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HOMOTHETIC MOTIONS VIA GENERALIZED BICOMPLEX NUMBERS

Ferdağ Kahraman Aksoyak[1], Sıddıka Özkaldı Karakuş[2]

[1] **Ahi Evran University, Faculty of Education,**
Division of Elementary Mathematics Education, 40100 Kırşehir, Turkey
[2] **Bilecik Şeyh Edebali University, Faculty of Arts and Science,**
Department of Mathematics, 11000 Bilecik, Turkey

Abstract. In this paper, by using the matrix representation of generalized bicomplex numbers, we have defined the homothetic motions on some hypersurfaces in four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$. Also, for some special cases we have given some examples of homothetic motions in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ and obtained some rotational matrices, too. Therefore, we have investigated some applications about kinematics of generalized bicomplex numbers.
Key words: Bicomplex number, Generalized Bicomplex numbers, Homothetic motion.

1. Introduction

In the middle of the 1800s, several mathematicians discussed the problem of whether a number system extended the field of complex numbers. In 1843, Sir William Rowan Hamilton defined a number system which is called quaternions in four dimensional space. Although quaternions and complex numbers have a lot of similar properties, quaternions are not commutative with respect to multiplication. So, in 1892, a new number system called bicomplex numbers was discovered by Corrado Segre [13]. Unlike quaternions, bicomplex numbers are commutative four dimensional real algebra.

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Corresponding Author: F. Kahraman Aksoyak, Ahi Evran University, Faculty of Education, Division of Elementary Mathematics Education, 40100 Kırşehir, Turkey | E-mail: ferdag.aksoyak@ahievran.edu.tr
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The set of bicomplex numbers denoted by $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij : i^2 = -1, j^2 = -1, ij = ji, x_k \in \mathbb{R}, 1 \leq k \leq 4 \right\}.$$

Any x bicomplex number can be rewritten as $x = z_1 + jz_2$, where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ are complex numbers and j is a different imaginer unit from the imaginer unit i satisfied $j^2 = -1$ and $ij = ji$. Hence, we can perceive a bicomplex number as a complex number whose components are complex numbers. There are some applications of bicomplex numbers on the algebra, geometry and analysis. A first theory of differentiability in $\mathbb{C}_2$ was developed by Price in [12]. Özkaldı Karakuş and Kahraman Aksoyak defined generalized bicomplex numbers and gave some algebraic properties. Also, they showed that some hypersurfaces in four dimensional generalized linear space are Lie groups by using generalized bicomplex number product and obtained Lie algebras of these Lie groups [10].

Kabadayı and Yaylı defined the homothetic motions with the help of bicomplex numbers in $\mathbb{R}^4$ [5]. They showed that this homothetic motion under some conditions holds all of the properties in [14], [15]. Alkaya studied the homothetic motion with bicomplex numbers in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ [1].

In this paper, by using the matrix representation of generalized bicomplex numbers, we shall define the homothetic motions on some hypersurfaces in four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$. Also, for some special cases we shall give some examples of homothetic motions in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ and obtain some rotational matrices, too. Therefore, we shall investigate some applications about kinematics of generalized bicomplex numbers.

2. Preliminaries

In this section we give some basic concepts about generalized bicomplex numbers defined by Özkaldı Karakuş and Kahraman Aksoyak [10].

A generalized bicomplex number x is defined as follows:

$$x = x_1 1 + x_2 i + x_3 j + x_4 ij,$$

where x_k for $1 \leq k \leq 4$ are real numbers and the basis $\{1, i, j, ij\}$ holds $i^2 = -\alpha$, $j^2 = -\beta$, $(ij)^2 = \alpha\beta$, $ij = ji$, $\alpha, \beta \in \mathbb{R}$. The set of generalized bicomplex numbers is denoted by $\mathbb{C}_{\alpha\beta}$. For any two generalized bicomplex numbers $x = x_1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 + y_2 i + y_3 j + y_4 ij$, addition and multiplication are as follows:

