FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 4 (2021), 735–748
https://doi.org/10.22190/FUMI191022054B
**Original Scientific Paper**

# BOUNDARY VALUE PROBLEM FOR NONLINEAR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATION WITH HADAMARD FRACTIONAL INTEGRAL AND ANTI-PERIODIC CONDITIONS

**Abdelatif Boutiara**[1], **Maamar Benbachir**[2] **and Kaddour Guerbati**[1]

[1] **Laboratoire de Mathématiques et Sciences appliquées**
**University of Ghardaia, 47000, Algeria**
[2] **Département de Mathématiques, Université Saad Dahlab**
**Blida1, Algérie**

**Abstract.** The aim of this work is to study a class of boundary value problem including a fractional order differential equation involving the Caputo-Hadamard fractional derivative. Sufficient conditions will be presented to guarantee the existence and uniqueness of solution of this fractional boundary value problem. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.
**Key words:** fractional differential equation, fractional derivatives and integrals, boundary value problem.

## 1. Introduction

Fractional differential equations is a subject of the domain of mathematics, which are basically used to describe the comportment of several complex and nonlocal systems with memory. Due to the effective memory function of fractional derivative, they have been widely used to describe many physical phenomena such as flow in porous media and in fluid dynamic traffic model. Moreover, fractional differential equations been widely used in engineering, physics, chemistry, biology, and other

Received October 22, 2019, accepted: April 9, 2021
Communicated by Hari Mohan Srivastava
Corresponding Author: Maamar Benbachir, Département de Mathématiques, Université Saad Dahlab, Blida1, Algérie | E-mail: mbenbachir2001@gmail.com
2010 *Mathematics Subject Classification.* Primary 26A33 ; Secondary 34B25, 34B15

fields; see the monographs of Kilbas et al. [24], F. Jarad et al. [22], Miller and Ross [26], Samko et al. [28] and the papers of Delbosco and Rodino [17], Hazarika et al. [16], Diethelm et al. [18], El-Sayed [19], Kilbas and Marzan [23], Mainardi [25], H.M. Srivastava [29] and Podlubny et al. [27]. Moreover, several papers have been devoted to the study of the existence, stability, existence and uniqueness of solutions for fractional differential equations, among others we refer to the papers [2, 3, 4, 5, 7, 9, 10, 15, 16, 30, 31].

In 2008, Benchohra et al. [10] studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ ay(0) + by(T) = c, & \end{cases}$$

where $J := [0, T]$, $D^{\alpha}$ is the caputo fractional derivative of order $\alpha$, $(0 < \alpha < 1)$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $a, b, c$ are real constants with $a + b \neq 0$.

In 2017, Asghar Ahmadkhanlu [6] studied the existence and uniqueness of solutions of the following boundary value problem of fractional differential equation is considered:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ y(0) = \eta I^{\beta}y(\tau), 0 < \tau < 1. & \end{cases}$$

Where $J := [0, 1]$, $D^{\alpha}$ is the caputo fractional derivative of order $\alpha$, $(0 < \alpha < 1)$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\eta \in \mathbb{R}$, $I^{\beta}$, $0 < \beta < 1$, is the Riemman-Liouville fractional integral of order $\beta$.

In 2018, Benhamida et al. [12, 13], studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ ay(1) + by(T) = c, & \end{cases}$$

where $J := [1, T]$, $D^{\alpha}$ is the caputo-Hadamard fractional derivative of order $\alpha$, $(0 < \alpha < 1)$, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a, b, c$ are real constants with $a + b \neq 0$.

In 2018, Benhamida et al. [11], studied the existence of solutions to the boundary value problem for fractional order differential equations

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ y(0) + y(T) = b\int_0^T y(s)ds, bT \neq 2, & \end{cases}$$

where $J := [0,T], T > 0$, $D^{\alpha}$ is the Caputo fractional derivative of order $\alpha$, $(0 < \alpha < 1)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $b$ are real constants .

In 2018, Abdo et al. [1] discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ y(0) = b\int_0^1 y(s)ds + d. & \end{cases}$$

Where $J := [0,1]$, $0 < \alpha \le 1$, $\lambda \ge 0$, $d > 0$ , $D^{\alpha}$ is the standard Caputo fractional operator and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function .

