FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 35, No 5 (2020), 1273–1290
https://doi.org/10.22190/FUMI2005273C

ON THE FIXED-CIRCLE PROBLEM

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Abstract. In this paper, we focus on the geometric properties of fixed-points of a self-mapping and obtain new solutions to a recent problem called "fixed-circle problem" in the setting of an S-metric space. For this purpose, we develop various techniques by defining new contractive conditions and using some auxiliary functions. Furthermore, we present new examples to support our theoretical results.
Keywords: fixed-points; S-metric space; self-mapping.

1. Introduction

It is known that the fixed-point theory has been generalized by various approaches. One of these approaches is to generalize the used contractive condition (for example see [2], [5]). The other is to generalize the used metric space (see [1, 8, 21, 23] and the references therein). For example, in [21], Sedghi, Shobe and Aliouche presented the notion of an S-metric space as the generalization of a metric space. Then, some fixed-point theorems have been extensively studied on S-metric spaces (see [6, 7, 9, 13, 15, 18, 19, 21, 22, 24, 25, 27] for more details).

On the other hand, fixed-point theorems have been widely studied with different aspects such as the uniqueness of a fixed-point, common fixed point, etc. If a fixed point is not unique then the investigation of the geometric properties of fixed points of a self-mapping is an interesting problem. As a recent approach, the concept of a fixed circle and the fixed-circle problem have been presented on a metric (resp. an S-metric) space as a new direction of the generalization of known fixed-point results (see [17] and [16]). Then, new fixed circle theorems have been given by various techniques on metric (resp. S-metric) spaces (see [11, 12, 20, 26] for the metric case; [10, 14, 24, 25] for the S-metric case).

Our aim in this paper is to obtain new fixed-circle theorems for self-mappings on an S-metric space. In Section 2., we recall some basic facts about S-metric spaces.

Received October 31, 2019; accepted January 05, 2020
2020 *Mathematics Subject Classification.* Primary 47H10; Secondary 47H09, 54H25

In Section 3., we give new fixed-circle theorems by introducing new types of the notion of an F_c^S-contraction introduced and used in [10]. In Section 4., we investigate new existence and uniqueness theorems for fixed circles of self-mappings using some auxiliary functions and contractive conditions. We support our theoretical results by illustrative examples.

2. Preliminaries

In this section, we recall some necessary notions and results on S-metric spaces with new examples.

Definition 2.1. [21] Let X be a nonempty set and $\mathcal{S} : X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

1. $\mathcal{S}(x,y,z) = 0$ if and only if $x = y = z$,
2. $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$.

Then $\mathcal{S}$ is called an S-metric on X and the pair $(X,\mathcal{S})$ is called an S-metric space.

Example 2.1. [21] Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z) = |x-z| + |y-z|,$$

for all $x, y, z \in \mathbb{R}$ (or $\mathbb{C}$). Then the function $\mathcal{S} : X^3 \to [0,\infty)$ is an S-metric and it is called the usual S-metric on $\mathbb{R}$ (or $\mathbb{C}$).

Lemma 2.1. *[21] Let $(X,\mathcal{S})$ be an S-metric space and $x, y \in X$. Then we have*

$$\mathcal{S}(x,x,y) = \mathcal{S}(y,y,x).$$

It was given the relationships between a metric and an S-metric in the following lemma [7].

Lemma 2.2. *[7] Let (X,d) be a metric space. Then the following properties are satisfied:*

1. *$\mathcal{S}_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X,d) if and only if $x_n \to x$ in $(X,\mathcal{S}_d)$.*
3. *$\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in $(X,\mathcal{S}_d)$.*
4. *(X,d) is complete if and only if $(X,\mathcal{S}_d)$ is complete.*

The metric $\mathcal{S}_d$ was called as the S-metric generated by d in [13].

Now we give a new example of an S-metric generated by a metric.

Example 2.2. Let $X \neq \varnothing$, $d : X^2 \to [0, \infty)$ be any metric on X and the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \min\{1, d(x, z)\} + \min\{1, d(y, z)\}.$$

Then the function $\mathcal{S} : X^3 \to [0, \infty)$ is an S-metric on X and the pair $(X, \mathcal{S})$ is an S-metric space. Clearly, this S-metric $\mathcal{S}$ is generated by the metric m defined as $m(x, y) = \min\{1, d(x, y)\}$.

There are some examples of an S-metric which is not generated by any metric (see [7], [10], [14] and [13]). We give a new example.

Example 2.3. Let $X = \mathbb{R}$, $d : X^2 \to [0, \infty)$ be any metric on X and the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \min\{1, d(x, z)\} + |y - z|.$$

Then the function $\mathcal{S} : X^3 \to [0, \infty)$ is an S-metric on X which is not generated by any metric and the pair $(X, \mathcal{S})$ is an S-metric space. Conversely, assume that there exists a metric d_1 such that

$$\mathcal{S}(x, y, z) = d_1(x, z) + d_1(y, z),$$

for all $x, y, z \in X$. Then we obtain

$$\mathcal{S}(x, x, z) = 2d_1(x, z) \Rightarrow d_1(x, z) = \frac{1}{2}\min\{1, d(x, z)\} + \frac{1}{2}|x - z|$$

and

$$\mathcal{S}(y, y, z) = 2d_1(y, z) \Rightarrow d_1(y, z) = \frac{1}{2}\min\{1, d(y, z)\} + \frac{1}{2}|y - z|,$$

for all $x, y, z \in X$. So we get

$$\begin{aligned} \min\{1, d(x, z)\} + |y - z| \neq \tfrac{1}{2}\min\{1, d(x, z)\} + \tfrac{1}{2}|x - z| \\ +\tfrac{1}{2}\min\{1, d(y, z)\} + \tfrac{1}{2}|y - z|, \end{aligned}$$

which is a contradiction. Hence $\mathcal{S}$ is not generated by any metric.

