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ANTI-INVARIANT RIEMANNIAN SUBMERSIONS FROM LOCALLY CONFORMAL KAEHLER MANIFOLDS *

Majid Ali Choudhary and Lamia Saeed Alqahtani

Abstract. Recently, Sahin [10] studied the anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. In present work, these notions of anti-invariant and Lagrangian Riemannian submersions have been extended to locally conformal Kaehler manifolds. Certain decomposition results and the geometry of foliation have also been investigated.
Keywords: anti-invariant Riemannian submersions; almost Hermitian manifolds; Riemannian manifolds; Kaehler manifolds.

1. Introduction

Locally conformal Kaehler manifolds (shortly, l.c.K. manifolds) have been rich source of attraction for many years. Many geometers considered these manifolds and their submanifolds in different settings (for details see, [3] and [13]). On the other side, for any Riemannian manifold $\mathcal{M}$ and Riemannian manifold $\mathcal{B}$, the Riemannian submersion π from $\mathcal{M}$ onto $\mathcal{B}$ was studied for very first time by B. O'Neil [6]. Gray [4], Ianus [5], Park ([7], [8]), Sahin ([11], [12]), Choudhary [2] etc. have also taken into consideration the geometry of Riemannian submersions for different structures on differentiable manifolds. Recently, anti-invariant Riemannian submersions have been taken into study from almost Hermitian manifolds onto Riemannian manifolds by B. Sahin [10].

In present work, these notions of anti-invariant and Lagrangian Riemannian submersions have been extended to locally conformal Kaehler manifolds. Certain decomposition results and the geometry of foliation have also been investigated.

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2. Preliminaries

This section is preliminary in nature wherein we collect definitions and formulas that are to be used. We start with l.c.K. manifold.

Definition 2.1. [3] For Hermitian manifold $(\tilde{\mathcal{M}}, g)$ of dimension-$2m$ and Kaehler 2-form Ω holding for the relation

$$\Omega(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, J\mathcal{Y}),$$

for all $\mathcal{X}, \mathcal{Y} \in \chi(\tilde{\mathcal{M}})$ and closed 1-form ω defined globally on manifold $\tilde{\mathcal{M}}$ such that

$$d\Omega = \omega \wedge \Omega,$$

the manifold $\tilde{\mathcal{M}}$ is known as locally conformal Kaehler manifold.

Here, ω is sign of the Lee form of $\tilde{\mathcal{M}}$. We have the following cases for ω:

- when ω is exact, $\tilde{\mathcal{M}}$ is globally conformal Kahler (g.c.K.) manifold,
- when $\omega = 0$, $\tilde{\mathcal{M}}$ is Kaehler manifold.

One can observe that any l.c.K. manifold becomes g.c.K. manifold provided it is simply connected. Let us use $\sharp$ to represent the rising of the indices in association with the metric g, then for any l.c.K. manifold $\tilde{\mathcal{M}}$, $B_1 = \omega^\sharp$ indicates the Lee vector field and satisfies

$$g(\mathcal{X}, B_1) = \omega(\mathcal{X}); \forall \mathcal{X} \in \chi(\tilde{\mathcal{M}}).$$

[3] When we use $\theta = \omega o J$ for anti-Lee form and $A = -JB_1$ for anti-Lee vector field, respectively. Then

$$(\tilde{\nabla}_{\mathcal{X}} J)\mathcal{Y} = \frac{1}{2}\{\theta(\mathcal{Y})\mathcal{X} - \omega(\mathcal{Y})J\mathcal{X} - g(\mathcal{X}, \mathcal{Y})A - \Omega(\mathcal{X}, \mathcal{Y})B_1\}, \tag{2.1}$$

$\forall \mathcal{X}, \mathcal{Y} \in \chi(\tilde{\mathcal{M}})$, where, $\tilde{\nabla}$ is used for the Levi Civita connection of $(\tilde{\mathcal{M}}, g)$.

Any map π of m-dimensional Riemannian manifold $(\mathcal{M}^m, g)$ onto a b'-dimensional Riemannian manifold $(\mathcal{B}^{b'}, g_{\mathcal{B}})$ with $m > b'$ stands for a Riemannian submersion if π has maximal rank and the lengths of horizontal vectors are preserved by differential π_*.

It is known that $\pi^{-1}(q'), q' \in \mathcal{B}$ is an $(m - b')$ dimensional submanifold of Riemannian manifold $\mathcal{M}$ and named as fibers. A vector field on $\mathcal{M}$ is said to be

- vertical provided it is always tangent to $\pi^{-1}(q')$;
- horizontal provided it is always orthogonal to $\pi^{-1}(q')$.

Next, we have

Definition 2.2. [10] Let $\mathcal{X}$ represents a vector field on a Riemannian manifold $\mathcal{M}$, then $\mathcal{X}$ is known as basic if

- it is horizontal
- it is π-related to a vector field $\mathcal{X}_*$ on $\mathcal{B}$, that is, $\pi_*\mathcal{X}_{p_1} = \mathcal{X}_{*\pi(p_1)}, \forall p_1 \in \mathcal{M}$.

