [39], Di Maio and Kočinac [14], Mursaleen et al. ([31],[30],[32]), Belen and Mohiuddine [3] and many authors investigated the arguments related to this notion.

A modulus \( f \) is a function from \([0, \infty)\) to \([0, \infty)\) such that

i) \( f(x) = 0 \) if and only if \( x = 0 \),

ii) \( f(x + y) \leq f(x) + f(y) \) for \( x, y \geq 0 \),

iii) \( f \) is increasing,

iv) \( f \) is continuous from the right at 0.

It follows that \( f \) must be continuous everywhere on \([0, \infty)\). A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined \( f \)-density of a subset \( E \subset \mathbb{N} \) for any unbounded modulus \( f \) by

\[
d_f(E) = \lim_{n \to \infty} \frac{f(|\{k \leq n : k \in E\}|)}{f(n)}, \text{ if the limit exists}
\]

Received July 10, 2019; accepted October 07, 2019

2010 Mathematics Subject Classification. Primary 40A05; Secondary 40C05, 46A45

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and defined $f-$statistical convergence for any unbounded modulus $f$ by
\[
d^f (\{ k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon \}) = 0
\]
i.e.
\[
\lim_{n \to \infty} \frac{1}{f(n)} f (\{ k \leq n : |x_k - \ell| \geq \varepsilon \}) = 0,
\]
and we write it as $S^f - \lim x_k = \ell$ or $x_k \to \ell (S^f)$.

Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f-$statistically convergent for every unbounded modulus $f$.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_r, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_r$, and $q_1 = k_1$ for convenience.

In [22], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence $(x_k)$ of real numbers is called lacunary statistically convergent to a real number $\ell$, if
\[
\lim_{r \to \infty} \frac{1}{h_r} |\{ k \in I_r : |x_k - \ell| \geq \varepsilon \}| = 0
\]
for every positive real number $\varepsilon$.

Lacunary sequence spaces were studied in ([7],[8],[9],[10],[18],[20],[22],[23],[25],[26],[28],[36],[43]).

A double sequence $x = (x_{j,k})_{j,k=0}^\infty$ has Pringsheim limit $\ell$ provided that given for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{j,k} - \ell| < \varepsilon$ whenever $j, k > N$. In this case, we write $P - \lim x = \ell$ (see Pringsheim [38]).

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K (m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double natural density of $K$ is defined by
\[
\delta_2 (K) = P - \lim_{m,n} \frac{1}{mn} |K (m,n)|, \text{ if the limit exists.}
\]

A double sequence $x = (x_{j,k})_{j,k \in \mathbb{N}}$ is said to be statistically convergent to a number $\ell$ if for every $\varepsilon > 0$ the set $\{(j,k) : j \leq m, k \leq n : |x_{j,k} - \ell| \geq \varepsilon\}$ has double natural density zero (see Mursaleen and Edely [31]).

In [35], Patterson and Savas introduced the concept of double lacunary sequence in the sense that double sequence $\theta'' = \{(k_r, l_s)\}$ is called double lacunary sequence, if there exists two increasing sequences of integers such that
\[
k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty
\]
and
\[
l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.
\]
where \( k_{r,s} = k_r l_s, h_{r,s} = h_r h_s \) and the following intervals are determined by \( \theta'' \), \( I_r = \{(k) : k_{r-1} < k \leq k_r\} \), \( I_s = \{(l) : l_{s-1} \leq l < l_s\} \), \( I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\} \), \( q_r = \frac{k_r}{k_{r-1}} \), \( q_s = \frac{l_s}{l_{s-1}} \) and \( q_{r,s} = q_r q_s \).

The double number sequence \( x \) is \( S_{\theta''} - \text{convergent} \) to \( \ell \) provided that for every \( \varepsilon > 0 \),

\[
P - \lim_{r,s \to \infty} \frac{1}{h_{r,s}} \left| \{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\} \right| = 0.
\]

In this case, we write \( S_{\theta''} - \lim x_{k,l} = \ell \) or \( x_{k,l} \to \ell (S_{\theta''}) \) (see [35]).

