FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 165-177
https://doi.org/10.22190/FUMI200421014K
Original Scientific Paper

(ω, c)- PSEUDO ALMOST PERIODIC FUNCTIONS, (ω, c)- PSEUDO ALMOST AUTOMORPHIC FUNCTIONS AND APPLICATIONS

Mohammed Taha Khalladi[1], Marko Kostić[2], Abdelkader Rahmani[3] and Daniel Velinov[4]

[1]**Department of Mathematics and Computer Sciences, University of Adrar, Adrar, Algeria**
[2]**Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6 73, 21125 Novi Sad, Serbia**
[3]**Laboratory of Mathematics, Modeling and Applications (LaMMA), University of Adrar, Adrar, Algeria**
[4]**Faculty of Civil Engineering, Ss. Cyril and Methodius University, Partizanski Odredi,24, P.O. box 560, 1000 Skopje, North Macedonia**

Abstract. In this paper, we introduce the classes of (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions. These collections include (ω, c)-pseudo periodic functions, pseudo almost periodic functions and their automorphic analogues. We present an application to the abstract semilinear first-order Cauchy inclusions in Banach spaces.
Keywords: (ω, c)-Pseudo almost periodic functions, (ω, c)-pseudo almost automorphic functions, abstract semilinear Cauchy inclusions

1. Introduction and Preliminaries

The theory of almost periodic functions and almost automorphic functions is an attractive field of investigation, which has a significant role in the qualitative theory of ordinary and partial differential equations, physics, mathematical biology and control theory.

The classes of (ω, c)-periodic functions and (ω, c)-pseudo periodic functions were introduced by Alvarez, Gómez, Pinto in [3] and Alvarez, Castillo, Pinto in [4],

Received April 21, 2020; accepted June 21, 2020.
Corresponding Author: Marko Kostić, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6 73, 21125 Novi Sad, Serbia | E-mail: marco.s@verat.net
2010 *Mathematics Subject Classification.* Primary 34C25; Secondary 42A75, 43A60

motivated by some known results regarding the qualitative properties of solutions to the Mathieu linear second-order differential equation

$$y''(t) + [a - 2q\cos 2t]y(t) = 0,$$

arising in seasonally forced population dynamics. The authors of [3] have analyzed the existence and uniqueness of mild (ω, c)-periodic solutions to the abstract semilinear integro-differential equation

$$D_{t,+}^{\gamma}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \quad t \in \mathbb{R},$$

where A is a closed linear operator, $a \in L^1([0,\infty))$ is a scalar-valued kernel and $f(\cdot,\cdot)$ satisfies some Lipschitz type conditions. Further on, Alvarez, Castillo and Pinto have analyzed in [4] the existence and uniqueness of mild (ω, c)-pseudo periodic solutions to the abstract semilinear differential equation of the first order:

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A generates a strongly continuous semigroup. The authors have proved the existence of positive (ω, c)-pseudo periodic solutions to the Lasota-Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0.$$

This equation describes the survival of red blood cells in the blood of an animal. (ω, c)-Pseudo periodic functions can be also solutions of the time varying impulsive differential equations and the linear delayed equations; for further information about applications of (ω, c)-pseudo periodic functions, we refer the reader to [8] and references cited therein.

In our recent paper [8], we have introduced and analyzed various generalizations of the concept of (ω, c)-periodicity. Among others, we have defined and analyzed the classes of (asymptotically) (ω, c)-almost periodic functions and (asymptotically) (ω, c)-almost automorphic functions. The main aim of this paper is to analyze the classes of (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions by taking into consideration the class of pseudo ergodic components introduced by C. Zhang [13]. We introduce two new types of (ω, c)-pseudo ergodic components and two new classes of (ω, c)-almost periodic ((ω, c)-almost automorphic) functions. It is our strong belief that these classes of functions will attract the attention of our readers and serve for some new applications in the theory of abstract differential equations soon.

The organization of paper is briefly described as follows. After recalling the basic definitions from the theory of almost periodic functions and almost automorphic functions in Subsection 1.2, we introduce the classes of (ω, c, i)-almost periodic functions, resp. (ω, c, i)-almost automorphic functions, and (ω, c, i)-pseudo ergodic vanishing components in Definition 2.2 and Definition 2.3; in Definition

2.4, we introduce the notion of an (ω, c)-pseudo almost periodic function, resp. an (ω, c)-pseudo almost automorphic function, and the notion of a two-parameter (ω, c, i)-pseudo almost periodic function, resp. two-parameter (ω, c, i)-pseudo almost automorphic function $(i = 1, 2)$. After that, we clarify some basic results about the class of (ω, c)-pseudo almost periodic functions, resp. (ω, c)-pseudo almost automorphic functions, depending on one variable. Subsection 2.1 investigates composition principles for introduced classes and Section 3 provides an interesting application in the qualitative analysis of (ω, c)-pseudo almost periodic solutions of the abstract semilinear Cauchy inclusions of the first order.