Definition 2.2. [16] Let $(X, \mathcal{S})$ be an S-metric space. Then a circle and a disc are defined on an S-metric space as follows, respectively:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = r\}$$

and

$$D^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\}.$$

Example 2.4. Let X be a nonempty set, the function $d : X^2 \to [0, \infty)$ be any metric on X and the S-metric space $(X, \mathcal{S})$ be defined as in Example 2.2. Let us consider the circle $C^S_{x_0,r}$ according to the S-metric $\mathcal{S}$:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = 2\min\{1, d(x, x_0)\} = r\}.$$

Then we have the following cases:

Case 1 : If $r = 2$ then $C^S_{x_0,r} = \{x \in X : d(x, x_0) \geq 1\}$.

Case 2 : If $r > 2$ then $C^S_{x_0,r} = \emptyset$.

Case 3 : If $r < 2$ then $C^S_{x_0,r} = C_{x_0,\frac{r}{2}}$, where $C_{x_0,\frac{r}{2}} = \left\{x \in X : d(x, x_0) = \frac{r}{2}\right\}$.

Example 2.5. Let X be a nonempty set, the function $d : X^2 \to [0, \infty)$ be any metric on X and the S-metric space be defined as in Example 2.3. Let us consider the circle $C^S_{x_0,r}$ according to the S-metric:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = \min\{1, d(x, x_0)\} + |x - x_0| = r\}.$$

Then we have the following cases:

Case 1 : If $x \in (X \backslash D_{x_0,1}) \cup C_{x_0,1}$ then $C^S_{x_0,r} = \{x \in (X \backslash D_{x_0,1}) \cup C_{x_0,1} : |x - x_0| = r - 1\}$.

Case 2 : If $x \in D_{x_0,1} \backslash C_{x_0,1}$ then $C^S_{x_0,r} = \{x \in D_{x_0,1} \backslash C_{x_0,1} : d(x, x_0) + |x - x_0| = r\}$.

In the following example, the S-metric is not generated by any metric but any circle on this S-metric space is the same as the circle on the usual metric space $\mathbb{R}$(or $\mathbb{C}$).

Example 2.6. Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\},$$

for all $x, y, z \in X$. Then the function $\mathcal{S} : X^3 \to [0, \infty)$ is an S-metric on X which is not generated by any metric. For any circle $C^S_{x_0,r}$ on this S-metric space we have $C^S_{x_0,r} = \{x_0 - r, x_0 + r\}$ which is correspond to the circle $C_{x_0,r}$ with the equation $|y - x_0| = r$ on the usual metric space $\mathbb{R}$.

3. Fixed-Circle Theorems via New Types of F^S_c-contractions

In this section, we give new fixed-circle theorems using new types of the notion of an F^S_c-contraction introduced in [10]. At first, we recall the definition of a fixed-circle and the following family of functions which was introduced by Wardowski in [28].

Definition 3.1. [16] Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for every $x \in C^S_{x_0,r}$ then the circle $C^S_{x_0,r}$ is called as the fixed circle of T.

Definition 3.2. [28] Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that

$(F1)$ F is strictly increasing,

$(F2)$ For each sequence $\{\alpha_n\}$ in $(0, \infty)$ the following holds $\lim \alpha_n = 0$ if and only if $\lim F(\alpha_n) = -\infty$,

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Some functions that satisfy the conditions $(F1)$, $(F2)$ and $(F3)$ of Definition 3.2 are given in the following example (see [28] for more details).

Example 3.1. [28] The following functions defined by

$$F_1 : (0, \infty) \to \mathbb{R}, F_1(x) = \ln(x),$$

$$F_2 : (0, \infty) \to \mathbb{R}, F_2(x) = \ln(x) + x,$$

$$F_3 : (0, \infty) \to \mathbb{R}, F_3(x) = -\frac{1}{\sqrt{x}}$$

and

$$F_4 : (0, \infty) \to \mathbb{R}, F_4(x) = \ln(x^2 + x)$$

are the examples of Definition 3.2.

Using this family of functions, in [4], some new fixed-point theorems was obtained on S-metric spaces. In [10], it was introduced the following new contraction type to obtain some fixed-circle results on an S-metric space.

Definition 3.3. [10] Let $(X, \mathcal{S})$ be an S-metric space. A self-mapping T on X is said to be an F_c^S-contraction if there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow t + F(\mathcal{S}(Tx, Tx, x)) \leq F(\mathcal{S}(x, x, x_0)).$$

In [24], Suzuki-Berinde type F_c^S-contractions were introduced for the same purpose. Now we define new types of F_c^S-contractions to get new fixed-circle results. To do this, we use some classical contraction conditions such as Ćirić-type, modified Hardy-Rogers type and Khan-type contractive conditions.

Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. We will use the number r defined by

$$r = \inf\{\mathcal{S}(Tx, Tx, x) : x \in X, x \neq Tx\}, \tag{3.1}$$

in all of our results.

3.1. Ćirić type fixed-circle results on S-metric spaces

At first, we introduce the following Ćirić type F_c^S-contraction.

Definition 3.4. Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow t + F(\mathcal{S}(Tx, Tx, x)) \leq F(m(x, x, x_0)),$$

where

$$m(x,x,y)=\max\left\{\begin{array}{c}\mathcal{S}(x,x,y),\mathcal{S}(x,x,Tx),\mathcal{S}(y,y,Ty),\\ \frac{1}{2}[\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)]\end{array}\right\},$$

then the self-mapping T is called a Ćirić type F_c^S-contraction on X.