The notion of a modulus was given by Nakano [33], Maddox [29] used a modulus function to construct some sequence spaces. Afterwards, different sequence spaces defined by modulus have been studied by Altın and Et [2], Et et al. [16], Işık [27], Gaur and Muravaleen [24], Nuray and Savaş [34], Pehlivan and Fisher [37], Şengül [42] and many others.

2. Main Results

In this section, we will introduce the concepts of \( f \)-lacunary statistical convergence of order \( \alpha \) and strong \( f \)-lacunary summability of order \( \alpha \) of double sequences, where \( f \) is an unbounded modulus and also give some results related to these concepts.

**Definition 2.1.** Let \( f \) be an unbounded modulus, \( \theta'' = \{(k_r, l_s)\} \) be a double lacunary sequence and \( \alpha \) be a real number such that \( 0 < \alpha \leq 1 \). We say that the double sequence \( x = (x_{k,l}) \) is \( f \)-lacunary statistically convergent of order \( \alpha \), if there is a real number \( \ell \) such that

\[
\lim_{r,s \to \infty} \frac{1}{f(h_{r,s})^\alpha} \int \left| \{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\} \right| = 0.
\]

This space will be denoted by \( S_{\theta''}^{f,\alpha} \). In this case, we write \( S_{\theta''}^{f,\alpha} - \lim x_{k,l} = \ell \) or \( x_{k,l} \to \ell (S_{\theta''}^{f,\alpha}) \). In the special case \( \theta'' = \{(2^r, 2^s)\} \), we shall write \( S_{\theta''}^{f,\alpha} \) instead of \( S_{\theta''}^{f,\alpha} \).

**Definition 2.2.** Let \( f \) be a modulus function, \( \theta'' = \{(k_r, l_s)\} \) be a double lacunary sequence, \( p = (p_k) \) be a sequence of strictly positive real numbers and \( \alpha \) be a positive real number. We say that the double sequence \( x = (x_{k,l}) \) is strongly \( w^\alpha \left[ \theta'', f, p \right] \)-summable to \( \ell \) (a real number), if there is a real number \( \ell \) such that

\[
\lim_{r,s \to \infty} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} f \left( |x_{k,l} - \ell| \right)^{p_k} = 0.
\]
In this case we write $w^\alpha \left[ \theta'' , f , p \right] - \lim x_{k,l} = \ell$. The set of all strongly $w^\alpha \left[ \theta'' , f , p \right] -$ summable sequences will be denoted by $w^\alpha \left[ \theta'' , f , p \right]$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, we write $w^\alpha \left[ \theta'' , f \right]$ instead of $w^\alpha \left[ \theta'' , f , p \right]$.

**Definition 2.3.** Let $f$ be an unbounded modulus, $\theta'' = \{(k_r , l_s)\}$ be a double lacunary sequence, $p = (p_k)$ be a sequence of strictly positive real numbers and $\alpha$ be a positive real number. We say that the double sequence $x = (x_{k,l})$ is strongly $w_{\theta''}^{\alpha} (p)$—summable to $\ell$ (a real number), if there is a real number $\ell$ such that

$$
\lim_{r,s \to \infty} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} |f(|x_{k,l} - \ell|)|^{p_k} = 0.
$$

In the present case, we write $w_{\theta''}^{\alpha} (p) - \lim x_{k,l} = \ell$. The set of all strongly $w_{\theta''}^{\alpha} (p)$—summable sequences will be denoted by $w_{\theta''}^{\alpha} (p)$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $w_{\theta''}^{\alpha} (p)$ instead of $w_{\theta''}^{\alpha} (p)$.

**Definition 2.4.** Let $f$ be an unbounded modulus, $\theta'' = \{(k_r , l_s)\}$ be a double lacunary sequence, $p = (p_k)$ be a sequence of strictly positive real numbers and $\alpha$ be a positive real number. We say that the double sequence $x = (x_{k,l})$ is strongly $w_{\theta''}^{\alpha,f} (p)$—summable to $\ell$ (a real number), if there is a real number $\ell$ such that

$$
\frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell|^{p_k} = 0.
$$

In the present case, we write $w_{\theta''}^{\alpha,f} (p) - \lim x_{k,l} = \ell$. The set of all strongly $w_{\theta''}^{\alpha,f} (p)$—summable sequences will be denoted by $w_{\theta''}^{\alpha,f} (p)$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $w_{\theta''}^{\alpha,f} (p)$ instead of $w_{\theta''}^{\alpha,f} (p)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

**Theorem 2.1.** The space $w_{\theta''}^{\alpha,f} (p)$ is paranormed by

$$
g(x) = \sup_{r,s} \left\{ \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} |f(|x_{k,l}|)|^{p_k} \right\}^{\frac{1}{p_k}}
$$

where, $M = \max(1, H)$.