We use the standard notation throughout the paper. Let $I = \mathbb{R}$ or $I = [0, \infty)$; unless stated otherwise, we will always assume henceforth that $f : I \to E$ is a continuous function. By $C(I : E)$, $C_b(I : E)$ and $C_0(I : E)$ we denote the vector spaces consisting of all continuous functions $f : I \to E$, all bounded continuous functions $f : I \to E$ and all bounded continuous functions $f : I \to E$ satisfying that $\lim_{|t| \to +\infty} \|f(t)\| = 0$. As is well known, $C_b(I : E)$ and $C_0(I : E)$ are Banach spaces equipped with the sup-norm, denoted by $\| \cdot \|_\infty$. If X is also a complex Banach space, then by $L(E, X)$ we denote the space consisting of all bounded continuous mappings from E into X; $L(E) \equiv L(E, E)$. The principal branches are always used for taking the powers of complex numbers.

1.1. Almost Periodic Functions, Almost Automorphic Functions and Their Generalizations

Let $I = [0, \infty)$ or $I = \mathbb{R}$. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t + \tau) - f(t)\| \leq \epsilon, \quad t \in I$. The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I : E)$.

Let $f : \mathbb{R} \to E$ be continuous. Then it is said that $f(\cdot)$ is almost automorphic if and only if for every real sequence (b_n) there exists a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to E$ such that $\lim_{n \to \infty} f(t + a_n) = g(t)$ and $\lim_{n \to \infty} g(t - a_n) = f(t)$, pointwise for $t \in \mathbb{R}$. The space consisting of all almost automorphic functions will be denoted by $AA(\mathbb{R} : E)$.

A function $f : I \times X \to E$ is called almost periodic if and only if $f(\cdot, \cdot)$ is bounded continuous as well as for every $\epsilon > 0$ and every compact $K \subseteq X$ there exists $l(\epsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\epsilon, K)$ contains a number τ with the property that $\|f(t + \tau, x) - f(t, x)\| \leq \epsilon$ for all $t \in I$, $x \in K$. The collection of such functions will be denoted by $AP(I \times X : E)$. Observe that we require the boundedness of function $f(\cdot, \cdot)$ a priori, which is not the common case in the existing literature. This is also not the case in the usual definition of an almost automorphic function depending on two variables, given as follows. A continuous function $F : \mathbb{R} \times X \to E$ is said to be almost automorphic if and only if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that $G(t,x) := \lim_{n\to\infty} F(t+s_n,x)$ is well defined for each $t \in \mathbb{R}$ and $x \in X$, and $\lim_{n\to\infty} G(t-s_n,x) = F(t,x)$ for each $t \in \mathbb{R}$ and $x \in X$. The vector space consisting of such functions will be denoted by $AA(\mathbb{R} \times X : E)$.

By $PAP_0(\mathbb{R} : E)$ we denote the space consisting of all bounded continuous functions $\Phi : \mathbb{R} \to E$ such that $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s)\|\, ds = 0$. For example, it is well known that $f \in PAP_0(\mathbb{R} : \mathbb{C})$ if and only if $f \cdot f \in PAP_0(\mathbb{R} : \mathbb{C})$. Moreover, let us define

$$f(t) := \frac{1}{2t}\int_{-t}^{t} s|\sin s|^{s^N}\, ds, \quad t \in \mathbb{R},$$

where $N > 6$. From [1, Example p. 1143] we know that $\lim_{t\to+\infty} f(t) = 0$ and therefore $\cdot|\sin\cdot|^{\cdot^N} \in PAP_0(\mathbb{R} : \mathbb{C})$ for $N > 6$.

