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Abstract. Functions defined in the form $g : \mathbb{N} \rightarrow [0, \infty)$ such that $\lim_{n \to \infty} g(n) = \infty$ and $\lim_{n \to \infty} \frac{n}{g(n)} = 0$ are called weight functions. Using the weight function, the concept of weighted density, which is a generalization of natural density, was defined by Balcerzak, Das, Filipczak and Swaczyna in the paper “Generalized kinds of density and the associated ideals”, Acta Mathematica Hungarica 147(1) (2015), 97-115.

In this study, the definitions of $g$-statistical convergence and $g$-statistical Cauchy sequence for any weight function $g$ are given and it is proved that these two concepts are equivalent. Also, some inclusions of the sets of all weight $g_1$-statistical convergent and weight $g_2$-statistical convergent sequences for $g_1, g_2$ which have the initial conditions are given.

Keywords: weight functions; natural density; statistical convergent sequences.

1. Introduction

In [5], Fast introduced the concept of statistical convergence. In [15], Schoenberg gave some basic properties of statistically convergence and also studied the concept as a summability method. After this works many Mathematician have used these concept in their studies [8, 9, 10, 11]. In [2, 3], the authors proposed a modified version of density by replacing $n$ by $n^\alpha$ where $0 < \alpha \leq 1$. In [1], the authors defined a more general kind of density by replacing $n^\alpha$ by a function $g : \mathbb{N} \rightarrow [0, \infty)$ with $\lim_{n \to \infty} g(n) = \infty$. In this paper, we will study the weighted $g$-statistically convergence concept.

Let $K$ be a subset of natural numbers. Natural density of $K$ is defined by

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K(n)|$$

where $K(n) = \{k \leq n : k \in K\}$ and the vertical bars denotes the number of elements of $K(n)$.

Received December 02, 2019; accepted February 19, 2020
2010 Mathematics Subject Classification. Primary 40A05; Secondary 46A45
Let \( g : \mathbb{N} \to [0, \infty) \) be a function with \( \lim_{n \to \infty} g(n) = \infty \). Let us remember that the definition of density of weight \( g(n) \).

**Definition 1.1.** The density of weight \( g \) defined by the formula

\[
d_g(A) = \lim_{n \to \infty} \frac{|A(n)|}{g(n)}
\]

for \( A \subset \mathbb{N} \) [1, 4].

After the study [1], the concept of \( g \)-density was applied to various problems related to sequences and interesting results were obtained in [4, 7, 12, 13, 14].

Basically in this study, it will be shown that the results given in [6] can be re-examined by using \( g \)-density.

In this paper, we are concerned with the subsets of natural numbers having weight \( g(n) \) density zero. To facilitate this, we have introduced the following notation: If \( x \) is a sequence such that \( x_k \) satisfies property \( P \) for all \( k \) except a set of weight \( g(n) \) density zero, then we say that \( x_k \) satisfies \( P \) for (weight \( g \) almost all \( k \)) and it is denoted by \( (g - a.a.k) \) for simplicity.

**Definition 1.2.** Let \( x = (x_k) \) be a real valued sequence. \( x \) is weight \( g \)-statistical convergent to the number \( L \) if for each \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0,
\]

i.e., \( |x_k - L| < \varepsilon \ (g - a.a.k) \). In this case we write \( g - st \lim x_k = L \).

\( C_g^{st} \) denotes the set of all weight \( g \)-statistical convergent sequences.

If we take the function \( g(n) = n \) we obtain the usual statistical convergence.

It is clear that every convergent sequence is also weight \( g \)-statistical convergent. But the converse is not true in general.

**Example 1.1.** Let us define the function \( g(n) = 2n \) and the sequence as

\[
x_k = \begin{cases} 3, & k = m^2, \ m \in \mathbb{N}, \\ 0, & k \neq 0. \end{cases}
\]

Then \( |k \leq n : x_k \neq 0| \leq \sqrt{n} \). So, \( g - st \lim x_k = 0 \).