**Proposition 2.1.** ([37]) Let $f$ be a modulus and $0 < \delta < 1$. Then for each $\|u\| \geq \delta$, we have $f (\|u\|) \leq 2 f (1) \delta^{-1} \|u\|$. 
\textbf{Theorem 2.2.} Let $f$ be an unbounded modulus, $\alpha$ be a real number such that $0 < \alpha \leq 1$ and $p > 1$. If $\lim_{u \to \infty} \inf \frac{f(u)}{u} > 0$, then $w_{0^*}^{f,\alpha} [p] = w_{0^*}^{\alpha} [p]$.

\textit{Proof.} Let $p > 1$ be a positive real number and $x \in w_{0^*}^{f,\alpha} [p]$. If $\lim_{u \to \infty} \inf \frac{f(u)}{u} > 0$ then there exists a number $c > 0$ such that $f(u) > cu$ for $u > 0$. Clearly

$$\frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} |f(|x_{k,l} - \ell|)|^p \geq \frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} [c|x_{k,l} - \ell|]^p = \frac{c^p}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell|^p,$$

and therefore $w_{0^*}^{f,\alpha} [p] \subseteq w_{0^*}^{\alpha} [p]$.

Now let $x \in w_{0^*}^{\alpha} [p]$. Then we have

$$\frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell|^p \to 0 \text{ as } r, s \to \infty.$$

Let $0 < \delta < 1$. We can write

$$\frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell|^p \geq \frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s} \atop |x_{k,l} - \ell| \geq \delta} |x_{k,l} - \ell|^p \geq \frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s} \atop |x_{k,l} - \ell| \geq \delta} \left[ \frac{f(|x_{k,l} - \ell|)}{2f(1)\delta^{-1}} \right]^p \geq \frac{\delta^p}{[f(h_{r,s})]^\alpha} 2^p f(1)^p \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell|^p$$

by Proposition 2.1. Therefore $x \in w_{0^*}^{f,\alpha} [p]$.

If $\lim_{u \to \infty} \inf \frac{f(u)}{u} = 0$, the equality $w_{0^*}^{f,\alpha} [p] = w_{0^*}^{\alpha} [p]$ cannot be hold as shown in the following example:

Let $f(x) = 2\sqrt{x}$ and define a double sequence $x = (x_{k,l})$ by

$$x_{k,l} = \begin{cases} \frac{3}{2} \sqrt{h_{r,s}}, & \text{if } k = k_r \text{ and } l = l_s, \\ 0, & \text{otherwise} \end{cases} \quad r, s = 1, 2, \ldots.$$ 

For $\ell = 0$, $\alpha = \frac{3}{4}$ and $p = \frac{6}{5}$, we have

$$\frac{1}{[f(h_{r,s})]^\alpha} \sum_{(k,l) \in I_{r,s}} |f(|x_{k,l}|)|^p = \left( \frac{2 [h_{r,s}]^{\frac{3}{4}}}{(2 \sqrt{h_{r,s}})^{\frac{3}{4}}} \right)^p \to 0 \quad \text{as } r, s \to \infty.$$
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hence \( x \in w^{f,\alpha}_{\theta''} [p] \), but

\[
\frac{1}{[f(h)]^\alpha} \sum_{(k,l) \in I_{r,s}} |x_{k,l}|^p = \left( \frac{(\sqrt{h_{r,s}})^{\frac{p}{\alpha}}}{(2\sqrt{h_{r,s}})^{\frac{p}{\alpha}}} \right) \to \infty \quad \text{as} \quad r, s \to \infty
\]

and so \( x \notin w^{\alpha,\theta''} f [p] \). \( \square \)

Maddox [29] showed that the existence of an unbounded modulus \( f \) for which there is a positive constant \( c \) such that

\( f(xy) \geq cf(x)f(y), \) for all \( x \geq 0, y \geq 0 \).