By $PAP_0(\mathbb{R} \times X : E)$ we denote the space consisting of all continuous functions $\Phi : \mathbb{R} \times X \to E$ such that $\{\Phi(t,x) : t \in \mathbb{R}\}$ is bounded for all $x \in X$, and $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s,x)\|\, ds = 0$, uniformly in bounded sets of X. A function $f \in C_b(\mathbb{R} : X)$ is said to be pseudo-almost periodic, resp. pseudo-almost automorphic, if and only if it admits a decomposition $f(t) = g(t)+q(t)$, $t \in \mathbb{R}$, where $g \in AP(\mathbb{R} : E)$, resp. $g \in AA(\mathbb{R} : E)$, and $q \in PAP_0(\mathbb{R} : E)$. The parts $g(\cdot)$ and $q(\cdot)$ are called the almost periodic part of $f(\cdot)$, resp. the almost automorphic part of $f(\cdot)$, and the ergodic perturbation of $f(\cdot)$. The vector space consisting of such functions is denoted by $PAP(\mathbb{R} : E)$, resp. $PAA(\mathbb{R} : E)$; the sup-norm turns $PAP(\mathbb{R} : E)$, resp. $PAA(\mathbb{R} : E)$, into a Banach space ([13]).

For more details about almost periodic type functions and almost automorphic type functions, we refer the reader to the research monographs [5, 6, 7, 9, 12].

2. (ω, c)-Pseudo Almost Periodic Functions and (ω, c)-Pseudo Almost Automorphic Functions

Unless specified otherwise, in the remainder of paper we will always assume that $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. The following definition has been recently introduced in [8].

Definition 2.1. It is said that a continuous function $f : I \to E$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, if and only if the function $f_{\omega,c}(\cdot)$, defined by $f_{\omega,c}(t) := c^{-(t/\omega)}f(t)$, $t \in I$, is almost periodic, resp. almost automorphic. By $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, we denote the space consisting of all (ω, c)-almost periodic functions, resp. all (ω, c)-almost automorphic functions.

Let us recall that $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, is a vector space with the usual operations of addition of functions and pointwise multiplication of functions with scalars ([8]). Furthermore, the space $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, equipped with the norm $\|\cdot\|_{\omega,c}$, where

$$\|f\|_{\omega,c} := \sup_{t\in I}\left\|c^{-\frac{t}{\omega}} f(t)\right\|,$$

is a Banach space.

With the exception of consideration preceding Definition 2.3, in the remainder of paper we will deal with the interval $I = \mathbb{R}$, only. Let us recall the (ω, c)-mean of a function $h : \mathbb{R} \to E$ is introduced in [4] by

$$\mathcal{M}_{\omega,c}(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c^{-\sigma/\omega} h(\sigma)\, d\sigma,$$

whenever the limit exists. For example, for $h_1(t) = c^{t/\omega}$ and $h_2(t) = c^{t/\omega} e^{it}$, we have that $\mathcal{M}_{\omega,c}(h_1) = 1$ and $\mathcal{M}_{\omega,c}(h_2) = 0$. Furthermore, $\mathcal{M}_{\omega,c}$ is a linear and continuous operator. Indeed, if $c^{-t/\omega} h_n(t) \to c^{-t/\omega} h(t)$ uniformly as $n \to \infty$, then $\mathcal{M}_{\omega,c}(h_n) \to \mathcal{M}_{\omega,c}(h)$ as $n \to \infty$.

Remark 2.1. If $h(\cdot)$ is (ω, c)-almost periodic in the sense of Definition 2.1, then the mean $\mathcal{M}_{\omega,c}(h)$ always exists, because the function $c^{-(\cdot/\omega)} f(\cdot)$ is almost periodic and the usual mean value of any almost periodic function exists.

In this paper, we will use the space

$$PAP_{0;\omega,c}(\mathbb{R} : E) := \big\{ h \in C(\mathbb{R} : E)\ ;\ c^{-\cdot/\omega} h(\cdot) \in PAP_0(\mathbb{R} : E) \big\}.$$

A function $h(\cdot)$ is said to be c-ergodic if and only if belongs to this space. Therefore, the ergodic space of Zhang ([13]) can be recovered by plugging $c = 1$ in the above definition.

Furthermore, we will use the following two types of (ω, c)-pseudo ergodic components:

Definition 2.2. Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$.

(i) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 1)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, \cdot) \in PAP_0(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,1}(\mathbb{R} \times X : E)$.

(ii) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 2)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in PAP_0(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,2}(\mathbb{R} \times X : E)$.

Similarly, we will use two different types of (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions, depending on two variables (albeit some composition principles for two-parameter (ω, c)-almost periodic functions have been clarified in [8], we have not explicitly defined the notion of a two-parameter (ω, c)-almost periodic function there; the notion introduced in Definition 2.3 should not be mistakenly identified with the notion of an (ω, c)-almost periodic function of type 1 (type 2), introduced and analyzed in [8, Section 3]).