$$x + y = (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij,$$

$$\begin{aligned} x \cdot y \;=\; & (x_1 y_1 - \alpha x_2 y_2 - \beta x_3 y_3 + \alpha\beta x_4 y_4) + (x_1 y_2 + x_2 y_1 - \beta x_3 y_4 - \beta x_4 y_3)\, i \\ & + (x_1 y_3 + x_3 y_1 - \alpha x_2 y_4 - \alpha x_4 y_2)\, j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2)\, ij \end{aligned} \tag{2.1}$$

and the scalar multiplication of an element in $\mathbb{C}_{\alpha\beta}$ by a real number c is as:

$$cx = cx_1 1 + cx_2 i + cx_3 j + cx_4 ij.$$

Hence, by means of these elementary arithmetic operations on $\mathbb{C}_{\alpha\beta}$, we have two important results. $\mathbb{C}_{\alpha\beta}$ is a four dimensional real vector space with respect to addition and scalar multiplication and it is a commutative real algebra according to generalized bicomplex number product.

Let us consider the following set of matrices

$$Q_{\alpha\beta} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_i \in \mathbb{R},\ 1 \leq i \leq 4 \right\},$$

where the set $Q_{\alpha\beta}$ is a vector space with matrix addition and scalar matrix product and it is an algebra together with matrix product. The algebras $\mathbb{C}_{\alpha\beta}$ and $Q_{\alpha\beta}$ are isomorphic. The isomorphism between two algebras is defined as:

$$h : \mathbb{C}_{\alpha\beta} \to Q_{\alpha\beta},$$

$$h\left(x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

With the help of this isomorphism, any generalized bicomplex number in $\mathbb{C}_{\alpha\beta}$ can be represent by a matrix in $Q_{\alpha\beta}$. Moreover, it is possible to express the generalized bicomplex number product which has been given by (2.1) by matrix product, that is,

$$x \cdot y = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

A generalized bicomplex number can be rewritten as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. There are three kinds of conjugations for generalized bicomplex numbers. They are given as follows:

$$\begin{aligned} x^{t_1} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_1} = (x_1 - x_2 i) + (x_3 - x_4 i)\, j, \\ x^{t_2} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_2} = (x_1 + x_2 i) - (x_3 + x_4 i)\, j, \\ x^{t_3} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_3} = (x_1 - x_2 i) - (x_3 - x_4 i)\, j, \end{aligned}$$

where x^{t_1}, x^{t_2} and x^{t_3} denote the conjugations of x with respect to i, j and both i and j, respectively. Also we can compute

$$\begin{aligned} x \cdot x^{t_1} &= \left(x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_3 + \alpha x_2 x_4\right) j, \\ x \cdot x^{t_2} &= \left(x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_2 + \beta x_3 x_4\right) i, \\ x \cdot x^{t_3} &= \left(x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2\right) + 2\left(x_1 x_4 - x_2 x_3\right) ij. \end{aligned}$$

3. One Parameter Homothetic Motion

Let the fixed space and the moving space be R_0 and R., respectively. The one-parameter homothetic motion of R_0 with respect to R is denoted by R_0/R. This motion is obtained by the following transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{3.1}$$

in which X_0 and X are the position vectors of the same point in R_0 and R, respectively and $B = hA$. Also, h, A and C are continuously differentiable functions depend on the real parameter t, where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t)$ is called homothetic scale of the motion, A is a real quasi-orthogonal matrix that holds $A^T \varepsilon A = \varepsilon$ (ε is a signature matrix according to metric), C is the translation matrix. To avoid the case of affine transformation we suppose that h is not constant and to avoid the cases of pure translation and pure rotation we also assume that $\frac{d(hA)}{dt} \neq 0$ and $\frac{dC}{dt} \neq 0$ [2].

