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# WEIGHTED STATISTICAL CONVERGENCE OF REAL VALUED SEQUENCES

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**Abstract.** Functions defined in the form "$g : \mathbb{N} \to [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{g(n)} = 0$" are called weight functions. Using the weight function, the concept of weighted density, which is a generalization of natural density, was defined by Balcerzak, Das, Filipczak and Swaczyna in the paper "Generalized kinsd of density and the associated ideals", Acta Mathematica Hungarica 147(1) (2015), 97-115.

In this study, the definitions of $g$-statistical convergence and $g$-statistical Cauchy sequence for any weight function $g$ are given and it is proved that these two concepts are equivalent. Also, some inclusions of the sets of all weight $g_1$-statistical convergent and weight $g_2$-statistical convergent sequences for $g_1, g_2$ which have the initial conditions are given.

**Keywords**: weight functions; natural density; statistical convergent sequences.

## 1. Introduction

In [5], Fast introduced the concept of statistical convergence. In [15], Schoenberg gave some basic properties of statistically convergence and also studied the concept as a summability method. After this works many Mathematician have used these concept in their studies [8, 9, 10, 11]. In [2, 3], the authors proposed a modified version of density by replacing $n$ by $n^\alpha$ where $0 < \alpha \leq 1$. In [1], the authors defined a more general kind of density by replacing $n^\alpha$ by a function $g : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} g(n) = \infty$. In this paper, we will study the weighted $g$-statistically convergence concept.

Let $K$ be a subset of natural numbers. Natural density of $K$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K(n)|$$

where $K(n) = \{k \leq n : k \in K\}$ and the vertical bars denotes the number of elements of $K(n)$.

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Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. Let us remember that the definition of density of weight $g(n)$.

**Definition 1.1.** The density of weight $g$ defined by the formula

$$d_g(A) = \lim_{n\to\infty} \frac{|A(n)|}{g(n)}$$

for $A \subset \mathbb{N}$ [1, 4].

After the study [1], the concept of $g$-density was applied to various problems related to sequences and interesting results were obtained in [4, 7, 12, 13, 14].

Basically in this study, it will be shown that the results given in [6] can be re-examined by using $g$-density.

In this paper, we are concerned with the subsets of natural numbers having weight $g(n)$ density zero. To facilitate this, we have introduced the following notation: If $x$ is a sequence such that $x_k$ satisfies property $P$ for all $k$ except a set of weight $g(n)$ density zero, then we say that $x_k$ satisfies $P$ for (weight $g$ almost all $k$) and it is denoted by $(g - a.a.k)$ for simplicity.

**Definition 1.2.** Let $x = (x_k)$ be a real valued sequence. $x$ is weight $g$-statistical convergent to the number $L$ if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - L| < \varepsilon \quad (g - a.a.k)$. In this case we write $g - st - \lim x_k = L$.

$C_g^{st}$ denotes the set of all weight $g$-statistical convergent sequences.

If we take the function $g(n) = n$ we obtain the usual statistical convergence.

It is clear that every convergent sequence is also weight $g$-statistical convergent. But the converse is not true in general.

**Example 1.1.** Let us define the function $g(n) = 2n$ and the sequence as

$$x_k = \begin{cases} 3, & k = m^2, \quad m \in \mathbb{N}, \\ 0, & k \neq 0. \end{cases}$$

Then $|k \leq n : x_k \neq 0| \leq \sqrt{n}$. So, $g - st - lim x_k = 0$.