Let us use $\mathcal{V}$ and $\mathcal{H}$ to denote the projection morphisms on $ker\pi_*$ and $(ker\pi_*)^\perp$, respectively. Then

Lemma 2.1. *[6] When $\pi : \mathcal{M} \to \mathcal{B}$ represents a Riemannian submersion from a Riemannian manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$. Then*

(a) $g(\mathcal{X}, \mathcal{Y}) = g_{\mathcal{B}}(\mathcal{X}_*, \mathcal{Y}_*)o\pi$,

(b) $\mathcal{H}[\mathcal{X}, \mathcal{Y}]$ *of* $[\mathcal{X}, \mathcal{Y}]$ *is basic vector field corresponding to* $[\mathcal{X}_*, \mathcal{Y}_*]$, *i.e.,* $([\mathcal{X}, \mathcal{Y}]^{\mathcal{H}}) = (\mathcal{X}_*, \mathcal{Y}_*)$,

(c) *when V is vertical vector, $[V, \mathcal{X}]$ is also vertical,*

(d) *when ∇^* be the Levi-Civita connection on $\mathcal{B}$, $\mathcal{H}(\nabla_{\mathcal{X}}\mathcal{Y})$ will be the basic vector field that corresponds to $\nabla^*_{\mathcal{X}_*}\mathcal{Y}_*$.*

Here, $\mathcal{X}, \mathcal{Y}$ are considered as basic vector fields on $\mathcal{M}$.

[6] Let us denote by the symbols $\mathcal{T}$ and $\mathcal{A}$, O'Neills tensors for vector fields E, F on $\mathcal{M}$ and by ∇ the Levi-Civita connection of g such that the following hold

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.2}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \tag{2.3}$$

The necessary and sufficient condition for Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$ to be totally geodesic fibres is that $\mathcal{T}$ vanishes identically. Now, let us suppose that $\Gamma(T\mathcal{M})$ denotes the set of all sections on the tangent bundle $T\mathcal{M}$, then for any $E \in \Gamma(T\mathcal{M}), \mathcal{T}_E$ and $\mathcal{A}_E$ represent skew-symmetric operators on $(\Gamma(T\mathcal{M}), g)$ reversing the horizontal and vertical distributions. One can observe that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$ and hold for the following ([6], [10])

$$\mathcal{T}_{\mathcal{U}}\mathcal{W} = \mathcal{T}_{\mathcal{W}}\mathcal{U}, \forall \mathcal{U}, \mathcal{W} \in \Gamma(ker\pi_*) \tag{2.4}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{Y} = -\mathcal{A}_{\mathcal{Y}}\mathcal{X} = \frac{1}{2}\mathcal{V}[\mathcal{X}, \mathcal{Y}], \forall \mathcal{X}, \mathcal{Y} \in (\Gamma(ker\pi_*)^\perp). \tag{2.5}$$

Now we state the following lemma [10]

Lemma 2.2. *When $\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$ and $\mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$, we have the following relations:*

(a) $\nabla_{\mathcal{W}}\mathcal{W}' = \mathcal{T}_{\mathcal{W}}\mathcal{W}' + \hat{\nabla}_{\mathcal{W}}\mathcal{W}'$

(b) $\nabla_{\mathcal{W}}\mathcal{X} = \mathcal{H}\nabla_{\mathcal{W}}\mathcal{X} + \mathcal{T}_{\mathcal{W}}\mathcal{X}$

(c) $\nabla_{\mathcal{X}}\mathcal{W} = \mathcal{A}_{\mathcal{X}}\mathcal{W} + \mathcal{V}\nabla_{\mathcal{X}}\mathcal{W}$

(d) $\nabla_{\mathcal{X}}\mathcal{Y} = \mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{Y}$

where $\hat{\nabla}_{\mathcal{W}}\mathcal{W}' = \mathcal{V}\nabla_{\mathcal{W}}\mathcal{W}'$. Moreover, $\mathcal{H}\nabla_{\mathcal{W}}\mathcal{X} = \mathcal{A}_{\mathcal{X}}\mathcal{W}$, when $\mathcal{X}$ is basic.

3. Anti-invariant and Lagrangian Riemannian submersions

This section deals with the anti-invariant and Lagrangian Riemannian submersion. Certain conditions to show these submersions to be totally geodesic maps are also discussed. A diffeomorphism f of Riemannian manifold $(\mathcal{M}, g)$ onto another Riemannian manifold $(\mathcal{B}, g')$ is said be geodesic map if image of any geodesic arc in $\mathcal{M}$ under f is a geodesic arc in $\mathcal{B}$ and image of any geodesic arc in $\mathcal{B}$ under f^{-1} is a geodesic arc in $\mathcal{M}$. A map is said to be totally geodesic if its hessian vanishes.

Now, recall anti-invariant Riemannian submersion by the following way.

Definition 3.1. [10] Let $(\mathcal{M}, g_{\mathcal{M}}, J)$ represents a complex almost Hermitian manifold of dimension m and $(\mathcal{B}, g_{\mathcal{B}})$ be a Riemannian manifold. Then, any Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$ is said to be anti-invariant Riemannian submersion if $J(ker\pi_*) \subseteq (ker\pi_*)^{\perp}$.

