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$I-$LACUNARY STATISTICAL CONVERGENCE OF ORDER β OF DIFFERENCE SEQUENCES OF FRACTIONAL ORDER

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Abstract. In this paper, we have introduced the concepts of ideal $\Delta^{\alpha}-$lacunary statistical convergence of order β with the fractional order α and ideal $\Delta^{\alpha}-$lacunary strongly convergence of order β with the fractional order α (where $0 < \beta \leq 1$) and given some relations about these concepts.
Keywords: I-convergence, lacunary sequence, difference sequence.

1. Introduction

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund [53] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [48] and Fast [24] and later reintroduced by Schoenberg [45]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Çakallı et al. ([7],[8],[9]). Caserta et al. [10], Çınar et al. [12], Connor [11], Et et al. ([20], [23]), Fridy [26], Fridy and Orhan [27], Isik et al. ([29],[30],[31]), Mursaleen [40], Salat [47], Mohiuddine et al. ([5],[6],[33],[38],[39],[41]) and many others.

The idea of statistical convergence depends upon the density of subsets of the

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set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \text{ provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

Recently, Çolak [13] have generalized the statistical convergence by ordering the interval $(0, 1]$ and defined the statistical convergence of order β and strong $p-$Cesàro summability of order β, where $0 < \beta \leq 1$ and p is a positive real number. Şengül and Et ([19],[49]) generalized the concepts such as lacunary statistical convergence of order β and lacunary strong $p-$Cesàro summability of order β for sequences of real numbers.

The notation of I-convergence is a generalization of the statistical convergence. Kostyrko et al. ([36]) introduced the notation of I-convergence. Some further results connected with the notation of I-convergence can be found in ([14],[15],[37],[43],[44],[52]).

Let X be non-empty set. Then a family sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in X, $X \neq \phi$, then the family of sets $F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with* I. Throughout this study, I will stand for a non-trivial admissible ideal of $\mathbb{N}$ and by a sequence we always mean a sequence of real numbers.

Difference sequence spaces were defined by Kızmaz [35] and the concept was generalized by Et et al. ([16],[17]) as follows:

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \left(\begin{array}{c} k-v-1 \\ m-1 \end{array} \right) y_v = \sum_{v=1}^{k} (-1)^m \left(\begin{array}{c} k+m-v-1 \\ m-1 \end{array} \right) y_{v-m}, \tag{1.1}$$

$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then, some properties of difference sequence spaces have been studied in ([1],[2],[21],[22],[34],[44]).

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \tag{1.2}$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} \cdots$

$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} \cdots$

$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6} \cdots$

$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6} \cdots$

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r \notin \{0, -1, -2, -3, ...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

From the definition, it is observed that:

(i) For any natural number n, $\Gamma(n+1) = n!$,

(ii) For any real number n and $n \notin \{0, -1, -2, -3, ...\}, \Gamma(n+1) = n\Gamma(n)$,

(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

Without loss of generality, we assume throughout that the series defined in (1.2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (1.2) reduces to a finite sum i.e., $\sum_{i=0}^{\alpha}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$. In fact, this operator is generalized the difference operator introduced by Et and Çolak [16].

Recently, using fractional operator Δ^α (fractional order of α) Baliarsingh et al. ([3],[4],[42]) defined the sequence space $\Delta^\alpha (X)$ such as:

$$\Delta^\alpha (X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\},$$

where X is any sequence space.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([7],[8],[9],[25],[27],[28],[32],[46],[50],[51]).

1.1. Definitions and Main Results

Definition 1 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. The sequence $x = (x_k)$ is said to be $(\Delta^\alpha, I)-$lacunary statistically convergent of order β (or $\Delta^\alpha(S_\theta^\beta, I)-$convergent) to the number L, if there is a real number L such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \delta \right\} \in I$$

for each $\varepsilon > 0$ and $\delta > 0$. In this case, we write $x_k \to L(\Delta^\alpha(S_\theta^\beta, I))$. The set of all $(\Delta^\alpha, I)-$lacunary statistically convergent of order β sequences will be denoted by $\Delta^\alpha(S_\theta^\beta, I)$. If $\theta = (2^r)$, then we write $\Delta^\alpha(S^\beta, I)$ instead of $\Delta^\alpha(S_\theta^\beta, I)$. In the special cases $\theta = (2^r)$ and $\beta = 1$, we write $\Delta^\alpha(S, I)$ instead of $\Delta^\alpha(S_\theta^\beta, I)$.