**Theorem 2.3.** Let \( f \) be an unbounded modulus and \( \alpha \) be a positive real number. If \( \lim_{u \to \infty} \frac{f(u)^{\alpha}}{u^\alpha} > 0 \), then \( w^{\alpha} \left[ \theta'', f \right] \subset S_{\theta''}^{f,\alpha} \).

**Proof.** Let \( x \in w^{\alpha} \left[ \theta'', f \right] \) and \( \epsilon > 0 \) be given and \( \sum_1, \sum_2 \) denote the sums over \( (k,l) \in I_{r,s} \), \( |x_{k,l} - \ell| \geq \epsilon \) and \( (k,l) \in I_{r,s} \), \( |x_{k,l} - \ell| < \epsilon \) respectively. Since

\[
\frac{1}{[h_{r,s}]^\alpha} \sum_{(k,l) \in I_{r,s}} f \left( |x_{k,l} - \ell| \right) \geq \frac{1}{[h_{r,s}]^\alpha} f \left( \sum_{(k,l) \in I_{r,s}} |x_{k,l} - \ell| \right) \geq \frac{1}{[h_{r,s}]^\alpha} f \left( \sum_{(k,l) \in I_{r,s}, |x_{k,l} - \ell| \geq \epsilon} |x_{k,l} - \ell| \right) \geq \frac{c}{[h_{r,s}]^\alpha} f \left( \sum_{(k,l) \in I_{r,s}, |x_{k,l} - \ell| \geq \epsilon} \right) \frac{1}{f(h)} f(\epsilon).
\]

Therefore, \( w^{\alpha} \left[ \theta'', f \right] - \lim x_{k,l} = \ell \) implies \( S_{\theta''}^{f,\alpha} - \lim x_{k,l} = \ell \). \( \square \)

**Theorem 2.4.** Let \( \alpha_1, \alpha_2 \) be two real numbers such that \( 0 < \alpha_1 \leq \alpha_2 \leq 1 \), \( f \) be an unbounded modulus function and let \( \theta'' = \{(k_r, l_s)\} \) be a double lacunary sequence, then we have \( w^{f,\alpha_2}_{\theta''} (p) \subset S_{\theta''}^{f,\alpha_2} \).

**Proof.** Let \( x \in w^{f,\alpha_1}_{\theta''} (p) \) and \( \epsilon > 0 \) be given and \( \sum_1, \sum_2 \) denote the sums over \( (k,l) \in I_{r,s} \), \( |x_{k,l} - \ell| \geq \epsilon \) and \( (k,l) \in I_{r,s} \), \( |x_{k,l} - \ell| < \epsilon \) respectively.
\[ f(h_{r,s})^{\alpha_1} \leq f(h_{r,s})^{\alpha_2} \] for each \( r \) and \( s \), we may write

\[
\begin{aligned}
1 \cdot \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} & = 1 \cdot \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} + \sum_{2} [f(|x_{k,l} - \ell|)]^{p_k} \\
& \geq 1 \cdot \sum_{(k,l) \in I_{r,s}} [f(\varepsilon)]^{p_k} \\
& \geq H. \cdot [f(h_{r,s})^{\alpha_2}] \cdot \sum_{1} [f(\varepsilon)]^{p_k} \\
& \geq H. \cdot [f(h_{r,s})^{\alpha_2}] \cdot \sum_{1} \min([\varepsilon]^h, [\varepsilon]^H) \\
& \geq c \cdot H. \cdot [f(h_{r,s})^{\alpha_2}] \cdot \sum_{1} \left( \min([\varepsilon]^h, [\varepsilon]^H) \right).
\end{aligned}
\]

Hence \( x \in S_{\theta''}^{I,\alpha_2} \). \( \square \)

**Theorem 2.5.** Let \( \theta'' = \{(k_r, l_s)\} \) be a double lacunary sequence and \( \alpha \) be a fixed real number such that \( 0 < \alpha < 1 \). If \( \liminf_{r} q_r > 1 \), \( \liminf_{s} q_s > 1 \) and \( \lim_{u \to \infty} \frac{f(u)\alpha}{u\alpha} > 0 \), then \( S''^{I,\alpha} \subset S_{\theta''}^{I,\alpha} \).