**Theorem 1.1.** *If the sequence $(x_n)$ is weight-$g$-statistical convergent to $L$ then there is a set $K = \{k_1 < k_2 < ...\}$ such that $d_g(K) = d_g(\mathbb{N})$ and $\lim_{n\to\infty} x_{k_n} = L$.*

*Proof.* Let us assume that $g-st-limx_k=L$. Take $K_i:=\{n\in\mathbb{N}:|x_n-L|<\frac{1}{i}\}$, $(i=1,2,...)$. Then by definition we have $d_g(K_i^c)=0$ and it is clear that $d_g(K_i)=d_g(\mathbb{N})$, $(i=1,2,...)$. Also it is easy to control that

$$...\subset K_{i+1}\subset K_i\subset ...\subset K_2\subset K_1 \tag{1.1}$$

Let $\{T_j\}_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers. Let choose an arbitrary number $a_1\in K_1$. By (1.1) we can choose an element $a_2\in K_2$, $a_2>a_1$ such that for each $n\geq a_2$ we have $\frac{K_2(n)}{g(n)}>T_2$. Moreover choose $a_3>a_2$, $a_3\in K_3$ such that for each $n\geq a_3$ we have $\frac{K_3(n)}{g(n)}>T_3$. If we proceed in this way we obtain a sequence $a_1<a_2...<a_i<...$ of positive integers such that

$$a_i\in K_i,\ (i=1,2,...)\text{ and }\frac{K_i(n)}{g(n)}>T_i \tag{1.2}$$

for each $n\geq a_i$, $i=1,2,...$

Let us establish the set $K$ as follows: each natural number of the interval $[1,a_1]$ belong to $K$, moreover, any natural number of the interval $[a_i,a_{i+1}]$ belongs to $K$ if and only if it belongs to $K_i$ $(i=1,2,...)$. From (1.1) and (1.2) we have

$$\frac{K(n)}{g(n)}\geq\frac{K_i(n)}{g(n)}>T_i$$

for each $n$, $a_i\leq n<a_{i+1}$. By last inequality it is clear that $\overline{d}_g(K)=\infty$.

Let $\varepsilon>0$, and choose $i$ such that $\frac{1}{i}<\varepsilon$. Let $n\geq a_i$, $n\in K$. There exists a number $t\geq i$ such that $a_t\leq n<a_{t+1}$. But from the definition of $K$, $n\in K_t$. Thus $|x_n-L|<\frac{1}{t}\leq\frac{1}{i}<\varepsilon$. Hence, $\lim_{n\to\infty}x_{k_n}=L$. □

**Remark 1.1.** The converse of Theorem 1.1 is not true.

**Example 1.2.** Let us consider the sequence

$$(x_k):=\begin{cases}1, & k=n^2,\\ 0, & k\neq n^2,\end{cases}$$

and $g(n)=n^{1/4}$. It is clear that the set $K=\{k:k=n^2,n\in\mathbb{N}\}\subset\mathbb{N}$ has the property $\overline{d}_g(K)=\infty$. But $g-st-\lim x_k\neq 1$.

Let us note that every statistical convergent sequence is also weight-$g$-statistical convergent to the same number. But the converse of this situation is not true.

**Example 1.3.** Let $a_k=2^{2^k}$, and

$$g(n):=\begin{cases}a_{2k}, & n\in[a_k,a_{k+1}),\ k=1,2,...\\ 1, & n<4.\end{cases}$$

Let $A_k := \{n \in \mathbb{N} : a_k \le n < 2a_k\}$ and $A := \cup_{k\ge 1} A_k$. Let us take account the sequence

$$x_n := \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}a_k \le |A_k| \le a_k$. Let us check that $x_n \nrightarrow 0(st)$. If we put $m_k = \max A_k$, we obtain

$$\frac{|\{k \le n : |x_k - 0| \ge \varepsilon\}|}{n} = \frac{|\{k \le n : x_k \in A\}|}{n} = \frac{|A|}{m_k} \ge \frac{|A_k|}{m_k} \ge \frac{\frac{1}{2}a_k}{2a_k} = \frac{1}{4}$$

for all $k \ge 1$.

Moreover, $g - st - \lim x_k = 0$. For sufficiently large n, we have

$$\begin{aligned} \frac{|\{k \le n : |x_k - 0| \ge \varepsilon\}|}{g(n)} &= \frac{|\{k \le n : x_k \in A\}|}{g(n)} = \frac{|A|}{g(n)} \\ &= \frac{|\{k \le m_k : x_k \in A\}|}{g(m_k)} \\ &\le \frac{|A_k|}{a_{2k}} \le \frac{a_k}{a_{2k}} \to 0. \end{aligned}$$

**Definition 1.3.** Let $x = (x_k)$ be a real valued sequence. $x$ is weight $g$-statistical Cauchy sequence if for each $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - x_N| \ge \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - x_N| < \varepsilon \quad (g - a.a.k)$. In this case we write $x$ is weight $g$-Cauchy sequence.