**Theorem 1.1.** If the sequence \( (x_n) \) is weight-\( g \)-statistical convergent to \( L \) then there is a set \( K = \{k_1 < k_2 < ...\} \) such that \( d_g(K) = d_g(\mathbb{N}) \) and \( \lim_{n \to \infty} x_{k_n} = L \).
Proof. Let us assume that \( g - st - \lim x_k = L \). Take \( K_i := \{ n \in \mathbb{N} : |x_n - L| < \frac{1}{i} \} \), \((i = 1, 2, \ldots)\). Then by definition we have \( d_g(K_i) = 0 \) and it is clear that \( d_g(K_i) = d_g(\mathbb{N}) \), \((i = 1, 2, \ldots)\). Also it is easy to control that

\[
\ldots \subset K_{i+1} \subset K_i \subset \ldots \subset K_2 \subset K_1
\]

Let \( \{ T_j \}_{j \in \mathbb{N}} \) be a strictly increasing sequence of positive real numbers. Let choose an arbitrary number \( a_1 \in K_1 \). By (1.1) we can choose an element \( a_2 > a_1 \), \( a_2 \in K_2 \) such that for each \( n \in a_2 \) we have \( K_2(n) > T_2 \). Moreover choose \( a_3 > a_2 \), \( a_3 \in K_3 \) such that for each \( n \in a_3 \) we have \( K_3(n) > T_3 \). If we proceed in this way we obtain a sequence \( a_1 < a_2 \ldots < a_i \lessdot \ldots \) of positive integers such that

\[
a_i \in K_i, \quad (i = 1, 2, \ldots) \quad \text{and} \quad \frac{K_i(n)}{g(n)} > T_i
\]

for each \( n \geq a_i, \quad i = 1, 2, \ldots \)

Let us establish the set \( K \) as follows: each natural number of the interval \([1, a_1]\) belong to \( K \), moreover, any natural number of the interval \([a_i, a_{i+1})\) belongs to \( K \) if and only if it belongs to \( K_i \) \((i = 1, 2, \ldots)\). From (1.1) and (1.2) we have

\[
\frac{K(n)}{g(n)} \geq \frac{K_i(n)}{g(n)} > T_i
\]

for each \( n, \quad a_i \leq n < a_{i+1} \). By last inequality it is clear that \( \overline{d}_g(K) = \infty \).

Let \( \varepsilon > 0 \), and choose \( i \) such that \( \frac{1}{i} < \varepsilon \). Let \( n \geq a_i, \quad n \in K \). There exists a number \( t \geq i \) such that \( a_t \leq n < a_{t+1} \). But from the definition of \( K, \quad n \in K_i \). Thus \( |x_n - L| < \frac{1}{i} \leq \frac{1}{t} < \varepsilon \). Hence, \( \lim_{n \to \infty} x_k = L \). \( \square \)

Remark 1.1. The converse of Theorem 1.1 is not true.

Example 1.2. Let us consider the sequence

\[
(x_k) := \begin{cases}
1, & k = n^2, \\
0, & k \neq n^2,
\end{cases}
\]

and \( g(n) = n^{1/4} \). It is clear that the set \( K = \{ k : k = n^2, n \in \mathbb{N} \} \subset \mathbb{N} \) has the property \( \overline{d}_g(K) = \infty \). But \( g - st - \lim x_k \neq 1 \).

Let us note that every statistical convergent sequence is also weight-\( g \)-statistical convergent to the same number. But the converse of this situation is not true.

