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ON PSEUDO-HERMITIAN MAGNETIC CURVES IN SASAKIAN MANIFOLDS

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Abstract. We define pseudo-Hermitian magnetic curves in Sasakian manifolds endowed with the Tanaka-Webster connection. After we have given a complete classification theorem, we shall construct parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$.

Keywords: magnetic curve; slant curve; Sasakian manifold; the Tanaka-Webster connection.

1. Introduction

The study of the motion of a charged particle in a constant and time-independent static magnetic field on a Riemannian surface is known as the Landau–Hall problem [16]. The main problem is to study the movement of a charged particle moving in the Euclidean plane $\mathbb{E}^2$. The solution of the Lorentz equation (called also the Newton equation) corresponds to the motion of the particle. The trajectory of a charged particle moving on a Riemannian manifold under the action of the magnetic field is a very interesting problem from a geometric point of view [16].

Let (N, g) be a Riemannian manifold, and F a closed 2-form, Φ the Lorentz force, which is a $(1, 1)$-type tensor field on N. F is called a *magnetic field* if it is associated to Φ by the relation

$$F(X, Y) = g(\Phi X, Y), \tag{1.1}$$

where X and Y are vector fields on N (see [1], [3] and [8]). Let ∇ be the Riemannian connection on N and consider a differentiable curve $\alpha : I \to N$, where I denotes an open interval of $\mathbb{R}$. α is said to be a *magnetic curve* for the magnetic field F, if it is a solution of the Lorentz equation given by

$$\nabla_{\alpha'(t)}\alpha'(t) = \Phi(\alpha'(t)). \tag{1.2}$$

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From the definition of magnetic curves, it is straightforward to see that their speed is constant. Specifically, unit-speed magnetic curves are called *normal magnetic curves* [9].

In [9], Druţă-Romaniuc, Inoguchi, Munteanu and Nistor studied magnetic curves in a Sasakian manifold. Magnetic curves in cosymplectic manifolds were studied in [10] by the same authors. In [13], 3-dimensional Berger spheres and their magnetic curves were considered by Inoguchi and Munteanu. Magnetic trajectories of an almost contact metric manifold were studied in [14], by Jleli, Munteanu and Nistor. The classification of all uniform magnetic trajectories of a charged particle moving on a surface under the action of a uniform magnetic field was obtained in [19], by Munteanu. Furthermore, normal magnetic curves in para-Kaehler manifolds were researched in [15], by Jleli and Munteanu. In [17], Munteanu and Nistor obtained the complete classification of unit-speed Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$. Moreover, in [18], they studied magnetic curves on $\mathbb{S}^{2n+1}$. 3-dimensional normal para-contact metric manifolds and their magnetic curves of a Killing vector field were investigated in [5], by Calvaruso, Munteanu and Perrone. In [20], the present authors studied slant curves in contact Riemannian 3-manifolds with pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian harmonic mean curvature vector field for the Tanaka-Webster connection in the tangent and normal bundles, respectively. The second author gave the parametric equations of all normal magnetic curves in the 3-dimensional Heisenberg group in [21]. Recently, the present authors have also considered slant magnetic curves in S-manifolds in [11].

These studies motivate us to investigate pseudo-Hermitian magnetic curves in $(2n+1)$-dimensional Sasakian manifolds endowed with the Tanaka-Webster connection. In Section 2, we summarize the fundamental definitions and properties of Sasakian manifolds and the unique connection, namely the Tanaka-Webster connection. We give the main classification theorems for pseudo-Hermitian magnetic curves in Section 3. We show that a pseudo-Hermitian magnetic curve cannot have osculating order greater than 3. In the last section, after a brief information on $\mathbb{R}^{2n+1}(-3)$, we obtain the parametric equations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection.

2. Preliminaries

Let N be a $(2n+1)$-dimensional Riemannian manifold satisfying the following equations

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \quad g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y on N, where ϕ is a $(1,1)$-type tensor field, η is a 1-form, ξ is a vector field and g is a Riemannian metric on N. In this case, (N, ϕ, ξ, η, g) is said to be an *almost contact metric manifold* [2]. Moreover, if $d\eta(X,Y) = \Phi(X,Y)$, where $\Phi(X,Y) = g(X,\phi Y)$ is the *fundamental* 2*-form* of the manifold, then N is said to be a *contact metric manifold* [2].

