ON BINOMIAL SUMS WITH THE TERMS OF SEQUENCES \( \{g_{kn}\} \) AND \( \{h_{kn}\} \)

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Abstract. In this paper, we derive sums and alternating sums of products of terms of the sequences \( \{g_{kn}\} \) and \( \{h_{kn}\} \) with binomial coefficients. For example,

\[
\sum_{i=0}^{n} \binom{n}{i} i^m g_{(n-i)t} h_{kt} = 2^n n^m g_{kn} - n^m \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{n(1-t)} h_{kt}^{n-m} g_{kn+m+n-n},
\]

where \(c\) is a nonzero real number, \(t\) is any integer and \(m\) is a nonnegative integer.

Keywords: Binomial sums, alternating sums, generalized Fibonacci numbers, recurrence relation.

1. Introduction

Define the second order linear recursive sequences \( \{u_n\} \) and \( \{v_n\} \) for \( n \geq 1 \) and nonzero integers \( p, q \), by

\[
\begin{align*}
\quad u_{n+1} &= pu_n + qu_{n-1} \quad \text{and} \quad v_{n+1} = pv_n + qv_{n-1},
\end{align*}
\]

with initials \( u_0 = 0, u_1 = 1 \) and \( v_0 = 2, v_1 = p \), respectively.

When \( q = 1 \), \( u_n = U_n \) (the \( n \)th generalized Fibonacci number) and \( v_n = V_n \) (the \( n \)th generalized Lucas number). Also, when \( p = q = 1 \), \( u_n = F_n \) (the \( n \)th Fibonacci number) and \( v_n = L_n \) (the \( n \)th Lucas number). If \( \alpha \) and \( \beta \) are the roots of the
equation $x^2 - px - q = 0$, the Binet formulae of the sequences $\{u_n\}$ and $\{v_n\}$ have the forms

$$ u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n, $$

respectively, where $\alpha, \beta = \left( p \pm \sqrt{\Delta} \right) / 2$ and $\Delta = p^2 + 4q$. From [3], Kılıç and Stanica derived the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0$, $n > 0$. It is clear that

$$ u_{k(n+1)} = u_k u_{kn} + (-1)^{k+1} q^k u_{k(n-1)} \quad \text{and} \quad v_{k(n+1)} = v_k v_{kn} + (-1)^{k+1} q^k v_{k(n-1)}, $$

where the initial conditions are $0$, $u_k$, and $2$, $v_k$, respectively. The Binet formulae of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$ u_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \quad \text{and} \quad v_{kn} = \alpha^{kn} + \beta^{kn}, $$

respectively. It is clearly seen that $u_{-kn} = (-1)^{kn+1} u_{kn}$ and $u_{2kn} = u_{kn} v_{kn}$.

In [9], Komatsu obtained the two binomial sums of the generalized Fibonacci numbers as follows:

$$ \sum_{i=0}^{n} \binom{n}{i}^c u_i = g_n \quad (n \geq 0) $$

which satisfies the recurrence relation

$$ g_{n+1} = (pc + 2) g_n + (qc^2 - pc - 1) g_{n-1} \quad (n \geq 1) $$

with $g_0 = 0$, $g_1 = c$ and

$$ \sum_{i=0}^{n} \binom{n}{i}^c u_i = h_n \quad (n \geq 0) $$

which satisfies the recurrence relation

$$ h_{n+1} = (pd + 2c) h_n + (qd^2 - pcd - c^2) h_{n-1} \quad (n \geq 1) $$

with $h_0 = 0$ and $h_1 = d$, where $c, d$ are nonzero real numbers. Also he gave several Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions.