Example 1.3. Let \( a_k = 2^{a_k} \), and

\[
g(n) := \begin{cases}
a_{2k}, & n \in [a_{k}, a_{k+1}), k = 1, 2, \ldots \\
1, & n < 4.
\end{cases}
\]
Let $A_k := \{n \in \mathbb{N} : a_k \leq n < 2a_k\}$ and $A := \cup_{k \geq 1} A_k$. Let us take account the sequence

$$x_n := \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}a_k \leq |A_k| \leq a_k$. Let us check that $x_n \nrightarrow 0$. If we put $m_k = \max A_k$, we obtain

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n} = \frac{|\{k \leq n : x_k \in A\}|}{n} = \frac{|A|}{m_k} \geq \frac{|A_k|}{m_k} \geq \frac{\frac{1}{2}a_k}{2a_k} = \frac{1}{4}$$

for all $k \geq 1$.

Moreover, $g - st - \lim x_k = 0$. For sufficiently large $n$, we have

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{g(n)} = \frac{|\{k \leq n : x_k \in A\}|}{g(n)} = \frac{|\{k \leq m_k : x_k \in A\}|}{g(m_k)} \leq \frac{|A_k|}{a_{2k}} \leq \frac{a_k}{a_{2k}} \rightarrow 0.$$

**Definition 1.3.** Let $x = (x_k)$ be a real valued sequence. $x$ is weight $g$-statistical Cauchy sequence if for each $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that

$$\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - x_N| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - x_N| < \varepsilon$ (g.a.a.k). In this case we write $x$ is weight $g$-Cauchy sequence.

**Lemma 1.1.** The following statements are equivalent:

(i) $x$ is a weight $g$-statistically convergent sequence,

(ii) $x$ is a weight $g$-statistically Cauchy sequence,

(iii) $x$ is a sequence for which there is a convergent sequence $y$ such that $x_k = y_k$ (g.a.a.k).

**Proof.** (i) $\Rightarrow$ (ii) Let us assume that $x$ is a weight $g$-statistical convergent sequence. Suppose $\varepsilon > 0$ and $g - st - \lim x = L$. Then $|x_k - L| < \frac{\varepsilon}{2}$ (g.a.a.k) holds.

If we choose a natural number $N$ such that $|x_N - L| < \frac{\varepsilon}{2}$, then we have

$$|x_k - x_N| < |x_k - L| + |x_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon (g.a.a.k).$$

Hence, $x$ is a weight $g$-statistical Cauchy sequence.

(ii) $\Rightarrow$ (iii) Let us assume that $x$ is a weight $g$-statistical Cauchy sequence. Choose $N(1)$ such that the interval $I = [x_N(1) - 1, x_N(1) + 1]$ contains $x_k$ (g.a.a.k). Choose $M$ such that $I' = [x_M - \frac{1}{2}, x_M + \frac{1}{2}]$ contains $x_k$ (g.a.a.k). We claim that

$$I_1 = I \cap I'$$

contains $x_k$ (g.a.a.k),
for
\[ \{ k \leq n : x_k \notin I \cap I' \} = \{ k \leq n : x_k \notin I \} \cup \{ k \leq n : x_k \notin I' \}. \]

Thus,
\[
\lim_{n \to \infty} \frac{1}{g(n)} | \{ k \leq n : x_k \notin I \cap I' \} | \leq \lim_{n \to \infty} \frac{1}{g(n)} | \{ k \leq n : x_k \notin I \} | + \lim_{n \to \infty} \frac{1}{g(n)} | \{ k \leq n : x_k \notin I' \} | = 0.
\]

So, \( I_1 \) is closed interval of length less than or equal to 1 and contains \( x_k \) \((g-a.a.k)\).