For an anti-invariant Riemannian submersion π from an almost Hermitian manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, above definition implies $J(ker\pi_*)^{\perp} \cap (ker\pi_*) \neq 0$, and that produces

$$(ker\pi_*)^{\perp} = J(ker\pi_*) \oplus \mu, \tag{3.1}$$

here μ is used for the orthogonal complementary distribution to $J(ker\pi_*)$ in $(ker\pi_*)^{\perp}$. So,

$$J\mathcal{X} = B\mathcal{X} + C\mathcal{X}, \quad \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), B\mathcal{X} \in \Gamma(ker\pi_*), C\mathcal{X} \in \Gamma(\mu). \tag{3.2}$$

For Riemannian submersion π, (3.2) and $\pi_*((ker\pi_*)^{\perp}) = T\mathcal{B}$ indicate

$$g_{\mathcal{B}}(\pi_* JV, \pi_* C\mathcal{X}) = 0, \quad \forall \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*),$$

implying

$$T\mathcal{B} = \pi_*(J(ker\pi_*)) \oplus \pi_*(\mu). \tag{3.3}$$

[1] Let $\phi^{'} : \mathcal{M} \to \mathcal{B}$ be smooth map from Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}})$ onto $(\mathcal{B}, g_{\mathcal{B}})$. Then, any section of the bundle $\mathrm{Hom}(T\mathcal{M}, \phi^{'-1}(T\mathcal{B})) \to \mathcal{M}$ can be thought by differential $\phi^{'}_{*}$, $\phi^{'-1}(T\mathcal{B})$ being the pullback bundle having fibres $(\phi^{'-1}(TB))_p = T_{\phi^{'}(p)}B, p \in \mathcal{M}$. Thanks to pullback connection and the Levi-Civita connection $\nabla^{\mathcal{M}}$, one can induce a connection ∇ for $Hom(T\mathcal{M}, \phi^{'-1}(T\mathcal{B}))$. Hence, define the second fundamental form of $\phi^{'}$ by

$$(\nabla\phi^{'}_{*})(\mathcal{X}, \mathcal{Y}) = \nabla^{\phi^{'}}_{\mathcal{X}}\phi^{'}_{*}(\mathcal{Y}) - \phi^{'}_{*}(\nabla^{\mathcal{M}}_{\mathcal{X}}\mathcal{Y}), \forall \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M}), \tag{3.4}$$

here, $\Gamma(T\mathcal{M})$ represents set of all sections on the tangent bundle $T\mathcal{M}$ and $\nabla^{\phi^{'}}$ is the pullback connection.

Next, we give the following result.

Lemma 3.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, and ω be closed 1-form defined globally on $\mathcal{M}$, then for all $\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$, we have*

(i) $g(C\mathcal{Y}, J\mathcal{W}) = 0$

(ii) $g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$

(iii) $g(\nabla_{\mathcal{W}}B\mathcal{Y}, C\mathcal{X}) = g(C\mathcal{X}, \mathcal{T}_{\mathcal{W}}B\mathcal{Y}) = -g(B\mathcal{Y}, \mathcal{T}_{\mathcal{W}}C\mathcal{X})$.

Proof (i) Let $\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$ and $\mathcal{W} \in \Gamma(ker\pi_*)$, then in the light of (3.2), we get

$$\begin{aligned} g(C\mathcal{Y}, J\mathcal{W}) &= g(J\mathcal{Y} - B\mathcal{Y}, J\mathcal{W}) \\ &= g(J\mathcal{Y}, J\mathcal{W}) \end{aligned}$$

where the fact $B\mathcal{Y} \in \Gamma(ker\pi_*)$ and $J\mathcal{W} \in \Gamma((ker\pi_*)^{\perp})$ was used. Moreover, $g(J\mathcal{Y}, J\mathcal{W}) = g(\mathcal{Y}, \mathcal{W}) = 0$ and this completes the proof.

(ii) Let us assume $B_1 \in \Gamma(ker\pi_*)$, then taking view of (2.1) and part (i), we get

$$\begin{aligned} g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) &= -g(C\mathcal{Y}, \nabla_{\mathcal{X}}J\mathcal{W}) \\ &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, J\mathcal{X}) \end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$. Thanks to (3.2), we arrive

$$\begin{aligned} g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, B\mathcal{X} + C\mathcal{X}) \\ &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \end{aligned}$$

because $C\mathcal{Y} \in \Gamma(\mu)$ and $B\mathcal{X} \in \Gamma(ker\pi_*)$. Taking use of Lemma 2.2 produces

$$g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) - (C\mathcal{Y}, J\mathcal{V}\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$$

that simplifies to

$$g(\nabla_{\mathcal{X}}C\mathcal{Y},J\mathcal{W})=-g(C\mathcal{Y},J\mathcal{A}_{\mathcal{X}}\mathcal{W})+\frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y},C\mathcal{X}),$$

here, we used $J\mathcal{V}\nabla_{\mathcal{X}}\mathcal{W}\in\Gamma(Jker\pi_*)$. □

From here, we assume that $B_1\in(ker\pi_*)$. We also assume horizontal vector fields to be basic whenever needed in the proofs. Now, let us move to study the integrability results of the horizontal distribution $(ker\pi_*)^\perp$. Also, note that $ker\pi_*$ is integrable.

Theorem 3.1. *When $\pi:\mathcal{M}\to\mathcal{B}$ represents anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M},g,J)$ onto a Riemannian manifold $(\mathcal{B},g_{\mathcal{B}})$, then the following are equivalent:*

(a) $(ker\pi_*)^\perp$ *is integrable*

(b) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y},B\mathcal{X}),\pi_*J\mathcal{W})=g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X},B\mathcal{Y}),\pi_*J\mathcal{W})+g(C\mathcal{Y},J\mathcal{A}_{\mathcal{X}}\mathcal{W})$
$-g(C\mathcal{X},J\mathcal{A}_{\mathcal{Y}}\mathcal{W})-\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W})$
$+\frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W})$

(c) $g(\mathcal{A}_{\mathcal{Y}}B\mathcal{X}-\mathcal{A}_{\mathcal{X}}B\mathcal{Y},J\mathcal{W})=-g(C\mathcal{Y},J\mathcal{A}_{\mathcal{X}}\mathcal{W})+g(C\mathcal{X},J\mathcal{A}_{\mathcal{Y}}\mathcal{W})$
$+\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W})-\frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W})$

$\forall\mathcal{X},\mathcal{Y}\in\Gamma((ker\pi_*)^\perp)$,$\mathcal{W}\in\Gamma(ker\pi_*)$.