In 2019, A. Ardjouni et al. [8] discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions:

$$\left\{ \begin{array}{ll} D_1^{\alpha}y(t) = f(t, y(t)), & \\ & t \in J, \\ y(1) = b\int_1^e y(s)ds + d, & \end{array} \right)$$

where $J := [1,e]$, $D_1^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0 < \alpha \le 1$, $\lambda \ge 0$, $d > 0$ and $f : J \times [0,\infty) \to [0,\infty)$ is a given continuous function.

Motivated by the studies above, among others, in this paper, we concentrate on the following boundary value problem, of nonlinear fractional differential equation with fractional integral as well as integer and fractional derivative:

$$ {}^{C}_{H}D^{r}_{1^+}x(t) = f(t, x(t)), \;\; t \in J := [1,T], \;\; 0 < r \le 1, \tag{1.1}$$

with fractional boundary conditions:

$$\alpha x(1) + \beta x(T) = \lambda I^q x(\eta) + \delta, \qquad q \in (0,1] \tag{1.2}$$

where ${}^{C}_{H}D^{r}_{1^+}$ denote the Caputo-Hadamard fractional derivative and $I^q$ denotes the standard Hadamard fractional integral. Throughout this paper, we always assume that $0 < r, q \le 1$, $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. $\alpha, \beta, \lambda, \delta$ are real constants, and $\eta \in (1,T)$.

The rest of the paper is organized as follows. We recall some basic concepts of fractional calculus and introduce the integral operator associated to the given problem in Sect.2. Existence results, which rely on Schauder's fixed point theorem nonlinear alternative for single valued maps, and Scheafer's fixed point theorem are given. Also, In Sect.3, we obtain uniqueness results by means of Boyd and Wong's and Banach's fixed point theorems. Example illustrating the obtained results are presented in Sect.4, and the paper concludes with some interesting observations in Sect.5.

## 2. Preliminaries and lemmas

At first, we recall some concepts on fractional calculus and present some additional properties that will be used later. For more details, we refer to [20, 22, 24, 32]. We present some basic definitions and results from fractional calculus theory.
Let $E = C([1,T],\mathbb{R})$ be the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ with the norm

$$\|u\| = \max_{t\in[1,T]} |u(t)|$$

Let bet the space

$$AC_\delta^n([a,b],\mathbb{R}) = \{h : [a,b] \to \mathbb{R} : \delta^{n-1}h(x) \in AC([a,b],\mathbb{R})\}.$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC([a,b],\mathbb{R})$ is the space of absolutely continuous functions on $[a,b]$.

**Definition 2.1.** (Hadamard fractional integral [24]) *The Hadamard fractional integral of order $\alpha > 0$ for a function $h : [1,+\infty) \to \mathbb{R}$ is defined as*

$$I_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\log \frac{t}{s})^{\alpha-1} h(s) \frac{ds}{s} \tag{2.1}$$

*where $\Gamma$ is the Gamma function.*

**Definition 2.2.** (Hadamard fractional derivative [24]) *For a function $h$ given on the interval $[1,+\infty)$, and $n-1 < \alpha < n$, the Hadamard derivative of order $\alpha$ is defined by*

$$\begin{aligned} D_{a^+}^{\alpha} h(t) &= \tfrac{1}{\Gamma(n-\alpha)} (t\tfrac{d}{dt})^n \textstyle\int_a^t (\log \tfrac{t}{s})^{n-\alpha-1} h(s) \tfrac{ds}{s} \\ &= \delta^n I_{a^+}^{n-\alpha} h(t). \end{aligned} \tag{2.2}$$

*where $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number $\alpha$ and $\delta = t\frac{d}{dt}$. provided the right integral converges.*

There is a recent generalization introduced by Jarad and al in [22], where the authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