**Lemma 1.1.** *The following statements are equivalent:*

*(i) $x$ is a weight $g$-statistically convergent sequence,*

*(ii) $x$ is a weight $g$-statistically Cauchy sequence,*

*(iii) $x$ is a sequence for which there is a convergent sequence $y$ such that $x_k = y_k$ $(g - a.a.k)$.*

*Proof.* $(i) \Rightarrow (ii)$ Let us assume that $x$ is a weight $g$-statistical convergent sequence. Suppose $\varepsilon > 0$ and $g - st - \lim x = L$. Then $|x_k - L| < \frac{\varepsilon}{2}$ $(g - a.a.k)$ holds.

If we choose a natural number $N$ such that $|x_N - L| < \frac{\varepsilon}{2}$, then we have

$$|x_k - x_N| < |x_k - L| + |x_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ (g - a.a.k).$$

Hence, $x$ is a weight $g$-statistical Cauchy sequence.

$(ii) \Rightarrow (iii)$ Let us assume that $x$ is a weight $g$-statistical Cauchy sequence. Choose $N(1)$ such that the interval $I = [x_{N(1)} - 1, x_{N(1)} + 1]$ contains $x_k$ $(g - a.a.k)$. Also apply (ii) to choose $M$ such that $I' = [x_M - \frac{1}{2}, x_M + \frac{1}{2}]$ contains $x_k$ $(g - a.a.k)$. We claim that

$$I_1 = I \cap I' \text{ contains } x_k \ (g - a.a.k),$$

for

$$\{k \leq n : x_k \notin I \cap I'\} = \{k \leq n : x_k \notin I\} \cup \{k \leq n : x_k \notin I'\}.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I \cap I'\}| \leq$$

$$\leq \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I\}| + \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I'\}| = 0.$$

So, $I_1$ is closed interval of length less than or equal to 1 and contains $x_k$ $(g - a.a.k)$. Now we continue by choosing $N(2)$ such that $I'' = [x_{N(2)} - \frac{1}{4}, x_{N(2)} + \frac{1}{4}]$ contains $x_k$ $(g - a.a.k)$, by the previously argument $I_2 = I_1 \cap I''$ contains $x_k$ $(g - a.a.k)$, and $I_2$ has length less than or equal to $\frac{1}{2}$. Proceeding inductively we construct a sequence$\{I_m\}_{m=1}^{\infty}$ of closed intervals such that for each $m$, $I_{m+1} \subseteq I_m$, and the length of $I_m$ is not greater than $2^{1-m}$, and $x_k \in I_m$ $(g - a.a.k)$. From the Nested Interval Theorem there is a number $\alpha$ such that $\alpha = \cap_{m=1}^{\infty} I_m$. If we use $x_k \in I_m$ $(g - a.a.k)$, we can choose an increasing positive sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{g(n)} |\{k \leq n : x_k \notin I_m\}| < \frac{1}{g(m)} \; if \; n > \; T_m. \tag{1.3}$$

Next define a subsequence $z$ of $x$ consisting of all terms $x_k$ such that $k > T_1$ and if $T_m < k \leq T_{m+1}$ then $x_k \notin I_m$.

Now define the sequence $y$ by

$$y_k = \begin{cases} \alpha, & \text{if } x_k \text{ is a term of } z, \\ x_k, & \text{otherwise.} \end{cases}$$

Then $\lim y_k = \alpha$; for , if $\varepsilon > \frac{1}{g(m)} > 0$ and $k > T_m$ then either $x_k$ is a term of $z$, which means $y_k = \alpha$ or $y_k = x_k \in I_m$ and $|y_k - \alpha| \leq$ length of $I_m < 2^{1-m}$. We also assert that $x_k = y_k$ $(g - a.a.k)$. To confirm this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \leq n : y_k \neq x_k\} \subseteq \{k \leq n : x_k \notin I_m\}$$

so from (1.3)

$$\frac{1}{g(n)} |\{k \leq n : y_k \neq x_k\}| \leq \frac{1}{g(n)} |\{k \leq n : x_k \notin I_m\}| < \frac{1}{g(m)}$$

is obtained. Thus, the limit as $n \to \infty$ is 0 and $x_k = y_k$ $(g - a.a.k)$.