Furthermore, if we denote the Nijenhuis torsion of ϕ by $[\phi,\phi]$, for all $X,Y \in \chi(N)$, the condition given by

$$[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi$$

is called the *normality condition* of the almost contact metric structure. An almost contact metric manifold turns into a *Sasakian manifold* if the normality condition is satisfied [2].

From Lie differentiation operator in the characteristic direction ξ, the operator h is defined by

$$h = \frac{1}{2}L_\xi\phi.$$

It is directly found that the structural operator h is symmetric. It also validates the equations below, where we denote the Levi-Civita connection by ∇:

$$h\xi = 0, \quad h\phi = -\phi h, \quad \nabla_X\xi = -\phi X - \phi hX, \tag{2.3}$$

(see [2]).

If we denote the Tanaka-Webster connection on N by $\widehat{\nabla}$ ([22], [24]), then we have

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\widehat{\nabla}_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi$$

for all vector fields X,Y on N. By the use of equations (2.3), the Tanaka-Webster connection can be calculated as

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi. \tag{2.4}$$

The torsion of the Tanaka-Webster connection is

$$\widehat{T}(X,Y) = 2g(X,\phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY. \tag{2.5}$$

In a Sasakian manifold, from the fact that $h = 0$ (see [2]), the equations (2.4) and (2.5) can be rewritten as:

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{2.6}$$

$$\widehat{T}(X,Y) = 2g(X,\phi Y)\xi.$$

The following proposition states why the Tanaka-Webster connection is unique:

Proposition 2.1. [23] *The Tanaka-Webster connection on a contact Riemannian manifold $N = (N,\phi,\xi,\eta,g)$ is the unique linear connection satisfying the following four conditions:*

(a) $\widehat{\nabla}\eta = 0$, $\widehat{\nabla}\xi = 0$;

(b) $\widehat{\nabla}g = 0$, $\widehat{\nabla}\phi = 0$;

(c) $\widehat{T}(X,Y) = -\eta([X,Y])\xi, \quad \forall X,Y \in D$;

(d) $\widehat{T}(\xi,\phi Y) = -\phi\widehat{T}(\xi,Y), \ \forall Y \in D.$

3. Magnetic Curves with respect to the Tanaka-Webster Connection

Let (N,ϕ,ξ,η,g) be an n-dimensional Riemannian manifold and $\alpha : I \to N$ a curve parametrized by arc-length. If there exists g-orthonormal vector fields $E_1, E_2, ..., E_r$ along α such that

$$\begin{aligned} E_1 &= \alpha', \\ \widehat{\nabla}_{E_1} E_1 &= \widehat{k}_1 E_2, \\ \widehat{\nabla}_{E_1} E_2 &= -\widehat{k}_1 E_1 + \widehat{k}_2 E_3, \\ &\cdots \\ \widehat{\nabla}_{E_1} E_r &= -\widehat{k}_{r-1} E_{r-1}, \end{aligned} \tag{3.1}$$

then α is called a *Frenet curve for* $\widehat{\nabla}$ *of osculating order* r , $(1 \leq r \leq n)$. Here $\widehat{k}_1, ..., \widehat{k}_{r-1}$ are called *pseudo-Hermitian curvature functions of* α and these functions are positive valued on I. A *geodesic for* $\widehat{\nabla}$ (or *pseudo-Hermitian geodesic*) is a Frenet curve of osculating order 1 for $\widehat{\nabla}$. If $r = 2$ and $\widehat{k}_1$ is a constant, then α is called a *pseudo-Hermitian circle.* A *pseudo-Hermitian helix of order* r $(r \geq 3)$ is a Frenet curve for $\widehat{\nabla}$ of osculating order r with non-zero positive constant pseudo-Hermitian curvatures $\widehat{k}_1, ..., \widehat{k}_{r-1}$. If we shortly state *pseudo-Hermitian helix,* we mean its osculating order is 3 [7].

Let $N = \left(N^{2n+1}, \phi, \xi, \eta, g\right)$ be a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Let us denote the fundamental 2-form of N by Ω. Then, we have

$$\Omega(X,Y) = g(X,\phi Y), \tag{3.2}$$

(see [2]). From the fact that N is a Sasakian manifold, we have $\Omega = d\eta$. Hence, $d\Omega = 0$, i.e., it is closed. Thus, we can define a magnetic field F_q on N by

$$F_q(X,Y) = q\Omega(X,Y),$$

namely the *contact magnetic field with strength* q, where $X, Y \in \chi(N)$ and $q \in \mathbb{R}$ [14]. We will assume that $q \neq 0$ to avoid the absence of the strength of magnetic field (see [4] and [9]).