**Definition 2.3.** (Caputo-Hadamard fractional derivative [22]) *Let $\alpha = 0$, and $n = [\alpha]+1$. If $h(x) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$ and*

$$AC_\delta^n[a,b] = \{h : [a,b] \to C : \delta^{n-1}h(x) \in AC[a,b]\}.$$

*The left-sided Caputo-type modification of left-Hadamard fractional derivatives of order $\alpha$ is given by*

$${}_H^C D_{a^+}^{\alpha} h(t) = D_{a^+}^{\alpha} \left( h(t) - \sum_{k=0}^{n-1} \frac{\delta^k h(a)}{k!} (\log \frac{t}{s})^k \right) \tag{2.3}$$

**Theorem 2.4.** (See [22]) *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $y(t) \in AC_\delta^n[a, b]$, where $0 < a < b < \infty$. Then ${}_H^C D_{a^+}^\alpha f(t)$ exist everywhere on $[a, b]$ and*
*(i) if $\alpha \notin \mathbb{N} - \{0\}$, ${}_H^C D_{a^+}^\alpha f(t)$ can be represented by*

$$\begin{aligned} {}_H^C D_{a^+}^\alpha h(t) &= \tfrac{1}{\Gamma(n-\alpha)} \int_a^t (\log \tfrac{t}{s})^{n-\alpha-1} \delta^n h(s) \tfrac{ds}{s} \\ &= I_{a^+}^{n-\alpha} \delta^n h(t). \end{aligned} \tag{2.4}$$

*(ii) if $\alpha \in \mathbb{N} - \{0\}$, then*

$${}_H^C D_{a^+}^\alpha h(t) = \delta^n h(t) \tag{2.5}$$

*In particular*

$${}_H^C D_{a^+}^0 h(t) = h(t) \tag{2.6}$$

*Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing $a$ by $0$ in formula (2.4) provided that $h(t) \in AC_\delta^n(\mathbb{R}^+)$. Thus one has*

$${}_H^C D_{a^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\log \frac{t}{s})^{n-\alpha-1} \delta^n h(s) \frac{ds}{s} \tag{2.7}$$

**Proposition 2.5.** (See [24]) *Let $\alpha > 0, \beta > 0, n = [\alpha] + 1$, and $a > 0$, then*

$$\begin{aligned} I_{a^+}^\alpha (\log \tfrac{t}{a})^{\beta-1}(x) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log \tfrac{x}{a})^{\beta+\alpha-1} \\ {}_H^C D_{a^+}^\alpha (\log \tfrac{t}{a})^{\beta-1}(x) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log \tfrac{x}{a})^{\beta-\alpha-1}, \beta > n, \\ {}_H^C D_{a^+}^\alpha (\log \tfrac{t}{a})^k &= 0, k = 0, 1, ..., n-1. \end{aligned} \tag{2.8}$$

**Theorem 2.6.** (See [20]) *Let $u(t) \in AC_\delta^n[a, b], 0 < a < b < \infty$ and $\alpha \geq 0, \beta \geq 0$, Then*

$$\begin{aligned} {}_H^C D_{a^+}^\alpha \left( I_{a^+}^\alpha u \right)(t) &= \left( I_{a^+}^{\beta-\alpha} u \right)(t), \\ {}_H^C D_{a^+}^\alpha \left( {}_H^C D_{a^+}^\beta u \right)(t) &= \left( {}_H^C D_{a^+}^{\alpha+\beta} u \right)(t). \end{aligned} \tag{2.9}$$

**Lemma 2.7.** (See [22]) *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $u(t) \in AC_\delta^n[a, b]$, then the Caputo-Hadamard fractional differential equation*

$${}_H^C D_{a^+}^\alpha u(t) = 0, \tag{2.10}$$

*has a solution:*

$$u(t) = \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{a} \right)^k, \tag{2.11}$$

*and the following formula holds:*

$$I_{a^+}^\alpha \left( {}_H^C D_{a^+}^\alpha u \right)(t) = u(t) + \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{a} \right)^k, \tag{2.12}$$

*where $c_k \in \mathbb{R}, k = 1, 2, ..., n-1$.*

## 3. Main Results

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo-Hadamard derivative.

**Definition 3.1.** *A function $x(t) \in AC^1_\delta(J, \mathbb{R})$ is said to be a solution of* (1.1), (1.2) *if $x$ satisfies the equation ${}^C_H D^r x(t) = f(t, x(t))$ on $J$ , and the conditions* (1.2).