Now we continue by choosing \( N(2) \) such that \( I'' = [x_{N(2)} - \frac{1}{4}, x_{N(2)} + \frac{1}{4}] \) contains \( x_k \) \((g-a.a.k)\), by the previously argument \( I_2 = I_1 \cap I'' \) contains \( x_k \) \((g-a.a.k)\), and \( I_2 \) has length less than or equal to \( \frac{1}{2} \). Proceeding inductively we construct a sequence \( \{I_m\}_{m=1}^{\infty} \) of closed intervals such that for each \( m, I_{m+1} \subseteq I_m \), and the length of \( I_m \) is not greater than \( 2^{1-m} \), and \( x_k \in I_m \) \((g-a.a.k)\). From the Nested Interval Theorem there is a number \( \alpha \) such that \( \alpha = \bigcap_{m=1}^{\infty} I_m \). If we use \( x_k \in I_m \) \((g-a.a.k)\), we can choose an increasing positive sequence \( \{T_m\}_{m=1}^{\infty} \) such that
\[
\frac{1}{g(m)} | \{ k \leq n : x_k \notin I_m \} | < \frac{1}{g(m)} \text{ if } n > T_m.
\]

Next define a subsequence \( z \) of \( x \) consisting of all terms \( x_k \) such that \( k > T_1 \) and if \( T_m < k \leq T_{m+1} \) then \( x_k \notin I_m \).

Now define the sequence \( y \) by
\[
y_k = \begin{cases} 
\alpha, & \text{if } x_k \text{ is a term of } z, \\
x_k, & \text{otherwise.}
\end{cases}
\]

Then \( \lim y_k = \alpha \); for \( \epsilon > \frac{1}{g(m)} > 0 \) and \( k > T_m \) then either \( x_k \) is a term of \( z \), which means \( y_k = \alpha \) or \( y_k = x_k \in I_m \) and \( |y_k - \alpha| \leq \text{length of } I_m < 2^{1-m} \). We also assert that \( x_k = y_k \) \((g-a.a.k)\). To confirm this we observe that if \( T_m < n < T_{m+1} \) then
\[
\{ k \leq n : y_k \neq x_k \} \subseteq \{ k \leq n : x_k \notin I_m \}
\]
so from (1.3)
\[
\frac{1}{g(m)} | \{ k \leq n : y_k \neq x_k \} | \leq \frac{1}{g(n)} | \{ k \leq n : x_k \notin I_m \} | < \frac{1}{g(m)}
\]

is obtained. Thus, the limit as \( n \to \infty \) is 0 and \( x_k = y_k \) \((g-a.a.k)\).

\( (iii) \Rightarrow (i) \) Let us assume that \( x_k = y_k \) \((g-a.a.k)\) and \( \lim y_k = L \). Suppose \( \epsilon > 0 \). Then for each \( n \),
\[
\{ k \leq n : |x_k - L| > \epsilon \} \subseteq \{ k \leq n : x_k \neq y_k \} \cup \{ k \leq n : |y_k - L| > \epsilon \}
\]
from the assumption $\lim y_k = L$, the second set contains a fixed number of integers, say $l = l(\varepsilon)$. So,

$$\lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : |x_k - L| > \varepsilon\}| \leq \lim_{n \to \infty} \frac{1}{g(n)} |\{k \leq n : x_k \neq y_k\}| + \lim_{n \to \infty} \frac{l}{g(n)} = 0$$

because $x_k = y_k$ $(g - a.a.k)$. Hence, $|x_k - L| \leq \varepsilon$ $(g - a.a.k)$. So, the proof is complete. \(\square\)

**Corollary 1.1.** Let $x$ be a real valued sequence. If $g - st - \lim x_k = L$, then $x$ has a subsequence $y$ such that $\lim y_k = L$.

## 2. Inclusion Between Two $g - st$–Convergence

Let $G$ denotes the set of all functions $g : \mathbb{N} \to [0, \infty)$ satisfying the condition $g(n) \to \infty$ and $\frac{n}{g(n)} \not\to 0$. In this section, we will introduce some inclusions between various $g \in G$.

**Lemma 2.1.** Let $g_1, g_2 \in G$ such that there exist $M, m > 0$ and $k_0 \in \mathbb{N}$ such that $m \leq g_1(n) g_2(n) \leq M$ for all $n \geq k_0$. Then $C_{st}^{g_1}(x) = C_{st}^{g_2}(x)$.