In [1], Cook et al. obtained some binomial summation identities including the terms of the sequence $\{g_n\}$ in [9]. For example,

$$ \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (qc^2 - pc - 1)^{2n-i} g_{2i+1} = (pc + 2)^n g_{2n+1}, $$
and
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (qc^2 - pc - 1)^{2(2n-i)} g_{4i} = c^{2n} (pc + 2)^{2n} (p^2 + 4q)^n g_{4n}.\]

In [10], Ömür et al. defined the subsequences \(\{g_{kn}\}\) and \(\{h_{kn}\}\) with binomial sums \(g_{kn} = \sum_{i=0}^{n} \binom{n}{i} c^i u_{ki}\) and \(h_{kn} = \sum_{i=0}^{n} \binom{n}{i} c^i v_{ki}\). These subsequences satisfy the following relations
\[g_{k(n+1)} = (c^k v_k + 2) g_{kn} - \left( c^{2k} (-q)^k + c^k v_k + 1 \right) g_{k(n-1)}\]
and
\[h_{k(n+1)} = (c^k v_k + 2) h_{kn} - \left( c^{2k} (-q)^k + c^k v_k + 1 \right) h_{k(n-1)}\]
in which \(g_0 = 0, g_k = c^k u_k\) and \(h_0 = 2, h_k = 2 + c^k v_k\), respectively. The Binet formulae of the sequences \(\{g_{kn}\}\) and \(\{h_{kn}\}\) are
\[g_{kn} = \frac{(c^k \alpha^k + 1)^n - (c^k \beta^k + 1)^n}{\alpha - \beta} \quad \text{and} \quad h_{kn} = (c^k \alpha^k + 1)^n + (c^k \beta^k + 1)^n,
\]
respectively. The authors obtained some binomial summation identities of sequence \(\{g_{kn}\}\). For example, for \(n > 0\),
\[\sum_{i=0}^{2n} \binom{2n}{i} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{2n-i} g_{k(2i+1)} = (c^k v_k + 2)^n g_{k(2n+1)},\]
and
\[\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{2n-i} g_{k(2i+1)} = c^{2kn} \left( v_k^2 + 4q^k (-1)^{k+1} \right)^n g_{k(2n+1)}.\]

In [7], Kılıç et al. introduced sums and alternating sums of products of terms of sequences \(\{U_{kn}\}\) and \(\{V_{kn}\}\) as follows: for odd number \(n\),
\[\sum_{i=0}^{n} \binom{n}{i} U_{k(a+bi)} U_{k(c+fi)} = D^{(n-1)/2} \left( U_{k(b+f)} U_{k(n(b+f))/2+a+e}^{n} + (-1)^{c+(b-f)/2} U_{k(b-f)} U_{k(n(b-f))/2-a-e}^{n} \right),\]
and
\[\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(a+bi)} U_{k(c+fi)} = \frac{1}{D} \left( (-1)^n V_{k(b+f)} U_{k(n(b+f))/2+a+e}^{n} + (-1)^{e+(b-f)/2} V_{k(b-f)} U_{k(n(b-f))/2+a-e}^{n} \right),\]
where \( a, b, c, f \) are any integers, \( b + f \equiv 2 \pmod{4} \) and \( D = p^2 + 4 \).

In [5, 6], Kılıç considered and computed the alternating binomial sums of the forms

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n, i, k, t) \quad \text{and} \quad \sum_{i=0}^{n} \binom{n}{i} g(n, i, k, t),
\]

where \( f(n, i, k, t) = U_{kti}V_k(n - ti) \) and \( g(n, i, k, t) = U_{kti}V_k(n + ti) \) for positive integers \( t \) and \( n \). For example, for odd \( k \),

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_k(n - ti) = U_{kt}^{n+1} \left\{ \begin{array}{ll} (-1)^{n+1}V_{kn(t-1)}D^{(n-1)/2} & \text{if } n \text{ is odd}, \\ U_{kn(t-1)}D^{n/2} & \text{if } n \text{ is even}, \end{array} \right.
\]

where \( D \) is defined as before.