Proof. Taking account of definition 3.1, we see $J\mathcal{Y}\in\Gamma(ker\pi_*\oplus\mu)$ and $J\mathcal{W}\in\Gamma((ker\pi_*)^\perp)$ and hence with the help of (2.1) for $\mathcal{X}\in\Gamma((ker\pi_*)^\perp)$, we reach at

$$\begin{aligned}g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(J[\mathcal{X},\mathcal{Y}],J\mathcal{W})\\ &= g(J\nabla_{\mathcal{X}}\mathcal{Y},J\mathcal{W})-g(J\nabla_{\mathcal{Y}}\mathcal{X},J\mathcal{W})\\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y},J\mathcal{W})-\frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X},J\mathcal{W})\\ &\quad -g(\nabla_{\mathcal{Y}}J\mathcal{X},J\mathcal{W})+\frac{1}{2}\theta(\mathcal{X})g(\mathcal{Y},J\mathcal{W}),\end{aligned}$$

$\forall\mathcal{Y}\in\Gamma((ker\pi_*)^\perp),\mathcal{W}\in\Gamma(ker\pi_*)$. Here $\theta=\omega oJ$, $\Omega(\mathcal{X},\mathcal{Y})=g(\mathcal{X},J\mathcal{Y})$ and $g(\mathcal{X},B_1)=\omega(\mathcal{X})$, then $B_1\in\Gamma(ker\pi_*)$ and (3.2) produce

$$\begin{aligned}g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(\nabla_{\mathcal{X}}J\mathcal{Y},J\mathcal{W})-g(\nabla_{\mathcal{Y}}J\mathcal{X},J\mathcal{W})-\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W})\\ &\quad +\frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W})\\ &= g(\nabla_{\mathcal{X}}B\mathcal{Y},J\mathcal{W})+g(\nabla_{\mathcal{X}}C\mathcal{Y},J\mathcal{W})-g(\nabla_{\mathcal{Y}}B\mathcal{X},J\mathcal{W})\\ &\quad -g(\nabla_{\mathcal{Y}}C\mathcal{X},J\mathcal{W})-\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W})+\frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}).\end{aligned}$$

Because π represents a Riemannian submersion, we conclude

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(\pi_*\nabla_{\mathcal{X}}B\mathcal{Y},\pi_*J\mathcal{W}) + g(\nabla_{\mathcal{X}}C\mathcal{Y},J\mathcal{W}) - g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{Y}}B\mathcal{X},\pi_*J\mathcal{W}) \\ &\quad -g(\nabla_{\mathcal{Y}}C\mathcal{X},J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}). \end{aligned}$$

Taking into account Lemma 3.1, we arrive at

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g_{\mathcal{B}}(-(\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) + (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}),\pi_*J\mathcal{W}) \\ &\quad -g(C\mathcal{Y},J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + g(C\mathcal{X},J\mathcal{A}_{\mathcal{Y}}\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}) \end{aligned}$$

proving (a)$\Leftrightarrow$(b).

Next, taking into consideration Lemma 2.2, we derive

$$\begin{aligned} (\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) &- (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}) \\ &= -\pi_*(\nabla_{\mathcal{X}}B\mathcal{Y}) + \pi_*(\nabla_{\mathcal{Y}}B\mathcal{X}) \\ &= -\pi_*(\nabla_{\mathcal{X}}B\mathcal{Y} - \nabla_{\mathcal{Y}}B\mathcal{X}) \\ &= \pi_*(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y}), \end{aligned}$$

$\forall \mathcal{X},\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$. Simplification reduces to

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) &- (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}),\pi_*J\mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y}),\pi_*J\mathcal{W}) \\ &= g(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y},J\mathcal{W}), \end{aligned}$$

moreover, $\mathcal{A}_{\mathcal{X}}B\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}B\mathcal{X} \in \Gamma((ker\pi_*)^{\perp})$, it establishes (b)$\Leftrightarrow$(c). □

Definition 3.2. [10] Let π represents an anti-invariant Riemannian submersion such that $J(ker\pi_*) = (ker\pi_*)^{\perp}$. Then, π is known as Lagrangian Riemannian submersion. Moreover, when $\mu \neq \{0\}$, π is called as proper anti-invariant Riemannian submersion.

Thanks to Theorem 3.1, we write the following.

Corollary 3.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) $(ker\pi_*)^{\perp}$ *is integrable*

(b) $(\nabla\pi_*)(\mathcal{X},J\mathcal{Y}) = (\nabla\pi_*)(\mathcal{Y},J\mathcal{X}) - \frac{1}{2}g(B\mathcal{Y},B_1)\mathcal{X} + \frac{1}{2}g(B\mathcal{X},B_1)\mathcal{Y}$

(c) $\pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}J\mathcal{X}) = \frac{1}{2}g(B\mathcal{Y},B_1)\mathcal{X} - \frac{1}{2}g(B\mathcal{X},B_1)\mathcal{Y}$, $\forall \mathcal{X},\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$.