An immediate consequence of this definition is the following proposition.

Proposition 3.1. *Let $(X,\mathcal{S})$ be an S-metric space. If a self-mapping T on X is a Ćirić-type F_c^S-contraction with $x_0\in X$ then we have $Tx_0=x_0$.*

Proof. Assume that $Tx_0\neq x_0$. From the definition of a Ćirić-type F_c^S-contraction and Lemma 2.1, we get

$$\begin{aligned}\mathcal{S}(Tx_0,Tx_0,x_0) &> 0\Longrightarrow t+F[\mathcal{S}(Tx_0,Tx_0,x_0)]\leq F(m(x_0,x_0,x_0))\\ &= F\left(\max\left\{\begin{array}{c}\mathcal{S}(x_0,x_0,x_0),\mathcal{S}(x_0,x_0,Tx_0),\mathcal{S}(x_0,x_0,Tx_0),\\ \frac{1}{2}[\mathcal{S}(x_0,x_0,Tx_0)+\mathcal{S}(x_0,x_0,Tx_0)]\end{array}\right\}\right)\\ &= F(\mathcal{S}(x_0,x_0,Tx_0)).\end{aligned}$$

This is a contradiction by the fact that $t>0$. Then we have $Tx_0=x_0$. □

Using Ćirić type F_c^S-contractions, we give the following fixed-circle theorem.

Theorem 3.1. *Let $(X,\mathcal{S})$ be an S-metric space, T be a Ćirić type F_c^S-contractive self-mapping with $x_0\in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0)=r$ for all $x\in C_{x_0,r}^S$ then the circle $C_{x_0,r}^S$ is a fixed circle of T. In particular, T fixes every circle $C_{x_0,\rho}^S$ where $\rho<r$ if $\mathcal{S}(Tx,Tx,x_0)=\rho$ for all $x\in C_{x_0,\rho}^S$.*

Proof. Since $\mathcal{S}(Tx,Tx,x_0)=r$, the self-mapping T maps $C_{x_0,r}^S$ into (or onto) itself. Let $x\in C_{x_0,r}^S$ be an arbitrary point. If $Tx\neq x$, by the definition of r we have $\mathcal{S}(Tx,Tx,x)\geq r$. Hence, using the Ćirić-type F_c^S-contractive property, Lemma 2.1, Proposition 3.1 and the fact that F is increasing, we get

$$\begin{aligned}F(r) &\leq F(\mathcal{S}(Tx,Tx,x))\leq F(m(x,x,x_0))-t<F(m(x,x,x_0))\\ &= F\left(\max\left\{\begin{array}{c}\mathcal{S}(x,x,x_0),\mathcal{S}(x,x,Tx),\mathcal{S}(x_0,x_0,Tx_0),\\ \frac{1}{2}[\mathcal{S}(x,x,Tx_0)+\mathcal{S}(x_0,x_0,Tx)]\end{array}\right\}\right)\\ &= F(\max\{r,\mathcal{S}(x,x,Tx),0,r\})=F(\mathcal{S}(Tx,Tx,x)),\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{S}(Tx,Tx,x)=0$ and so $Tx=x$. Consequently, $C_{x_0,r}^S$ is a fixed circle of T.

Using the similar arguments, it is easy to see that T also fixes any circle $C_{x_0,\rho}^S$ where $\rho<r$. □

Remark 3.1. 1) Notice that, in Theorem 3.1, Ćirić type F_c^S-contractive self-mapping T fixes the disc $D_{x_0,r}^S$ if $\mathcal{S}(Tx,Tx,x_0)=\rho$ for all $x\in C_{x_0,\rho}^S$ and each $\rho\leq r$.

2) In Theorem 3.1, if $r=0$, then we have $C_{x_0,r}^S=\{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.1.

In the following example, we see that the converse statement of Theorem 3.1 is not always true.

Example 3.2. Let $X=\mathbb{C}$ be the S-metric space with the usual S-metric defined in Example 2.1, $z_0\in\mathbb{C}$ be any point and the self-mapping $T:X\to X$ be defined as

$$Tz=\begin{cases} z & , \quad |z-z_0|\leq\frac{\mu}{2} \\ z_0 & , \quad |z-z_0|>\frac{\mu}{2}\end{cases},$$

for all $z\in\mathbb{C}$ with $\mu>0$. We show that T is not a Ćirić-type F_c^S-contractive self-mapping. Indeed, if $|z-z_0|>\frac{\mu}{2}$ for $z\in\mathbb{C}$, then using Lemma 2.1 and the Ćirić-type F_c^S-contractive property, we get

$$\mathcal{S}(Tz,Tz,z)=\mathcal{S}(z_0,z_0,z)>0\Longrightarrow t+F(\mathcal{S}(z_0,z_0,z))\leq F(m(z,z,z_0)),$$

$$t+F(\mathcal{S}(z_0,z_0,z))\leq F(\mathcal{S}(z,z,z_0))$$

and so

$$t+F(r)\leq F(r)\Longrightarrow t\leq 0.$$

This is a contradiction since $t>0$. Hence T is not a Ćirić-type F_c^S-contractive self-mapping for any $z_0\in\mathbb{C}$. But T fixes every circle $C_{x_0,\rho}^S$ where $\rho\leq\mu$.

Now we give some illustrative examples of Theorem 3.1.

