ON $I_2$-CONVERGENCE AND $I_2$-CAUCHY DOUBLE SEQUENCES OF FUNCTIONS IN 2-NORMED SPACES

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Abstract. In this study, firstly, we studied some properties of $I_2$-convergence. Then, we introduced $I_2$-Cauchy and $I^*_2$-Cauchy sequence of double sequences of functions in 2-normed space. Also, we investigated the relationships between them for double sequences of functions in 2-normed spaces.

Keywords: $I_2$-Convergence, $I_2$-Cauchy, Double sequences of Functions, 2-normed Spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [15] and Schoenberg [36]. Gökhan et al. [20] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. The idea of $I$-convergence was introduced by Kostyrko et al. [28] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subset of $\mathbb{N}$ [15, 16]. Gezer and Karakuş [19] investigated $I$-pointwise and uniform convergence and $I^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [5] investigated $I$-convergence and $I^*$-continuity of real functions. Das et al. [7] introduced the concept of $I$-convergence of double sequences in a metric space and studied some properties of this convergence. Dündar and Altay [8, 10] studied the concepts of pointwise and uniformly $I_2$-convergence and $I^*_2$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [13] investigated some results of $I_2$-convergence of double sequences of functions. Also, a lot of development has been made about double sequences of functions (see [9, 11, 14, 30, 34, 40–42]).
The concept of 2-normed spaces was initially introduced by Gähler [17,18] in the 1960’s. Statistical convergence and statistical Cauchy sequence of functions in 2-normed space were studied by Yegül and Dündar [43]. Yegül and Dündar [44] introduced concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Also, Yegül and Dündar [45] introduced concepts of $I_2$-convergence and $I^*_2$-convergence of double sequences of functions in 2-normed space. Recently, Arslan and Dündar [1,2] introduced $I$-convergence and $I$-Cauchy sequences of functions in 2-normed spaces. Furthermore, there has been a lot of development in this area (see [3,4,6,26,27,29,31–33,37–39]).

2. Definitions and Notations

Now, we recall the concept of density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [1,2,7,12,16,18–25,28,31,35,43–45]).

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-norm on $X$ is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if $x$ and $y$ are linearly dependent.

(ii) $\|x,y\| = \|y,x\|$.

(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors $x$ and $y$ which may be given explicitly by the formula $\|x,y\| = |x_1y_2 - x_2y_1|; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$.

In this study, we suppose $X$ to be a 2-normed space having dimension $d$; where $2 \leq d < \infty$.

Throughout the paper, we let $X$ and $Y$ be two 2-normed spaces, $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences of functions and $f,g$ be two functions from $X$ to $Y$.

The sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is said to be convergent to $f$ if $f_n(x) \to f(x)(\|\cdot,\cdot\|_Y)$ for each $x \in X$. We write $f_n \to f(\|\cdot,\cdot\|_Y)$. This can be expressed by the formula $(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x),y\| < \varepsilon$.

A family of sets $\mathcal{I} \subseteq 2^\mathbb{N}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^\mathbb{N}$ is called a filter if and only if
(i) \( \emptyset \notin \mathcal{I} \).  (ii) For each \( A, B \in \mathcal{I} \) we have \( A \cap B \in \mathcal{I} \).  (iii) For each \( A \in \mathcal{I} \) and each \( B \supseteq A \) we have \( B \in \mathcal{I} \).

\( \mathcal{I} \) is nontrivial ideal in \( \mathbb{N} \) if and only if \( \mathcal{F}(\mathcal{I}) = \{ M \subseteq \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \setminus A) \} \) is a filter in \( \mathbb{N} \).

A nontrivial ideal \( \mathcal{I}_2 \) of \( \mathbb{N} \times \mathbb{N} \) is called strongly admissible ideal if \( \{ i \} \times \mathbb{N} \) and \( \mathbb{N} \times \{ i \} \) belong to \( \mathcal{I}_2 \) for each \( i \in \mathbb{N} \).

Throughout the paper we take \( \mathcal{I}_2 \) as a strongly admissible ideal in \( \mathbb{N} \times \mathbb{N} \).

It is evident that a strongly admissible ideal is admissible also.

\( \mathcal{I}_2^2 = \{ A \subseteq \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A) \} \). Then \( \mathcal{I}_2^2 \) is a strongly admissible ideal and clearly an ideal \( \mathcal{I}_2 \) is strongly admissible if and only if \( \mathcal{I}_2^2 \subseteq \mathcal{I}_2 \).