Definition 2.3. Let $c \in \mathbb{C}\setminus\{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 1)$-almost periodic, resp. $(\omega, c, 1)$-almost automorphic, if and only if $c^{-t/\omega} f(t, \cdot) \in AP(\mathbb{R} \times X : E)$, resp. $c^{-t/\omega} f(t, \cdot) \in AA(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $AP_{\omega,c,1}(\mathbb{R} \times X : E)$, resp. $AA_{\omega,c,1}(\mathbb{R} \times X : E)$.

(ii) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 2)$-almost periodic, resp. $(\omega, c, 2)$-almost automorphic, if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AP(\mathbb{R} \times X : E)$, resp. $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AA(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $AP_{\omega,c,2}(\mathbb{R} \times X : E)$, resp. $AA_{\omega,c,2}(\mathbb{R} \times X : E)$.

In [8], we have analyzed the classes of asymptotically (ω, c)-almost periodic functions, resp. asymptotically (ω, c)-almost automorphic functions, defined on the non-negative real axis by adding the usual ergodic components from the space $C_0([0, \infty) : E)$ to the principal components, which are (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions. In order to stay consistent with the notion introduced in [4, Definition 2.5], we will slightly change the approach obeyed in [8] and use the following notion in case $I = \mathbb{R}$:

Definition 2.4. Let $c \in \mathbb{C} \backslash \{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} : E)$ is said to be (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if it admits a decomposition $f(t) = g(t) + h(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, and $h \in PAP_{0;\omega,c}(\mathbb{R} : E)$. The space of all such functions will be denoted by $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$.

(ii) A function $f(\cdot, \cdot) \in C(\mathbb{R} \times X : E)$ is said to be (ω, c, i)-pseudo almost periodic, resp. (ω, c, i)-pseudo almost automorphic, if and only if it admits a decomposition $f(t, x) = g(t, x) + h(t, x)$, $t \in \mathbb{R}$, $x \in X$, where $g(\cdot, \cdot)$ is (ω, c, i)-almost periodic, resp. (ω, c, i)-almost automorphic, and $h(\cdot, \cdot) \in PAP_{0;\omega,i}(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{\omega,c,i}(\mathbb{R} \times X : E)$, resp. $PAA_{\omega,c,i}(\mathbb{R} \times X : E)$.

For simplicity, we will not consider here the class of (ω, c)-pseudo compactly almost automorphic functions; for some applications of compactly almost automorphic functions, the reader may consult the article [2] by Ait Dads, Boudchich, Es-sebbar and references cited therein.

Theorem 2.1. *Let $f \in C(\mathbb{R} : E)$. Then $f(\cdot)$ is (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if:*

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega}, \ u \in PAP(\mathbb{R} : E), \tag{2.1}$$

resp.

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega}, \ u \in PAA(\mathbb{R} : E).$$

Proof. We will consider only (ω, c)-pseudo almost periodic functions for simplicity. It is clear that if $f(\cdot)$ satisfies (2.1), then $f(\cdot)$ is an (ω, c)-pseudo almost periodic function. In order to show the converse statement, let $f \in PAP_{\omega,c}(\mathbb{R} : E)$. Then there exists $g \in AP_{\omega,c}(\mathbb{R} : E)$ and $PAP_{0;\omega,c}(\mathbb{R} : E)$ such that $f = g + h$. Therefore,

$$u(t) = c^{-t/\omega}g(t) + c^{-t/\omega}h(t) = F_1(t) + F_2(t), \quad t \in \mathbb{R}.$$

So, $u(t)$ is written as a sum of $F_1(\cdot)$ which is almost periodic and $F_2(\cdot)$ which belongs to $PAP_{0;\omega,c}(\mathbb{R} : E)$. $\square$

Remark 2.2. Let us note that the decompositions given in Definition 2.4 are unique; see also [4, Remark 2.9]. The proof of this simple fact can be left to the interested readers.

It can be simply shown that:

(i) We have $f + g \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f + g \in PAA_{\omega,c}(\mathbb{R} : E)$, and $\alpha h \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $\alpha h \in PAA_{\omega,c}(\mathbb{R} : E)$, provided $f,\ g,\ h \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f,\ g,\ h \in PAA_{\omega,c}(\mathbb{R} : E)$, and $\alpha \in \mathbb{C}$.

(ii) If $\tau \in \mathbb{R}$ and $f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : E)$, then $f_\tau(\cdot) \equiv f(\cdot + \tau) \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f_\tau(\cdot) \in PAA_{\omega,c}(\mathbb{R} : E)$.