Example 3.3. Let $X=\{z\in\mathbb{C}:|z|=2\}$. Let us consider the S-metric $\mathcal{S}$ defined in Example 2.6 on X and define the self-mapping $T:X\to X$ by

$$Tz=\begin{cases} -2 & , \quad \frac{\pi}{3}\leq\arg(z)\leq\frac{\pi}{2} \\ z & , \quad otherwise\end{cases}.$$

Then the self-mapping T is a Ćirić-type F_c^S-contractive self-mapping with $F=\ln x$, $t=\ln\left(\frac{\sqrt{8+4\sqrt{3}}}{2\sqrt{3}}\right)$ and $z_0=-2i$. Indeed, we obtain

$$\begin{aligned} r &= \inf\{\mathcal{S}(z,z,Tz):z\in X, z\neq Tz\} \\ &= 2\sqrt{2}.\end{aligned}$$

In the case $\mathcal{S}(z,z,Tz)>0$, we find

$$\begin{aligned} m(z,z,-2i) &= \max\left\{\begin{array}{c}\mathcal{S}(z,z,-2i),\mathcal{S}(z,z,-2),\mathcal{S}(-2i,-2i,-2i),\\ \frac{1}{2}[\mathcal{S}(z,z,-2i)+\mathcal{S}(-2i,-2i,-2)]\end{array}\right\} \\ &= \max\left\{|z+2i|,|z+2|,0,\frac{1}{2}[|z+2i|+|2i-2|]\right\} \\ &= \sqrt{8+4\sqrt{3}}\end{aligned}$$

and hence

$$t+\ln(|z+2|)\leq \ln\left(\sqrt{8+4\sqrt{3}}\right).$$

Clearly, T fixes the circle $C^S_{-2i,2\sqrt{2}}=\{-2,2\}$ and the disc $D^S_{-2i,2\sqrt{2}}=\left\{z\in X:\mathcal{S}\left(z,z,-2i\right)\leq 2\sqrt{2}\right\}$.

3.2. Modified Hardy–Rogers type fixed-circle results on S-metric spaces

Now we introduce the following modified Hardy-Rogers type F_c^S-contraction.

Definition 3.5. Let $(X,\mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F\in\mathbb{F}$, $t>0$ and $x_0\in X$ such that for all $x\in X$ the following holds

$$\begin{aligned}\mathcal{S}(Tx,Tx,x) &> 0\Longrightarrow t+F(\mathcal{S}(Tx,Tx,x))\leq \\ & F\left[\begin{array}{c}\alpha\mathcal{S}(x,x,x_0)+\beta\mathcal{S}(Tx_0,Tx_0,x)+\gamma\mathcal{S}(Tx,Tx,x_0)\\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx,Tx,x)]}{[1+\mathcal{S}(Tx_0,Tx_0,x)]}+\lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx,Tx,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x_0,x_0,x)}\\ +\mu\frac{\mathcal{S}(Tx,Tx,x)[1+\mathcal{S}(Tx,Tx,x_0)]}{1+\mathcal{S}(x,x,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}\end{array}\right],\end{aligned}$$

where $\alpha+\beta+\gamma+\eta+\lambda+\mu<\frac{1}{2}$, $\alpha,\beta,\gamma,\eta,\lambda,\mu\geq 0$ and $a\neq 0$, then the self-mapping T is called a modified Hardy-Rogers type F_c^S-contraction on X.

Proposition 3.2. *Let $(X,\mathcal{S})$ be an S-metric space. If a self-mapping T on X is a modified Hardy-Rogers type F_c^S-contraction with $x_0\in X$ then we have $Tx_0=x_0$.*

Proof. Assume that $Tx_0\neq x_0$. By the hypothesis, we obtain

$$\begin{aligned}\mathcal{S}(Tx_0,Tx_0,x_0) &> 0\Longrightarrow t+F(\mathcal{S}(Tx_0,Tx_0,x_0))\leq \\ & F\left[\begin{array}{c}\alpha\mathcal{S}(x_0,x_0,x_0)+\beta\mathcal{S}(Tx_0,Tx_0,x_0)+\gamma\mathcal{S}(Tx_0,Tx_0,x_0)\\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}{[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}+\lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x_0,x_0,x_0)}\\ +\mu\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}{1+\mathcal{S}(x_0,x_0,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}\end{array}\right]\\ &= F\left[(\beta+\gamma+\eta+2\lambda+\mu)\mathcal{S}(Tx_0,Tx_0,x_0)\right]\\ &< F\left[\mathcal{S}(Tx_0,Tx_0,x_0)\right].\end{aligned}$$

This is a contradiction since $t>0$. Hence we get $Tx_0=x_0$. □

Now using the notion of a modified Hardy-Rogers type F_c^S-contraction condition, we prove the following fixed-circle theorem.

Theorem 3.2. *Let $(X,\mathcal{S})$ be an S-metric space, T be a modified Hardy-Rogers type F_c^S-contractive self-mapping with $x_0\in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0)=r$ for all $x\in C^S_{x_0,r}$ then $C^S_{x_0,r}$ is a fixed circle of T. In particular, T fixes every circle $C^S_{x_0,\rho}$ where $\rho<r$ if $\mathcal{S}(Tx,Tx,x_0)=\rho$ for all $x\in C^S_{x_0,\rho}$.*