**Proof.** Suppose the sequence $x$ is weight $g_1$-statistical convergence to $L$. This implies that for each $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = 0.$$ 

Together with the fact that $\frac{g_1(n)}{g_2(n)} \leq M$, this implies that

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{M g_2(n)} \leq \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)}.$$

for all $n \geq k_0$. This implies

$$\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{M g_2(n)} \leq \lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = 0.$$ 

From the hypothesis we obtain

$$\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_2(n)} = 0.$$ 

Thus, the sequence $x$ is weight $g_2$-statistical convergent to $L$. So, $C_{st}^{g_1}(x) \subset C_{st}^{g_2}(x)$. We can prove the inclusion $C_{st}^{g_2}(x) \subset C_{st}^{g_1}(x)$ by similar way. \(\square\)
Lemma 2.2. For each function \( f \in G \) there exists a nondecreasing function \( g \in G \) such that \( C_{g}^{st}(x) = C_{f}^{st}(x) \). Moreover,

\[
(2.1) \quad g(n) \leq f(n)
\]

for all \( n \in \mathbb{N} \).

Proof. If \( f \) is nondecreasing, it is clear. Otherwise, define the related function \( g : \mathbb{N} \to [0, \infty) \) as follows. Let \( a_{1} = \min\{f(n) : n \in \mathbb{N}\}, i_{1} = \max\{i \in \mathbb{N} : f(i) = a_{1}\} \) and \( g(i) = a_{1} \) for \( 0 \leq i \leq i_{1} \). Next, let \( a_{2} = \min\{f(n) : n > i_{1}\}, i_{2} = \max\{i \in \mathbb{N} : f(i) = a_{2}\} \) and \( g(i) = a_{2} \) for \( i_{1} < i \leq i_{2} \). Rest of the function \( g \) is established by induction.

Obviously, the function \( g \) is nondecreasing and \( g(n) \to \infty \). By the construction, \( g(n) \leq f(n) \), for all \( n \in \mathbb{N} \). Hence \( \frac{n}{g(n)} \leq \frac{n}{f(n)} \) for all \( n \) which implies that \( \frac{n}{g(n)} \to 0 \). Thus \( g \in G \).

Let \( (x_{n}) \) be a weight \( g \)-statistical convergent sequence to \( L \). So, for each \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{|\{k \leq n : |x_{k} - L| \geq \varepsilon\}|}{g(n)} = 0
\]

holds. From (2.1) we have following inequality

\[
\left| \frac{\{k \leq n : |x_{k} - L| \geq \varepsilon\}}{f(n)} \right| \leq \frac{|\{k \leq n : |x_{k} - L| \geq \varepsilon\}|}{g(n)}.
\]

If we take limit when \( n \to \infty \) we obtain \( f - st - \lim x_{k} = L \). Thus, the inclusion \( C_{g}^{st} \subseteq C_{f}^{st} \).

By construction, for each \( n \in \mathbb{N} \) there exist \( m \geq n \) such that \( g(n) = g(m) = f(m) \). Suppose that \( x_{n} \to L \) (\( g - st \)). Then there exists \( a \), where \( a \in \mathbb{R}^{+} \cup \{+\infty\} \)

and an increasing sequence \( (n_{i}) \) of indices such that

\[
\lim_{i \to \infty} \frac{|\{k \leq n_{i} : |x_{k} - L| \geq \varepsilon\}|}{g(n_{i})} = a > 0.
\]

For each \( i \in \mathbb{N} \) we can find \( m_{i} \geq n_{i} \) such that \( g(n_{i}) = g(m_{i}) = f(m_{i}) \). Hence

\[
\left| \frac{\{k \leq n_{i} : |x_{k} - L| \geq \varepsilon\}}{g(n_{i})} \right| \leq \frac{|\{k \leq m_{i} : |x_{k} - L| \geq \varepsilon\}|}{f(m_{i})}
\]

holds. So, \( x_{n} \to L \) (\( f - st \)). \( \Box \)