From (1.1) and (3.2), the Lorentz force Φ associated to the contact magnetic field F_q can be written as

$$\Phi = -q\phi.$$

So the Lorentz equation (1.2) is

$$\nabla_{E_1} E_1 = -q\phi E_1, \tag{3.3}$$

where $\alpha : I \to N$ is a curve with arc-length parameter, $E_1 = \alpha'$ is the tangent vector field and ∇ is the Levi-Civita connection (see [9] and [14]). By the use of equations (2.6) and (3.3), we have

$$\widehat{\nabla}_{E_1} E_1 = \left[-q + 2\eta(E_1)\right] \phi E_1. \tag{3.4}$$

Definition 3.1. Let $\alpha : I \to N$ be a unit-speed curve in a Sasakian manifold $N = \left(N^{2n+1}, \phi, \xi, \eta, g\right)$ endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then it is called a *normal magnetic curve with respect to the Tanaka-Webster connection* $\widehat{\nabla}$ (or shortly a *pseudo-Hermitian magnetic curve*) if it satisfies equation (3.4).

If $\eta(E_1) = \cos\theta$ is a constant, then α is called a *slant curve* [6]. From the definition of pseudo-Hermitian magnetic curves, we have the following direct result as in the Levi-Civita case:

Proposition 3.1. *If α is a pseudo-Hermitian magnetic curve in a Sasakian manifold, then it is a slant curve.*

Proof. Let $\alpha : I \to N$ be a pseudo-Hermitian magnetic curve. Then, we find

$$\begin{aligned} \frac{d}{dt} g(E_1, \xi) &= g(\widehat{\nabla}_{E_1} E_1, \xi) + g(E_1, \widehat{\nabla}_{E_1} \xi) \\ &= g([-q + 2\eta(E_1)]\, \phi E_1, \xi) \\ &= 0. \end{aligned}$$

So we obtain

$$\eta(E_1) = \cos\theta = constant,$$

which completes the proof. $\square$

As a result, we can rewrite equation (3.4) as

$$\widehat{\nabla}_{E_1} E_1 = (-q + 2\cos\theta)\, \phi E_1, \tag{3.5}$$

where θ is the contact angle of α. Now, we can state the following theorem:

Theorem 3.1. *Let $\left(N^{2n+1}, \phi, \xi, \eta, g\right)$ be a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then $\alpha : I \to N$ is a pseudo-Hermitian magnetic curve if and only if it belongs to the following list:*

(a) *pseudo-Hermitian non-Legendre slant geodesics (including pseudo-Hermitian geodesics as integral curves of ξ);*

(b) *pseudo-Hermitian Legendre circles with $\widehat{k}_1 = |q|$ and having the Frenet frame field (for $\widehat{\nabla}$)*

$$\{E_1, -sgn(q)\phi E_1\};$$

(c) *pseudo-Hermitian slant helices with*

$$\widehat{k}_1 = |-q + 2\cos\theta| \sin\theta, \; \widehat{k}_2 = |-q + 2\cos\theta|\, \varepsilon \cos\theta$$

and having the Frenet frame field (for $\widehat{\nabla}$)

$$\left\{E_1, \frac{\delta}{\sin\theta}\phi E_1, \frac{\varepsilon}{\sin\theta}\left(\xi - \cos\theta E_1\right)\right\},$$

where $\delta = sgn(-q + 2\cos\theta)$, $\varepsilon = sgn(\cos\theta)$ and $\cos\theta \neq \frac{q}{2}$.

Proof. Let us assume that $\alpha : I \to N$ is a normal magnetic curve with respect to $\widehat{\nabla}$. Consequently, equation (3.5) must be validated. Let us assume $\widehat{k}_1 = 0$. Hence, we have $\cos\theta = \frac{q}{2}$ or $\phi E_1 = 0$. If $\cos\theta = \frac{q}{2}$, then α is a pseudo-Hermitian non-Legendre slant geodesic. Otherwise, $\phi E_1 = 0$ gives us $E_1 = \pm\xi$. Thus, α is a pseudo-Hermitian geodesic as an integral curve of $\pm\xi$. So we have just proved that α belongs to (a) from the list, if the osculating order $r = 1$. Now, let $\widehat{k}_1 \neq 0$. From equation (3.5) and the Frenet equations for $\widehat{\nabla}$, we find

$$\widehat{\nabla}_{E_1} E_1 = \widehat{k}_1 E_2 = (-q + 2\cos\theta)\, \phi E_1. \tag{3.6}$$