In [4], inspired by the works of [5, 6, 8], Kılıç et al. gave rising factorial of the summation index instead of its powers. Clearly, they considered and computed the generalized alternating weighted binomial sums:

\[
\sum_{i=0}^{n} \binom{n}{i} x^m (-1)^i f(n, i, k, t),
\]

where \( f(n, i, k, t) \) as before and \( m \) is a nonnegative integer and \( x^m \) stands for the falling factorial defined by \( x^m = x(x - 1)(x - 2) \ldots (x - m + 1) \). These kinds of binomial sums (except some special cases of \( k \) and \( t \)) have not been considered according to our best literature acknowledgement. For example, for any integers \( k \) and \( t \),

\[
\sum_{i=0}^{n} \binom{n}{i} x^m (-1)^i U_{kti}V_k(n - ti) = (-1)^{kn(t+1)+m} U_{kn(t-1)}^{n+1} \left\{ \begin{array}{ll} D^{(n-1)/2} & \text{if } n \equiv 0 \pmod{2}, \\ -V_k^{n+1}D^{(n-1)/2} & \text{if } n \equiv 1 \pmod{2}, \end{array} \right.
\]

where \( m \) is a nonnegative integer.

2. Sums of certain products with the terms of \( \{g_{kn}\} \) and \( \{h_{kn}\} \)

In this section, firstly, we will start with some lemmas for further use.

**Lemma 2.1.** For any integers \( m \) and \( n \), we have

\[
g_{k(m+n)} + g_{k(m-n)} \left( q^k (-1)^{k+1} e^{2k} - e^{k}v_k - 1 \right)^n = \left\{ \begin{array}{ll} h_{km}g_{kn} & \text{if } n \text{ is odd}, \\ g_{km}h_{kn} & \text{if } n \text{ is even}, \end{array} \right.
\]
where $c$ is a nonzero real number.

Proof. By the Binet formulae of \{g_{kn}\} and \{h_{kn}\}, the claimed equalities are obtained.

We recall some facts for the readers convenience: For any real numbers $m$ and $n$,

$$\begin{align*}
(m + n)^t &= \begin{cases}
\sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i \left( m^{t-2i} + n^{t-2i} \right) & \text{if } t \text{ is odd}, \\
\sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i \left( m^{t-2i} + n^{t-2i} \right) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even},
\end{cases}
\end{align*}$$

(2.1)

and

$$\begin{align*}
(m - n)^t &= \begin{cases}
\sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i \left( m^{t-2i} - n^{t-2i} \right) & \text{if } t \text{ is odd}, \\
\sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i \left( m^{t-2i} + n^{t-2i} \right) + \binom{t}{t/2} (-1)^{t/2} (mn)^{t/2} & \text{if } t \text{ is even},
\end{cases}
\end{align*}$$

(2.2)

where $t$ is a positive integer.

Lemma 2.2. For any integers $r$ and $s$, we have

$$\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} h_{k(2r_i+s)} &= h_{k(2n+s)} h_{kr}^n, \\
\sum_{i=0}^{n} \binom{n}{i} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} g_{k(2r_i+s)} &= g_{k(2n+s)} h_{kr}^n, \\
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} g_{k(2r_i+s)} &= \begin{cases}
-\Delta^{(n-1)/2} g_{kr} h_{k(2n+s)} & \text{if } n \text{ is odd}, \\
\Delta^{n/2} g_{kr} g_{k(2n+s)} & \text{if } n \text{ is even},
\end{cases}
\end{align*}$$
and

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{r(n-i)} h_{k(2r+i)} = \begin{cases} 
-\Delta^{(n+1)/2} g_k n g_k (r n + s) & \text{if } n \text{ is odd}, \\
\Delta^{n/2} g_k n h_k (r n + s) & \text{if } n \text{ is even},
\end{cases}
\]

where \( c \) is a nonzero real number.