Proof. Let us assume that $\mathcal{X},\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$ and $\mathcal{W} \in \Gamma(ker\pi_*)$. Then, $J\mathcal{X} \in \Gamma(ker\pi_*)$ and $J\mathcal{W} \in \Gamma((ker\pi_*)^{\perp})$. Hence, taking into light (2.1), we derive

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(J[\mathcal{X},\mathcal{Y}], J\mathcal{W}) \\ &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W}) - g(J\nabla_{\mathcal{Y}}\mathcal{X}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{Y}}J\mathcal{X}, J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}). \end{aligned}$$

Use of (3.4) produces

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) - g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{Y}}J\mathcal{X}, \pi_*J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}) \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X},J\mathcal{Y}), \pi_*J\mathcal{W}) + g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y},J\mathcal{X}), \pi_*J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}) \end{aligned}$$

thus, $(ker\pi_*)^{\perp}$ is integrable iff

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X},J\mathcal{Y}), \pi_*J\mathcal{W}) &= g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y},J\mathcal{X}), \pi_*J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) \\ &\quad +\frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}) \end{aligned}$$

establishing (a)⇔(b).

Next, with the help of (3.4) we get

$$\begin{aligned} (\nabla\pi_*)(\mathcal{Y},J\mathcal{X}) &- (\nabla\pi_*)(\mathcal{X},J\mathcal{Y}) \\ &= -\pi_*(\nabla_{\mathcal{Y}}J\mathcal{X}) + \pi_*(\nabla_{\mathcal{X}}J\mathcal{Y}) \\ &= \pi_*(\mathcal{H}(\nabla_{\mathcal{X}}J\mathcal{Y}) - \mathcal{H}(\nabla_{\mathcal{Y}}J\mathcal{X})) \\ &= \pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}J\mathcal{X}), \end{aligned}$$

$\forall \mathcal{X},\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$. This concludes (b)⇔(c). □

4. Geometry of leaves

The geometry of leaves of $(ker\pi_*)$ and $(ker\pi_*)^{\perp}$ of anti-invariant and Lagrangian Riemannian submersions are studies here. We have

Theorem 4.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)^{\perp}$*

(b) $g(\mathcal{A}_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) = g(C\mathcal{Y}, J\mathcal{A}_\mathcal{X}\mathcal{W}) - \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$
$+ \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

(c) $g_\mathcal{B}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_\mathcal{X}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$
$- \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$.

Proof. Taking into account (2.1), (3.2), Lemma 2.2 and Lemma 3.1, we write the following

$$\begin{aligned} g(\nabla_\mathcal{X}\mathcal{Y}, \mathcal{W}) &= g(J\nabla_\mathcal{X}\mathcal{Y}, J\mathcal{W}) \\ &= g(\nabla_\mathcal{X} J\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g(\nabla_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) + g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) \\ &\quad -\frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g(\mathcal{A}_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) - g(C\mathcal{Y}, J\mathcal{A}_\mathcal{X}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$. In this way, a totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)^\perp$ iff

$$\begin{aligned} g(\mathcal{A}_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) &= g(C\mathcal{Y}, J\mathcal{A}_\mathcal{X}\mathcal{W}) - \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \\ &\quad +\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \end{aligned}$$

concluding (a)⇔(b). Next, with the help of (3.4), we derive

$$\begin{aligned} g(\mathcal{A}_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) &= g(\nabla_\mathcal{X} B\mathcal{Y}, J\mathcal{W}) \\ &= g(\nabla_\mathcal{X} J\mathcal{Y}, J\mathcal{W}) - g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) \\ &= g_\mathcal{B}(\pi_*\nabla_\mathcal{X} J\mathcal{Y}, \pi_* J\mathcal{W}) - g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) \\ &= -g_\mathcal{B}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) + g_\mathcal{B}(\nabla^\pi_\mathcal{X}\pi_*(J\mathcal{Y}), \pi_* J\mathcal{W}) \\ &\quad -g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) \\ &= -g_\mathcal{B}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) + g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) - g(\nabla_\mathcal{X} C\mathcal{Y}, J\mathcal{W}) \\ &= -g_\mathcal{B}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) \end{aligned}$$

proving (b)⇔(c). □

For Lagrangian Riemannian submersion, we have the following corollary.

Corollary 4.1. *When π denotes a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_\mathcal{B})$, then the following are equivalent:*

(a) *totally geodesic foliation is defined by* $(ker\pi_*)^\perp$ *on manifold* $\mathcal{M}$

(b) $g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

(c) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) = -\frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

$\forall\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$.

Proof. Thanks to (2.1), we write

$$\begin{aligned} g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{W}) &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}), \pi_*J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}), \end{aligned}$$

$\forall\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$. This way, a totally geodesic foliation is defined by $(ker\pi_*)^\perp$ on the manifold $\mathcal{M}$ iff

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}).$$

Therefore, (a)⇔(b). Next, taking help of (3.4) it follows

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) &= g_{\mathcal{B}}(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) \end{aligned}$$

establishing

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) = -\frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

and that proves (b)⇔(c). □

Now, taking into consideration (3.4) to get

$$(\nabla\pi_*)(\mathcal{W}, \mathcal{X}) = \nabla^{\pi}_{\mathcal{W}}\pi_*\mathcal{X} - \pi_*\nabla_{\mathcal{W}}\mathcal{X}, \quad \mathcal{X} \in \Gamma(\mu), \mathcal{W} \in \Gamma(ker\pi_*).$$

Also,

$$(\nabla\pi_*)(\mathcal{X}, \mathcal{W}) = \nabla^{\pi}_{\mathcal{X}}\pi_*\mathcal{W} - \pi_*\nabla_{\mathcal{X}}\mathcal{W}.$$

We use above two equations and symmetric property of second fundamental form to get

$$\nabla^{\pi}_{\mathcal{W}}\pi_*\mathcal{X} = 0. \tag{4.1}$$

Next, we state the following Theorem.