4. Pole Points and Pole Curves of the Homothetic Motion

If we take the derivative of (3.1) with respect to t, we obtain the following equality

$$\dot{X} = \dot{B}X_0 + \dot{C} + B\dot{X}_0,$$

where $\dot{X}$ is the absolute velocity, $\dot{B}X_0 + \dot{C}$ is the sliding velocity and $B\dot{X}_0$ is the relative velocity of the point X_0. The points at which the sliding velocity of the motion vanishes at all time t are called pole points of the motion in R_0. In that case, to determine the pole points of the motion, we solve the following equality

$$\dot{B}X_0 + \dot{C} = 0. \tag{4.1}$$

For more details see[2].

5. Homothetic Motions on Some Hypersurfaces via Generalized Bicomplex Numbers

In this section we have defined the homothetic motions on some hypersurfaces at $\mathbb{R}^4_{\alpha\beta}$ with the help of generalized bicomplex numbers and given some examples about the homothetic motions.

5.1. Homothetic Motion on Hypersurface M_1

Let us consider the hypersurface M_1 as follows:

$$M_1 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1 x_3 + \alpha x_2 x_4 = 0,\ x \neq 0 \right\}.$$

By using generalized bicomplex numbers, M_1 can be rewritten as:

$$M_1 = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1 x_3 + \alpha x_2 x_4 = 0,\ x \neq 0 \right\},$$

or the hypersurface M_1 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M_1} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1 x_3 + \alpha x_2 x_4 = 0,\ x \neq 0 \right\},$$

where M_x is the matrix representiation of generalized bicomplex number x. The metric on hypersurface M_1 is defined by $g_1(x,x) = x \cdot x^{t_1} = x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2$ and the norm of any element x on M_1 is defined by $\|x\| = \sqrt{|g_1(x,x)|} = \sqrt{|x \cdot x^{t_1}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides with four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$.

Proposition 5.1. *There are following properties about the norm on the hypersurface M_1.*

i) For $x, y \in M_1$, $\|x \cdot y\| = \|x\| \|y\|$,

ii) $\|x\|^4 = \det(M_x)$.

Proof. These properties can be easily seen with direct calculations. □

Corollary 5.1. *A unit generalized bicomplex number on the hypersurface M_1 determines a rotation motion.*

Proof. It is obvious from Proposition 5.1. □

Theorem 5.1. *M_1 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex numbers on M_1 by M_1^*. M_1^* is as:

$$\begin{aligned} M_1^* &= \{ x \in M_1 : g_1(x,x) = 1 \} \\ &= \left\{ x \in M_1 : x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2 = 1 \right\}. \end{aligned}$$

Theorem 5.2. *M_1^* is Lie subgroup of M_1.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_1. In that case, it can be expressed as

$$\begin{array}{ccl} \gamma & : & I \subset \mathbb{R} \rightarrow M_1 \\ t & \rightarrow & \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij, \ \gamma_1(t)\gamma_3(t) + \alpha\gamma_2(t)\gamma_4(t) = 0. \end{array}$$

Then the matrix B corresponding to the curve γ is obtained as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.1}$$

Now by using this matrix B, we can define the one parameter motion on M_1 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.1. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_1 is defined by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.2}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_1, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0.

Theorem 5.3. *The equation given by* (5.2) *determines a homothetic motion on M_1.*

Proof. Since the curve γ lies on M_1, it does not pass through the origin. So, the matrix given by (5.1) can be expressed as:

$$B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA, \tag{5.3}$$

where $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $t \rightarrow h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2}$. Because of $\gamma(t) \in M_1$, $\gamma_1(t)\gamma_3(t) + \alpha\gamma_2(t)\gamma_4(t) = 0$. By using this equality, we obtain that the matrix A in (5.3) is a real quasi-orthogonal matrix. In that case it satisfies $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_1 is as:

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.2) is a homothetic motion. $\square$

Remark 5.1. The norm of $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as $\|\gamma(t)\| = \sqrt{|\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.2. *Let $\gamma(t)$ be a curve on M_1^*. Then one-parameter motion on M_1 given by (5.2) is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_1^*. Then $\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2 = 1$. In that case the matrix B given by (5.1) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T \varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. $\square$