**Proof.** Suppose first that \( \liminf_{r} q_r > 1 \) and \( \liminf_{s} q_s > 1 \); then there exists \( a, b > 0 \) such that \( q_r \geq 1 + a \) and \( q_s \geq 1 + b \) for sufficiently large \( r \) and \( s \), which implies that

\[
\frac{h_r}{k_r} \geq \frac{a}{1 + a} \Rightarrow \left( \frac{h_r}{k_r} \right)^\alpha \geq \left( \frac{a}{1 + a} \right)^\alpha
\]

and

\[
\frac{\ell_r}{l_s} \geq \frac{b}{1 + b} \Rightarrow \left( \frac{\ell_r}{l_s} \right)^\alpha \geq \left( \frac{b}{1 + b} \right)^\alpha.
\]

If \( S''^{I,\alpha} - \lim x_{k,l} = \ell \), then for every \( \varepsilon > 0 \) and for sufficiently large \( r \) and \( s \), we
Hence we have
\[
\frac{1}{[f(k_r, l_s)]^\alpha} f (\{ k \leq k_r, l \leq l_s : |x_{k,l} - \ell| \geq \varepsilon \}) \leq \frac{1}{[f(h_r, l_s)]^\alpha} f (\{ k \leq k_r, l \leq l_s : |x_{k,l} - \ell| \geq \varepsilon \}) \geq \frac{1}{[f(h_r, l_s)]^\alpha} \frac{k_r^\alpha} {k_r^\alpha} \frac{h_r^\alpha} {h_r^\alpha} \frac{f (\{ k \leq k_r, l \leq l_s : |x_{k,l} - \ell| \geq \varepsilon \})}{[f(h_r)]^\alpha} \geq \frac{1}{[f(h_r, l_s)]^\alpha} \frac{(k_r/l_s)^\alpha} {f(h_r, l_s)^\alpha} \frac{h_r^\alpha} {h_r^\alpha} \frac{f (\{ k \leq k_r, l \leq l_s : |x_{k,l} - \ell| \geq \varepsilon \})}{[f(h_r)]^\alpha}. \]

This proves the sufficiency. \(\square\)

**Theorem 2.6.** Let \( f \) be an unbounded modulus, \( \theta = (k_r) \) and \( \theta' = (l_s) \) be two lacunary sequences, \( \theta'' = \{ (k_r, l_s) \} \) be a double lacunary sequence and \( 0 < \alpha \leq 1 \). If \( S_{f, \theta}^\alpha - \lim x_k = \ell \) and \( S_{f, \theta'}^\alpha - \lim x_l = \ell \), then \( S_{f, \theta''}^\alpha - \lim x_{k,l} = \ell \).

**Proof.** Suppose \( S_{f, \theta}^\alpha - \lim x_k = \ell \) and \( S_{f, \theta'}^\alpha - \lim x_l = \ell \). Then for \( \varepsilon > 0 \) we can write
\[
\lim_{r} \frac{1}{[f(h_r)]^\alpha} \{ k \in I_r : |x_k - \ell| \geq \varepsilon \} = 0
\]
and
\[
\lim_{s} \frac{1}{[f(h_s)]^\alpha} \{ l \in I_s : |x_l - \ell| \geq \varepsilon \} = 0.
\]
So we have
\[
\frac{1}{[f(h_r, l_s)]^\alpha} |\{ (k, l) \in I_r, s : |x_{k,l} - \ell| \geq \varepsilon \}| \leq \frac{1}{[c f(h_r)]^\alpha} \{ (k, l) \in I_r, s : |x_{k,l} - \ell| \geq \varepsilon \} \leq \frac{1}{c^\alpha [f(h_r)]^\alpha} \{ (k, l) \in I_r, s : |x_{k,l} - \ell| \geq \varepsilon \} \leq \left[ \frac{1}{[f(h_r)]^\alpha} \{ k \in I_r : |x_k - \ell| \geq \varepsilon \} \right] \left[ \frac{1}{[f(h_s)]^\alpha} \{ l \in I_s : |x_l - \ell| \geq \varepsilon \} \right].
\]
Hence \( S_{f, \theta''}^\alpha - \lim x_{k,l} = \ell \). \(\square\)
Theorem 2.7. Let $f$ be an unbounded modulus. If $\lim p_k > 0$, then $w_{\theta_0}^{f, \alpha} (p) - \lim x_{k,l} = \ell$ uniquely.