For the existence of solutions for the problem (1.1), (1.2), we need the following auxiliary lemma.

**Lemma 3.2.** *Let $h : [1, +\infty) \to \mathbb{R}$ be a continuous function. A function $x$ is a solution of the fractional integral equation*

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\} \tag{3.1}$$

*if and only if $x$ is a solution of the fractional BVP*

$${}^C_H D^r x(t) = h(t), t \in J, r \in (0, 1] \tag{3.2}$$

$$\alpha x(1) + \beta x(T) = \lambda I^q x(\eta) + \delta, q \in (0, 1] \tag{3.3}$$

*Proof.* Assume $x$ satisfies (3.2). Then Lemma 2.7 (2.12) implies that

$$x(t) = I^r h(t) + c_1. \tag{3.4}$$

By applying the boundary conditions (3.3) in (3.4), we obtain

$$\alpha c_1 + \beta I^r h(T) + \beta c_1 = \lambda I^{r+q} h(\eta)) + c_1 \frac{\lambda (\log \eta)^q}{\Gamma(q+1)} + \delta.$$

Thus,

$$c_1 \left(\alpha + \beta - \frac{\lambda(\log \eta)^q}{\Gamma(q+1)}\right) = \lambda I^{r+q} h(\eta)) - \beta I^r h(T) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta)) - \beta I^r h(T) + \delta\right\},$$

where,

$$\Lambda = \left(\alpha + \beta - \frac{\lambda(\log \eta)^q}{\Gamma(q+1)}\right).$$

Finally, we obtain the solution (3.1)

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\}.$$

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1), (1.2) by using a variety of fixed point theorems.

### 3.1. Existence and uniqueness result via Banach's fixed point theorem

**Theorem 3.3.** *Assume the following hypothesis:*
*(H1) There exists a constant $L > 0$ such that*

$$|f(t, x) - f(t, y)| \leq L|x - y|.$$

*If*

$$LM < 1, \tag{3.5}$$

*with*

$$M := \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\},$$

*then the problem (1.1) has a unique solution on $J$.*

*Proof.* Transform the problem 1.1), (1.2) into a fixed point problem for the operator $\mathfrak{F}$ defined by

$$\mathfrak{F}x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\}. \tag{3.6}$$

Applying the Banach contraction mapping principle, we shall show that $\mathfrak{F}$ is a contraction.

Now let $x, y \in C(J, \mathbb{R})$. Then, for $t \in J$, we have

(3.7)

Thus

$$\|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)\|_\infty \leq LM\|x - y\|_\infty.$$

We deduce that $\mathfrak{F}$ is a contraction mapping. As a consequence of Banach contraction principle. the problem (1.1)-(1.2) has a unique solution on $J$. This completes the proof. □

### 3.2. Existence result via Schaefer's fixed point theorem

**Theorem 3.4.** *Assume the hypotheses:*
*(H2): The function $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.*
*(H3) There exists a constant $K > 0$, such that*

$$|f(t, 0)| \leq K, \text{ for a.e. } t \in J.$$

*Then, the problem (1.1)-(1.2) has a least one solution in $J$.*

*Proof.* We shall use Schaefer's fixed point theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. The proof will be given in several steps.
**Step 1:** $\mathfrak{F}$ is continuous Let $x_n$ be a sequence such that $x_n \to x$ in $C(J, \mathbb{R})$. Then for each $t \in J$ ,

$$\begin{aligned}\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\| &\leq \tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &\leq\left\{\tfrac{(\log T)^r}{\Gamma(r+1)}+\tfrac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\times\\ &\quad\|f(s,x_n(s))-f(s,x(s)).\|\end{aligned}$$

Since $f$ is continuous, we have $\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\|_\infty\to 0$ as $n\to\infty$.