In particular, $\Delta^\alpha(S_\theta^\beta, I)-$convergence includes many special cases; for example, in case of $\alpha = m \in \mathbb{N}$, $(\Delta^\alpha, I)-$lacunary statistical convergence of order β reduces to the $(\Delta^m, I)-$lacunary statistical convergence which was defined and studied by Et and Şengül [18].

Definition 2 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a fixed proper fraction and $p \geq 1$ be a real number. A sequence $x = (x_k)$ is said to be $\Delta^\alpha(N_\theta^\beta, I)-$summable to L (or ideal $\Delta^\alpha-$lacunary strongly summable of order β) if

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p \geqslant \varepsilon \right\} \in I.$$

In this case we write $x_k \to L(\Delta^\alpha(N_\theta^\beta, p, I))$. We denote the class of all ideal $\Delta^\alpha-$lacunary strongly summable sequences of order β by $\Delta^\alpha(N_\theta^\beta, p, I)$.

Theorem 1 Let $0 < \beta \leqslant \gamma \leqslant 1$. If $x_k \to L(\Delta^\alpha(S_\theta^\beta, I))$, then $x_k \to L(\Delta^\alpha(S_\theta^\gamma, I))$.

Proof. The inclusion part of the proof is trivial. The following example shows that the inclusion is strict. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^\alpha x_k$ by

$$\Delta^\alpha x_k = \begin{cases} k & k = n^3 \\ \frac{1}{3} & \text{otherwise} \end{cases} .$$

Then $x \in (\Delta^\alpha(S_\theta^\gamma, I))$ for $\frac{1}{3} < \gamma \leqslant 1$ but $x \notin (\Delta^\alpha(S_\theta^\beta, I))$ for $0 < \beta \leqslant \frac{1}{3}$ by (1.1) .

Theorem 2 If $x_k \to L(\Delta^\alpha(N_\theta^\beta, p, I))$, then $x_k \to L(\Delta^\alpha(N_\theta^\gamma, p, I))$ and the inclusion is proper.

Proof. The inclusion part of the proof is easy. The following example shows that the inclusion is strict. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^\alpha x_k$ by

$$\Delta^\alpha x_k = \begin{cases} 1 & k = n^2 \\ 0 & \text{otherwise} \end{cases} .$$

Then $x \in (\Delta^{\alpha}(N_{\theta}^{\gamma}, p, I))$ for $\frac{1}{2} < \gamma \leqslant 1$ but $x \notin (\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$ for $0 < \beta \leqslant \frac{1}{2}$ by (1.1).

Theorem 3 If $x_k \to L(\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$, then $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$ and the inclusion is proper.

Proof. Taking $p = 1$ and $L = 0$, we show the strictness of the inclusion. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^{\alpha} x_k$ by

$$\Delta^{\alpha} x_k = \begin{cases} \left[\sqrt[3]{h_r}\right] & k = 1, 2, 3, \cdots, \left[\sqrt[3]{h_r}\right] \\ 0 & \text{otherwise} \end{cases}.$$

Then we have for every $\varepsilon > 0$ and $\frac{1}{3} < \beta \leqslant 1$,

$$\frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta^{\alpha} x_k - 0| \geq \varepsilon\}| \leqslant \frac{\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}},$$

and for any $\delta > 0$ we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta^{\alpha} x_k - 0| \geq \varepsilon\}| \geqslant \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \geqslant \delta \right\}$$

and so $x_k \to 0(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$ for $\frac{1}{3} < \beta \leqslant 1$ by (1.1). On the other hand, for $0 < \beta \leqslant \frac{2}{3}$,

$$\frac{1}{h_r^{\beta}} \sum_{k \in I_r} |\Delta^{\alpha} x_k - 0| = \frac{\left[\sqrt[3]{h_r}\right] \left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \to \infty$$

and for $\alpha = \frac{2}{3}$,

$$\frac{\left[\sqrt[3]{h_r}\right] \left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \to 1.$$

$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\beta}} \sum_{k \in I_r} |\Delta^{\alpha} x_k - 0| \geqslant 1 \right\} = \left\{ r \in \mathbb{N} : \frac{\left[\sqrt[3]{h_r}\right]\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \geqslant 1 \right\} = \{a, a+1, a+2, ...\} \in F(I)$ for some $a \in \mathbb{N}$, since I is admissible. Thus $x_k \nrightarrow 0(\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$ by (1.1).

The proof of the following theorems is straightforward, so we choose to state these results without proof.

Theorem 4 If $\liminf_r q_r > 1$, then $x_k \to L(\Delta^{\alpha}(S^{\beta}, I))$ implies $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$.

Theorem 5 If $\liminf_r \frac{h_r^{\alpha}}{k_r} > 0$, then $x_k \to L(\Delta^{\alpha}(S, I))$ implies $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$.