$(iii) \Rightarrow (i)$ Let us assume that $x_k = y_k$ $(g - a.a.k)$ and $\lim y_k = L$. Suppose $\varepsilon > 0$. Then for each $n$,

$$\{k \leq n : |x_k - L| > \varepsilon\} \subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n : |y_k - L| > \varepsilon\}$$

from the assumption $\lim y_k = L$, the second set contains a fixed number of integers, say $l = l(\varepsilon)$. So,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{g(n)}|\{k \le n : |x_k - L| > \varepsilon\}| &\le \lim_{n\to\infty} \frac{1}{g(n)}|\{k \le n : x_k \ne y_k\}| + \\ &+ \lim_{n\to\infty} \frac{l}{g(n)} = 0\end{aligned}$$

because $x_k = y_k$ $(g - a.a.k)$. Hence, $|x_k - L| \le \varepsilon$ $(g - a.a.k)$. So, the proof is complete. □

**Corollary 1.1.** *Let $x$ be a real valued sequence. If $g - st - \lim x_k = L$, then $x$ has a subsequence $y$ such that $\lim y_k = L$.*

## 2. Inclusion Between Two $g - st-$Convergence

Let $G$ denotes the set of all functions $g : \mathbb{N} \to [0, \infty)$ satisfying the condition $g(n) \to \infty$ and $\frac{n}{g(n)} \nrightarrow 0$. In this section, we will introduce some inclusions between various $g \in G$.

**Lemma 2.1.** *Let $g_1, g_2 \in G$ such that there exist $M,\ m > 0$ and $k_0 \in \mathbb{N}$ such that $m \le \frac{g_1(n)}{g_2(n)} \le M$ for all $n \ge k_0$. Then $C^{st}_{g_1}(x) = C^{st}_{g_2}(x)$.*

*Proof.* Suppose the sequence $x$ is weight $g_1$-statistical convergence to $L$. This implies that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)} = 0.$$

Together with the fact that $\frac{g_1(n)}{g_2(n)} \le M$, this implies that

$$\frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{M g_2(n)} \le \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)}.$$

for all $n \ge k_0$. This implies

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{M g_2(n)} \le \lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)} = 0.$$

From the hypothesis we obtain

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_2(n)} = 0.$$

Thus, the sequence $x$ is weight $g_2$-statistical convergent to $L$. So, $C^{st}_{g_1}(x) \subset C^{st}_{g_2}(x)$. We can prove the iclusion $C^{st}_{g_2}(x) \subset C^{st}_{g_1}(x)$ by similar way. □

**Lemma 2.2.** *For each function $f \in G$ there exists a nondecreasing function $g \in G$ such that $C_f^{st}(x) = C_g^{st}(x)$. Moreover,*

$$g(n) \leq f(n) \tag{2.1}$$

*for all $n \in \mathbb{N}$.*

*Proof.* If $f$ is nondecreasing, it is nclear. Otherwise, define the related function $g : \mathbb{N} \to [0, \infty)$ as follows. Let $a_1 = \min\{f(n) : n \in \mathbb{N}\}$, $i_1 = \max\{i \in \mathbb{N} : f(i) = a_1\}$ and $g(i) = a_1$ for $0 \leq i \leq i_1$. Next, let $a_2 = \min\{f(n) : n > i_1\}$, $i_2 = \max\{i \in N : f(i) = a_2\}$ and $g(i) = a_2$ for $i_1 < i \leq i_2$. Rest of the function $g$ is established by induction.