**Step 2:** $\mathfrak{F}$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$
Indeed, it is enough to show that for any $r>0$, we take

$$u\in B_r=\{x\in C(J,\mathbb{R}),\|x\|_\infty\leq r\}.$$

From (H1) and (H3), Then we have

$$|f(s,x(s))|\leq|f(s,x(s))-f(t,0)|+|f(t,0)|\leq Lr+K.$$

For $x\in B_r$ and for each $t\in[1,T]$, we have

$$\begin{aligned}|(\mathfrak{F}x)(t)|&\leq\tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s}+\tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}|f(s,x(s))|\tfrac{ds}{s}\\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq\tfrac{Lr+K}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}\tfrac{ds}{s}+\tfrac{|\lambda|(Lr+K)}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}\tfrac{ds}{s}\\ &+\tfrac{|\beta|(Lr+K)}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}\tfrac{ds}{s}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq(Lr+K)\left\{\tfrac{(\log T)^r}{\Gamma(r+1)}+\tfrac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq(Lr+K)M+\tfrac{|\delta|}{|\Lambda|}.\end{aligned}$$

Thus,

$$\|(\mathfrak{F}x)(t)\|\leq(Lr+K)M+\frac{|\delta|}{|\Lambda|}.$$

**Step 3:** $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1,t_2\in J,t_1<t_2$, $B_r$ be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $x\in B_r$. Then

$$\begin{aligned}\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|&\leq\tfrac{1}{\Gamma(r)}\int_1^{t_1}\left[(\log\tfrac{t_2}{s})^{r-1}-(\log\tfrac{t_1}{s})^{r-1}\right]\|f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{1}{\Gamma(r)}\int_{t_1}^{t_2}(\log\tfrac{t_2}{s})^{r-1}\|f(s,x(s))\|\tfrac{ds}{s}\\ &\leq\tfrac{Lr+K}{\Gamma(r)}\int_1^{t_1}\left[(\log\tfrac{t_2}{s})^{r-1}-(\log\tfrac{t_1}{s})^{r-1}\right]\tfrac{ds}{s}+\tfrac{K}{\Gamma(r)}\int_{t_1}^{t_2}(\log\tfrac{t_2}{s})^{r-1}\tfrac{ds}{s}\\ &\leq\tfrac{Lr+K}{\Gamma(r+1)}\left[(\log t_2)^r-(\log t_1)^r\right],\end{aligned}$$

which implies $\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|_\infty\to 0$ as $t_1\to t_2$, as consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{F}$ is continuous and completely continuous.

**Step 4**: A priori bounds.
Now it remains to show that the set

$$\Lambda = \{x \in C(J,\mathbb{R}) : x = \rho\mathfrak{F}(x) \text{ for some } 0 < \rho < 1\}$$

is bounded.

For such a $x \in \Lambda$. Thus, for each $t \in J$, we have

$$\begin{aligned} x(t) &\leq \rho \Big\{ \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} f(s,x(s)) \tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} f(s,x(s)) \tfrac{ds}{s} \\ &+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} f(s,x(s)) \tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|} \Big\} \end{aligned}$$

For $\rho \in [0,1]$, let $x$ be such that for each $t \in J$

$$\begin{aligned} \|\mathfrak{F}x(t)\| &\leq \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} |f(s,x(s))| \tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} |f(s,x(s))| \tfrac{ds}{s} \\ &+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} |f(s,x(s))| \tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|} \\ &\leq (Lr + K)M + \tfrac{|\delta|}{|\Lambda|}. \end{aligned}$$

Thus

$$\|\mathfrak{F}x(t)\| \leq \infty$$

This implies that the set $\Lambda$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathfrak{F}$ has a fixed point which is a solution on $J$ of the problem (1.1)-(1.2). $\square$

### 3.3. Existence via the Leray-Schauder nonlinear alternative

**Theorem 3.5.** *Assume the following hypotheses:*
*(H4) There exist $\omega \in L^1(J,\mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that*

$$|f(t,x)| \leq \omega(t)\psi(\|x\|), \text{ for a.e. } t \in J \text{ and each } x \in \mathbb{R}.$$

*(H5) There exists a constant $\epsilon > 0$ such that*

$$\frac{\epsilon}{\|\omega\|\psi(\epsilon)M + \frac{|\delta|}{|\Lambda|}} > 1.$$

*Then the boundary value problem (1.1)-(1.2) has at least one solution on $J$.*

*Proof.* We shall use the Leray-Schauder theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. As shown in Theorem 3.4, we see that the operator $\mathfrak{F}$ is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the