Now we would like to endow the introduced space of (ω, c)-pseudo almost periodic functions, resp. (ω, c)-pseudo almost automorphic functions, with a certain norm.

Proposition 2.1. *The space $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$, equipped with the norm $\| \cdot \|_{\omega,c}$ is a Banach space.*

Proof. We will consider the space $PAP_{\omega,c}(\mathbb{R} : E)$, only. Let (f_n) be a Cauchy sequence in $PAP_{\omega,c}(\mathbb{R} : E)$. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m,\ n \geq N$, we have

$$\|f_n - f_m\|_{\omega,c} < \epsilon.$$

Since $f_m,\ f_n \in PAP_{\omega,c}(\mathbb{R} : E)$, Theorem 2.1 implies that there exists $u_m,\ u_n \in PAP(\mathbb{R} : E)$ such that $f_m(t) \equiv c^\wedge(t)u_m(t)$ and $f_n(t) \equiv c^\wedge(t)u_n(t)$ for all $t \in \mathbb{R}$. Now, for $m,\ n \geq N$ we have $\|u_m - u_n\|_\infty \leq \|f_n - f_m\|_{\omega,c} < \epsilon$. It follows that (u_n) is a Cauchy sequence in $PAP(\mathbb{R} : E)$. Since $PAP(\mathbb{R} : E)$ is complete, there exists $u \in PAP(\mathbb{R} : E)$ such that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t)u(t)$, $t \in \mathbb{R}$. We claim that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Indeed, $\|f_n - f\|_{\omega,c} = \sup_{t\in\mathbb{R}} \|u_n(t) - u(t)\| \to 0 \quad (n \to \infty)$. Hence, $PAP_{\omega,c}(\mathbb{R} : E)$ is a Banach space with the norm $\| \cdot \|_{\omega,c}$. $\square$

Lemma 2.1. ([4]) *Assume that $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$. Then $h \in PAP_{0;\omega,c}(\mathbb{R} : E)$ implies that $k * h \in PAP_{0;\omega,c}(\mathbb{R} : E)$.*

Theorem 2.2. *Let $f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : E)$, with $f(\cdot) = c^{\wedge}(\cdot)p(\cdot)$, $p \in PAP(\mathbb{R} : E)$, resp. $p \in PAA(\mathbb{R} : E)$. If for some $k(\cdot)$ we have that $k^{\sim}(\cdot) := c^{\wedge}(-\cdot)k(\cdot) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds = c^{\wedge}(t)\,(k^{\sim} * p)\,(t), \quad t \in \mathbb{R}.$$

In particular, $k * f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $k * f \in PAA_{\omega,c}(\mathbb{R} : E)$.

Proof. As before, we will consider the space $PAP_{\omega,c}(\mathbb{R} : E)$ only, because the proof is quite analogous for the space $PAA_{\omega,c}(\mathbb{R} : E)$. Since $p \in PAP(\mathbb{R} : E)$, we have that there exists $p_1 \in AP(\mathbb{R} : E)$ and $p_2 \in PAP_0(\mathbb{R} : E)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$, where $f_1(\cdot) = c^{\wedge}(\cdot)p_1(\cdot) \in AP_{\omega,c}(\mathbb{R} : E)$ and $f_2(\cdot) = c^{\wedge}(\cdot)p_1(\cdot) \in PAP_{0;\omega,c}(\mathbb{R} : E)$. For every $t \in \mathbb{R}$, we have

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds$$

$$= \int_{-\infty}^{\infty} k(t-s)f_1(s)\,ds + \int_{-\infty}^{\infty} k(t-s)f_2(s)\,ds$$

$$= (k * f_1)\,(t) + (k * f_2)\,(t) =: I_1(t) + I_2(t).$$

We have that $I_1 \in AP_{\omega,c}(\mathbb{R} : E)$; see [8]. Next, by Lemma 2.1, we have that $I_2 \in PAP_{0;\omega,c}(\mathbb{R} : E)$. Moreover, by definition of $f(\cdot)$, we have $(k * f)(\cdot) = c^{\wedge}(\cdot)\,(k^{\sim} * p)\,(\cdot)$ so that $k * f \in PAP_{\omega,c}(\mathbb{R} : E)$. □

Example 2.1. ([12]) Let us consider the heat equation $u_t(x,t) = u_{xx}(x,t)$, $t > 0$, $x \in \mathbb{R}$, with the initial value condition $u(x,0) = f(x)$. Let $u(x,t)$ be a regular solution satisfying the initial value condition. It is well known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\,ds, \quad t > 0,\, x \in \mathbb{R}.$$

Fix $t_0 > 0$ and assume that $f(\cdot)$ is an (ω,c)-pseudo almost periodic function. Then, by Theorem 2.2, the solution $u(x,t_0)$ is (ω,c)-pseudo almost periodic with respect to x.