Obviously, the function $g$ is nondecreasing and $g(n) \to \infty$. By the construction, $g(n) \leq f(n)$, for all $n \in \mathbb{N}$. Hence $\frac{n}{f(n)} \leq \frac{n}{g(n)}$ for all $n$ which implies that $\frac{n}{g(n)} \nrightarrow 0$. Thus $g \in G$.

Let $(x_n)$ be a weight $g$-statistical convergent sequence to $L$. So, for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0$$

holds. From (2.1) we have following inequality

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \leq \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)}.$$

If we take limit when $n \to \infty$ we obtain $f - st - \lim x_k = L$. Thus, the inclusion $C_g^{st} \subset C_f^{st}$.

By construction, for each $n \in \mathbb{N}$ there exist $m \geq n$ such that $g(n) = g(m) = f(m)$. Suppose that $x_n \nrightarrow L$ $(g - st)$. Then there exists $a$, where $a \in \mathbb{R}^+ \cup \{+\infty\}$ and an inreasing sequence $(n_i)$ of indices such that

$$\lim_{i\to\infty} \frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} = a > 0.$$

For each $i \in \mathbb{N}$ we can find $m_i \geq n_i$ such that $g(n_i) = g(m_i) = f(m_i)$. Hence

$$\frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} \leq \frac{|\{k \leq m_i : |x_k - L| \geq \varepsilon\}|}{f(m_i)}$$

holds. So, $x_n \nrightarrow L$ $(f - st)$. $\square$

**Lemma 2.3.** *Let $f \in G$ be such that $\frac{n}{f(n)} \to \infty$, $L, \varepsilon$ real numbers with $\varepsilon > 0$. Then there exists a sequence $(x_n)$ such that $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded but not convergent to zero.*

*Proof.* Firstly, let us assume that $f$ is nondecreasing. Take to the smallest non negative integer, $k_0$, such that for $n \geq k_0$, $f(n) > 2$. Let us define a set $A \subset \mathbb{N} \backslash \{0, 1, 2, ...k_0 - 1\}$ inductively, deciding whether $n \geq k_0$ should belong to $A$ or not. Let $n \notin A$ for all $n < k_0$. Suppose that $n \geq k_0$ and then we have defined $A(n)$. If $\frac{|A(n)|}{f(n+1)} < 1$ then let $n \in A$. Otherwise, let $n \notin A$. So, we construct the set $A$. From this construction and the condition $f(n) \to \infty$, $A$ is infinite.

We assert that $\mathbb{N} \backslash A$ is also infinite. Let us assume that it is finite and choose $n_0 \in \mathbb{N}$ such that $n \in A$ for all $n \geq n_0$. Then, we have

$$\frac{n - n_0}{f(n+1)} \leq \frac{|A(n)|}{f(n+1)} < 1$$

for all $n \geq n_0$. But this is impossible because of the assumption, $\frac{n-n_0}{f(n+1)} \to \infty$. Now, we will show that $\frac{|A(n)|}{f(n)} < 2$ for each $n \geq k_0$. It is clear that if $n = k_0$ it is true. Suppose that $\frac{|A(n)|}{f(n)} < 2$ for a fixed $n \geq k_0$.

If $\frac{|A(n)|}{f(n+1)} < 1$, we have

$$\begin{aligned} \frac{|A(n+1)|}{f(n+1)} &= \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n+1)} \\ &\leq \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n)} \\ &\leq 1 + \frac{1}{2} < 2. \end{aligned}$$

If $\frac{|A(n)|}{f(n+1)} > 1$ , then $n \notin A$ and so,

$$\frac{|A(n+1)|}{f(n+1)} = \frac{|A(n)|}{f(n+1)} \leq \frac{|A(n)|}{f(n)} < 2.$$

Now, let us define a sequence $(x_n)$ as follows:

$$x_n := \begin{cases} n & n \in A \\ L & n \notin A \end{cases}$$

where $L \in \mathbb{R}$ is a fixed number. It is clear that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded from the first part of this proof.