Lemma 2.3. Let \( f \in G \) be such that \( \frac{n}{f(n)} \to \infty \), \( L, \varepsilon \) real numbers with \( \varepsilon > 0 \). Then there exists a sequence \( (x_{n}) \) such that \( \left( \frac{|\{k \leq n : |x_{k} - L| \geq \varepsilon\}|}{f(n)} \right) \) is bounded but not convergent to zero.
Proof. Firstly, let us assume that $f$ is nondecreasing. Take to the smallest nonnegative integer, $k_0$, such that for $n \geq k_0$, $f(n) > 2$. Let us define a set $A \subset \mathbb{N} \setminus \{0, 1, 2, \ldots, k_0 - 1\}$ inductively, deciding whether $n \geq k_0$ should belong to $A$ or not.

Let $n \notin A$ for all $n < k_0$. Suppose that $n \geq k_0$ and then we have defined $A(n)$. If \( \frac{|A(n)|}{f(n+1)} < 1 \) then let $n \in A$. Otherwise, let $n \notin A$. So, we construct the set $A$. From this construction and the condition $f(n) \to \infty$, $A$ is infinite.

We assert that $\mathbb{N} \setminus A$ is also infinite. Let us assume that it is finite and choose $n_0 \in \mathbb{N}$ such that $n \in A$ for all $n \geq n_0$. Then, we have

\[
\frac{n - n_0}{f(n+1)} \leq \frac{|A(n)|}{f(n+1)} < 1
\]

for all $n \geq n_0$. But this is impossible because of the assumption, \( \frac{n - n_0}{f(n+1)} \to \infty \).

Now, we will show that \( \frac{|A(n)|}{f(n+1)} < 2 \) for each $n \geq k_0$. It is clear that if $n = k_0$ it is true. Suppose that \( \frac{|A(n)|}{f(n+1)} < 2 \) for a fixed $n \geq k_0$.

If \( \frac{|A(n)|}{f(n+1)} < 1 \), we have

\[
\frac{|A(n+1)|}{f(n+1)} = \frac{|A(n)|}{f(n+1)} + 1 \leq \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n+1)}
\]

\[
\leq 1 + \frac{1}{2} < 2.
\]

If \( \frac{|A(n)|}{f(n+1)} > 1 \), then $n \notin A$ and so,

\[
\frac{|A(n+1)|}{f(n+1)} = \frac{|A(n)|}{f(n+1)} \leq \frac{|A(n)|}{f(n)} < 2.
\]

Now, let us define a sequence \((x_n)\) as follows:

\[
x_n := \begin{cases} n & n \in A \\ L & n \notin A \end{cases}
\]

where $L \in \mathbb{R}$ is a fixed number. It is clear that the sequence \( \left( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \right) \) is bounded from the first part of this proof.

Now, we will show that the sequence \( \left( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \right) \) is not convergent to $0$. For this aim consider any $n \geq k_0$. We will find $m \geq n$ such that \( \frac{|A(m)|}{f(m)} \geq 1 \). If \( \frac{|A(n)|}{f(n)} \geq 1 \), put $m := n$. Otherwise, choose the smallest $m \geq n$ such that $m \in \mathbb{N} \setminus A$.

Then \( \frac{|A(m)|}{f(m+1)} \geq 1 \) and so, \( \frac{|A(m)|}{f(m)} \geq 1 \). Thus, the sequence \( \left( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \right) \) is not convergent to $0$. 
Now, let us back to the general case where \( f \in G \) need not be nondecreasing. Then we assume the associated function \( g \in G \) from Lemma 2.2. Note that \( \frac{n}{g(n)} \to \infty \) since \( \frac{n}{f(n)} \geq \frac{n}{f(n)} \) for all \( n \) and \( \frac{n}{f(n)} \to \infty \). By the above reasons we obtain the respective set \( A \) for \( g \). Thus, \( \frac{|A(n)|}{g(n)} \to 0 \) and the sequence \( \frac{|A(n)|}{g(n)} \) is bounded. Then \( \frac{|A(n)|}{f(n)} \to 0 \), and the sequence \( \left( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \right) \) is bounded since \( g(n) \leq f(n) \) for all \( n \in \mathbb{N} \).