The sequence of functions \( \{ f_n \} \) is said to be \( \mathcal{I} \)-convergent (pointwise) to \( f \), if for each \( \varepsilon > 0 \) and each nonzero \( z \in Y A(\varepsilon, z) = \{ n \in \mathbb{N} : \| f_n(x) - f(x), z \| \geq \varepsilon \} \in \mathcal{I} \) or \( \mathcal{I} = \lim_{n \to \infty} \| f_n(x) - f(x), z \| \leq \varepsilon \), for each \( x \in X \). This can be expressed by the formula \((\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathcal{I})(\forall n \in \mathbb{N} \setminus M)(\forall x \in X)(\forall \varepsilon \geq m_0)\| f_n(x) - f(x), z \| \leq \varepsilon \). In this case, we write \( f_n \to f \) \( ||\cdot||_Y \).

The sequence of functions \( \{ f_n \} \) is said to be \( \mathcal{I}^* \)-convergent (pointwise sense) to \( f \), if there exists a set \( M \in \mathcal{F}(\mathcal{I}) \), (i.e., \( \mathbb{N} \setminus M \in \mathcal{I} \)), \( M = \{ m_1 < m_2 < \cdots < m_k < \cdots \} \), such that for each \( x \in X \) and each nonzero \( z \in Y \) \( \lim_{k \to \infty} \| f_{n_k}(x), z \| = \| f(x), z \| \) and we write \( \mathcal{I}^* = \lim_{n \to \infty} \| f_n(x), z \| = \| f(x), z \| \) or \( f_n \to f \) \( ||\cdot||_Y \).

The sequence of functions \( \{ f_n \} \) is said to be \( \mathcal{I}^* \)-Cauchy sequence, if for each \( \varepsilon > 0 \) and each nonzero \( z \in Y \) there exists \( s = s(\varepsilon, x) \in \mathbb{N} \) such that \( \{ n \in \mathbb{N} : \| f_n(x) - f_s(x), z \| \geq \varepsilon \} \in \mathcal{I} \), for each nonzero \( z \in X \).

The sequence of functions \( \{ f_n \} \) is said to be \( \mathcal{I}^* \)-convergent (pointwise sense) to \( f \), if there exists a set \( M = \{ m_1 < m_2 < \cdots < m_k < \cdots \} \subseteq \mathbb{N} \), such that the subsequence \( \{ f_{n_k} \} = \{ f_{m_k} \} \) is a Cauchy sequence, i.e., \( \lim_{k, n \to \infty} \| f_{m_k}(x) - f_{m_n}(x), z \| = 0 \), for each \( x \in X \) and each nonzero \( z \in Y \).

An admissible ideal \( \mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}} \) satisfies the property (AP2) if for every countable family of mutually disjoint sets \( \{ E_1, E_2, \ldots \} \) belonging to \( \mathcal{I}_2 \), there exists a countable family \( \{ F_1, F_2, \ldots \} \) such that \( E_j \Delta F_j \in \mathcal{I}_2 \), i.e., \( E_j \Delta F_j \) is included in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \) for each \( j \in \mathbb{N} \) and \( F = \bigcup_{j=1}^{\infty} F_j \subseteq \mathcal{I}_2 \) (hence \( F_j \in \mathcal{I}_2 \) for each \( j \in \mathbb{N} \)).

Throughout the paper, we let \( \mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal, \( X \) and \( Y \) be two 2-normed spaces, \( \{ f_{mn} \}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \), \( \{ g_{mn} \}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \) and \( \{ h_{mn} \}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \) be three double sequences of functions, \( f, g \), and \( k \) are three functions from \( X \) to \( Y \).

A double sequence \( \{ f_{mn} \} \) is said to be convergent (pointwise) to \( f \) if, for each point \( x \in X \) and for each \( \varepsilon > 0 \), there exists a positive integer \( k_0 = k_0(x, \varepsilon) \) such that for all \( m, n \geq k_0 \) implies \( \| f_{mn}(x) - f(x), z \| < \varepsilon \), for every \( z \in Y \). In this case, we write \( f_{mn} \to f \) \( ||\cdot||_Y \).