Theorem 4.2. *When π denotes an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)$*

(b) $\mathcal{T}_{\mathcal{W}}B\mathcal{X} + \mathcal{A}_C\mathcal{X}\mathcal{W} = 0$ *or* $\mathcal{T}_{\mathcal{W}}B\mathcal{X} + \mathcal{A}_C\mathcal{X}\mathcal{W} \in \Gamma(\mu)$

(c) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_*J\mathcal{W}') = 0,\ \forall\mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. Taking into use (2.1), We obtain

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}', \mathcal{X}) &= g(J\nabla_{\mathcal{W}}\mathcal{W}', J\mathcal{X}) \\ &= g(\nabla_{\mathcal{W}}J\mathcal{W}', J\mathcal{X}) \\ &= -g(J\mathcal{W}', \nabla_{\mathcal{W}}J\mathcal{X}), \quad \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*), \end{aligned}$$

where orthogonality between $(ker\pi_*)$ and $(ker\pi_*)^{\perp}$ has been used. Taking help of (3.2) and Lemma 2.2, above equation reduces to

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}', \mathcal{X}) &= -g(J\mathcal{W}', \nabla_{\mathcal{W}}B\mathcal{X}) - g(J\mathcal{W}', \nabla_{\mathcal{W}}C\mathcal{X}) \\ &= -g(J\mathcal{W}', \mathcal{T}_{\mathcal{W}}B\mathcal{X}) - g(J\mathcal{W}', \mathcal{A}_{C\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{W}', \mathcal{T}_{\mathcal{W}}B\mathcal{X} + \mathcal{A}_{C\mathcal{X}}\mathcal{W}) \end{aligned}$$

implying (a)⇔(b). Furthermore, (3.4) produces

$$\begin{aligned} g(\mathcal{T}_{\mathcal{W}}B\mathcal{X}, J\mathcal{W}') &+ g(\mathcal{A}_{C\mathcal{X}}\mathcal{W}, J\mathcal{W}') \\ &= g(\mathcal{H}(\nabla_{\mathcal{W}}B\mathcal{X}), J\mathcal{W}') + g(\mathcal{H}(\nabla_{\mathcal{W}}C\mathcal{X}), J\mathcal{W}') \\ &= g(\nabla_{\mathcal{W}}B\mathcal{X}, J\mathcal{W}') + g(\nabla_{\mathcal{W}}C\mathcal{X}, J\mathcal{W}') \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{W}}B\mathcal{X}, \pi_*J\mathcal{W}') + g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{W}}C\mathcal{X}, \pi_*J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, B\mathcal{X}), \pi_*J\mathcal{W}') - g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, C\mathcal{X}), \pi_*J\mathcal{W}') \\ &+ g_{\mathcal{B}}(\nabla^{\pi}_{\mathcal{W}}\pi_*C\mathcal{X}, \pi_*J\mathcal{W}'). \end{aligned}$$

Taking into consideration (4.1), we get

$$\begin{aligned} g(\mathcal{T}_{\mathcal{W}}B\mathcal{X}, J\mathcal{W}') &+ g(\mathcal{A}_{C\mathcal{X}}\mathcal{W}, J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, B\mathcal{X}), \pi_*J\mathcal{W}') - g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, C\mathcal{X}), \pi_*J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_*J\mathcal{W}') \end{aligned}$$

concluding (b)⇔(c). □

Now, for a Lagrangian Riemannian submersion π, (3.3) interprets $T\mathcal{B} = \pi_*(J(ker\pi_*))$.

Corollary 4.2. *When π represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on* $\mathcal{M}$ *is defined by* $(ker\pi_*)$

(b) $\mathcal{T}_{\mathcal{W}} J\mathcal{W}' = 0$

(c) $(\nabla\pi_*)(\mathcal{W}, J\mathcal{X}) = 0$

for $\mathcal{X} \in \Gamma((ker\pi_*)^{\perp})$ *and* $\mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. In the light of Theorem 4.2, $(a) \Leftrightarrow (b)$ is obvious. For the proof of $(b) \Leftrightarrow (c)$, consider that $(ker\pi_*)$ and $(ker\pi_*)^{\perp}$ are orthogonal, then we write

$$\begin{aligned} g(\nabla_V J\mathcal{W}, J\mathcal{X}) &= -g(J\mathcal{W}, \nabla_V J\mathcal{X}) \\ &= -g_{\mathcal{B}}(\pi_* J\mathcal{W}, \pi_* \nabla_V J\mathcal{X}) \\ &= g_{\mathcal{B}}(\pi_* J\mathcal{W}, (\nabla\pi_*)(V, J\mathcal{X})) \\ g(\mathcal{T}_V J\mathcal{W}, J\mathcal{X}) &= g_{\mathcal{B}}(\pi_* J\mathcal{W}, (\nabla\pi_*)(V, J\mathcal{X})), \end{aligned}$$

here, we have taken help of (3.4) and Lemma 2.2. Further, $\mathcal{T}_V J\mathcal{W} \in \Gamma(ker\pi_*)$ that provides the required result $(b) \Leftrightarrow (c)$. □

Definition 4.1. [1] For a differential map π from a Riemannian manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$, if $\nabla\pi_* = 0$ holds, then π is said to be is called totally geodesic.