**Theorem 2.1.** If \( g_1, g_2 \) belong to \( G \) such that \( \frac{g_2(n)}{g_1(n)} \to \infty \) then \( C_{g_1}^{st}(x) \subset C_{g_2}^{st}(x) \). If \( g \in G \) and \( \frac{g}{g(n)} \to \infty \) then \( C_{g}^{st}(x) \subset C^{st}(x) \).

**Proof.** To prove the first claim note that the inclusion \( C_{g_1}^{st}(x) \subset C_{g_2}^{st}(x) \) follows from Lemma 2.1. Set \( f := \sqrt{g_1 \cdot g_2} \). Then

\[
\lim_{n \to \infty} \frac{f(n)}{g_1(n)} = \lim_{n \to \infty} \frac{g_2(n)}{f(n)} = \infty.
\]

Also we have

\[
\frac{n}{g_1(n)} = \frac{n}{g_2(n)} \rightarrow \infty.
\]

So \( \frac{n}{f(n)} = \sqrt{\frac{n^2}{g_1(n) \cdot g_2(n)}} \to \infty \). Hence \( f \) have the assumption of Lemma 2.3. Take the sequence \( (x_n) \) obtained in this lemma. Then \( x_n \in C_{g_1}^{st}(x) \) but \( x_n \notin C_{g_2}^{st}(x) \). Indeed, using (2.2) we have

\[
\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_2(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \cdot \frac{f(n)}{g_2(n)} \to 0
\]

because \( \left( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \right) \) is bounded from Lemma 2.3. Thus, \( x_n \in C_{g_2}^{st}(x) \).

To prove that \( x_n \notin C_{g_1}^{st}(x) \) observe that

\[
\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \cdot \frac{f(n)}{g_1(n)}
\]

So, \( x_n \notin C_{g_1}^{st}(x) \) because \( \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \to 0 \), and \( \frac{f(n)}{g_1(n)} \to \infty \) from (2.2).

If we take \( g_2(n) = n \), for all \( n \in \mathbb{N} \), second assertion proved easily from the same way.

**Corollary 2.1.** Let \( 0 < \alpha < \beta \leq 1 \) and \( g_1(n) = n^\alpha \), \( g_2 = n^\beta \) for \( n \in \mathbb{N} \). Then \( C_{g_1}^{st}(x) \subset C_{g_2}^{st}(x) \).

**Example 2.1.** Let \( g_1(n) = \left\{ \begin{array}{ll} n, & \text{for even } n \in \mathbb{N} \\ \sqrt{n}, & \text{for odd } n \in \mathbb{N} \end{array} \right. \)
For $k > 0$ we have
\[
\limsup_{n \to \infty} \frac{g(n)}{a(n)^{1/3}} = \infty.
\] However, $C_{g^*}^{st}(x) = C_{g^1}^{st}(x)$. Indeed, construct a nondecreasing function $g \in G$ such that $C_{g^*}^{st}(x) = C_{g^1}^{st}(x)$, according to the method used in the proof of Lemma 2.1. Then it follows from simple calculations that $g$ is given by
\[
g(n) = \begin{cases} \sqrt{n} + 1 & \text{for even } n \in \mathbb{N} \\ \sqrt{n} & \text{for odd } n \in \mathbb{N}. \end{cases}
\]

Obviously, $\frac{1}{2} \leq \frac{g(n)}{a(n)^{1/3}} \leq 2$ for all $n \geq 1$. Therefore, by Lemma 2.1 we have $C_{g^*}^{st}(x) = C_{g^1}^{st}(x)$.