Now, we will show that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0. For this aim consider any $n \geq k_0$. We will find $m \geq n$ such that $\frac{|A(m)|}{f(m)} \geq 1$. If $\frac{|A(n)|}{f(n)} \geq 1$, put $m := n$. Otherwise, choose the smallest $m \geq n$ such that $m \in \mathbb{N} \backslash A$. Then $\frac{|A(m)|}{f(m+1)} \geq 1$ and so, $\frac{|A(m)|}{f(m)} \geq 1$. Thus, the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0.

Now, let us back to the general case where $f \in G$ need not be nondecreasing. Then we assume the associated function $g \in G$ from Lemma 2.2. Note that $\frac{n}{g(n)} \to \infty$ since $\frac{n}{g(n)} \geq \frac{n}{f(n)}$ for all $n$ and $\frac{n}{f(n)} \to \infty$. By the above reasons we obtain the respective set $A$ for $g$. Thus, $\frac{|A(n)|}{g(n)} \nrightarrow 0$ and the sequence $(\frac{|A(n)|}{g(n)})$ is bounded. Then $\frac{|A(n)|}{f(n)} \nrightarrow 0$, and the sequence $\left(\frac{|\{k \leq n: |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded since $g(n) \leq f(n)$ for all $n \in \mathbb{N}$. $\square$

**Theorem 2.1.** *If $g_1$, $g_2$ belong to $G$ such that $\frac{g_2(n)}{g_1(n)} \to \infty$ then $C^{st}_{g_1}(x) \subsetneq C^{st}_{g_2}(x)$. If $g \in G$ and $\frac{n}{g(n)} \to \infty$ then $C^{st}_{g}(x) \subsetneq C^{st}(x)$.*

*Proof.* To prove the first claim note that the inclusion $C^{st}_{g_1}(x) \subset C^{st}_{g_2}(x)$ follows from Lemma 2.1. Set $f := \sqrt{g_1.g_2}$. Then

$$\lim_{n\to\infty} \frac{f(n)}{g_1(n)} = \lim_{n\to\infty} \frac{g_2(n)}{f(n)} = \infty. \tag{2.2}$$

Also we have

$$\frac{n}{g_1(n)} = \frac{n}{g_2(n)}.\frac{g_2(n)}{g_1(n)} \to \infty.$$

So $\frac{n}{f(n)} = \sqrt{\frac{n^2}{g_1(n)g_2(n)}} \to \infty$. Hence $f$ have the assumption of Lemma 2.3. Take the sequence $(x_n)$ obtained in this lemma. Then $x_n \in C^{st}_{g_2}(x)$ but $x_n \notin C^{st}_{g_1}(x)$. Indeed, using (2.2) we have

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_2(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}.\frac{f(n)}{g_2(n)} \to 0$$

because $\left(\frac{|\{k \leq n: |x_k - L| \geq \varepsilon\}|}{f(n)}\right)_{n\in\mathbb{N}}$ is bounded from Lemma 2.3. Thus, $x_n \in C^{st}_{g_2}(x)$. To prove that $x_n \notin C^{st}_{g_1}(x)$ observe that

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \frac{f(n)}{g_1(n)}.$$

So, $x_n \notin C^{st}_{g_1}(x)$ because $\frac{|\{k \leq n: |x_k - L| \geq \varepsilon\}|}{f(n)} \nrightarrow 0$, and $\frac{f(n)}{g_1(n)} \to \infty$ from (2.2).

If we take $g_2(n) = n$, for all $n \in \mathbb{N}$, second assertion proved easily from the same way. $\square$

**Corollary 2.1.** *Let $0 < \alpha < \beta \leq 1$ and $g_1(n) = n^\alpha$, $g_2 = n^\beta$ for $n \in \mathbb{N}$. Then $C^{st}_{g_1}(x) \subsetneq C^{st}_{g_2}(x)$.*