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. If $r = 0$, then we have $C^S_{x_0,r} = \{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.2. Assume that $r > 0$. Using the modified Hardy-Rogers type F^S_c-contraction property, Proposition 3.2, Lemma 2.1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(\mathcal{S}(Tx,Tx,x)) \\ &\leq F\left[\begin{array}{c} \alpha\mathcal{S}(x,x,x_0) + \beta\mathcal{S}(Tx_0,Tx_0,x) + \gamma\mathcal{S}(Tx,Tx,x_0) \\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx,Tx,x)]}{[1+\mathcal{S}(Tx_0,Tx_0,x)]} + \lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx,Tx,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x,x,x_0)} \\ +\mu\frac{\mathcal{S}(Tx,Tx,x)[1+\mathcal{S}(Tx,Tx,x_0)]}{1+\mathcal{S}(x,x,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)} \end{array}\right] - t \\ &< F[\alpha r + \beta r + \gamma r + \lambda r + \mu\mathcal{S}(Tx,Tx,x)] \\ &\leq F[(\alpha+\beta+\gamma+\lambda+\mu)\mathcal{S}(Tx,Tx,x)] \\ &\leq F[\mathcal{S}(Tx,Tx,x)], \end{aligned}$$

which is a contradiction. Therefore, $\mathcal{S}(Tx,Tx,x) = 0$ and so $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T. Using the similar arguments, it is easy to see that T also fixes any circle $C^S_{x_0,\rho}$ where $\rho < r$. □

Remark 3.2. 1) Let (X,S) be an S-metric space, T be a modified Hardy-Rogers type F^S_c-contractive self-mapping with $x_0 \in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$ and each $\rho \leq r$, then T fixes the disc $D^S_{x_0,r}$.

2) Let us consider the self-mapping T given in Example 3.2. Then it can be easily seen that T is not a modified Hardy-Rogers type F^S_c-contractive self-mapping. But, T fixes every circle $C^S_{x_0,\rho}$ where $\rho \leq r$. Hence the converse statement of Theorem 3.2 is not always true.

Example 3.4. Let $X = \mathbb{R}^+$ and the S-metric $\mathcal{S}$ be the usual S-metric. Let us define the self-mapping $T : X \to X$ as

$$Tx = \left\{\begin{array}{ccc} 2x + \frac{4}{x} & , & x \in [1,4) \\ x & , & otherwise \end{array}\right.,$$

for all $x \in X$. Then the self-mapping T is a modified Hardy-Rogers type F^S_c-contractive self-mapping with $\alpha = \frac{1}{4}$, $\beta = \frac{1}{25}$, $\gamma = \frac{1}{25}$, $\lambda = \frac{1}{25}$, $\mu = \frac{1}{25}$, $F = \ln x$, $t = \ln\frac{9}{8}$ and $x_0 = 35$. Indeed, in the cases $\mathcal{S}(Tx,Tx,x) > 0$ we find

$$8 \leq \mathcal{S}(Tx,Tx,x) \leq 10$$

and

$$62 \leq \mathcal{S}(x,x,x_0) \leq 68$$

and hence

$$\begin{aligned} t + F\left(2\left|x + \frac{4}{x}\right|\right) &\leq F\left[2\alpha\left|x - 35\right|\right] \\ &\leq F\left[\begin{array}{c} 2\alpha\left|x-35\right| + 2\beta\left|x-35\right| + 2\gamma\left|Tx-35\right| \\ +\eta.0 + 2\lambda\left|Tx-35\right| \\ +\mu\frac{2\left|x+\frac{4}{x}\right|[1+|Tx-35|]}{1+2|x-35|} \end{array}\right]. \end{aligned}$$

Also we have

$$r = \inf \{\mathcal{S}(Tx, Tx, x) : x \neq Tx\} = 8.$$

Therefore, the self-mapping T fixes the circle $C^S_{35,8} = \{31, 39\}$ and the disc $D^S_{35,8} = \{x \in \mathbb{R}^+ : 31 \leq x \leq 39\}$.

3.3. Khan-type fixed-circle results on S-metric spaces

Now we introduce the following Khan-type F^S_c-contraction.

Definition 3.6. Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\begin{aligned} \mathcal{S}(Tx, Tx, x) &> 0 \Longrightarrow t + F(\mathcal{S}(Tx, Tx, x)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx, Tx, x)\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx_0, Tx_0, x)\mathcal{S}(Tx, Tx, x_0)}{\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0)}\right], \end{aligned}$$

where

$$h \in [0, 1), \mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0) \neq 0.$$

Then the self-mapping T is called Khan-type F^S_c-contraction on X.

Proposition 3.3. *Let $(X, \mathcal{S})$ be an S-metric space. If a self-mapping T on X is a Khan-type F^S_C-contraction with $x_0 \in X$. Then we have $Tx_0 = x_0$.*

Proof. Assume that $Tx_0 \neq x_0$. By the hypothesis, we have

$$\begin{aligned} \mathcal{S}(Tx_0, Tx_0, x_0) &> 0 \Longrightarrow t + F(\mathcal{S}(Tx_0, Tx_0, x_0)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx_0, Tx_0, x_0)\mathcal{S}(Tx_0, Tx_0, x_0) + \mathcal{S}(Tx_0, Tx_0, x_0)\mathcal{S}(Tx_0, Tx_0, x_0)}{\mathcal{S}(Tx_0, Tx_0, x_0) + \mathcal{S}(Tx_0, Tx_0, x_0)}\right] \\ &= F\left[h\frac{\mathcal{S}^2(Tx_0, Tx_0, x_0) + \mathcal{S}^2(Tx_0, Tx_0, x_0)}{2\mathcal{S}(Tx_0, Tx_0, x_0)}\right] \\ &= F\left[h\frac{2\mathcal{S}^2(Tx_0, Tx_0, x_0)}{2\mathcal{S}(Tx_0, Tx_0, x_0)}\right] \\ &< F\left[\mathcal{S}(Tx_0, Tx_0, x_0)\right], \end{aligned}$$

which is contradiction since $t > 0$. Then we have $Tx_0 = x_0$. □

Now using the notion of a Khan-type F^S_c-contraction condition, we prove the following fixed-circle theorem.