The double sequence of functions \( \{ f_{mn} \} \) is said to be \( \mathcal{I}_2 \)-convergent (pointwise
sense) to $f$, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in X$. This can be expressed by the formula

$$(\forall z \in Y) (\forall x \in X) (\forall \varepsilon > 0) (\exists H \in \mathcal{I}_2) (\forall (m, n) \notin H) \|f_{mn}(x) - f(x), z\| < \varepsilon.$$

In this case, we write $\mathcal{I}_2 - \lim_{m,n \to \infty} f_{mn}(x), z = \|f(x), z\|$, or $f_{mn} \to \mathcal{I}_2^* f(\|\cdot\|, \|\cdot\| Y)$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X, \|\cdot\|, \|\cdot\| Y)$ is said to be $\mathcal{I}_2^*$-convergent (pointwise) to $f$, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ $(H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2)$ such that for each $x \in X$, each nonzero $z \in Y$ and all $(m, n) \in M$

$$\lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

and we write $\mathcal{I}_2^* - \lim_{m,n \to \infty} f_{mn}(x), z = \|f(x), z\|$ or $f_{mn} \to \mathcal{I}_2^* f(\|\cdot\|, \|\cdot\| Y)$.

**Lemma 2.1.** [45] For each $x \in X$ and nonzero $z \in Y$,

$$\mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

**Lemma 2.2.** [45] Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be an admissible ideal having the property $(AP2)$.

For each $x \in X$ and nonzero $z \in Y$,

$$\mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I}_2^* - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

**Lemma 2.3.** [11] Let $\{P_i\}_{i=1}^\infty$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $\{P_i\}_{i=1}^\infty \subset F(\mathcal{I}_2)$ for each $i$, where $F(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in F(\mathcal{I}_2)$ and the set $P \setminus P_i$ is finite for all $i$.

**Lemma 2.4.** [45] For each $x \in X$ and each nonzero $z \in Y$, if

$$\mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathcal{I}_2 - \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

then

(i) $\mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|,$

(ii) $\mathcal{I}_2 - \lim_{m,n \to \infty} \|c f_{mn}(x), z\| = \|c f(x), z\|$, $c \in \mathbb{R},$

(iii) $\mathcal{I}_2 - \lim_{m,n \to \infty} \|f_{mn}(x) g_{mn}(x), z\| = \|f(x) g(x), z\|.$

**3. Main Results**

In this study, firstly, we studied some properties of $\mathcal{I}_2$-convergence. Then, we introduced $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy sequence of double sequences of functions in 2-normed space. Also, we investigated relationships between them for double sequences of functions in 2-normed spaces.
Theorem 3.1. Let $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal with the property (AP2). Then, for each $x \in X$ and each nonzero $z \in Y$, following conditions are equivalent

(i) $I_2 - \underbrace{\lim}_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$

(ii) There exists $\{g_{mn}(x)\}$ and $\{h_{mn}(x)\}$ be two sequences of functions from $X$ to $Y$ such that $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, $\lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|$ and $\text{supp}{h_{mn}(x)} \in I_2$,

where $\text{supp}{h_{mn}(x)} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\}$.

Proof.

(i) $\Rightarrow$ (ii): $I_2 - \underbrace{\lim}_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, by Lemma 2.2 there exists a set $M \in F(I_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \setminus M \in I_2$) such that for each $x \in X$, each nonzero $z \in Y$ and all $(m, n) \in M$

$$\lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Let us define the sequence $\{g_{mn}(x)\}$ by

$$g_{mn}(x) = \begin{cases} f_{mn}(x), & (m, n) \in M, \\ f(x), & (m, n) \in \mathbb{N} \times \mathbb{N} \setminus M. \end{cases}$$

(3.1)

It is clear that $\{g_{mn}(x)\}$ is a double sequence of functions and $\lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Besides let

$$h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \ (m, n) \in \mathbb{N} \times \mathbb{N}$$

(3.2)

for each $x \in X$. Since

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\} \subset \mathbb{N} \times \mathbb{N} \setminus M \in I_2,$$

for each $x \in X$, so we have

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in I_2.$$

It follows that $\text{supp}{h_{mn}(x)} \in I_2$ and by (3.1) and (3.2) we get $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, for each $x \in X$.

