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ARENS REGULARITY OF PROJECTIVE TENSOR PRODUCT

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Abstract. In this paper, we study some of approximate identity properties, and its application in the Arens regularity of tensor products of Banach algebras with some results in group algebras. W e consider under which sufficient and necessary conditions the Banach algebra $A\widehat{\otimes}B$ is Arens regular.
Keywords: Arens regularity; tensor product; Banach algebras; group algebras.

1. Introduction

Suppose that A and B are Banach algebras. Since 1988 the Arens regularity of $A\widehat{\otimes}B$ has received a great deal of attention by many researchers. Among them, Ülger in [19, 21] showed that $A\widehat{\otimes}B$ is not Arens regular, in general, even when A and B are Arens regular. He introduced a new concept of biregular mapping and showed that a bounded bilinear mapping $m : A \times B \to \mathbb{C}$ is biregular if and only if $A\widehat{\otimes}B$ is Arens regular, where $\mathbb{C}$ is the space of complex numbers. Let X, Y and Z be normed spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} that he called m is Arens regular whenever $m^{***} = m^{t***t}$, for more information see [9, 10, 14]. Let A be a Banach algebra, regarding A as a Banach A-bimodule, the operation $\pi : A \times A \longrightarrow A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. The first (left) and second (right) Arens products of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and $a''ob''$, respectively. Let B be a Banach A-bimodule, and let

$$\pi_\ell : \ A \times B \longrightarrow B \quad \text{and} \quad \pi_r : \ B \times A \longrightarrow B,$$

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be the right and left module actions of A on B. By above notation, the transpose of π_r denoted by $\pi_r^t : A \times B \to B$. Then

$$\pi_\ell^* : B^* \times \ A \longrightarrow B^* \quad \text{and} \quad \pi_r^{t*t} : A \times B^* \longrightarrow B^*.$$

Thus B^* is a left Banach A-module and a right Banach A-module with respect to the module actions π_r^{t*t} and π_ℓ^*, respectively. The the second dual B^{**} is a Banach A^{**}-bimodule with the following module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{***} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the first Arens product. Similarly, B^{**} is a Banach A^{**}-bimodule with the module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{t***t} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the second Arens product.

Let B be a left Banach A-module and e be a left unit element of A. Then e is a left unit (resp. weakly left unit) for B, if $\pi_\ell(e, b) = b$ (resp. $\langle b', \pi_\ell(e, b)\rangle = \langle b', b\rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) is similar. A Banach A-bimodule B is called unital, if B has the same left and right unit. In this way, B is called a unitary Banach A-bimodule.

Suppose that A is a Banach algebra and B is a Banach A-bimodule. Since B^{**} is a Banach A^{**}-bimodule, where A^{**} is equipped with the first Arens product, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$\begin{aligned} Z^\ell_{A^{**}}(B^{**}) = Z(\pi_r) \quad &= \quad \{b'' \in B^{**} : \ \text{the map} \ \ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**} \\ &\qquad \text{is} \quad \text{weak}^*\text{-weak}^* \text{ continuous}\}. \end{aligned}$$

In this way, we write $Z^\ell_{B^{**}}(A^{**}) = Z(\pi_\ell)$, $Z^r_{A^{**}}(B^{**}) = Z(\pi_\ell^t)$ and $Z^r_{B^{**}}(A^{**}) = Z(\pi_r^t)$, where $\pi_\ell : \ A \times B \to B \quad \textit{and} \quad \pi_r : \ B \times A \to B$ are the left and right module actions of A on B, for more information related to the Arens regularity of module actions on Banach algebras, see [2, 4, 9, 10]. If we set $B = A$, we write $Z^\ell_{A^{**}}(A^{**}) = Z_1(A^{**}) = Z_1^\ell(A^{**})$ and $Z^r_{A^{**}}(A^{**}) = Z_2(A^{**}) = Z_2^r(A^{**})$, for more information see [12]. Let A be a Banach algebra, A^* and A^{**} be the first and second dual of A, respectively. For $a \in A$ and $a' \in A^*$, by $a'a$ and aa', we mean the functionals in A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$, respectively. A Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle$ - $\langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$.

2. Main Results

Consider the tensor product, $X \otimes Y$, of the vector space X and Y which can be constructed as a space of linear functional on $B(X \times Y)$. By $X \widehat{\otimes} Y$ we shall denote the projective tensor products of X and Y, where $X \widehat{\otimes} Y$ is the completion of $X \otimes Y$ for the norm

$$\|u\| = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\|,$$

where the infimum is taken over all the representations of u as a finite sum of the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ [5].