**Theorem 2.2.** There exists a function $g \in G$ such that $C_{g^*}^{st}$ is different from $C_{g^1}^{st}$ with $0 < \alpha < 1$.

**Proof.** Let $a_k$ and $g(n)$ defined as in Example 1.3. Let $A_k := \{ n \in \mathbb{N} : a_k+1 - (a_{k+1})^{1/4} \leq n < a_{k+1} \}$ and $A = \bigcup_{k \geq 2} A_k$. Let us take account the sequence
\[
x_n = \begin{cases} n, & n \in A \\ 0, & n \notin A. \end{cases}
\]

It is clear that $\frac{1}{2}(a_{k+1})^{1/4} \leq |B_k| \leq (a_{k+1})^{1/4}$. Let us check that $g - st - \lim x_k \neq 0$. For $k > 0$ we have
\[
\frac{|\{ k \leq a_{k+1} - 1 : |x_k - 0| \leq \varepsilon \}|}{g(a_{k+1} - 1)} \geq \frac{1}{2} \frac{|\mathbb{N}|}{g(a_k)} \geq \frac{1}{2} \frac{(a_{k+1})^{1/4}}{g(a_k)} \geq 1,
\]
so, $g - st - \lim x_k \neq 0$. Furthermore,
\[
|\{ k \leq a_{k+1} : |x_k - 0| \geq \varepsilon \}| \leq (a_k)^{1/4} + (a_{k+1})^{1/4} \leq 2(a_{k+1})^{1/4}
\]
and so,
\[
\frac{|\{ k \leq a_{k+1} : |x_k - 0| \geq \varepsilon \}|}{(a_{k+1})^{1/4}} \leq 2(a_{k+1})^{1/4} \leq 2(a_{k+1})^{-1/12} \to 0, \quad (k \to \infty)
\]
holds.

Now, fix any $n \geq 4$ and choose a unique $k \in \mathbb{N}$ such that $n \in [a_k, a_{k+1})$. If $n < a_{k+1} - (a_{k+1})^{1/4}$ then
\[
|\{ k \leq n : |x_k - 0| \geq \varepsilon \}| \leq \frac{|\{ k \leq a_k : |x_k - 0| \geq \varepsilon \}|}{n^{1/3}} \leq 2(a_k)^{-1/12}.
\]
If $a_{k+1} - (a_{k+1})^{1/4} \leq n < a_{k+1}$ then for $b > a > 0$, the function
\[
f(x) := \frac{a + x}{(b + x)^{1/3}}, \quad x \geq 0
\]
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is increasing, thus

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon \}|}{n^{1/3}} \leq \frac{|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon \}|}{(a_{k+1})^{1/3}}.$$ 

So, $x_n \in C_{n^{1/3}}^{x^*}(x)$.

Now, let $0 < \alpha < 1$, $\alpha \neq \frac{1}{3}$. If $\alpha < \frac{1}{3}$ then from Corollary 2.1 $C_{n^{\alpha}}^{x^*} \subset C_{n^{1/3}}^{x^*}$ and $C_{\frac{1}{3}}^{x^*} \setminus C_{n^{1/3}}^{x^*} \neq \emptyset$ because $C_{\frac{1}{3}}^{x^*} \setminus C_{n^{1/3}}^{x^*} \neq \emptyset$. If $\alpha > \frac{1}{3}$ then $C_{n^{1/3}}^{x^*} \setminus C_{\frac{1}{3}}^{x^*} \neq \emptyset$. By the same way we can show that $x_n \in C_{\frac{1}{3}}^{x^*} \setminus C_{\frac{1}{3}}^{x^*}$. So $C_{\frac{1}{3}}^{x^*} \subset C^{x^*}$. □

Acknowledgement

The authors would like to thank Professor Mehmet Küçükaslan for his discussions some steps during the preparation of this paper.

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