Since E_1 is unit, the equation (2.2) gives us

$$g(\phi E_1, \phi E_1) = \sin^2\theta. \tag{3.7}$$

By the use of (3.6) and (3.7), we obtain

$$\widehat{k}_1 = |-q + 2\cos\theta| \sin\theta, \tag{3.8}$$

which is a constant. Let us denote $\delta = sgn(-q + 2\cos\theta)$. From (3.8), we can write

$$\phi E_1 = \delta \sin\theta E_2. \tag{3.9}$$

Let us assume $\widehat{k}_2 = 0$, that is, $r = 2$. From the fact that $\widehat{k}_1$ is a constant, α is a pseudo-Hermitian circle. (3.9) gives us

$$\eta(\phi E_1) = 0 = \delta \sin\theta \eta(E_2),$$

which is equivalent to

$$\eta(E_2) = 0.$$

Differentiating this last equation with respect to $\widehat{\nabla}$, we obtain

$$\widehat{\nabla}_{E_1} \eta(E_2) = 0 = g\left(\widehat{\nabla}_{E_1} E_2, \xi\right) + g\left(E_2, \widehat{\nabla}_{E_1} \xi\right).$$

Since $\widehat{\nabla}\xi = 0$ and $r = 2$, we have

$$g(-\widehat{k}_1 E_1, \xi) = 0,$$

that is, $\eta(E_1) = 0$. Hence, α is Legendre and $\cos\theta = 0$. From equation (3.8), we get $\widehat{k}_1 = |q|$. In this case, we also obtain $\delta = -sgn(q)$ and $E_2 = -sgn(q)\phi E_1$. We have proved that α belongs to (b) from the list, if the osculating order $r = 2$. Now, let us assume $\widehat{k}_2 \neq 0$. If we use $\widehat{\nabla}\phi = 0$, we calculate

$$\widehat{\nabla}_{E_1} \phi E_1 = \widehat{k}_1 \phi E_2. \tag{3.10}$$

From (2.1) and (3.9), we find

$$\phi^2 E_1 = -E_1 + \cos\theta\xi = \delta \sin\theta \phi E_2, \tag{3.11}$$

which gives us

$$\phi E_2 = \frac{\delta}{\sin\theta}\left(-E_1 + \cos\theta\xi\right).$$

So equation (3.10) becomes

$$\widehat{\nabla}_{E_1}\phi E_1 = \widehat{k}_1 \frac{\delta}{\sin\theta}\left(-E_1 + \cos\theta\xi\right). \tag{3.12}$$

If we differentiate the equation (3.9) with respect to $\widehat{\nabla}$, we also have

$$\begin{aligned} \widehat{\nabla}_{E_1}\phi E_1 &= \delta\sin\theta\widehat{\nabla}_{E_1}E_2 \\ &= \delta\sin\theta\left(-\widehat{k}_1 E_1 + \widehat{k}_2 E_3\right). \end{aligned} \tag{3.13}$$

By the use of (3.12) and (3.13), we obtain

$$\widehat{k}_1 \cot\theta\left(\xi - \cos\theta E_1\right) = \widehat{k}_2 \sin\theta E_3. \tag{3.14}$$

One can easily see that

$$g(\xi - \cos\theta E_1, \xi - \cos\theta E_1) = \sin^2\theta.$$

From (3.14), we calculate

$$\widehat{k}_2 = \left|-q + 2\cos\theta\right| \varepsilon\cos\theta,$$

where we denote $\varepsilon = sgn(\cos\theta)$. As a result, we get

$$E_3 = \frac{\varepsilon}{\sin\theta}\left(\xi - \cos\theta E_1\right), \tag{3.15}$$

$$E_2 = \frac{\delta}{\sin\theta}\phi E_1.$$

If we differentiate (3.15) with respect to $\widehat{\nabla}$, since $\phi E_1 \parallel E_2$, we find $\widehat{k}_3 = 0$. So we have just completed the proof of (c). Considering the fact that $\widehat{k}_3 = 0$, the Gram-Schmidt process ends. Thus, the list is complete.