The natural multiplication of $A \widehat{\otimes} B$ is the linear extension of the following multiplication on decomposable tensors $(a \otimes b)(\tilde{a} \otimes \tilde{b}) = a\tilde{a} \otimes b\tilde{b}$. For more details, see [16].

A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. Pym in [15] showed that this definition is equivalent with the following condition:

$$\lim_i \lim_j \langle a', a_i b_j \rangle = \lim_j \lim_i \langle a', a_i b_j \rangle,$$

whenever both iterated limits exist, for any two net $(a_i)_i$ and $(b_j)_j$ in $\{a \in A : \|a\| \leq 1\}$. The collection of all weakly almost periodic functionals on A is denoted by $wap(A)$. Also, $a' \in wap(A)$ if and only if $\langle a''b'', a' \rangle = \langle a''ob'', a' \rangle$ for every $a'', b'' \in A^{**}$. Thus, it is clear that A is Arens regular if and only if $wap(A) = A^*$ [9, Theorem 2.6.17]. In the sequel, to show that the projective tensor products $A \widehat{\otimes} B$ is Arens regular, it is sufficient that we show that $wap(A \widehat{\otimes} B) = (A \widehat{\otimes} B)^*$. In all of this section, we regard $A^* \widehat{\otimes} B^*$ as a subset of $(A \widehat{\otimes} B)^*$ and by A_1 and B_1 we mean all elements of $a \in A$ and $b \in B$ such that $\|a\| \leq 1$ and $\|b\| \leq 1$.

Theorem 2.1. *Suppose that A and B are Banach algebras and for every sequence $(x_i)_i$, $(y_j)_j \subseteq A_1$, $(z_i)_i$, $(w_j)_j \subseteq B_1$ and $f \in B(A \times B)$, we have*

$$\lim_j \lim_i f(x_i z_i, y_j w_j) = \lim_i \lim_j f(x_i z_i, y_j w_j).$$

Then $A \widehat{\otimes} B$ is Arens regular.

Proof. Assume that $f \in B(A \times B)$. Since $B(A \times B) = (A \widehat{\otimes} B)^*$, it is enough to show that $f \in wap(A \widehat{\otimes} B)$. Let $(x_i)_i$, $(y_j)_j \subseteq A_1$ and $(z_i)_i$, $(w_j)_j \subseteq B_1$, then we have the following equality

$$\begin{aligned}
\lim_j \lim_i \langle f, (x_i \otimes y_j)(z_i \otimes w_j) \rangle &= \lim_j \lim_i \langle f, x_i z_i \otimes y_j w_j \rangle \\
&= \lim_j \lim_i f(x_i z_i, y_j w_j) \\
&= \lim_i \lim_j f(x_i z_i, y_j w_j) \\
&= \lim_i \lim_j \langle f, (x_i \otimes y_j)(z_i \otimes w_j) \rangle,
\end{aligned}$$

for every $f \in (A\widehat{\otimes}B)^*$. This means that $f \in wap(A\widehat{\otimes}B)$, and proof is complete. $\square$

Definition 2.1. Let A be a Banach algebra and let B be a Banach A-bimodule and let $\pi : A\widehat{\otimes}B \longrightarrow B$ such that $\pi(a \otimes b) = ab$ for every $a \in A$, $b \in B$. We say that B is non-trivial on A, if π is surjective and has a bounded right inverse.

Remark 2.1. In the above definition, if A is unital, then π will be surjective. Now, suppose π has a continuous right inverse ρ, e_A and e_B are units of A and B, respectively. Let $\varphi \in (A\widehat{\otimes}B)^*$, then $\varphi \circ \rho$ belongs to B^*. Hence, there is a $\phi \in B^*$ such that $\varphi \circ \rho = \phi$. In other word, in the following diagram, we have $\varphi \circ \rho = \phi \circ \mathrm{id}_B$.

$$\begin{tikzcd}
 & B \arrow[dl, "\rho"'] \arrow[d, "\mathrm{id}_B"] & \\
A\widehat{\otimes}B \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & B \arrow[r] \arrow[d, "\phi"] & 0 \\
 & \mathbb{C} &
\end{tikzcd}$$