(ii) $\Rightarrow$ (i): Assume that there exist two sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ such that

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \ \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|$$

and $\text{supp}{h_{mn}(x)} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in I_2$

(3.3)

for each $x \in X$ and each nonzero $z \in Y$. We show that $I_2 - \underbrace{\lim}_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Let

$$M = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) = 0\} = \mathbb{N} \times \mathbb{N} \setminus \text{supp}{h_{mn}(x)}.$$

(3.4)
Since supp $h_{mn}(x) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2$, then from (3.3) and (3.4), we have $M \in \mathcal{F}(\mathcal{I}_2)$ and $f_{mn}(x) = g_{mn}(x)$ for $(m, n) \in M$. Hence, we conclude that exists a set $M \in \mathcal{F}(\mathcal{I}_2)$, (i.e., $H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$) such that $\lim_{m, n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and so

$$\mathcal{I}_2^* - \lim_{m, n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for $(m, n) \in M$, each $x \in X$ and each nonzero $z \in Y$. By Lemma 2.2 it follows that

$$\mathcal{I}_2^* - \lim_{m, n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. This completes the proof.

Corollary 3.1. Let $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$. Then, $\mathcal{I}_2 - \lim_{m, n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$ if and only if there exist $\{g_{mn}\}$ and $\{h_{mn}\}$ be two sequences of functions from $X$ to $Y$ such that

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x), \quad \lim_{m, n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathcal{I}_2^* - \lim_{m, n \to \infty} \|h_{mn}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Proof. Let $\mathcal{I}_2 - \lim_{m, n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and $\{g_{mn}(x)\}$ is sequence defined by (3.1). Consider the sequence

$$h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \quad (m, n) \in \mathbb{N} \times \mathbb{N}$$

for each $x \in X$. Then, we have

$$\lim_{m, n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|$$

and since $\mathcal{I}_2$ is a strongly admissible ideal so

$$\mathcal{I}_2^* - \lim_{m, n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. By Lemma 2.4 and by (3.5) we have

$$\mathcal{I}_2^* - \lim_{m, n \to \infty} \|h_{mn}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$. Now, let

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x),$$

where

$$\lim_{m, n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathcal{I}_2^* - \lim_{m, n \to \infty} \|h_{mn}(x), z\| = 0,$$
for each $x \in X$ and each nonzero $z \in Y$. Since $I_2$ is a strongly admissible ideal so

$$I_2 - \lim_{m,n \to \infty} \|g_{mn}(x), z\| = \|f(x), z\|$$

and by Lemma 2.4 we get

$$I_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. □

**Remark 3.1.** In Theorem 3.1 if (ii) is satisfied then the admissible ideal $I_2$ need not have the property (AP2). Since for each $x \in X$ and each nonzero $z \in Y$,

$$\{(m,n) \in N \times N : \|h_{mn}(x), z\| \geq \varepsilon\} \subset \{(m,n) \in N \times N : h_{mn}(x) \neq 0\} \in I_2,$$

for each $\varepsilon > 0$, then

$$I_2 - \lim_{m,n \to \infty} \|h_{mn}(x), z\| = 0.$$

Hence, we have

$$I_2 - \lim_{m,n \to \infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.

**Definition 3.1.** A double sequence of functions $\{f_{mn}\}$ is said to be $I_2$-Cauchy sequence, if for every $\forall \varepsilon > 0$ and each $x \in X$ there exist $s = s(\varepsilon, x), t = t(\varepsilon, x) \in N$ such that

$$\{(m,n) \in N \times N : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in I_2,$$

for each nonzero $z \in Y$.

**Theorem 3.2.** If $\{f_{mn}\}$ is $I_2$-convergent if and only if it is $I_2$-Cauchy sequence in 2-normed spaces.

**Proof.** Assume that $\{f_{mn}\}$ is $I_2$-convergent to $f$. Then, for $\varepsilon > 0$

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in N \times N : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in I_2,$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in N \times N : \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \in F(I_2).$$

for each $x \in X$ and each nonzero $z \in Y$ and thus $A^c\left(\frac{\varepsilon}{2}, z\right)$ is non-empty. So we can select a positive integers $k, l$ such that $(k, l) \notin A\left(\frac{\varepsilon}{2}, z\right)$ and $\|f_{kl}(x) - f(x), z\| < \frac{\varepsilon}{2}$. Now, we define the set

$$B(\varepsilon, z) = \{(m,n) \in N \times N : \|f_{mn}(x) - f_{kl}(x), z\| \geq \varepsilon\},$$
for each \( x \in X \) and each nonzero \( z \in Y \), such that we show that \( B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z) \).