2.1. Composition principles

In this subsection, we will use two lemmae. The first one is a slight extension of the well known result of H.-X. Li, F.-L. Huang and J.-Y. Li [10, Theorem 2.1], clarified recently in [9, Lemma 2.12.2]:

Lemma 3.1. *Let $f \in PAP(\mathbb{R} \times X : E)$ and $u \in PAP(\mathbb{R} : X)$. Then the mapping $t \mapsto f(t,u(t))$, $t \in \mathbb{R}$ belongs to the space $PAP(\mathbb{R} : E)$ provided that the following conditions hold:*

(i) *The set* $\{f(t,x) : t \in \mathbb{R},\ x \in B\}$ *is bounded for every bounded subset* $B \subseteq X$.

(ii) $f(t,x)$ *is uniformly continuous in each bounded subset of* X *uniformly in* $t \in \mathbb{R}$. *That is, for any* $\epsilon > 0$ *and* $B \subseteq X$ *bounded, there exists* $\delta > 0$ *such that* $x,\ y \in B$ *and* $\|x - y\| \leq \delta$ *imply* $\|f(t,x) - f(t,y)\| \leq \epsilon$ *for all* $t \in \mathbb{R}$.

The second lemma is the following slight extension of the composition principle established by J. Liang et al. in [11, Theorem 2.4]:

Lemma 3.2. (see [9, Theorem 3.2.4]) *Suppose that* $f = g + \phi \in PAA(\mathbb{R} \times X : E)$ *with* $g \in AA(\mathbb{R} \times X : E)$, $\phi \in PAP_0(\mathbb{R} \times X : E)$ *and the following holds:*

(i) *the mapping* $(t,x) \mapsto g(t,x)$ *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

(ii) *the mapping* $(t,x) \mapsto \phi(t,x)$ *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

Then for each $u \in PAA(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA(\mathbb{R} : E)$.

For simplicity, we will not consider Stepanov p-almost periodic functions and Stepanov p-almost automorphic functions depending on two variables here (see [8, Section 3] for some composition principles for Stepanov (p, ω, c)-almost periodic functions).

Suppose now that a continuous function $g : \mathbb{R} \times X \to E$ satisfies $g(t + \omega, x) = cg(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$, resp. $g(t+\omega, cx) = cg(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$. Define the functions

$$G_1(t,x) := c^{-\frac{t}{\omega}} g(t,x), \quad t \in \mathbb{R},\ x \in X \tag{2.2}$$

and

$$G_2(t,x) := c^{-\frac{t}{\omega}} g\big(t, c^{t/\omega} x\big), \quad t \in \mathbb{R},\ x \in X. \tag{2.3}$$

Then, for every $t \in \mathbb{R}$ and $x \in X$, we have

$$G_1(t+\omega, x) = c^{-\frac{t+\omega}{\omega}} g(t+\omega, x) = c^{-\frac{t+\omega}{\omega}} cg(t+\omega, x) = c^{-\frac{t}{\omega}} g(t,x) = G_1(t,x)$$

and

$$G_2(t+\omega, x) = c^{-\frac{t+\omega}{\omega}} g\Big(t+\omega, c^{\frac{t+\omega}{\omega}} x\Big) = c^{-\frac{t+\omega}{\omega}} cg\big(t, c^{t/\omega} x\big)$$
$$= c^{-t/\omega} g\big(t, c^{t/\omega} x\big) = G_2(t,x).$$

In both cases, the function $G_i(\cdot,\cdot)$ is ω-periodic in time variable ($i = 1, 2$). Furthermore, if the requirements of [4, Theorem 2.24] hold (case $i = 2$), then condition (i) of Lemma 3.2 holds with the function $g(\cdot,\cdot)$ replaced therein with the function $G_2(\cdot,\cdot)$, and condition (ii) of Lemma 3.2 holds with the function $\phi(\cdot,\cdot)$ replaced therein with the function $h_2(t,\cdot) \equiv c^{-t/\omega} h(t, c^{t/\omega} \cdot)$, $t \in \mathbb{R}$. Furthermore, $G_2 \in AA(\mathbb{R} \times X : E)$ and