As well as, $\phi \circ \pi$ is in $(A\widehat{\otimes}B)^*$. Thus, there is a $\psi \in (A\widehat{\otimes}B)^*$ such that $\phi \circ \pi = \psi$. Then $\phi = \psi \circ \rho$. For given $a \otimes b \in A\widehat{\otimes}B$ we have

$$\begin{aligned}
\phi \circ \pi(a \otimes b) &= \phi(ab) = \varphi \circ \rho(ab) = \varphi \circ \rho(e_A a e_B b) \\
&= \varphi \circ \rho((e_A a)(e_B b)) = \psi \circ \rho((e_A a)(e_B b)) \\
&= \psi(a \otimes b). \qquad (2.1)
\end{aligned}$$

Then, by (2.1), for every $\varphi \in (A\widehat{\otimes}B)^*$ there is a $\psi \in (A\widehat{\otimes}B)^*$ such that $\varphi \circ \rho = \psi \circ \rho$ and $\psi(a \otimes b) = \psi \circ \rho(ab)$, for every $a \in A$ and $b \in B$. Since A is unital, every element c of B can be written as $c = ab$ where $a \in A$ and $b \in B$. We can define $\rho : B \longrightarrow A\widehat{\otimes}B$ by $\rho(b) = e_A \otimes b$ and $\rho(ab) = \rho((e_A a)b) = a \otimes b$, for every $a \in A$ and $b \in B$. By this definition ρ is injective and it is a unique way to define of ρ. By this definition the above diagram commutes and we have $\phi \circ \pi(a \otimes b) = \varphi(a \otimes b)$, for every $a \in A$ and $b \in B$.

A wide class of Banach algebras which satisfy in the Definition 2.1, are projective and biprojective Banach algebras. A Banach algebra A-bimodule B is called projective if $\pi : A^\sharp\widehat{\otimes}B \longrightarrow B$ has bounded right inverse in ${}_AB(B, A^\sharp\widehat{\otimes}B)$ and the Banach algebra A is called biprojective if $\pi : A\widehat{\otimes}A \longrightarrow A$ has bounded right inverse in ${}_AB(A, A\widehat{\otimes}A)$ (for more details see [18]).

Theorem 2.2. *Let A and B be Banach algebras and B is unital. Suppose B is a Banach A-bimodule. Then*

1. *if $A\widehat{\otimes}B$ is Arens regular, then A is Arens regular.*
2. *if B is non-trivial on A and B be a unitary Banach A-bimodule. Then A and B are Arens regular if and only if $A\widehat{\otimes}B$ is Arens regular.*

Proof. 1. Assume that $A\widehat{\otimes}B$ is Arens regular and $u \in B$ is the unit element of B. We show that $wap(A) = A^*$. Get $(a_i)_i \subseteq A$, $(c_j)_j \subseteq A$ and $a' \in A^*$. Define $\phi = a' \otimes b'$ where $b' \in B^*$ and $b'(u) = 1$. Since $A^* \otimes B^* \subseteq (A\widehat{\otimes}B)^*$ and $A\widehat{\otimes}B$ is Arens regular, we have $a' \otimes b' \in wap(A\widehat{\otimes}B)$. Hence it follows that

$$\begin{aligned}\lim_i \lim_j \langle a', a_i c_j\rangle &= \lim_i \lim_j \langle a' \otimes b', a_i c_j \otimes u\rangle \\ &= \lim_i \lim_j \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle \\ &= \lim_j \lim_i \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle \\ &= \lim_j \lim_i \langle a', a_i c_j\rangle.\end{aligned}$$

This means that $a' \in wap(A)$, and so A is Arens regular.

2. Let u be a unit element of B and let B be Arens regular. Then $wap(B) = B^*$. Suppose that $(a_i)_i \subseteq A_1$ and $(b_j)_j \subseteq B_1$ whenever both iterated limits exist. Then $(a_i u)_i \subseteq B_1$, and so for every $b' \in B^*$, we have the following equality

$$\lim_i \lim_j \langle b', (a_i u) b_j\rangle = \lim_j \lim_i \langle b', (a_i u) b_j\rangle.$$