Arzela-Ascoli theorem $\mathfrak{F}$ is completely continuous.
Let $x$ be such that for each $t \in J$, we take the equation $x = \lambda \operatorname{Im} x$ for $\lambda \in (0,1)$ and let $x$ be a solution. After that, the following is obtained.

$$\begin{aligned} |x(t)| &\leq \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} \\ &+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|} \\ &\leq \|\omega\|\psi(\|x\|)M + \tfrac{|\delta|}{|\Lambda|}. \end{aligned}$$

and consequently

$$\frac{\|x\|_\infty}{\|\omega\|\psi(\|x\|)M + \frac{|\delta|}{|\Lambda|}} \leq 1.$$

Then by condition (H5), there exists $\epsilon$ such that $\|x\|_\infty \neq \epsilon$. Let us set

$$\kappa = \{x \in C(J,\mathbb{R}) : \|x\| < \epsilon\}.$$

Obviously, the operator $\operatorname{Im} : \overline{\kappa} \to C(J,\mathbb{R})$ is completely continuous. From the choice of $\kappa$ , there is no $x \in \partial\kappa$ such that $x = \lambda \operatorname{Im}(x)$ for some $\lambda \in (0,1)$ . As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathfrak{F}$ has a fixed point $x \in \kappa$ which is a solution of the (1.1)-(1.2).
The proof is completed. □

Now we present another variant of existence-uniqueness result.

### 3.4. Existence and uniqueness result via Boyd-Wong nonlinear contraction

**Definition 3.6.** *Assume that $E$ is a Banach space and $T : E \to E$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property: $\|Tx - Ty\| \leq \psi(\|x-y\|), \forall x, y \in E$. then, we say that $T$ is a nonlinear contraction.*

**Theorem 3.7.** *(Boyd-Wong Contraction Principle)[14]*
*Suppose that $B$ is a Banach space and $T : B \to B$ is a nonlinear contraction. Then $T$ has a unique fixed point in $B$.*

**Theorem 3.8.** *Assume that $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $H > 0$ satisfying the condition*

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{H + |x-y|}, \text{ for } t \in J, x, y \in \mathbb{R}. \tag{3.8}$$

*Then the fractional BVP (1.1)-(1.2) has a unique solution on $J$ .*

*Proof.* We define an operator $\mathfrak{F} : \chi \to \chi$ as in (3.6) and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \forall \varepsilon > 0,$$

where $M \leq H$ . We notice that the function $\psi$ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in \chi$ , and for each $t \in J$ , we obtain

$$\begin{aligned}
|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)| &\leq \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} \|f(s,x(s)) - f(s,y(s))\| \tfrac{ds}{s} \\
&+ \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} \|f(s,x(s)) - f(s,y(s))\| \tfrac{ds}{s} \\
&+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} \|f(s,x(s)) - f(s,y(s))\| \tfrac{ds}{s} \\
&\leq \tfrac{|x-y|}{H+|x-y|} \left\{ \tfrac{(\log T)^r}{\Gamma(r+1)} + \tfrac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \\
&:= M \tfrac{|x-y|}{H+|x-y|} \\
&\leq \psi(\|x-y\|).
\end{aligned}$$

Then, we get $\|\mathfrak{F}x - \mathfrak{F}y\| \leq \psi(\|x-y\|)$. Hence, $\mathfrak{F}$ is a nonlinear contraction. Thus, by Theorem 3.9 (Boyd-Wong Contraction Principle) the operator $\mathfrak{F}$ has a unique fixed point which is the unique solution of the fractional BVP (1.1)-(1.2). The proof is completed. □

## 4. Example

We consider the problem for Caputo-Hadamard fractional differential equations of the form:

$$\text{(4.1)} \qquad \begin{cases} {}^{C}_{H}D^{\frac{2}{3}} x(t) = f(t, x(t)), (t, x) \in ([1, e], \mathbb{R}^+), \\[2ex] x(1) + x(e) = \frac{1}{2} \left( I^{\frac{1}{2}} x(2) \right) + \frac{3}{4}. \end{cases}$$