Theorem 6 $\Delta^{\alpha}(S_{\theta}^{\beta}, I) \cap \ell_{\infty}(\Delta^{\alpha})$ is closed subset of $\ell_{\infty}(\Delta^{\alpha})$ for $0 < \beta \leqslant 1$.

Theorem 7 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ (for all $r \in \mathbb{N}$) and $\beta, \gamma \in (0, 1]$ be real numbers such that $\beta \leqslant \gamma$ and α be a proper fraction.

Theorem 8 i) If

$$\lim_{r\to\infty} \inf \frac{h_r^\beta}{\ell_r^\gamma} > 0, \tag{1.3}$$

then $\Delta^\alpha(S_{\theta'}^\gamma, I) \subseteq \Delta^\alpha(S_\theta^\beta, I)$

ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^\gamma} = 1, \tag{1.4}$$

then $\Delta^\alpha(S_\theta^\beta, I) \subseteq \Delta^\alpha(S_{\theta'}^\gamma, I)$.

Proof. i) Omitted.

ii) Let $x = (x_k) \in \Delta^\alpha(S_\theta^\beta, I)$ and be (1.4) satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\frac{1}{\ell_r^\gamma}|\{k \in J_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| = \frac{1}{\ell_r^\gamma}|\{s_{r-1} < k \leqslant k_{r-1} : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

$$+\frac{1}{\ell_r^\gamma}|\{k_r < k \leqslant s_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{1}{\ell_r^\gamma}|\{k_{r-1} < k \leqslant k_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

$$\leqslant \frac{k_{r-1} - s_{r-1}}{\ell_r^\gamma} + \frac{s_r - k_r}{\ell_r^\gamma} + \frac{1}{\ell_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

$$= \frac{\ell_r - h_r}{\ell_r^\gamma} + \frac{1}{\ell_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

$$\leqslant \frac{\ell_r - h_r^\gamma}{h_r^\gamma} + \frac{1}{h_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) + \frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $\ell_r = s_r - s_{r-1}$. Thus

$$\left\{r \in \mathbb{N} : \frac{1}{\ell_r^\beta}|\{k \in J_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \delta\right\} \subseteq$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \delta\right\} \in I.$$

This implies that $\Delta^\alpha(S_\theta^\beta, I) \subseteq \Delta^\alpha(S_{\theta'}^\gamma, I)$.

Theorem 9 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, β and γ be fixed real numbers such that $0 < \beta \leqslant \gamma \leqslant 1$ and $0 < p < \infty$. Then we have,

i) If (1.3) holds then $\Delta^{\alpha}(N_{\theta'}^{\gamma},p,I)\subset\Delta^{\alpha}(N_{\theta}^{\beta},p,I)$,

ii) If (1.4) holds and $x\in\Delta^{\alpha}(\ell_{\infty})$ then $\Delta^{\alpha}(N_{\theta}^{\beta},p,I)\subset\Delta^{\alpha}(N_{\theta'}^{\gamma},p,I)$.

Proof. Omitted.

Theorem 10 Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ (for all $r\in\mathbb{N}$), β and γ be fixed real numbers such that $0<\beta\leqslant\gamma\leqslant 1$ and $0<p<\infty$. Then,

i) Let (1.3) holds, if a sequence is strongly $\Delta^{\alpha}(N_{\theta'}^{\gamma},p,I)$-summable to L, then it is $\Delta^{\alpha}(S_{\theta}^{\beta},I)$-statistically convergent to L.

ii) Let (1.4) holds and $x=(x_k)$ be a $\Delta^{\alpha}-$bounded sequence if $\Delta^{\alpha}(S_{\theta}^{\beta},I)$-statistically convergent to L, then it is strongly $\Delta^{\alpha}(N_{\theta'}^{\gamma},p,I)-$summable to L.

Proof. i) For any sequence $x=(x_k)$ and $\varepsilon>0$, we have

$$\begin{aligned}\sum_{k\in J_r}|\Delta^{\alpha}x_k-L|^p &= \sum_{\substack{k\in J_r\\ |\Delta^{\alpha}x_k-L|\geqslant\varepsilon}}|\Delta^{\alpha}x_k-L|^p+\sum_{\substack{k\in J_r\\ |\Delta^{\alpha}x_k-L|<\varepsilon}}|\Delta^{\alpha}x_k-L|^p\\ &\geqslant \sum_{\substack{k\in I_r\\ |\Delta^{\alpha}x_k-L|\geqslant\varepsilon}}|\Delta^{\alpha}x_k-L|^p\\ &\geqslant |\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\varepsilon^p\end{aligned}$$

and so that

$$\begin{aligned}\frac{1}{\ell_r^{\gamma}}\sum_{k\in J_r}|\Delta^{\alpha}x_k-L|^p &\geqslant \frac{1}{\ell_r^{\gamma}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\varepsilon^p\\ &\geqslant \frac{h_r^{\beta}}{\ell_r^{\gamma}}\frac{1}{h_r^{\beta}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\varepsilon^p.\end{aligned}$$