Now; let $\varphi \in (A\widehat{\otimes}B)^*$. Then by Remark 2.1, $\pi : A\widehat{\otimes}B \longrightarrow B$ has a continuous right inverse ρ such that $\varphi \circ \rho$ belongs to B^* and there is a $\phi \in B^*$ such that $\varphi \circ \rho = \phi$, and $\phi \circ \pi(a \otimes b) = \varphi(a \otimes b)$, for every $a \otimes b \in A\widehat{\otimes}B$. Now we have

$$\begin{aligned}\lim_i \lim_j \langle \varphi, a_i \otimes b_j\rangle &= \lim_i \lim_j \langle \phi \circ \pi, a_i \otimes b_j\rangle = \lim_i \lim_j \langle \phi, \pi(a_i \otimes b_j)\rangle \\ &= \lim_i \lim_j \langle \phi, a_i b_j\rangle = \lim_i \lim_j \langle \phi, a_i(u b_j)\rangle \\ &= \lim_j \lim_i \langle \phi, (a_i u) b_j\rangle = \lim_j \lim_i \langle \phi, \pi(a_i \otimes b_j)\rangle \\ &= \lim_j \lim_i \langle \varphi, a_i \otimes b_j\rangle.\end{aligned}$$

It follows that $\varphi \in wap(A\widehat{\otimes}B)$, and so $A\widehat{\otimes}B$ is Arens regular. The converse by using the part (1) holds.

□

Corollary 2.1. *Suppose that A and B are unital Banach algebras and B is a unitary Banach A-bimodule. Assume that B is non-trivial on A. If A and B are Arens regular, then every bilinear form $m : A \times B \to \mathbb{C}$ is weakly compact.*

Proof. Apply Theorem 2.2 and Theorem 3.4 of [19]. □

Let A and B be Banach algebras. A bilinear form $m : A \times B \to \mathbb{C}$ is said to be biregular, if for any two pairs of sequence $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j)$$

provided that these limits exist. There are some examples of biregular non regular bilinear form for more information see [19].

Corollary 2.2. *Suppose that A and B are Banach algebras. Then we have the following assertions:*

1. *By the conditions of Theorem 2.1, every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.*
2. *By the conditions of Theorem 2.2 (2), every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.*

In the following, we give a simple proof of Theorem 3.4 of [19].

Theorem 2.3. *[19, Theorem 3.4] Let A and B be Banach algebras and $u : A \to B^*$ be a continuous linear operator. Then the bilinear form $m : A \times B \to \mathbb{C}$ defined by $m(a, b) = \langle u(a), b\rangle$ is biregular.*

Proof. Let $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1 such that the following iterated limits exist:

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) \quad \text{and} \quad \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

There are $(a_\alpha)_\alpha$, $(\tilde{a}_\beta)_\beta$ in A and $(b_\alpha)_\alpha$, $(\tilde{b}_\beta)_\beta$ in B such that $a_\alpha \overset{w^*}{\to} a''$ and $\tilde{a}_\beta \overset{w^*}{\to} \tilde{a}''$ in A^{**} and we have $b_\alpha \overset{w^*}{\to} b''$ and $\tilde{b}_\beta \overset{w^*}{\to} \tilde{b}''$ in B^{**}. Since A and B are Arens regular, we have

$$\lim_\alpha \lim_\beta a_\alpha\tilde{a}_\beta = \lim_\beta \lim_\alpha a_\alpha\tilde{a}_\beta = a''\tilde{a}'',$$

and

$$\lim_\alpha \lim_\beta b_\alpha\tilde{b}_\beta = \lim_\beta \lim_\alpha b_\alpha\tilde{b}_\beta = b''\tilde{b}''.$$

Then, since u is continuous, we have

$$\begin{aligned} \lim_\alpha \lim_\beta m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) &= \lim_\alpha \lim_\beta \langle u(a_\alpha\tilde{a}_\beta), b_\alpha\tilde{b}_\beta\rangle \\ &= \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle. \end{aligned}$$

Similarly, we have

$$\lim_\beta \lim_\alpha m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Consequently, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

It follows that m is biregular. $\square$

Example 2.1. [19] Let A be a Banach algebra and $1 < p < \infty$. Then

1. $\ell^p \widehat{\otimes} A$ is Arens regular if and only if A is Arens regular.
2. Let G be a locally compact group. Then, $L^p(G)\widehat{\otimes} A$ is Arens regular if and only if A is Arens regular.

Proof. For prove, we apply Theorem 3.4 of [19] and Theorem 2.7. $\square$

We finish this section with the following problems:

Problem 2.1. Let G be a locally compact group. What can say for the following sets?

$Z^{\ell}_{L^1(G)^{**}}((L^1(G)\widehat{\otimes} L^1(G))^{**}) =?$ $\qquad$ $Z^{\ell}_{L^1(G)^{**}}(L^1(G)^{**}\widehat{\otimes} L^1(G)^{**}) =?$

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