Let \( (m, n) \in B(\varepsilon, z) \), then we have

\[
\varepsilon \leq \|f_{mn}(x) - f_{kl}(x), z\| \leq \|f_{mn}(x) - f(x), z\| + \|f_{kl}(x) - f(x), z\| < \|f_{mn}(x) - f(x), z\| + \frac{\varepsilon}{2},
\]

for each \( x \in X \) and each nonzero \( z \in Y \). This implies that \( \frac{\varepsilon}{2} < \|f_{mn}(x) - f(x), z\| \) and so, \((m, n) \in A(\frac{\varepsilon}{2}, z) \). Hence, we have \( B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z) \) and so \( \{f_{mn}\} \) is \( \mathcal{I}_2 \)-Cauchy sequence.

Conversely, assume that \( \{f_{mn}\} \) is \( \mathcal{I}_2 \)-Cauchy sequence. We prove that \( \{f_{mn}\} \) is \( \mathcal{I}_2 \)-convergent. Let \( (\varepsilon_{pq}) \) be a strictly decreasing sequence of number converging to zero since \( \{f_{mn}\} \) is \( \mathcal{I}_2 \)-Cauchy sequence, there exist two strictly increasing sequences \( (k_p) \) and \( (l_q) \) of positive integers such that

\[
A(\varepsilon_{pq}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_p l_q}(x), z\| \geq \varepsilon_{pq}\} \in \mathcal{I}_2, (p, q = 1, 2, ...),
\]

for each \( x \in X \) and each nonzero \( z \in Y \). This implies that

\[
(p, q = 1, 2, ...), \text{ for each } x \in X \text{ and each nonzero } z \in Y. \text{ Let } p, q, s \text{ and } t \text{ be four positive integers such that } p \neq q \text{ and } s \neq t. \text{ By (3.6), both the sets }
\]

\[
C(\varepsilon_{pq}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_p l_q}(x), z\| \leq \varepsilon_{pq}\}
\]

and

\[
D(\varepsilon_{st}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_s l_t}(x), z\| < \varepsilon_{st}\}
\]

are non empty sets in \( \mathcal{F}(\mathcal{I}_2) \), for each \( x \in X \) and each nonzero \( z \in Y \). Since \( \mathcal{F}(\mathcal{I}_2) \) is a filter on \( \mathbb{N} \times \mathbb{N} \), so

\[
\emptyset \neq C(\varepsilon_{pq}, z) \cap D(\varepsilon_{st}, z) \in \mathcal{F}(\mathcal{I}_2).
\]

Therefore, for each pair \((p, q)\) and \((s, t)\) of positive integers with \( p \neq q \) and \( s \neq t \), we can select a pair \((m_{(p,q),(s,t)}, n_{(p,q),(s,t)})\) \( \in \mathbb{N} \times \mathbb{N} \) such that

\[
\|f_{m_{pq} n_{pq}}(x) - f_{k_p l_q}(x), z\| < \varepsilon_{pq} \text{ and } \|f_{m_{pq} n_{pq}}(x) - f_{k_s l_t}(x), z\| < \varepsilon_{st},
\]

for each \( x \in X \) and each nonzero \( z \in Y \). It follows that

\[
\|f_{k_p l_q}(x) - f_{k_s l_t}(x), z\| \leq \|f_{m_{pq} n_{pq}}(x) - f_{k_p l_q}(x), z\| + \|f_{m_{pq} n_{pq}}(x) - f_{k_s l_t}(x), z\| \leq \varepsilon_{pq} + \varepsilon_{st} \to 0,
\]

as \( p, q, s, t \to \infty \). This implies that \( \{f_{k_p l_q}\} \ (p, q = 1, 2, ...) \) is a Cauchy sequence and therefore it satisfies the Cauchy convergence criterion. Thus, the sequence \( \{f_{k_p l_q}\} \) converges to a limit \( f \) (say) i.e.,

\[
\lim_{p,q \to \infty} \|f_{k_p l_q}, z\| = \|f(x), z\|,
\]
for each \( x \in X \) and each nonzero \( z \in Y \). Also, we have \( \varepsilon_{pq} \to 0 \) as \( p, q \to \infty \), so for each \( \varepsilon > 0 \) we can choose positive integers \( p_0, q_0 \) such that

\[
(3.7) \quad \varepsilon_{pq_0} < \frac{\varepsilon}{2} \quad \text{and} \quad \|f_{k_{pq_0}} - f(x), z\| < \frac{\varepsilon}{2}, \quad (\text{for } p > p_0 \text{ and } q > q_0).
\]