**Proof.** From (2.1), (2.2) and Lemma 2.1, the proof is obtained. \( \square \)

**Lemma 2.3.** \[2\]Let \( n \) and \( m \) be integers such that \( 0 \leq m < n \). For \( z \neq -1 \),

\[
\sum_{k=0}^{n} \binom{n}{k} n^m z^k = z^m n^m (1+z)^{n-m}.
\]

**Theorem 2.1.** Let \( a, b \) and \( e \) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(a+i+b)} g_{k(a+i+e)}
\]

\[
= \frac{1}{\Delta} \left( h_{k(a+b+c)} h_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an}
\right.
\]

\[
- 2^n h_{k(b-c)} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^c
\]

\[
= h_{k(a+b+c)} h_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an}
\]

\[
+ 2^n h_{k(b-c)} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^c,
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-ai} g_{k(a+i+b)} h_{k(a+i+c)}
\]

\[
= g_{k(a+b+c)} h_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an}
\]

\[
+ 2^n g_{k(b-c)} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^c,
\]

where \( c \) is a nonzero real number.
Proof. Consider that

\[
\sum_{i=0}^{n} \binom{n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{-ai} g_{k(a+i)} g_{k(a+i+c)}
= \sum_{i=0}^{n} \binom{n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{-ai}
\times \left[ (c^k \alpha^k + 1)^{ai+b} - (c^k \beta^k + 1)^{ai+b} \right]
\times \left[ (c^k \alpha^k + 1)^{ai+c} - (c^k \beta^k + 1)^{ai+c} \right]
\times \frac{1}{(\alpha - \beta)^2 \sum_{i=0}^{n} \binom{n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{-ai}}
\times \left( (c^k \alpha^k + 1)^{2ai+b+c} - (c^k \alpha^k + 1)^{ai+b} (c^k \beta^k + 1)^{ai+c} + (c^k \beta^k + 1)^{2ai+b+c} - (c^k \alpha^k + 1)^{ai+c} (c^k \beta^k + 1)^{ai+b} \right)
= \frac{1}{\Delta} \sum_{i=0}^{n} \binom{n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{-ai} h_{k(2ai+b+c)}
- \frac{1}{\Delta} \sum_{i=0}^{n} \binom{n}{i} (c^{2k} (-q)^k + c^k v_k + 1)^{c} h_{k(b-c)}.
\]

From Lemma 2.2, the desired result is obtained. Similarly, the other cases are given. \(\square\)

Theorem 2.2. Let \(a\), \(b\) and \(c\) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i (c^{2k} (-q)^k + c^k v_k + 1)^{-ai} g_{k(a+i)} g_{k(a+i+c)}
= \begin{cases} 
-\Delta^{(n-1)/2} g_{k(an+b+c)} g_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is odd}, \\
\Delta^{(n-2)/2} h_{k(an+b+c)} g_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is even}, 
\end{cases}
\]

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i (c^{2k} (-q)^k + c^k v_k + 1)^{-ai} h_{k(a+i)} h_{k(a+i+c)}
= \begin{cases} 
-\Delta^{(n+1)/2} g_{k(an+b+c)} g_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is odd}, \\
\Delta^{n/2} h_{k(an+b+c)} g_{ka} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-an} & \text{if } n \text{ is even}, 
\end{cases}
\]
and

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-a_i} g_{k(a+i+b)} h_{k(a+i+d)}
\]

where \( c \) is a nonzero real number.

**Proof.** Consider that

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-a_i} g_{k(a+i+b)} g_{k(a+i+c)}
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-a_i}
\]

\[
\times \left[ (c^k \alpha^k + 1)^{a_i+b} - (c^k \beta^k + 1)^{a_i+b} \right] \left[ (c^k \alpha^k + 1)^{a_i+c} - (c^k \beta^k + 1)^{a_i+c} \right]
\]

\[
= \frac{1}{(\alpha - \beta)^2} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-a_i}
\]

\[
\times \left( (c^k \alpha^k + 1)^{2a_i+b+c} - (c^k \alpha^k + 1)^{a_i+b} (c^k \beta^k + 1)^{a_i+c} \right.
\]

\[
+ (c^k \beta^k + 1)^{2a_i+b+c} - (c^k \alpha^k + 1)^{a_i+c} (c^k \beta^k + 1)^{a_i+b} \bigg)
\]

\[
= \frac{1}{\Delta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-a_i} h_{k(2a_i+b+c)}
\]

\[- \frac{1}{\Delta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{c} h_{k(b-c)}.
\]