Conversely, let $\alpha : I \to N$ belong to the given list. It is easy to show that equation (3.5) is satisfied. Hence, α is a pseudo-Hermitian magnetic curve. □

A pseudo-Hermitian geodesic is said to be a pseudo-Hermitian ϕ-curve if the set $sp\left\{E_1, \phi E_1, \xi\right\}$ is ϕ-invariant. A Frenet curve of osculating order $r = 2$ is said to be a pseudo-Hermitian ϕ-curve if $sp\left\{E_1, E_2, \xi\right\}$ is ϕ-invariant. A Frenet curve of osculating order $r \geq 3$ is said to be a pseudo-Hermitian ϕ-curve if $sp\left\{E_1, E_2, ..., E_r\right\}$ is ϕ-invariant.

Theorem 3.2. *Let $\alpha : I \to N$ be a pseudo-Hermitian ϕ-helix of order $r \leq 3$, where $N = \left(N^{2n+1}, \phi, \xi, \eta, g\right)$ is a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then:*

(a) *If* $\cos\theta = \pm 1$*, then it is an integral curve of* ξ*, i.e. a pseudo-Hermitian geodesic and it is a pseudo-Hermitian magnetic curve for* F_q *for arbitrary* q*;*

(b) *If* $\cos\theta \notin \{-1,0,1\}$ *and* $\widehat{k}_1 = 0$*, then it is a pseudo-Hermitian non-Legendre slant geodesic and it is a pseudo-Hermitian magnetic curve for* $F_{2\cos\theta}$*;*

(c) *If* $\cos\theta = 0$ *and* $\widehat{k}_1 \neq 0$*, i.e.* α *is a Legendre* ϕ*-curve, then it is a pseudo-Hermitian magnetic circle generated by* $F_{-\delta\widehat{k}_1}$*, where* $\delta = sgn(g(\phi E_1, E_2))$*;*

(d) *If* $\cos\theta = \frac{\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$ *and* $\widehat{k}_2 \neq 0$*, then it is a pseudo-Hermitian magnetic curve for* $F_{-\delta\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}+\frac{2\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}}$*, where* $\delta = sgn(g(\phi E_1, E_2))$ *and* $\varepsilon = sgn(\cos\theta)$*.*

(e) *Except above cases,* α *cannot be a pseudo-Hermitian magnetic curve for any* F_q*.*

Proof. Firstly, let us assume $\cos\theta = \pm 1$, that is, $E_1 = \pm\xi$. As a result, we have

$$\widehat{\nabla}_{E_1}E_1 = 0, \ \phi E_1 = 0.$$

Hence, equation (3.5) is satisfied for arbitrary q. This proves (a). Now, let us take $\cos\theta \notin \{-1,0,1\}$ and $\widehat{k}_1 = 0$. In this case, we obtain

$$\widehat{\nabla}_{E_1}E_1 = 0, \ \phi E_1 \neq 0.$$

So equation (3.5) is valid for $q = 2\cos\theta$. The proof of (b) is over. Next, let us assume $\cos\theta = 0$ and $\widehat{k}_1 \neq 0$. One can easily see that α has the Frenet frame field (for $\widehat{\nabla}$)

$$\{E_1, \delta\phi E_1\}$$

where δ corresponds to the sign of $g(\phi E_1, E_2)$. Consequently, we get

$$\widehat{\nabla}_{E_1}E_1 = \delta\widehat{k}_1\phi E_1,$$

that is, α is a pseudo-Hermitian magnetic curve for $q = -\delta\widehat{k}_1$. We have just proven (c). Finally, let $\cos\theta = \frac{\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$ and $\widehat{k}_2 \neq 0$. So α has the Frenet frame field (for $\widehat{\nabla}$)

$$\left\{E_1, \frac{\delta}{\sin\theta}\phi E_1, \frac{\varepsilon}{\sin\theta}(\xi - \cos\theta E_1)\right\},$$

where $\delta = sgn(g(\phi E_1, E_2))$ and $\varepsilon = sgn(\cos\theta)$. After calculations, it is easy to show that equation (3.5) is satisfied for $q = -\delta\sqrt{\widehat{k}_1^2+\widehat{k}_2^2} + \frac{2\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$. Hence, the proof of (d) is completed. Except above cases, from Theorem 3.1, α cannot be a pseudo-Hermitian magnetic curve for any F_q. $\square$

4. Parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$

In this section, our aim is to obtain parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$. To do this, we need to recall some notions from [2]. Let $N = \mathbb{R}^{2n+1}$. Let us denote the coordinate functions of N with $(x_1, ..., x_n, y_1, ..., y_n, z)$. One may define a structure on N by $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y_i dx_i)$, which is a contact structure, since $\eta \wedge (d\eta)^n \neq 0$. This contact structure has the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$. Let us also consider a $(1,1)$-type tensor field ϕ given by the matrix form as

$$\phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

Finally, let us take the Riemannian metric on N given by $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2)$. It is known that (N, ϕ, ξ, η, g) is a Sasakian space form and its ϕ-sectional curvature is $c = -3$. This special Sasakian space form is denoted by $\mathbb{R}^{2n+1}(-3)$ [2]. One can easily show that the vector fields

$$X_i = 2\frac{\partial}{\partial y_i},\ X_{n+i} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}),\ i = \overline{1, n},\ \xi = 2\frac{\partial}{\partial z} \tag{4.1}$$

are g-unit and g-orthogonal. Hence, they form a g-orthonormal basis [2]. Using this basis, the Levi-Civita connection of $\mathbb{R}^{2n+1}(-3)$ can be obtained as

$$\nabla_{X_i} X_j = \nabla_{X_{m+i}} X_{m+j} = 0,\ \nabla_{X_i} X_{m+j} = \delta_{ij}\xi,\ \nabla_{X_{m+i}} X_j = -\delta_{ij}\xi,$$

$$\nabla_{X_i}\xi = \nabla_\xi X_i = -X_{m+i},\ \nabla_{X_{m+i}}\xi = \nabla_\xi X_{m+i} = X_i,$$

(see [2]). As a result, the Tanaka-Webster connection of $\mathbb{R}^{2n+1}(-3)$ is

$$\begin{aligned} \widehat{\nabla}_{X_i} X_j &= \widehat{\nabla}_{X_{m+i}} X_{m+j} = \widehat{\nabla}_{X_i} X_{m+j} = \widehat{\nabla}_{X_{m+i}} X_j = \\ \widehat{\nabla}_{X_i}\xi &= \widehat{\nabla}_\xi X_i = \widehat{\nabla}_{X_{m+i}}\xi = \widehat{\nabla}_\xi X_{m+i} = 0, \end{aligned}$$

which was calculated in [12]. Now, we can investigate the parametric equations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection.

Let $N = \mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Let $\alpha : I \subseteq \mathbb{R} \to N$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}, ..., \alpha_{2n}, \alpha_{2n+1})$ be a pseudo-Hermitian magnetic curve. Then, the tangential vector field of α can be written as

$$E_1 = \sum_{i=1}^{n} \alpha_i' \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \alpha_{n+i}' \frac{\partial}{\partial y_i} + \alpha_{2n+1}' \frac{\partial}{\partial z}.$$

In terms of the g-orthonormal basis, E_1 is rewritten as

$$E_1 = \frac{1}{2}\left[\sum_{i=1}^{n} \alpha_{n+i}' X_i + \sum_{i=1}^{n} \alpha_i' X_{n+i} + \left(\alpha_{2n+1}' - \sum_{i=1}^{n} \alpha_i' \alpha_{n+i}\right)\xi\right].$$

From Proposition 3.1, α is a slant curve. Hence, we have

$$\eta(E_1) = \cos\theta = constant,$$

which is equivalent to

$$\alpha'_{2n+1} = 2\cos\theta + \sum_{i=1}^{n} \alpha'_i \alpha_{n+i}. \tag{4.2}$$

From the fact that α is parametrized by arc-length, we also have

$$g(E_1, E_1) = 1,$$

that is,

$$\sum_{i=1}^{2n} (\alpha'_i)^2 = 4\sin^2\theta. \tag{4.3}$$

Differentiating E_1 with respect to $\widehat{\nabla}$, we obtain

$$\widehat{\nabla}_{E_1} E_1 = \frac{1}{2}\left(\sum_{i=1}^{n} \alpha''_{n+i} X_i + \sum_{i=1}^{n} \alpha''_i X_{n+i} \right).$$

We also easily find

$$\phi E_1 = \frac{1}{2}\left(-\sum_{i=1}^{n} \alpha'_i X_i + \sum_{i=1}^{n} \alpha'_{n+i} X_{n+i} \right).$$

Since α is a pseudo-Hermitian magnetic curve, it must satisfy

$$\widehat{\nabla}_{E_1} E_1 = (-q + 2\cos\theta)\, \phi E_1.$$

Then, we can write

$$\frac{\alpha''_{n+1}}{-\alpha'_1} = ... = \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_1}{\alpha'_{n+1}} = ... = \frac{\alpha''_n}{\alpha'_{2n}} = -\lambda,$$

where $\lambda = q - 2\cos\theta$. From the last equations, we can select the pairs

$$\frac{\alpha''_{n+1}}{-\alpha'_1} = \frac{\alpha''_1}{\alpha'_{n+1}},\ ...\ ,\ \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_n}{\alpha'_{2n}}. \tag{4.4}$$