Theorem 3.3. *Let $(X, \mathcal{S})$ be an S-metric space, T be a Khan-type F^S_c-contraction with $x_0 \in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx, Tx, x_0) = r$ for all $x \in C^S_{x_0,r}$ then $C^S_{x_0,r}$ is a fixed circle of T. In particular, T fixes every circle $C^S_{x_0,\rho}$ with $\rho < r$ if $\mathcal{S}(Tx, Tx, x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$.*

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. If $r = 0$, then we have $C^S_{x_0,r} = \{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.3.

Assume that $r > 0$. Using the Khan-type F^S_C-contractive property, Proposition 3.3, Lemma 2.1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(\mathcal{S}(Tx,Tx,x)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}\right]-t \\ &< F\left[h\frac{\mathcal{S}(Tx,Tx,x)r+r^2}{2r}\right] = F\left[h\frac{\mathcal{S}(Tx,Tx,x)+r}{2}\right] \\ &\leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)+\mathcal{S}(Tx,Tx,x)}{2}\right] = F\left[h\mathcal{S}(Tx,Tx,x)\right] \\ &< F[\mathcal{S}(Tx,Tx,x)], \end{aligned}$$

which is a contradiction. Therefore we have $\mathcal{S}(Tx,Tx,x) = 0$ and so $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T.

By the similar arguments, it is easy to verify that T also fixes any circle $C^S_{x_0,\rho}$ where $\rho < r$. □

Remark 3.3. Notice that, in Theorem 3.3, Khan-type F^S_c-contractive self-mapping T fixes the disc $D^S_{x_0,r}$ if $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$ and each $\rho \leq r$. Therefore, the center of any fixed circle is also fixed by T.

Now we give the following illustrative example.

Example 3.5. Let $X = \{e^k : k \in \mathbb{N}\}$ and the S-metric be defined as in [14] such that

$$\mathcal{S}(x,y,z) = \left|\ln\frac{x}{y}\right| + \left|\ln\frac{xy}{z^2}\right|$$

for all $x,y,z \in X$ (see Example 2.6 on page 12 in [14]). Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} ex^2 & , \quad x \in \{e^1,e^2,e^3,e^4,e^5,e^6,e^7\} \\ x & , \quad \textit{otherwise} \end{cases},$$

for all $x \in X$. Then the self-mapping T is a Khan-type F^S_c-contractive self-mapping with $F = -\frac{1}{\sqrt{x}}$, $t = \frac{1}{8} - \frac{1}{4\sqrt{5}}$ and $x_0 = e^{23}$. Indeed, in the case $\mathcal{S}(Tx,Tx,x) > 0$, we find

$$\mathcal{S}(Tx,Tx,x) \in \{4,6,8,10,12,14,16\}$$

and

$$20 < h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)},$$

where $h = \frac{20}{21}$. Then we have

$$t + F(\mathcal{S}(Tx,Tx,x)) \leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}\right].$$

We obtain

$$r=\inf\{\mathcal{S}(Tx,Tx,x):x\neq Tx\}=4$$

and therefore, the self-mapping T fixes the circle $C^S_{e^{23},4}=\{e^{21},e^{25}\}$ and the disc $D^S_{e^{23},4}=\{e^{21},e^{22},e^{23},e^{24},e^{25}\}$.

4. Fixed-Circle Theorems via Auxiliary Functions

In this section, we investigate the existence and uniqueness theorems for fixed circles of self-mappings using some auxiliary functions. Let $r>0$ be any real number. We consider the function $\varphi_r:\mathbb{R}^+\cup\{0\}\to\mathbb{R}$ defined as

$$\varphi_r(u)=\begin{cases} u-r & , \quad u>0 \\ 0 & , \quad u=0 \end{cases}, \tag{4.1}$$

for all $u\in\mathbb{R}^+\cup\{0\}$ [12]. Using the function φ_r we give the following theorem.

Theorem 4.1. *Let $(X,\mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Consider the function φ_r defined in (4.1). If there exists a self-mapping $T:X\to X$ satisfying the conditions*

1. *$\mathcal{S}(Tx,Tx,x_0)=r$ for each $x\in C^S_{x_0,r}$,*
2. *$\mathcal{S}(Tx,Tx,Ty)>r$ for each $x,y\in C^S_{x_0,r}$ and $x\neq y$,*
3. *$\mathcal{S}(Tx,Tx,Ty)\leq\mathcal{S}(x,x,y)-\varphi_r(\mathcal{S}(x,x,Tx))$ for each $x,y\in C^S_{x_0,r}$,*

then the circle $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x\in C^S_{x_0,r}$ be an arbitrary point. By the condition (1), we have $Tx\in C^S_{x_0,r}$ for all $x\in C^S_{x_0,r}$. Now we prove that x is a fixed point of T. On the contrary, let us assume that $Tx\neq x$. Taking $y=Tx$ and using the condition (2), we find

$$\mathcal{S}(Tx,Tx,T^2x)>r. \tag{4.2}$$

Using the condition (3), we have

$$\begin{aligned}\mathcal{S}(Tx,Tx,T^2x) &\leq \mathcal{S}(x,x,Tx)-\varphi_r(\mathcal{S}(x,x,Tx)) \\ &= \mathcal{S}(x,x,Tx)-\mathcal{S}(x,x,Tx)+r=r.\end{aligned} \tag{4.3}$$

Combining the inequalities (4.2) and (4.3), we get a contradiction. Hence it should be $Tx=x$. Consequently, the circle $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 4.1. Notice that the condition (1) in Theorem 4.1 guarantees that Tx is on the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$, the condition (2) shows that the distance of the images of any two elements on the circle $C^S_{x_0,r}$ can not be less than (or equal to) r.

Now we give an example of a self-mapping which has a fixed-circle on an S-metric space.