$h_2 \in PAP_0(\mathbb{R} \times X : E)$ so that repeating verbatim the arguments used in the proof of [11, Theorem 2.4] with appealing to [3, Theorem 2.11] in place of [11, Lemma 2.2] immediately yields a much simpler proof of [4, Theorem 2.24]. Furthermore, the statement of [3, Theorem 2.11] can be formulated for continuous functions which maps the space $\mathbb{R} \times X$ into E; in other words, we can use two different pivot spaces X and E. Keeping in mind this observation, we can immediately clarify an extension of [4, Theorem 2.24] in this context (the interested reader may try to reexamine [4, Theorem 2.25] for (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions). Furthermore, using Lemma 3.2 we can immediately clarify the following result:

Proposition 3.1.

(i) *Suppose that* $f = g+\phi$ *with* $g \in AA_{\omega,c,1}(\mathbb{R}\times X : E)$, $\phi \in PAP_{0;\omega,c,1}(\mathbb{R}\times X : E)$ *and the following holds:*

  (a) *the mapping* $(t, x) \mapsto G_1(t, x)$ *given by (2.2) is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

  (b) *the mapping* $(t, x) \mapsto \phi_1(t, x)$ *given by (2.2), with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in PAA(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : E)$.

(ii) *Suppose that* $f = g+\phi$ *with* $g \in AA_{\omega,c,2}(\mathbb{R}\times X : E)$, $\phi \in PAP_{0;\omega,c,2}(\mathbb{R}\times X : E)$ *and the following holds:*

  (c) *the mapping* $(t, x) \mapsto G_2(t, x)$ *given by (2.2) is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

  (d) *the mapping* $(t, x) \mapsto \phi_2(t, x)$ *given by (2.2), with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in PAA_{\omega,c}(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : E)$.

Concerning possible applications of Lemma 3.1, we can immediately clarify the following result:

Proposition 3.2.

(i) *Let* $f \in PAP_{\omega,c,1}(\mathbb{R} \times X : E)$ *and* $u \in PAP(\mathbb{R} : X)$. *Then the mapping* $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ *belongs to the space* $PAP_{\omega,c}(\mathbb{R} : E)$ *provided that the following conditions hold:*

  (a) *The set* $\{c^{-t/\omega} f(t, x) : t \in \mathbb{R},\ x \in B\}$ *is bounded for every bounded subset* $B \subseteq X$.

(b) $c^{-t/\omega} f(t,x)$ *is uniformly continuous in each bounded subset of* X *uniformly in* $t \in \mathbb{R}$.

(ii) *Let* $f \in PAP_{\omega,c,2}(\mathbb{R} \times X : E)$ *and* $u \in PAP_{\omega,c}(\mathbb{R} : X)$. *Then the mapping* $t \mapsto f(t,u(t))$, $t \in \mathbb{R}$ *belongs to the space* $PAP_{\omega,c}(\mathbb{R} : E)$ *provided that the following conditions hold:*

(a) *The set* $\{c^{-t/\omega} f(t, c^{t/\omega} x) : t \in \mathbb{R},\ x \in B\}$ *is bounded for every bounded subset* $B \subseteq X$.

(b) $c^{-t/\omega} f(t, c^{t/\omega} x)$ *is uniformly continuous in each bounded subset of* X *uniformly in* $t \in \mathbb{R}$.

3. An Application to the Abstract Semilinear Cauchy Inclusions in Banach Spaces

Consider the semilinear fractional Cauchy inclusion

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t,u(t)),\ t \in \mathbb{R}, \tag{3.1}$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to E$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator in E satisfying the condition

(P) There exists finite constants $a,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -a\big(|\operatorname{Im}\lambda| + 1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

Then there exists a finite constant $M_0 > 0$ such that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(E)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leq M_0 e^{-at} t^{\beta-1}$, $t > 0$; cf. [9] for more details. By a mild solution of problem (3.1), we mean any continuous function $t \mapsto u(t)$, $t \in \mathbb{R}$ satisfying

$$u(t) = \int_{-\infty}^{t} T(t-s) f(s,u(s))\, ds, \quad t \in \mathbb{R}.$$

We will use the following auxiliary result:

Lemma 4.1. (see the proof of [9, Lemma 2.12.3]) *Suppose that* $f : \mathbb{R} \to E$ *is pseudo-almost periodic (pseudo-almost automorphic) and* $(R(t))_{t>0} \subseteq L(E,X)$ *is a strongly continuous operator family satisfying that* $\|R(t)\| \leq Me^{-bt} t^{\beta-1}$, $t > 0$ *for some finite numbers* $M \geq 1$, $b > 0$ *and* $\beta \in (0,1]$. *Then the function* $F(t) := \int_{-\infty}^{t} R(t-s) f(s)\, ds$, $t \in \mathbb{R}$ *is well-defined and pseudo-almost periodic (pseudo-almost* *automorphic).*