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^{\beta}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geq\varepsilon\}|\geqslant\delta\right\}\subseteq$$

$$\subseteq\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\gamma}}\sum_{k\in J_r}|\Delta^{\alpha}x_k-L|^p\geqslant\frac{h_r^{\beta}}{\ell_r^{\gamma}}\delta\varepsilon^p\right\}\in I.$$

Hence $x=(x_k)$ is $\Delta^{\alpha}(S_{\theta}^{\beta},I)-$statistically convergent to L.

ii) Suppose that $\Delta^{\alpha}(S_{\theta}^{\beta},I)$-statistically convergent to L and $x=(x_k)\in\Delta^{\alpha}(\ell_{\infty})$. Then there exists some $M>0$ such that $|\Delta^{\alpha}x_k-L|\leqslant M$ for all k. Then for every $\varepsilon>0$ we may write

$$\frac{1}{\ell_r^{\gamma}}\sum_{k\in J_r}|\Delta^{\alpha}x_k-L|^p=\frac{1}{\ell_r^{\gamma}}\sum_{k\in J_r-I_r}|\Delta^{\alpha}x_k-L|^p+\frac{1}{\ell_r^{\gamma}}\sum_{k\in I_r}|\Delta^{\alpha}x_k-L|^p$$

$$\leqslant \left(\frac{\ell_r - h_r}{\ell_r^\gamma}\right) M^p + \frac{1}{\ell_r^\gamma} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r - h_r^\gamma}{\ell_r^\gamma}\right) M^p + \frac{1}{\ell_r^\gamma} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{1}{h_r^\gamma} \sum_{\substack{k \in I_r \\ |\Delta^\alpha x_k - L| \geqslant \varepsilon}} |\Delta^\alpha x_k - L|^p + \frac{1}{h_r^\gamma} \sum_{\substack{k \in I_r \\ |\Delta^\alpha x_k - L| < \varepsilon}} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{M^p}{h_r^\gamma} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{h_r}{h_r^\gamma} \varepsilon^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{M^p}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{\ell_r}{h_r^\gamma} \varepsilon^p$$

for all $r \in \mathbb{N}$.

$$\left\{ r \in \mathbb{N} : \frac{1}{\ell_r^\gamma} \sum_{k \in J_r} |\Delta^\alpha x_k - L|^p \geqslant \delta \right\} \subseteq$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \frac{\delta}{M^p} \right\} \in I.$$

Using (1.4) we obtain that $\Delta^\alpha(N_{\theta'}^\gamma, p, I)$-statistically convergent to L, whenever $\Delta^\alpha(S_\theta^\beta, I)$-summable to L.

Definition 3 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0, 1]$, α be a proper fraction. The sequence $x = (x_k)$ is said to be $(\Delta^\alpha, I)-$lacunary statistically Cauchy sequence of order β (or $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy) if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in J_r$ for each $r \in \mathbb{N}$, $x_{k'(r)} \to L(\Delta^\alpha)$ (i.e. $\lim_r |\Delta^\alpha x_{k'(r)} - L| = 0$)

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon \right\} \in I$$

for each $\varepsilon > 0$.

Theorem 11 If $x = (x_k)$ is a $\Delta^\alpha(N_\theta^\beta, I)-$summable if and only if it is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence.

Proof. Assume that (x_k) is a $\Delta^\alpha(N_\theta^\beta, I)-$summable sequence to L. Then there exists L such that $x_k \to L(\Delta^\alpha(N_\theta^\beta, I)$. Therefore,

$$H_i = \left\{ i \in \mathbb{N} : |\Delta^\alpha x_k - L| < \frac{1}{i} \right\}$$

for each $i \in \mathbb{N}$. Hence for each i, $H_{i+1} \subseteq H_i$ and

$$\left\{ r \in \mathbb{N} : \frac{|H_r \cap J_r|}{h_r^\beta} \geqslant \frac{1}{r} \right\} \in I.$$

We choose k_1, such that $r \geqslant k_1$, then

$$\left\{ r \in \mathbb{N} : \frac{|H_1 \cap J_r|}{h_r^\beta} < 1 \right\} \notin I.$$