Here

$$\begin{array}{llll} r = \frac{2}{3}, & q = \frac{1}{2}, & \alpha = 1, & \beta = 1, \\ \delta = \frac{3}{4}, & \lambda = \frac{1}{2}, & \eta = 2, & T = e. \end{array}$$

With

$$f(t, y(t)) = \frac{1}{t^2 + 4} cos x, \qquad t \in [1, e]$$

Clearly, the function $f$ is continuous.
For each $x \in \mathbb{R}^+$ and $t \in [1, e]$, we have

$$|f(t, x(t)) - f(t, y(t))| \leq \frac{1}{4} |x - y|$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{4}$.
Further,

$$M := \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \simeq 2.0286$$

and

$$LM \simeq 0.5071 < 1.$$

Therefore, by the conclusion of Theorem 3.3, It follows that the problem (4.1) has a unique solution defined on $[1, e]$.

## 5. Conclusion

In this paper, we have obtained some existence results for nonlinear Caputo-Hadamard type fractional differential equations with Hadamard integral boundary conditions by means of some standard fixed point theorems and nonlinear alternative of Leray-Schauder type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting some examples. Our results are new and generalize some available results on the topic. For instance,

✓ We remark that when $\alpha = \beta = 1$, $\lambda = 0$, problem (1.1)-(1.2) reduces to the case considered in [12, 13].

✓ If we take $\alpha = q = 1$, $\beta = 0$ , in (1.2), then our results correspond to the case integral boundary conditions considered in [8].

✓ By fixing $\beta = \lambda = 0$, in (1.2), our results correspond to the ones for initial value problem take the form:$x(1) = \delta$ .

✓ In case we choose $\alpha = \beta = 1$, $\lambda = \delta = 0$, in (1.2), our results correspond to anti-periodic type boundary conditions take the form: $x(1) = -x(T)$.

✓ When, $\alpha = \beta = 1$, $\delta = 0$, the (1.2), our results correspond to Fractional integral and anti-periodic type boundary conditions.

✓ If we take $\alpha = 1$, $\beta = \delta = 0$ , in (1.2), then our results correspond to the case Fractional integral boundary conditions.

In the nutshell, the boundary value problem studied in this paper is of fairly general nature and covers a variety of special cases.

## REFERENCES

1. M. A. Abdo, H. A. Wahash and S. K. Panchat: *Positive solutions of a fractional differential equation with integral boundary conditions.* Journal of Applied Mathematics and Computational Mechanics **17**(3) (2018), 5-15.
2. R. P. Agarwal, M. Meehan and D. O'Regan: *Fixed Point Theory and Applications, Cambridge Tracts in Mathematics*, 141, Cambridge University Press, Cambridge, 2001.