From (2.3), the desired result is obtained. Similarly, using Lemma 2.2, the other cases can be obtained. \( \square \)

**Theorem 2.3.** Let \( k \) and \( t \) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i g_{k(n-ti)} h_{kti} = (-1)^m n^m (c^{2k} (-q)^k + c^k v_k + 1)^{n(1-t)} g_{k^m}
\]

\[
\times \left\{ \begin{array}{ll}
- \Delta^{(n-m)/2} g_{k(n+t+m-n)} & \text{if } n \equiv m \text{ (mod 2)}, \\
\Delta^{(n-m-1)/2} h_{k(n+t+m-n)} & \text{if } n \equiv m + 1 \text{ (mod 2)},
\end{array} \right.
\]
and
\[
\sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i g_{kl} h_{k(n-ti)} = (-1)^m n^{m} \left( e^{2k} (-q)^k + e^k v_k + 1 \right) \left(1 - \left( e^k \alpha^k + 1 \right)\right)^{n-1} \Delta^{(n-m)/2} g_{kl}^{m} h_{k(n-tm-n)}
\]

\[
\times \begin{cases} 
\Delta^{(n-m)/2} g_{kl}^{m} h_{k(n-tm-n)} & \text{if } n \equiv m \pmod{2}, \\
-\Delta^{(n-m-1)/2} g_{kl}^{m} h_{k(n-tm-n)} & \text{if } n \equiv m + 1 \pmod{2},
\end{cases}
\]

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

\[\text{Proof.}\] Observe that
\[
\sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i g_{kl} h_{k(n-ti)}
\]

\[
= \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \left( (e^k \alpha^k + 1)^{n-ti} - (e^k \beta^k + 1)^{n-ti} \right)\times \left( (e^k \alpha^k + 1)^{ti} + (e^k \beta^k + 1)^{ti} \right)
\]

\[
= \frac{(e^k \alpha^k + 1)^n}{\alpha - \beta} \left( \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \right) \left( \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \right)
\]

\[
= \frac{(e^k \alpha^k + 1)^n}{\alpha - \beta} \left( \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \right) \left( \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \right)
\]

which by Lemma 2.3, equals
\[
\frac{(e^k \alpha^k + 1)^n}{\alpha - \beta} \left( \sum_{i=0}^{n} \binom{n}{i} i^{m} (-1)^i \right) \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{n-1}
\]

\[
= (-1)^m n^{m} \frac{1}{\alpha - \beta} \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{n-1} \left( 1 - \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{t} \right)^{n-m}
\]

\[
\times \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{tm} \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{tm} \left( 1 - \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{t} \right)^{n-m}
\]

\[
= (-1)^m n^{m} \frac{1}{\alpha - \beta} \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{n-1} \left( 1 - \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{t} \right)^{n-m}
\]

\[
\times \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{tm} \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{tm} \left( 1 - \left( \frac{(e^k \alpha^k + 1)}{(e^k \beta^k + 1)} \right)^{t} \right)^{n-m}
\]

which by the Binet formulae, gives us the claimed result. \(\square\)
Similar to the proof method of Theorem just above, we have the following results without proof.