Theorem 5.4. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_1. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve. Then $\dot{\gamma}_1^2 + \alpha\dot{\gamma}_2^2 - \beta\dot{\gamma}_3^2 - \alpha\beta\dot{\gamma}_4^2 = 1$. Also, since the tangent vector of γ is on M_1, it implies that $\dot{\gamma}_1(t)\dot{\gamma}_3(t) + \alpha\dot{\gamma}_2(t)\dot{\gamma}_4(t) = 0$. Thus $\dot{B}^T \varepsilon \dot{B} = \varepsilon$ and $\det \dot{B} = 1$. $\square$

Theorem 5.5. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_1. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.4, $\det \dot{B} = 1$ and thus the value of $\det \dot{B}$ is independent of h. $\square$

Theorem 5.6. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_1. Then the pole points of the motion given by (5.2) are $X_0 = -\dot{B}^{-1}\dot{C}$.*

Proof. Since the position vector of the curve γ is on M_1, from Theorem 5.3, the equation (5.2) becomes a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot{\gamma}(t) \in M_1$, from Theorem 5.4 $\det \dot{B} = 1$ and it implies that there is only one solution of the equation (4.1). Then the pole points of the motion given by (5.2) are obtained as $X_0 = -\dot{B}^{-1}\dot{C}$. $\square$

Corollary 5.3. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_1. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot{B}^{-1}$ of the speed vector of translation vector at the opposite direction $\left(-\dot{C}\right)$.*

Proof. From Theorem 5.4, the matrix $\dot{B}$ is a real quasi-orthogonal matrix. Then the matrix $\dot{B}^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_1 according to the situations of real numbers α and β.

Example 5.1. For $\alpha = \beta = 1$, M_1 becomes a hypersurface in $\mathbb{R}^4_2$. Let $\gamma : I \subset \mathbb{R} \rightarrow M_1 \subset \mathbb{R}^4_2$ be a curve given by

$$\gamma(t) = h(t)\begin{pmatrix} \cosh(at)\cos(bt) + \cosh(at)\sin(bt)\, i \\ -\sinh(at)\sin(bt)\, j + \sinh(at)\cos(bt)\, ij \end{pmatrix}, \tag{5.4}$$

where a and b are real numbers. By using (5.1) and (5.4), the matrix B associated with the curve γ becomes a homothetic matrix, where $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$ in (5.4), then γ is a curve on M_1^* and the matrix B determines a rotation matrix in $\mathbb{R}^4_2$. In (5.4), if we choose as $h(t) = 1$, $a = 0$ and $b = 1$, then we get

$$\gamma(t) = \cos t + i \sin t. \tag{5.5}$$

By using (5.1) and (5.5), we get the matrix B as follows:

$$B = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

where B is a rotational matrix in $\mathbb{R}^4_2$. Since this curve given by (5.5) is unit speed curve and its tangent vector belongs to M_1, the derivation of the above matrix $\dot{B}$ is a real quasi-orthogonal matrix, too. Then it is a rotational matrix in $\mathbb{R}^4_2$. Similarly, in (5.4) if we take as $h(t) = 1$, $a = 1$ and $b = 0$, then we get

$$\gamma(t) = \cosh t + ij \sinh t. \tag{5.6}$$

By using (5.1) and (5.6), we have the matrix B as follows:

$$B = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & -\sinh t & 0 \\ 0 & -\sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$

where B is a rotational matrix in $\mathbb{R}^4_2$.