Proof. Let $\lim p_k = s > 0$. Assume that $w_{\theta_0}^{f, \alpha} (p) - \lim x_{k,l} = \ell_1$ and $w_{\theta_0}^{f, \alpha} (p) - \lim x_{k,l} = \ell_2$. Then

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} = 0,$$

and

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} = 0.$$

By definition of $f$, we have

$$\frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} \leq \frac{D}{[f(h_{r,s})]^{\alpha}} \left( \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} \right)$$

$$= \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k}$$

where $\sup_k p_k = H$ and $D = \max (1, 2^{H-1})$. Hence

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l) \in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} = 0.$$

Since $\lim_{k \to \infty} p_k = s$ we have $\ell_1 - \ell_2 = 0$. Thus the limit is unique. $\square$

Theorem 2.8. Let $\theta_1' = \{(r_1, l_s)\}$ and $\theta_2' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and $\alpha_1, \alpha_2$ two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If

$$\lim_{r,s \to \infty} \inf \frac{[f(h_{r,s})]^{\alpha_1}}{[f(h_{r,s})]^{\alpha_2}} > 0$$

then $w_{\theta_1'}^{f, \alpha_2} (p) \subset w_{\theta_1'}^{f, \alpha_1} (p)$, where $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $k_{r,s} = k_{r}l_{s}$, $h_{r,s} = h_{r}h_{s}$ and $J_{r,s} = \{(s,t) : s_{r-1} < s \leq s_r \text{ and } t_{s-1} < t \leq t_s\}$, $s_{r,s} = s_{r}t_{s}$, $\ell_{r,s} = \ell_{r} \ell_{s}$. 

\[\text{(2.1)}\]
Proof. Let $x \in w_{\theta_2}^{f,\alpha_2}(p)$. We can write
\[
\frac{1}{[f((r,s))]^{\alpha_2}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} = \frac{1}{[f((r,s))]^{\alpha_2}} \sum_{(k,l) \in J_{r,s}-I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} \n
+ \frac{1}{[f((r,s))]^{\alpha_2}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} \n
\geq \frac{1}{[f((r,s))]^{\alpha_2}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k} \n
\geq \frac{[f((h_{r,s})]^{\alpha_1}}{[f((r,s))]^{\alpha_2}} \frac{1}{[f((h_{r,s})]^{\alpha_1}} \sum_{(k,l) \in I_{r,s}} [f(|x_{k,l} - \ell|)]^{p_k}.
\]
Thus if $x \in w_{\theta_2}^{f,\alpha_2}(p)$, then $x \in w_{\theta_1}^{f,\alpha_1}(p)$.

From Theorem 2.8, we have the following results.

Corollary 2.1. Let $\theta''_1 = \{(k_r,l_s)\}$ and $\theta''_2 = \{(s_r,t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and $\alpha_1, \alpha_2$ two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If (2.1) holds then
\begin{itemize}
  \item[(i)] $w_{\theta_2}^{f,\alpha}(p) \subset w_{\theta_1}^{f,\alpha}(p)$, if $\alpha_1 = \alpha_2 = \alpha$,
  \item[(ii)] $w_{\theta_2}^{f,\alpha}(p) \subset w_{\theta_1}^{f,\alpha_1}(p)$, if $\alpha_2 = 1$,
  \item[(iii)] $w_{\theta_2}^{f}(p) \subset w_{\theta_1}^{f}(p)$, if $\alpha_1 = \alpha_2 = 1$.
\end{itemize}

REFERENCES


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