**Theorem 2.4.** Let \( k \) and \( t \) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} i^m g_k (n-t) h_{kti} = 2^{n-m} n^m g_k n - n^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} h^{n-m}_{kt} g_k (tm+tn-n),
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} i^m g_{kti} h_k (n-t) = 2^{n-m} n^m g_k n + n^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{n(1-t)} h^{n-m}_{kt} g_k (tm+tn-n),
\]

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

**Theorem 2.5.** Let \( k \) and \( t \) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} i^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_k (n(t+1) - k(t+2)) = n^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m h^{n-m}_{k} g_k (nt-m) + n^m h^{n-m}_{k(t+1)} g_k (tm+1),
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} i^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{k(n(t+1) - k(t+2)) h_{kti}} = n^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m h^{n-m}_{k} g_k (nt-m) - n^m h^{n-m}_{k(t+1)} g_k (tm+1),
\]

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

**Theorem 2.6.** Let \( k \) and \( t \) be any integers. Then

\[
\sum_{i=0}^{n} \binom{n}{i} i^m (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_k (n(t+1) - k(t+2)) = (-1)^m n^m \Delta^{(n-m)/2}
\]

\[
\times \left\{ \begin{array}{ll}
\left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m \pmod{2}, \\
\times g_k^{n-m} g_k^{(nt-m)} + g_k^{n-m} g_k^{(nt+1)} & \\
\Delta^{-1/2} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m + 1 \pmod{2}, \\
\times g_k^{n-m} h_k^{(nt-m)} - g_k^{n-m} h_k^{(nt+1)} & \end{array} \right.
\]

and

\[ \sum_{i=0}^{n} \binom{n}{i} i^m (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn(t+1)} - g_{kn} h_{kti} \]

\[ = (-1)^m n^m \Delta^{(n-m)/2} \]

\[ \times \left\{ \begin{array}{ll}
\Delta^{-1/2} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m \pmod{2}, \\
\times g_{k(t+1)}^{n-m} g_{k(t+1)} g_{kn} + g_{k(t+1)}^{n-m} h_{kn(t+1)} & \text{if } n \equiv m + 1 \pmod{2},
\end{array} \right. \]

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

**Theorem 2.7.** Let \( k \) and \( t \) be any integers. Then

\[ \sum_{i=0}^{n} \binom{n}{i} i^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn}^{-nt} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^n g_{k(t+1)} - g_{kn} h_{kti} \]

and

\[ \sum_{i=0}^{n} \binom{n}{i} i^m \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn} h_{kn(t+2)}^{-nt} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^n g_{k(t+1)} - g_{kn} h_{kti} \]

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

**Theorem 2.8.** Let \( k \) and \( t \) be any integers. Then

\[ \sum_{i=0}^{n} \binom{n}{i} i^m (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kti} h_{kn}^{-nt} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^n g_{k(t+1)} - g_{kn} h_{kti} \]

\[ = (-1)^m n^m \Delta^{(n-m)/2} \]

\[ \times \left\{ \begin{array}{ll}
\Delta^{-1/2} \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^m & \text{if } n \equiv m \pmod{2}, \\
\times g_{k(t+1)}^{n-m} g_{k(t+1)} g_{kn} - g_{k(t+1)}^{n-m} h_{km} & \text{if } n \equiv m + 1 \pmod{2},
\end{array} \right. \]
and

\[
\sum_{i=0}^{n} \binom{n}{i} i^m (-1)^i \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^i g_{kn-ki(t+2)h_{kti}} \\
= (-1)^m n^m \Delta^{(n-m)/2} \times \left\{ \begin{array}{l}
- \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} \\
\times g_{k(t+1)}^{n-m} g_{k(n^2+nt+t)} - g_{k}^{n-m} g_{km} \\
\Delta^{-1/2} \left\{ \left( c^{2k} (-q)^k + c^k v_k + 1 \right)^{-nt} \\
\times g_{k(t+1)}^{n-m} h_{k(n^2+nt+t)} + g_{k}^{n-m} h_{km} \right\}
\end{array} \right\}
\]

if \( n \equiv m \pmod{2} \),

where \( c \) is a nonzero real number and \( m \) is a nonnegative integer.

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