Firstly, let $\lambda \neq 0$. Solving the ODEs, we have

$$(\alpha'_i)^2 + \left(\alpha'_{n+i}\right)^2 = c_i^2, i = 1, ..., n$$

for some arbitrary constants c_i $(i = 1, ..., n)$ such that

$$\sum_{i=1}^{n} c_i^2 = 4\sin^2\theta.$$

So we have

$$\alpha_i' = c_i \cos f_i,\ \alpha_{n+i}' = c_i \sin f_i$$

for some differentiable functions $f_i : I \to \mathbb{R}$ $(i = 1, ..., n)$. From (4.4), we get

$$\frac{\alpha_{n+i}''}{-\alpha_i'} = -f_i' = -\lambda,$$

which gives us

$$f_i = \lambda t + d_i$$

for some arbitrary constants d_i $(i = 1, ..., n)$. Here, t denotes the arc-length parameter. Then, we find

$$\alpha_i' = c_i \cos(\lambda t + d_i),\ \alpha_{n+i}' = c_i \sin(\lambda t + d_i).$$

Finally, we obtain

$$\alpha_i = \frac{c_i}{\lambda} \sin(\lambda t + d_i) + h_i,$$

$$\alpha_{n+i} = \frac{-c_i}{\lambda} \cos(\lambda t + d_i) + h_{n+i},$$

$$\begin{aligned} \alpha_{2n+1} &= 2t\cos\theta + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[2(\lambda t + d_i) + \sin(2(\lambda t + d_i))\right] \right. \\ &\qquad \left. + \frac{c_i h_{n+i}}{\lambda} \sin(\lambda t + d_i) \right\} + h_{2n+1} \end{aligned}$$

for some arbitrary constants h_i $(i = 1, ..., 2n + 1)$.

Secondly, let $\lambda = 0$. In this case, $q = 2\cos\theta$ and $\widehat{k}_1 = 0$. Hence, we have

$$\widehat{\nabla}_{E_1} E_1 = \frac{1}{2} \left(\sum_{i=1}^{n} \alpha_{n+i}'' X_i + \sum_{i=1}^{n} \alpha_i'' X_{n+i} \right) = 0,$$

which gives us

$$\alpha_i = c_i t + d_i,\ i = 1, ..., 2n,$$

$$\alpha_{2n+1} = 2t\cos\theta + \sum_{i=1}^{n} c_i \left(\frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + c_{2n+1},$$

where c_i $(i = 1, 2, ..., 2n + 1)$ and d_i $(i = 1, 2, ..., 2n)$ are arbitrary constants such that

$$\sum_{i=1}^{2n} c_i^2 = 4\sin^2\theta.$$

To conclude, we can state the following theorem:

Theorem 4.1. *The pseudo-Hermitian magnetic curves on* $\mathbb{R}^{2n+1}(-3)$ *endowed with the Tanaka-Webster connection have the parametric equations*

$$\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^{2n+1}(-3), \alpha = (\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}, ..., \alpha_{2n}, \alpha_{2n+1}),$$

where α_i $(i = 1, ..., 2n+1)$ *satisfies either*

(a)

$$\alpha_i = \frac{c_i}{\lambda} \sin(\lambda t + d_i) + h_i,$$

$$\alpha_{n+i} = \frac{-c_i}{\lambda} \cos(\lambda t + d_i) + h_{n+i},$$

$$\begin{aligned} \alpha_{2n+1} &= 2\cos\theta t + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[2(\lambda t + d_i) + \sin(2(\lambda t + d_i))\right] \right. \\ &\qquad \left. + \frac{c_i h_{n+i}}{\lambda} \sin(\lambda t + d_i) \right\} + h_{2n+1}, \end{aligned}$$

where $\lambda = q - 2\cos\theta \neq 0$, c_i, d_i $(i = 1, ..., n)$ *and* h_i $(i = 1, ..., 2n+1)$ *are arbitrary constants such that*

$$\sum_{i=1}^{n} c_i^2 = 4\sin^2\theta;$$

or

(b)