Example 4.1. Let $X = \mathbb{R}$ and the metric function $d : X^2 \to [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ |x| + |y| & , \quad x \neq y \end{cases},$$

for all $x, y \in X$. Let us consider the S-metric defined in Example 2.2. The circle $C^S_{\frac{1}{2},1} = \left\{x \in X : \mathcal{S}(x, x, \frac{1}{2}) = 1\right\} = \{0\}$. If we consider the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$T_1x = \begin{cases} 4 & , \quad x = \frac{1}{2} \\ 0 & , \quad x \neq \frac{1}{2} \end{cases},$$

for all $x \in \mathbb{R}$ then the self-mapping T_4 satisfies the conditions of Theorem 4.1 and T_4 fixes the circle $C^S_{\frac{1}{2},1}$.

In the following example, we see that the converse statement of Theorem 4.1 is not always true.

Example 4.2. Let $X = \mathbb{C}$ and consider the S-metric defined in Example 2.6. Let us consider the circle $C^S_{0,\frac{1}{3}}$ and define the self-mapping $T_2 : \mathbb{C} \to \mathbb{C}$

$$T_2z = \begin{cases} \frac{1}{9\overline{z}} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$, where $\overline{z}$ denotes the complex conjugate of the complex number z. Clearly, we have $T_2(C^S_{0,\frac{1}{3}}) = (C^S_{0,\frac{1}{3}})$. It can be easily checked that the self mapping T_2 does not satisfy the condition (2) of Theorem 4.1. But, an easy computation shows that T_2 fixes the circle $C^S_{0,\frac{1}{3}}$.

In the following example we see that the circle need not to be fixed even if $T(C^S_{x_0,r}) = C^S_{x_0,r}$.

Example 4.3. Let $(\mathbb{C}, \mathcal{S})$ be the usual S-metric space. Let us consider the circle $C^S_{0,\frac{1}{8}}$ and define the self-mapping $T_3 : \mathbb{C} \to \mathbb{C}$ as

$$T_3z = \begin{cases} \frac{1}{16z} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then we have $T_3(C^S_{0,\frac{1}{8}}) = C^S_{0,\frac{1}{8}}$. But the self-mapping T_3 does not satisfy the conditions (2) and (3) of Theorem 4.1. Clearly, the circle $C^S_{0,\frac{1}{8}}$ is not a fixed circle of T_3 since $T_3(\frac{i}{4}) = -\frac{i}{4}$ and $T_3(-\frac{i}{4}) = \frac{i}{4}$. More precisely, T_3 fixes only the points $\frac{1}{4}$ and $-\frac{1}{4}$ on the circle $C^S_{0,\frac{1}{8}}$.

In the following example we see that a self mapping can be fix more than one circle.

Example 4.4. Let $X=\mathbb{R}$ and $(X,\mathcal{S})$ be the S-metric space defined in Example 2.6. Let us consider the circles $C_{0,4}^{S}$ and $C_{6,2}^{S}$ and the self-mapping $T_4:\mathbb{R}\to\mathbb{R}$ as

$$T_4x=\begin{cases} \frac{2x+4}{x+5} & , \quad x\in(-\infty,4) \\ \frac{17x+56}{24} & , \quad x\in(4,\infty) \\ 4 & , \quad x=4 \end{cases},$$

for all $x\in\mathbb{R}$. It can be easily checked that the self-mapping T_4 satisfies the conditions of Theorem 4.1 and that both of the circles $C_{0,4}^{S}$ and $C_{6,2}^{S}$ are the fixed circles of T_4.

Now we give another existence theorem for fixed circles.

Theorem 4.2. *Let $(X,\mathcal{S})$ be an S-metric space and $C_{x_0,r}^{S}$ be any circle on X. Let us define the mapping*

$$\varphi:X\to[0,\infty)\,,\varphi(x)=\mathcal{S}\left(x,x,x_0\right),$$

for all $x\in X$. If there exists a self-mapping $T:X\to X$ satisfying

1. $\mathcal{S}(x,x,Tx)\le\max\{\varphi(x),\varphi(Tx)\}-r$,
2. $\mathcal{S}(Tx,Tx,x_0)-h\mathcal{S}(x,x,Tx)\le r$,

for all $x\in C_{x_0,r}^{S}$ and $h\in[0,1)$, then $C_{x_0,r}^{S}$ is a fixed circle of T.

Proof. Let $x\in C_{x_0,r}^{S}$. On the contrary, assume that $Tx\neq x$. Then we have the following cases:

Case 1. If $\max\{\varphi(x),\varphi(Tx)\}=\varphi(x)$ then using the condition (1) we have

$$\mathcal{S}(x,x,Tx)\le\max\{\varphi(x),\varphi(Tx)\}-r=\varphi(x)-r=r-r=0$$

and so $\mathcal{S}(x,x,Tx)=0$, a contradiction. Hence we get $Tx=x$.

Case 2. If $\max\{\varphi(x),\varphi(Tx)\}=\varphi(Tx)$ then we obtain

$$\mathcal{S}(x,x,Tx)\le\max\{\varphi(x),\varphi(Tx)\}-r=\varphi(Tx)-r,$$

and using the condition (2) we find

$$\mathcal{S}(x,x,Tx)\le\varphi(Tx)-r\le h\mathcal{S}(x,x,Tx)+r-r=h\mathcal{S}(x,x,Tx),$$

a contradiction since $h\in[0,1)$. Hence we get $Tx=x$.

Consequently, $C_{x_0,r}^{S}$ is a fixed circle of T. $\square$

Remark 4.2. (1) Notice that the condition (1) in Theorem 4.2 guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly the condition (2) guarantees that Tx is not exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$.