Now, we define the set

\[
A(\varepsilon, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},
\]

for each \( x \in X \) and each nonzero \( z \in Y \). We prove that \( A(\varepsilon, z) \subset A(\varepsilon_{pq_0}, z) \). Let \((m, n) \in A(\varepsilon, z)\), then by second half of (3.7) we have

\[
\varepsilon < \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2} + \|f_{k_{pq_0}} - f(x), z\| + \frac{\varepsilon}{2},
\]

for each \( x \in X \) and each nonzero \( z \in Y \). This implies that

\[
\frac{\varepsilon}{2} < \|f_{mn}(x) - f_{k_{pq_0}}(x), z\|
\]

and therefore by first half of (3.7)

\[
\varepsilon_{pq_0} < \|f_{mn}(x) - f_{k_{pq_0}}(x), z\|
\]

for each \( x \in X \) and each nonzero \( z \in Y \). Thus, we have \((m, n) \in A(\varepsilon_{pq_0}, z)\) and therefore \( A(\varepsilon, z) \subset A(\varepsilon_{pq_0}, z) \). Since \( A(\varepsilon_{pq_0}, z) \in \mathcal{I}_2 \) so \( A(\varepsilon, z) \in \mathcal{I}_2 \) by property of ideal. Hence \( \{f_{k_{pq_0}}\} \) is \( \mathcal{I}_2 \)-convergent. \( \square \)

**Definition 3.2.** A double sequence of functions \( \{f_{mn}\} \) is said to be \( \mathcal{I}_2 \)-Cauchy sequence, if there exists a set \( M \in \mathcal{F}(\mathcal{I}_2) \) (i.e., \( H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2 \)) and for every \( \varepsilon > 0 \) and each \( x \in X \), \( k_0 = k_0(\varepsilon, x) \in \mathbb{N} \) such that for all \( (m, n), (s, t) \in M \) and each \( z \in Y \)

\[
\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,
\]

whenever \( m, n, s, t > k_0 \). In this case, we write

\[
\lim_{m,n,s,t \to \infty} \|f_{mn}(x) - f_{st}(x), z\| = 0.
\]

**Theorem 3.3.** If double sequence of functions \( \{f_{mn}\} \) is a \( \mathcal{I}_2 \)-Cauchy sequence, then it is \( \mathcal{I}_2 \)-Cauchy sequence in 2-normed spaces.

**Proof.** Let \( \{f_{mn}\} \) is a \( \mathcal{I}_2 \)-Cauchy sequence in 2-normed spaces. Then, by definition there exists a set \( M \in \mathcal{F}(\mathcal{I}_2) \) (i.e., \( H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2 \)) and for every \( \varepsilon > 0 \) and each \( x \in X \), \( k_0 = k_0(\varepsilon, x) \in \mathbb{N} \) such that for all \( (m, n), (s, t) \in M \) and each \( z \in Y \)

\[
\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,
\]

for each \( x \in X \) and each nonzero \( z \in Y \). Also, we have \( \varepsilon_{pq} \to 0 \) as \( p, q \to \infty \), so
whenever \( m, n, s, t > k_0 \). Then, for each \( x \in X \) and nonzero each \( z \in Y \) we have

\[
A(\varepsilon, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\}
\]

\[
\subset H \cup \{M \cap ((1, 2, 3, ..., (k_0 - 1)) \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, 3, ..., (k_0 - 1)\})\}
\]

Since \( I_2 \) is an admissible ideal, then

\[
H \cup \{M \cap ((1, 2, 3, ..., (k_0 - 1)) \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, 3, ..., (k_0 - 1)\})\} \in I_2.
\]

Therefore, we have \( A(\varepsilon, z) \in I_2 \) i.e., \( \{f_{mn}\} \) is a \( I_2 \)-Cauchy sequence. \( \square \)

**Theorem 3.4.** If \( I_2^* - \lim_{m,n \to \infty} \|f_{mn}(x) - f(x), z\| = 0 \), then \( \{f_{mn}\} \) is \( I_2 \)-Cauchy sequence in 2-normed spaces.