Next we choose $k_2 > k_1$ such that $r > k_2$ implies

$$\left\{ r \in \mathbb{N} : \frac{|H_2 \cap J_r|}{h_r^\beta} < 1 \right\} \notin I.$$

Proceeding this way, we can choose $k_{p+1} > k_p$ such that $r > k_{p+1}$, implies that $H_{p+1} \cap J_r \neq \emptyset$. Also, we can choose $k'(r) \in H_p \cap J_r$ for each r satisfying $k_p \leqslant r < k_{p+1}$ such that

$$|\Delta^\alpha x_{k'(r)} - L| < \frac{1}{p}.$$

This implies that $x_{k'(r)} \to L(\Delta^\alpha)$. Therefore, for every $\varepsilon > 0$, we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k'(r) \in J_r} |\Delta^\alpha x_{k'(r)} - L| \geq \frac{\varepsilon}{2} \right\}.$$

Then,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon \right\} \in I.$$

Therefore (x_k) is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence.

Conversely suppose (x_k) is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence. Then for every $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k'(r) \in J_r} |\Delta^\alpha x_{k'(r)} - L| \geq \frac{\varepsilon}{2} \right\}$$

and so (x_k) is a $(\Delta^\alpha(N_\theta^\beta, I)-$summable sequence to L.

Definition 4 A lacunary sequence $\rho = (\bar{k}(r))$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (\bar{k}(r))$.

Theorem 12 If $\rho = (\bar{k}(r))$ is a lacunary refinement of a lacunary sequence θ and $x_k \to L(\Delta^\alpha(N_\rho^\beta, I))$, then $x_k \to L(\Delta^\alpha(N_\theta^\beta, I))$.

Proof. Suppose that for each J_r of θ contains the points $(\bar{k}_{r,t})_{t=1}^{\nu(r)}$ of ρ such that $k_{r-1} < \bar{k}_{r,1} < \bar{k}_{r,2} < \cdots < \bar{k}_{r,\nu(r)} = k_r$, where $\bar{J}_{r,t} = (\bar{k}_{r,t-1}, \bar{k}_{r,t}]$. For all r and let $\nu(r) \geqslant 1$ this implies $k_r \subseteq (\bar{k}(r))$. Let $(J_j^*)_{j=1}^\infty$ be the sequence of intervals $(\bar{J}_{r,t})$ ordered by increasing right end points. Since $x_k \in L(\Delta^\alpha(N_\rho^\beta, I))$, then for each $\varepsilon > 0$,

$$\left\{ j \in \mathbb{N} : \frac{1}{(h_j^*)^\beta} \sum_{J_j^* \subset J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \in I.$$

Also since $h_r = k_r - k_{r-1}$, so $\bar{h}_{r,t} = \bar{k}_{r,t} - \bar{k}_{r,t-1}$. For each $\varepsilon > 0$, we get

$$\left\{ r \in \mathbb{N} : \frac{1}{(h_r)^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{(h_r)^\beta} \sum_{k \in J_r} \left\{ j \in \mathbb{N} : \frac{1}{(h_j^*)^\beta} \sum_{\substack{J_j^* \subset J_r \\ k \in J_j^*}} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \right\}.$$

Therefore $\left\{ r \in \mathbb{N} : (h_r)^{-\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \in I$. Thus $x_k \in (\Delta^\alpha(N_\theta^\beta, I))$.

Theorem 13 Let ψ be set of lacunary sequences.

a) If ψ is closed under arbitrary union, then $\Delta^\alpha(N_\mu^\beta, I) = \bigcap_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$, where $\mu = \bigcup_{\theta \in \psi} \theta$,

b) If ψ closed under arbitrary intersection, then $\Delta^\alpha(N_\tau^\beta, I) = \bigcup_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$, where $\tau = \bigcap_{\theta \in \psi} \theta$,

c) If ψ is closed under union and intersection, then $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I) \subseteq \Delta^\alpha(N_\tau^\beta, I)$.

Proof. a) By hypothesis, we have $\mu \in \psi$ which is a refinement of each $\theta \in \psi$. Then from Theorem 12, we have if $x_k \in \Delta^\alpha(N_\mu^\beta, I)$ implies that $x_k \in \Delta^\alpha(N_\theta^\beta, I)$. Therefore, for each $\theta \in \psi$, we have $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I)$. The reverse inclusion is obvious. Hence $\Delta^\alpha(N_\mu^\beta, I) = \bigcap_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$.

b) By part a) and Theorem 12, we have $\Delta^\alpha(N_\tau^\beta, I) = \bigcup_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$.

c)By part a) and b)we get $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I) \subseteq \Delta^\alpha(N_\tau^\beta, I)$.

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