**Example 2.1.** Let

$$g_1(n) = \begin{cases} n, & \text{for even } n \in \mathbb{N} \\ \sqrt{n}, & \text{for odd } n \in \mathbb{N} \end{cases}$$

and $g_2(n) = \sqrt{n}$ for $n \in \mathbb{N}$. It is clear that, $\limsup_{n\to\infty} \frac{g_1(n)}{g_2(n)} = \infty$. However, $C^{st}_{g_1}(x) = C^{st}_{g_2}(x)$. Indeed, construct a nondecreasing function $g \in G$ such that $C^{st}_g(x) = C^{st}_{g_1}(x)$, according to the method used in the proof of Lemma 2.1. Then it follows from simple calculations that $g$ is given by

$$g(n) = \begin{cases} \sqrt{n+1} & \text{for even } n \in \mathbb{N} \\ \sqrt{n} & \text{for odd } n \in \mathbb{N}. \end{cases}$$

Obviously, $\frac{1}{2} \leq \frac{g(n)}{g_2(n)} \leq 2$ for all $n \geq 1$. Therefore, by Lemma 2.1 we have $C^{st}_g(x) = C^{st}_{g_1}(x)$.

**Theorem 2.2.** *There exists a function $g \in G$ such that $C^{st}_g$ is different from $C^{st}_{n^\alpha}$ with $0 < \alpha < 1$.*

*Proof.* Let $a_k$ and $g(n)$ defined as in Example 1.3. Let $A_k := \{n \in \mathbb{N} : a_{k+1} - (a_{k+1})^{1/4} \leq n < a_{k+1}\}$ and $A = \cup_{k\geq 2} A_k$. Let us take account the sequence

$$x_n = \begin{cases} n, & n \in A \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}(a_{k+1})^{1/4} \leq |B_k| \leq (a_{k+1})^{1/4}$. Let us check that $g - st - \lim x_k \neq 0$. For $k > 0$ we have

$$\frac{|\{k \leq a_{k+1} - 1 : |x_k - 0| \geq \varepsilon\}|}{g(a_{k+1} - 1)} \geq \frac{\frac{1}{2}|B_k|}{g(a_k)} \geq \frac{\frac{1}{4}(a_{k+1})^{1/4}}{(a_{k+1})^{1/4}} = \frac{1}{4},$$

so, $g - st - \lim x_k \neq 0$. Furthermore,

$$|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}| \leq (a_k)^{1/4} + (a_{k+1})^{1/4} \leq 2(a_{k+1})^{1/4}$$

and so,

$$\frac{|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}|}{(a_{k+1})^{1/3}} \leq \frac{2(a_{k+1})^{1/4}}{(a_{k+1})^{1/3}} = 2(a_{k+1})^{-1/12} \to 0, \ (k \to \infty)$$

holds.

Now, fix any $n \geq 4$ and choose a unique $k \in \mathbb{N}$ such that $n \in [a_k, a_{k+1})$. If $n < a_{k+1} - (a_{k+1})^{1/4}$ then

$$\begin{aligned} \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} &= \frac{|\{k \leq a_k : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} \\ &\leq \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{(a_k)^{1/3}} \leq 2(a_k)^{-1/12}. \end{aligned}$$

If $a_{k+1} - (a_{k+1})^{1/4} \leq n < a_{k+1}$ then for $b > a > 0$, the function

$$f(x) := \frac{a + x}{(b + x)^{1/3}}, \ x \geq 0$$

is increasing, thus

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} \leq \frac{|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}|}{(a_{k+1})^{1/3}}.$$

So, $x_n \in C^{st}_{n^{1/3}}(x)$.

Now, let $0 < \alpha < 1$, $\alpha \neq \frac{1}{3}$. If $\alpha < \frac{1}{3}$ then from Corollary 2.1 $C^{st}_{n^\alpha} \subsetneq C^{st}_{n^{1/3}}$ and $C^{st}_g \backslash C^{st}_{n^\alpha} \neq \emptyset$ because $C^{st}_g \backslash C^{st}_{n^{1/3}} \neq \emptyset$. If $\alpha > \frac{1}{3}$ then $C^{st}_{n^\alpha} \backslash C^{st}_g \neq \emptyset$. By the same way we can show that $x_n \in C^{st} \backslash C^{st}_g$. So $C^{st}_g \subsetneq C^{st}$. $\square$

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