(2) Notice that the conditions of Theorem 4.2 are satisfied by the self-mapping T_2.

Now we give the following example.

Example 4.5. Let $X = \mathbb{R}$ be the S-metric space with the usual S-metric defined in Example 2.1. Let us consider the circle $C^S_{0,8}$ and define the self-mapping $T_5 : \mathbb{R} \to \mathbb{R}$ as

$$T_5x = \begin{cases} 2 & , \quad x \in \left\{-\frac{8}{\sqrt{3}}, 2\right\} \\ \frac{8x+16\sqrt{3}}{\sqrt{3}x+8} & , \quad x \in \mathbb{R} \backslash \left\{-\frac{8}{\sqrt{3}}, 2\right\} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_5 satisfies the conditions (1) and (2) in Theorem 4.2. Hence $C^S_{0,8}$ is a fixed circle of T_5. Notice that $C^S_{3,2}$ is another fixed circle of T_5 and so the number of the fixed circles need not to be unique for a giving self-mapping.

Now, in the following example, we give an example of a self-mapping which satisfies the condition (1) and does not satisfy the condition (2) of Theorem 4.2.

Example 4.6. Let $X = \mathbb{R}$ and the S-metric be defined as in Example 2.6. Let us consider the circle $C^S_{0,6}$ and define the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ as

$$T_6x = \begin{cases} \frac{4x+48\sqrt{3}}{\sqrt{3}x+3} & , \quad x \in (-7, 7) \\ 20 & , \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_6 satisfies the conditions (1) but does not satisfy the conditions (2) in Theorem 4.2. Consequently $C^S_{0,6}$ is not a fixed circle of T_6.

In the following, we give an example of a self-mapping which satisfies the condition (2) and does not satisfy the condition (1) in Theorem 4.2.

Example 4.7. Let $X = \mathbb{C}$ be the S-metric space with the usual S-metric defined in Example 2.1. Let us consider the circle $C^S_{0,12}$ and define the self-mapping $T_7 : \mathbb{C} \to \mathbb{C}$ as

$$T_7z = \begin{cases} \frac{Re(z)}{2} & if \quad Re(z) \geq 0 \\ -\frac{Re(z)}{2} & if \quad Re(z) < 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then the self-mapping T_7 satisfies the condition (2) and does not satisfy the condition (1) in Theorem 4.2.

Now we use the following corollaries to obtain a uniqueness theorem for fixed circles of self-mappings.

Corollary 4.1. *[22] Let $(X, \mathcal{S})$ be a complete S-metric space and T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y), \tag{4.4}$$

for some $a, b, c \geq 0, a + b + c < 1$, and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$ then T is continuous at the fixed point.

Corollary 4.2. *[22] Let $(X, \mathcal{S})$ be a complete S-metric space and T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, y), S(Ty, Ty, x)\}, \tag{4.5}$$

for some $h \in \left[0, \frac{1}{3}\right)$ and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

We give the following theorem.

Theorem 4.3. *Let $(X, \mathcal{S})$ be an S-metric space and $T : X \to X$ be a self-mapping with the fixed circle $C^S_{x_0,r}$. If one of the contractive conditions (4.4) or (4.5) is satisfied for all $x \in C^S_{x_0,r}, y \in X \setminus C^S_{x_0,r}$ by T then $C^S_{x_0,r}$ is the unique fixed circle of T.*

Proof. Assume that there exists two fixed circles $C^S_{x_0,r}$ and $C^S_{x_0,\rho}$ of the self-mapping T. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_0,\rho}$ be arbitrary points with $x \neq y$. If the contractive condition (4.4) is satisfied by T , then we obtain

$$\begin{aligned} S(x,x,y) &= S(Tx,Tx,Ty) \leq aS(x,x,y) + bS(Tx,Tx,x) + cS(Ty,Ty,y) \\ &= aS(x,x,y), \end{aligned}$$

which is a contradiction since $a+b+c < 1$. Hence it should be $x = y$. Consequently $C^S_{x_0,r}$ is the unique fixed circle of T. Similarly, if the contractive condition (4.5) is satisfied by T then we get

$$S(x,x,y) = S(Tx,Tx,Ty) \leq h \max\{S(Tx,Tx,y), S(Ty,Ty,x)\} = hS(x,x,y),$$

which is a contradiction since $h \in \left[0, \frac{1}{3}\right)$. Hence it should be $x = y$. Consequently $C^S_{x_0,r}$ is the unique fixed circle of T. □

Now we consider the identity map $I_X : X \to X$ defined as $I_X(x) = x$ for all $x \in X$. We note that the identity map satisfies the conditions of Theorem 4.2 but can not satisfy the condition (2) of Theorem 4.1 everywhen. Therefore, we investigate a condition which excludes the identity map in Theorem 4.2 (resp. Theorem 4.1). For this purpose, we obtain the following theorem.

Theorem 4.4. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self mapping having a fixed circle $C^S_{x_0,r}$ and the mapping φ_r be defined as in (4.1). The self-mapping $T : X \to X$ satisfies the condition*

$$\mathcal{S}(x,x,Tx) < \varphi_r(\mathcal{S}(x,x,Tx)) + r, \tag{4.6}$$

for all $x \in X$ if and only if $T = I_X$.

Proof. Let $x \in X$ be any point and assume that $Tx \neq x$. Then using the inequality (4.6), we get

$$\mathcal{S}(x, x, Tx) < \varphi_r\left(\mathcal{S}(x, x, Tx)\right) + r = \mathcal{S}(x, x, Tx) - r + r,$$

which is a contradiction. Hence we have $Tx = x$ and $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (4.6). □

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