**Proof.** By assumption there exists a set \( M \in F(I_2) \) (i.e., \( H = \mathbb{N} \times \mathbb{N} M \in I_2 \)) such that \( \lim_{m,n \to \infty} \|f_{mn}(x) - f(x), z\| = 0 \) for each \( x \in X \) and each \( z \in Y \). It shows that for each \( \varepsilon > 0 \) there exists \( k_0 = k_0(\varepsilon, x) \in \mathbb{N} \) such that for each \( x \in X \), each \( z \in Y \)

\[
\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}
\]

for all \((m, n) \in M \) and \( m, n > k_0 \). Since for each \( \varepsilon > 0 \),

\[
\|f_{mn}(x) - f_{st}(x), z\| \leq \|f_{mn}(x) - f(x), z\| + \|f_{st}(x) - f(x), z\|
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

for each \( x \in X \), each \( z \in Y \) and \( m, n, s, t \geq k_0 \) we have

\[
\lim_{m,n,s,t \to \infty} \|f_{mn}(x) - f_{st}(x), z\| = 0,
\]

i.e., \( \{f_{mn}\} \) is a \( I_2^* \)-Cauchy sequence. Then, by Theorem 3.3 \( \{f_{mn}\} \) is \( I_2 \)-Cauchy sequence. \( \square \)

**Theorem 3.5.** Let \( I_2 \) be an admissible ideal with property (AP2) and a double sequence of functions \( \{f_{mn}\} \). Then, the concepts \( I_2 \)-Cauchy double sequence and \( I_2^* \)-Cauchy double sequence of functions coincide in 2-normed spaces.

**Proof.** By Theorem 3.3 \( I_2^* \)-Cauchy sequence implies \( I_2 \)-Cauchy sequence (in this case \( I_2 \) need not to have (AP2) condition).

Now, it is sufficient to prove that a double sequence \( \{f_{mn}\} \) is a \( I_2^* \)-Cauchy double sequence under assumption that \( \{f_{mn}\} \) is a \( I_2 \)-Cauchy double sequence. Let \( \{f_{mn}\} \) is a \( I_2 \)-Cauchy double sequence. Then, for every \( \varepsilon > 0 \) and each \( x \in X \) there exists \( s = s(\varepsilon, z), t = t(\varepsilon, z) \in \mathbb{N} \) such that

\[
A(\varepsilon, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in I_2
\]
for each nonzero $z \in Y$. Let 

$$P_i = \left\{ (m, n) \in N \times N : \|f_{mn}(x) - f_{si,ti}(x), z\| < \frac{1}{i} \right\}, \quad (i = 1, 2,...),$$

where $s = s(\frac{1}{i})$, $t = t(\frac{1}{i})$. It is clear that 

$$P_i \in F(I_2), \quad (i = 1, 2,...).$$

Since $I_2$ has $(AP2)$ property then by Lemma 2.3 there exists a set $P \subset N \times N$ such that $P \in F(I_2)$, and $P \setminus P_i$ is finite for all $i$. Now we show that 

$$\lim_{m,n,s,t \to \infty} \|f_{mn}(x) - f_{st}(x), z\| = 0$$

for each $x \in X$, $(m, n), (s, t) \in P$ and each nonzero $z \in Y$. Let $\varepsilon > 0$ and $j \in N$ such that $j > \frac{2}{\varepsilon}$, if $(m, n), (s, t) \in P$ then $P \setminus P_j$ is a finite set, so there exists $k = k(j)$ such that $(m, n), (s, t) \in P_j$ for all $m, n, s, t > k(j)$. Therefore, for each $x \in X$

$$\|f_{mn}(x) - f_{si,ti}(x), z\| < \frac{1}{j} \quad \text{and} \quad \|f_{st}(x) - f_{si,ti}(x), z\| < \frac{1}{j},$$

for each nonzero $z \in Y$ and all $m, n, s, t > k(j)$. Hence, for each $x \in X$ it follows that 

$$\|f_{mn}(x) - f_{st}(x), z\| \leq \|f_{mn}(x) - f_{si,ti}(x), z\| + \|f_{st}(x) - f_{si,ti}(x), z\| < \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon$$

for all $m, n, s, t > k(j)$ and each nonzero $z \in Y$. Therefore, for any $\varepsilon > 0$ and each $x \in X$ there exists $k = k(\varepsilon, x)$ such that for $m, n, s, t > k$ and $(m, n), (s, t) \in P \in F(I_2)$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for each nonzero $z \in Y$ and so, the sequence $\{f_{mn}\}$ is a $I_2$-Cauchy sequence in 2-normed space. □

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On $\mathcal{I}$-Convergence and $\mathcal{I}$-Cauchy Double Sequences


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