Next, we have

Theorem 4.3. *When* π *is used to denote a Lagrangian Riemannian submersion from l.c.K. manifold* $(\mathcal{M}, g, J)$ *onto a Riemannian manifold* $(\mathcal{B}, g_{\mathcal{B}})$*. Then* π *represents a totally geodesic map iff*

$$\mathcal{T}_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A = 0$$

and

$$\mathcal{A}_{\mathcal{X}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X}, \mathcal{W}')B_1 = 0,$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. The following holds for a Riemannian submersion π

$$(\nabla\pi_*)(\mathcal{X}, \mathcal{Y}) = 0 \qquad \forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}) \tag{4.2}$$

In the light of (2.1), (3.4) and (4.1), we derive

$$\begin{aligned} (\nabla\pi_*)(\mathcal{W}, \mathcal{W}') &= \nabla^{\pi}_{\mathcal{W}} \pi_*(\mathcal{W}') - \pi_*(\nabla_{\mathcal{W}} \mathcal{W}') \\ &= -\pi_*(\nabla_{\mathcal{W}} \mathcal{W}') \\ &= \pi_*(J(J\nabla_{\mathcal{W}} \mathcal{W}')) \\ &= \pi_*(J(\nabla_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A)) \\ &= \pi_*(J(\mathcal{T}_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A)), \qquad (4.3) \end{aligned}$$

$\forall \mathcal{W}, \mathcal{W}' \in (ker\pi_*)$.

Further, use of (3.4) produces

$$\begin{aligned}(\nabla\pi_*)(\mathcal{X}, \mathcal{W}') &= \nabla^{\pi}_{\mathcal{X}}\pi_*(\mathcal{W}') - \pi_*(\nabla_{\mathcal{X}}\mathcal{W}') \\ &= -\pi_*(\nabla_{\mathcal{X}}\mathcal{W}') \\ &= \pi_*(J(J\nabla_{\mathcal{X}}\mathcal{W}')) \\ &= \pi_*(J(\nabla_{\mathcal{X}}J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X}, \mathcal{W}')B_1)) \\ &= \pi_*(J(\mathcal{A}_{\mathcal{X}}J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X}, \mathcal{W}')B_1)) \qquad (4.4)\end{aligned}$$

$\forall \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}' \in (ker\pi_*)$. Hence, the result holds in view of (4.2),(4.3) and (4.4) and singularity of J. □

5. Decomposition theorems

[14] Let us use $\mathcal{M}$ to represent a manifold whose dimension is m and by (χ^t) a system of coordinate neighborhoods used to cover $\mathcal{M}$ in such a way that if (χ^t) and (χ^{t_1}) be any two coordinate neighborhoods, then in their intersection we obtain

$$\chi^{a_1} = \chi^{a_1}(\chi^a), \chi^{x_1} = \chi^{x_1}(\chi^x),$$

with

$$|\delta_a\chi^{a_1}| \neq 0, |\delta_x\chi^{x_1}| \neq 0,$$

here all the indices $a, b, ...$ run over $1, 2, ..., p$ and $x, y, z, ...$ over $p+1, ..., p+q = m$. This type of system of coordinate neighborhoods is known as separating coordinate system and if such a system of coordinate neighborhoods exists then it defines a locally product structure on the manifold $\mathcal{M}$. A manifold $\mathcal{M}$ equipped with a locally product structure is known as locally product manifold.

Next, we define

Definition 5.1. [9] When $N = \mathcal{M} \times \mathcal{B}$ is a manifold with Riemannian metric tensor g and $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ be the canonical foliations intersecting perpendicularly everywhere. Then

(i) the necessary and sufficient condition for g to represent the metric tensor of a warped product $\mathcal{M} \times_{f'} \mathcal{B}$ is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ denote the totally geodesic and spherical foliations, respectively.

(ii) the necessary and sufficient condition for g to be metric tensor of a twisted product $\mathcal{M} \times_{f'} \mathcal{B}$ is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ represent the totally geodesic and totally umbilical foliations, respectively

(iii) the necessary and sufficient condition for g to be metric tensor of a usual product of Riemannian manifolds is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ are totally geodesic foliations.

Thanks to Theorems 4.1 and 4.2, we have

Theorem 5.1. *When π is used to denote an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally product manifold is that the following hold*

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) &= -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \end{aligned}$$

and

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_* J\mathcal{W}') = 0$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Thanks to Corollaries 4.1 and 4.2, we have

Theorem 5.2. *When π is used to denote a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally product manifold is that the following hold*

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) = -\frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

and

$$\mathcal{T}_{\mathcal{W}} J\mathcal{W}' = 0$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

For twisted product manifold, we get

Theorem 5.3. *When π represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally twisted product manifold of the form $\mathcal{M}_{(ker\pi_*)^{\perp}} \times_{f'} \mathcal{M}_{(ker\pi_*)}$ is that the following relations hold*

$$\mathcal{T}_{\mathcal{W}} J\mathcal{X} = -g(\mathcal{X}, \mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2} J\mathcal{W}$$

and

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}} J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$. *Here, $\mathcal{M}_{(ker\pi_*)^{\perp}} \times_{f'} \mathcal{M}_{(ker\pi_*)}$ denote the integral manifold of the distributions $(ker\pi_*)^{\perp}$ and $(ker\pi_*)$.*