Example 5.2. For $\alpha = 1$, $\beta = -1$, M_1 is a hypersurface in $\mathbb{R}^4$. Let $\gamma : I \subset \mathbb{R} \rightarrow M_1 \subset \mathbb{R}^4$ be a curve given by

$$\gamma(t) = h(t)\begin{pmatrix} \cos(at)\cos(bt) + \cos(at)\sin(bt)\, i \\ -\sin(at)\sin(bt)\, j + \sin(at)\cos(bt)\, ij \end{pmatrix}, \tag{5.7}$$

where a and b are real numbers. By using (5.1) and (5.7), the matrix representation of the curve γ is a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_1, that is, $\gamma(t) \in M_1^* = M_1 \cap S^3$. The matrix representation of it is a rotation matrix in $\mathbb{R}^4$. Even, if we take as $h(t) = 1$, $a = 0$, $b = 1$, then

$$\gamma(t) = \cos t + i \sin t. \tag{5.8}$$

From (5.1) and (5.8), by determining the matrix representation of the above curve, we obtain

$$B = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}.$$

This matrix is a general rotational matrix in $\mathbb{R}^4$ which is defined by Moore [8]. Also, from Theorem 5.4, $\dot{B}$ is a real orthogonal matrix, too.

Example 5.3. For $\alpha = \beta = -1$, the hypersurface M_1 lies in $\mathbb{R}^4_2$ and the following curve lies on M_1

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cosh(bt) + \cosh(at)\sinh(bt)\, i \\ +\sinh(at)\sinh(bt)\, j + \cosh(at)\sinh(bt)\, ij \end{pmatrix}, \tag{5.9}$$

in which a and b are real numbers. From (5.1) and (5.9), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_1^* and the matrix B gives a rotation matrix in $\mathbb{R}^4_2$.

5.2. Homothetic Motion on Hypersurface M_2

Let us consider the hypersurface M_2 as follows:

$$M_2 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\}.$$

By using the generalized bicomplex numbers, M_2 can be rewritten as:

$$M_2 = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\},$$

or the hypersurface M_2 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M_2} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\},$$

where M_x is the matrix representiation of the generalized bicomplex number x on M_2. The metric on hypersurface M_2 is defined by $g_2(x, x) = x \cdot x^{t_2} = x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2$ and the norm of any element x on M_2 is given by $\|x\| = \sqrt{|g_2(x, x)|} = \sqrt{|x \cdot x^{t_2}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides with four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$.

Proposition 5.2. *There are following properties about the norm on the hypersurface M_2.*

i) For $x, y \in M_2$, $\|x \cdot y\| = \|x\| \, \|y\|$,

ii) $\|x\|^4 = \det(M_x)$.

Proof. These properties can be easily seen with directly calculations. □

Corollary 5.4. *A unit generalized bicomplex number on the hypersurface M_2 determines a rotation motion.*

Proof. It is obvious from Proposition 5.2. □

Theorem 5.7. *M_2 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex number on M_2 by M_2^*. M_2^* is given as:

$$\begin{aligned} M_2^* &= \{x \in M_2 : g_2(x, x) = 1\} \\ &= \left\{x \in M_2 : x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2 = 1\right\}. \end{aligned}$$

Theorem 5.8. *M_2^* is Lie subgroup of M_2.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_2. In that case, it can be expressed as:

$$\begin{aligned} \gamma &: I \subset \mathbb{R} \to M_2 \\ t &\to \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij, \ \gamma_1(t)\gamma_2(t) + \beta\gamma_3(t)\gamma_4(t) = 0. \end{aligned}$$

Then the matrix B corresponding to the curve γ is given as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.10}$$

Now by using this matrix B, we can define the one parameter motion on M_2 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.2. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_2 is given by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.11}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_2, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0.

Theorem 5.9. *The equation given by* (5.11) *determines a homothetic motion on* M_2.