$$\alpha_i = c_i t + d_i,$$

$$\alpha_{2n+1} = 2t\cos\theta + \sum_{i=1}^{n} c_i \left(\frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + c_{2n+1},$$

where $q = 2\cos\theta$ *and* c_i $(i = 1, 2, ..., 2n+1)$, d_i $(i = 1, 2, ..., 2n)$ *are arbitrary constants such that*

$$q^2 + \sum_{i=1}^{2n} c_i^2 = 4.$$

REFERENCES

1. T. Adachi: *Curvature bound and trajectories for magnetic fields on a Hadamard surface*. Tsukuba J. Math. **20** (1996), 225–230.
2. D. E. Blair: *Riemannian geometry of contact and symplectic manifolds, Second edition*. Birkhauser Boston, Inc., Boston, MA, 2010.
3. M. Barros, A. Romero, J. L. Cabrerizo and M. Fernández: *The Gauss-Landau-Hall problem on Riemannian surfaces*. J. Math. Phys. **46** (2005), 112905 15 pp.

4. J. L. Cabrerizo, M. Fernandez and J. S. Gomez: *On the existence of almost contact structure and the contact magnetic field*. Acta Math. Hungar. **125** (2009), 191–199.

5. G. Calvaruso, M. I. Munteanu and A. Perrone: *Killing magnetic curves in three-dimensional almost paracontact manifolds*. J. Math. Anal. Appl. **426** (2015), 423–439.

6. J. T. Cho, J. Inoguchi and J. E. Lee: *On slant curves in Sasakian 3-manifolds*. Bull. Austral. Math. Soc. **74** (2006), 359–367.

7. J. T. Cho and J. E. Lee: *Slant curves in contact pseudo-Hermitian 3-manifolds*. Bull. Austral. Math. Soc. **78** (2008), 383–396.

8. A. Comtet: *On the Landau levels on the hyperbolic plane*. Ann. Physics **173** (1987), 185–209.

9. S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor: *Magnetic curves in Sasakian manifolds*. J. Nonlinear Math. Phys. **22** (2015), 428–447.

10. S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor: *Magnetic curves in cosymplectic manifolds*. Rep. Math. Phys. **78** (2016), 33–48.

11. Ş. Güvenç and C. Özgür: *On slant magnetic curves in S-manifolds*. J. Nonlinear Math. Phys. **26** (2019), 536–554.

12. Ş. Güvenç: *On pseudo-Hermitian biharmonic slant curves in Sasakian space forms endowed with the Tanaka–Webster connection*. Bull. Iran. Math. Soc. **46** (2020), 207–223.

13. J. Inoguchi and M. I. Munteanu: *Periodic magnetic curves in Berger spheres*. Tohoku Math. J. **69** (2017), 113–128.

14. M. Jleli, M. I. Munteanu and A. I. Nistor: *Magnetic trajectories in an almost contact metric manifold* $\mathbb{R}^{2N+1}$. Results Math. **67** (2015), 125–134.

15. M. Jleli and M. I. Munteanu: *Magnetic curves on flat para-Kahler manifolds*. Turkish J. Math. **39** (2015), 963–969.

16. L. D. Landau and E. M. Lifshitz: *Course of theoretical physics*. Vol. 1. Mechanics. Third edition. Pergamon Press, Oxford-New York-Toronto, Ont., 1976.

17. M. I. Munteanu and A. I. Nistor: *The classification of Killing magnetic curves in* $S^2 \times R$. J. Geom. Phys. **62** (2012), 170–182.

18. M. I. Munteanu and A. I. Nistor: *A note on magnetic curves on* S^{2n+1}. C. R. Math. Acad. Sci. Paris **352** (2014), 447–449.

19. M. I. Munteanu: *The Landau-Hall problem on canal surfaces*. J. Math. Anal. Appl. **414** (2014), 725–733.

20. C. Özgür and Ş. Güvenç: *On some types of slant curves in contact pseudo-Hermitian 3-manifolds*. Ann. Polon. Math. **104** (2012), 217–228.

21. C. Özgür: *On magnetic curves in the 3-dimensional Heisenberg group*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **43** (2017), 278–286.

22. N. Tanaka: *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*. Japan. J. Math. (N.S.) **2** (1976), 131–190.

23. S. Tanno: *Variational problems on contact Riemannian manifolds*. Trans. Amer. Math. Soc. **314** (1989), 349–379.

24. S. M. Webster: *Pseudo-Hermitian structures on a real hypersurface*. J. Differential Geom. **13** (1978), 25–41.

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