Suppose now that

$$0 < M_0/(a + (\ln|c|/\omega)) < 1 \tag{3.2}$$

and define the mapping

$$Pu : PAP_{\omega,c}(\mathbb{R} : E) \to PAP_{\omega,c}(\mathbb{R} : E), \text{ resp. } Pu : PAA_{\omega,c}(\mathbb{R} : E) \to PAA_{\omega,c}(\mathbb{R} : E),$$

by

$$(Pu)(t) := \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

If the mapping $f(\cdot, \cdot)$ satisfies the requirements of Proposition 3.2(ii), resp. Proposition 3.1(ii), then we have that the mapping $f(\cdot, u(\cdot))$ belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$. Using the decomposition

$$\int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds = \int_{-\infty}^{t} \Big[c^{-\frac{t-s}{\omega}} T(t-s)\Big] \Big[c^{-\frac{s}{\omega}} f(s, u(s))\Big]\, ds, \quad t \in \mathbb{R},$$

the estimate (3.2) yields that the mapping $t \mapsto \int_{-\infty}^{t} T(t-s) f(s, u(s))\, ds$, $t \in \mathbb{R}$ belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$. Hence, the mapping $P(\cdot)$ is well defined. Using a simple calculation, we get that (see also Proposition 3.1):

$$\|Pu\|_{\omega,c} \leq \frac{M_0}{a + (\ln|c|/\omega)} \|Pu\|_{\omega,c}, \quad u \in PAP_{\omega,c}(\mathbb{R} : E) \quad \big[u \in PAA_{\omega,c}(\mathbb{R} : E)\big].$$

Applying the Banach contraction principle, we get that the mapping $P(\cdot)$ has a unique fixed point, so that there exists a unique solution of the abstract semilinear Cauchy inclusion (3.1) which belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$.

Acknowledgements

This research is partially supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia and Bilateral project between MANU and SANU.

REFERENCES

1. E. Ait Dads, K. Ezzinbi, O. Arino: *Pseudo almost periodic solutions for some differential equations in a Banach space.* Nonlinear Anal. **28** (1997), 1145–1155.
2. E. H. Ait Dads, F. Boudchich, B. Es-sebbar: *Compact almost automorphic solutions for some nonlinear integral equations with time-dependent and state-dependent delay.* Advances Diff. Equ. (2017) 2017:**307** doi 10.1186/s13662-017-1364-2.

3. E. Alvarez, A. Gómez, M. Pinto: *(ω, c)-Periodic functions and mild solution to abstract fractional integro-differential equations.* Electron. J. Qual. Theory Differ. Equ. **16** (2018), 1–8.

4. E. Alvarez, S. Castillo, M. Pinto: *(ω, c)-Pseudo periodic functioins, first order Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells.* Bound. Value Probl. **106** (2019), 1–20.

5. T. Diagana: *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces.* Springer, New York, 2013.

6. A. M. Fink: *Almost Periodic Differential Equations.* Springer-Verlag, Berlin, 1974.

7. G. M. N'Guérékata: *Almost Automorphic and Almost Periodic Functions in Abstract Spaces.* Kluwer Acad. Publ., Dordrecht, 2001.

8. M. T. Khalladi, M. Kostić, A. Rahmani, D. Velinov: *(ω, c)-Almost periodic type functions and applications.* Filomat, submitted.

9. M. Kostić: *Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations.* W. de Gruyter, Berlin, 2019.

10. H.-X. Li, F.-L. Huang, J.-Y. Li: *Composition of pseudo almost-periodic functions and semilinear differential equations.* J. Math. Anal. Appl. **255** (2001), 436–446.

11. J. Liang, J. Zhang, T.-J. Xiao: *Composition of pseudo-almost automorphic and asymptotically almost automorphic functions.* J. Math. Anal. Appl. **340** (2008), 1493–1499.

12. S. Zaidman: *Almost-Periodic Functions in Abstract Spaces.* Pitman Research Notes in Math., Vol. **126**, Pitman, Boston, 1985.

13. C. Zhang: *Pseudo almost periodic solutions of some differential equations.* J. Math. Anal. Appl. **181** (1994), 62–76.