3. B. Ahmad, M. Alghanmi, H. M. Srivastava and S. K. Ntouyas: *The Langevin equation in terms of generalized Liouville-Caputo derivatives with nonlocal boundary conditions involving a generalized fractional integral*, Mathematics **7** (2019), Article ID 533,1-10.
4. B. Ahmad and S. K. Ntouyas: *On Hadamard fractional integro-differential boundary value problems*, J. Appl. Math. Comput **47** (2015).
5. B. Ahmad and S. K. Ntouyas: *Initial value problems of fractional order Hadamard-type functional differential equations*, Electron. J. Differ. Equ. **77** (2015).
6. A. Ahmadkhanlu: *Existence and Uniqueness Results for a Class of Fractional Differential Equations with an Integral Fractional Boundary Condition*, Filomat **31:5** (2017), 1241-1249.
7. A. Alsaedi, M. Alsulami, H. M. Srivastava, B. Ahmad and S. K. Ntouyas: *Existence theory for nonlinear third-order ordinary differential equations with nonlocal multi-point and multi-strip boundary conditions*, Symmetry 11(2019), Article ID 281,1-18.
8. A. Ardjouni and A. Djoudi: *Positive solutions for nonlinear Caputo-Hadamard fractional differential equations with integral boundary conditions*, Open J. Math. Anal. 2019, **3**(1), 62-69.
9. Z. Baitiche, M. Benbachir and K. Guerbati: *Solvability for multi-point bvp of nonlinear fractional differential equations at resonance with three dimensional kernels*, Kragujevac Journal of Mathematics Volume **45**(5) (2021), pp: 761–780.
10. M. Benchohra, S. Hamani and S. K. Ntouyas: *Boundary value problems for differential equations with fractional order*, Surveys in Mathematics and its Applications vol, **3** (2008), 1-12.
11. W. Benhamida, J. R. Graef and S. Hamani: *Boundary Value Problems for Fractional Differential Equations with Integral and Anti-Periodic Conditions in a Banach Space*, Progr. Fract. Differ. Appl. **4**, No. 2, 65-70 (2018)
12. W. Benhamida and S. Hamani: *Measure of Noncompactness and Caputo-Hadamard Fractional Differential Equations in Banach Spaces*, Eurasian Bulletin Of Mathematics EBM (2018), Vol. **1**, No. 3, 98-106
13. W. Benhamida, S. Hamani and J. Henderson: *Boundary Value Problems For Caputo-Hadamard Fractional Differential Equations*, Advances in the Theory of Nonlinear Analysis and its Applications**2** (2018) No. 3, 138-145.
14. D. W. Boyd and J. S. W. Wong: *On nonlinear contractions.* Proc. Am. Math. Soc. **20**, 458-464 (1969).
15. D. Boutiara, K. Guerbati and M. Benbachir: *Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces*; AIMS Mathematics, **5**(1): 259–272., (2019).
16. B. Hazarika, H. M. Srivastava, R. Arab and M. Rabbani: *Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it*, J. Comput. Appl. Math. **343** (2018), 341-352.
17. D. Delbosco and L. Rodino: *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **204** (1996), 609-625.
18. K. Diethelm and A. D. Freed: *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, Scientific Computing in Chemical*

*Engineering II. Computational Fluid Dynamics, Reaction Engineering and Molecular Properties (F. Keil, W. Mackens, H. Voss and J. Werther, eds.)*, Springer-Verlag, Heidelberg, 1999, pp. 217-224.

19. A. M. A. EL-Sayed and E. O. Bin-Taher: *Positive solutions for a nonlocal multi-point boundary-value problem of fractional and second order*, Electron. J. Differential Equations, Number **64**, (2013), 1-8.
20. Y. Gambo et al: *On Caputo modification of the Hadamard fractional derivatives*, Adv. Difference Equ. 2014 (2014), Paper No. 10, 12 p.
21. A. Granas and J. dugundji: *Fixed Point Theory*, Springer-Verlag, New York, 2003.
22. F. Jarad, D. Baleanu and A. Abdeljawad: *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Differ. Equ. 2012 (2012).
23. A. A. Kilbas and S. A. Marzan: *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differential Equations **41** (2005), 84-89.
24. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo: *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V. Amsterdam, 2006.
25. F. Mainardi: *Fractional calculus: some basic problems in continuum and statistical mechanics*, Fractals and Fractional Calculus in Continuum Mechanics (A. Carpinteri and F. Mainardi, eds.), Springer-Verlag, Wien, 1997, pp. 291-348.
26. K. S. Miller and B. Ross: *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
27. I. Podlubny, I. Petrás and B. M. Vinagre and P. O'Leary and L. Dorcak: *Analogue realizations of fractional-order controllers. Fractional order calculus and its applications*, Nonlinear Dynam. **29** (2002), 281-296.
28. S. G. Samko, A. A. Kilbas and O. I. Marichev: *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon 1993.
29. H.M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, Kyungpook Math. J. **60** (2020), 73-116
30. H.M. Srivastava, Diabetes and its resulting complications: Mathematical modeling via fractional calculus, Public Health Open Access **4** (3) (2020), Article ID 2, 1-5.
31. P. Thiramanus and S. K. Ntouyas and J. Tariboon: *Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions*, Abstr. Appl. Anal. (2014).
32. A. Yacine and B. Nouredine: *boundary value problem for Caputo-Hadamard fractional differential equations*, Surveys in Mathematics and its Applications, Volume **12** (2017), 103-115.
33. H. Zhang: *Nonlocal boundary value problems of fractional order at resonance with integral conditions.* Adv. Differ. Equ.2017, **326** (2017)