Proof. With the help of (2.1) and Lemma 2.2, we write

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}',\mathcal{X}) &= -g(\nabla_{\mathcal{W}}\mathcal{X},\mathcal{W}') \\ &= -g(J\nabla_{\mathcal{W}}\mathcal{X},J\mathcal{W}') \\ &= -g(\nabla_{\mathcal{W}}J\mathcal{X},J\mathcal{W}') \\ &= -g(\mathcal{T}_{\mathcal{W}}J\mathcal{X},J\mathcal{W}'),\ \ \forall\mathcal{X}\in\Gamma((ker\pi_*)^{\perp}),\mathcal{W},\mathcal{W}'\in\Gamma(ker\pi_*), \end{aligned}$$

where orthogonality of $(ker\pi_*)^{\perp}$ and $(ker\pi_*)$ has been used. Hence, we conclude that for any function λ on $\mathcal{M}$, the condition of totally umbilicity holds for $(ker\pi_*)$ iff

$$\mathcal{T}_{\mathcal{W}}J\mathcal{X} = -\mathcal{X}(\lambda)J\mathcal{W}. \tag{5.1}$$

Therefore, taking in use (2.1), we obtain

$$\begin{aligned} g(-\mathcal{X}(\lambda)J\mathcal{W},J\mathcal{W}) &= g(\mathcal{T}_{\mathcal{W}}J\mathcal{X},J\mathcal{W}) \\ -\mathcal{X}(\lambda)\|\mathcal{W}\|^2 &= g(\mathcal{T}_{\mathcal{W}}J\mathcal{X},J\mathcal{W}) \\ &= g(\nabla_{\mathcal{W}}J\mathcal{X},J\mathcal{W}) \\ &= g(J\nabla_{\mathcal{W}}\mathcal{X},J\mathcal{W}) \\ &= -g(\mathcal{X},\mathcal{T}_{\mathcal{W}}\mathcal{W}) \\ \mathcal{X}(\lambda) &= g(\mathcal{X},\mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2}. \end{aligned} \tag{5.2}$$

In this way, (5.1) and (5.2) produce

$$\mathcal{T}_{\mathcal{W}}J\mathcal{X} = -g(\mathcal{X},\mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2}J\mathcal{W}$$

and that proves the result with the help of Corollary 4.1. □

Next, we give a non existence result of a twisted product manifold $\mathcal{M}_{(ker\pi_*)^{\perp}}\times_{f'}\mathcal{M}_{(ker\pi_*)}$.

Theorem 5.4. *There does not exist Lagrangian Riemannian submersion π from l.c.K. manifold $(\mathcal{M},g,J)$ onto a Riemannian manifold $(\mathcal{B},g_{\mathcal{B}})$ such that $\mathcal{M}$ is a locally proper twisted product manifold $\mathcal{M}_{(ker\pi_*)^{\perp}}\times_{f'}\mathcal{M}_{(ker\pi_*)}$.*

Proof. Let π denotes a Lagrangian Riemannian submersion from l.c.K. manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$ and $\mathcal{M}$ be representing a locally twisted product $\mathcal{M}_{(ker\pi_*)^{\perp}}\times_{f'}\mathcal{M}_{(ker\pi_*)}$. Then, due to definition 5.1, $\mathcal{M}_{(ker\pi_*)}$ and $\mathcal{M}_{(ker\pi_*)^{\perp}}$ will be representing totally geodesic and totally umbilical foliations, respectively. When h denotes the second fundamental form of $\mathcal{M}_{(ker\pi_*)^{\perp}}$, we write

$$g(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) = g(h(\mathcal{X},\mathcal{Y}),\mathcal{W}),\ \ \forall\mathcal{X},\mathcal{Y}\in\Gamma((ker\pi_*)^{\perp}),\mathcal{W}\in\Gamma(ker\pi_*).$$

When H is used for the mean curvature vector field of $\mathcal{M}_{(ker\pi_*)^{\perp}}$, then we deduce

$$g(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) = g(H,\mathcal{W})g(\mathcal{X},\mathcal{Y}). \tag{5.3}$$

Taking (2.1) and lemma 2.2 into consideration, we present

$$\begin{aligned} g(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) &= -g(\mathcal{Y},\nabla_{\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{Y},J\nabla_{\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}), \end{aligned} \tag{5.4}$$

here we used the orthogonal property between $(ker\pi_*)^{\perp}$ and $(ker\pi_*)$. Therefore, (5.3) and (5.4) generate the following

$$\begin{aligned} g(H,\mathcal{W})g(\mathcal{X},\mathcal{Y}) &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}) \\ g(H,\mathcal{W})g(J\mathcal{Y},J\mathcal{X}) &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}) \\ -g(H,\mathcal{W})\|\mathcal{X}\|^2 &= g(\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X},J\mathcal{X}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X},J\mathcal{X}) \\ &= g(J\nabla_{\mathcal{X}}\mathcal{W},J\mathcal{X}) \\ &= -g(\mathcal{W},\nabla_{\mathcal{X}}\mathcal{X}) \end{aligned}$$

Finally, we reach to

$$g(H,\mathcal{W})\|\mathcal{X}\|^2 = g(\mathcal{W},\mathcal{A}_{\mathcal{X}}\mathcal{X}).$$

So, use of (2.5) shows $\mathcal{A}_{\mathcal{X}}\mathcal{X}=0$, that is $g(H,\mathcal{W})\|\mathcal{X}\|^2=0$. But, $H\in\Gamma(ker\pi_*)$ with Riemannian metric g supply $H=0$ and that that means $(ker\pi_*)^{\perp}$ is totally geodesic. That proves $\mathcal{M}$ to be usual product of Riemannian manifolds. □

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Majid Ali Choudhary
Department of Mathematics
School of Sciences
Maulana Azad National Urdu University
Hyderabad, India
majid_alichoudhary@yahoo.co.in

Lamia Saeed Alqahtani
Department of Mathematics
Faculty of Science
King Abdulaziz University
Jeddah 21589, Saudi Arabia
lalqahtani@kau.edu.sa