Proof. Since the curve γ is on M_2, it does not pass through the origin. So, the matrix given by (5.10) can be expressed as:

$$B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA, \tag{5.12}$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2}$. Since $\gamma(t) \in M_2$, $\gamma_1(t)\gamma_2(t) + \beta\gamma_3(t)\gamma_4(t) = 0$. By using this equality, we obtain that the matrix A in (5.12) is a real quasi-orthogonal matrix. In that case it satisfies $A^T\varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_2 given by

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.11) determines a homothetic motion. $\square$

Remark 5.2. The norm of the curve $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as $\|\gamma(t)\| = \sqrt{|\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.5. *Let $\gamma(t)$ be a curve on M_2^*. Then one-parameter motion on M_2 given by (5.11) is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_2^*. Then $\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2 = 1$. In that case the matrix B in (5.11) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T\varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. $\square$

Theorem 5.10. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_2. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve then $\dot\gamma_1^2 - \alpha\dot\gamma_2^2 + \beta\dot\gamma_3^2 - \alpha\beta\dot\gamma_4^2 = 1$. Also since the tangent vector of the curve γ is on M_2, we have $\dot\gamma_1(t)\dot\gamma_2(t) + \beta\dot\gamma_3(t)\dot\gamma_4(t) = 0$. Thus $\dot B^T \varepsilon \dot B = \varepsilon$ and $\det \dot B = 1$. □

Theorem 5.11. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot\gamma(t)$ be on M_2. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.10, $\det \dot B = 1$ and thus the value of $\det \dot B$ is independent of h. □

Theorem 5.12. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_2. Then the pole points of the motion given by* (5.11) *are $X_0 = -\dot B^{-1}\dot C$.*

Proof. Since the position vector of the curve γ is on M_2, from Theorem 5.9, the equation (5.11) becomes a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot\gamma(t) \in M_2$, from Theorem 5.10 $\det \dot B = 1$ and it implies that there is only one solution of the equation (4.1). Then the pole points of the motion given by (5.11) are found as $X_0 = -\dot B^{-1}\dot C$. □

Corollary 5.6. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_2. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot B^{-1}$ of the speed vector of translation vector at the opposite direction $\left(-\dot C\right)$.*

Proof. From Theorem 5.10, the matrix $\dot B$ is a real quasi-orthogonal matrix. Then the matrix $\dot B^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_2 according to the situations of real numbers α and β.

Example 5.4. For $\alpha = \beta = 1$, M_2 becomes a hypersurface in $\mathbb{R}_2^4$. Let $\gamma : I \subset \mathbb{R} \to M_2 \subset \mathbb{R}_2^4$ be a curve as:

$$\gamma(t) = h(t)\begin{pmatrix} \cosh(at)\cos(bt) - \sinh(at)\sin(bt)\,i \\ +\cosh(at)\sin(bt)\,j + \sinh(at)\cos(bt)\,ij \end{pmatrix}, \tag{5.13}$$

where a and b are real numbers. By using (5.10) and (5.13), the matrix B is a homothetic matrix and $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, the matrix B becomes a rotation matrix in $\mathbb{R}_2^4$.

Example 5.5. For $\alpha = -1$, $\beta = 1$, M_2 is a hypersurface in $\mathbb{R}^4$. Let us consider the curve $\gamma : I \subset \mathbb{R} \to M_2 \subset \mathbb{R}^4$ as follows:

$$\gamma(t) = h(t)\begin{pmatrix} \cos(at)\cos(bt) - \sin(at)\sin(bt)\,i \\ +\cos(at)\sin(bt)\,j + \sin(at)\cos(bt)\,ij \end{pmatrix}, \tag{5.14}$$

where a and b are real numbers. From (5.10) and (5.14) we obtain the matrix representation of the curve γ and it determines a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_2, that is, $\gamma(t) \in M_2^* = M_2 \cap S^3$. From Corollary 5.5, the matrix representation of it is a rotation matrix in $\mathbb{R}^4$.

Example 5.6. For $\alpha = \beta = -1$, the hypersurface M_2 becomes a subset of $\mathbb{R}^4_2$ and the following curve lies on M_2

$$\gamma(t) = h(t)\begin{pmatrix} \cosh(at)\cosh(bt) + \sinh(at)\sinh(bt)\,i \\ +\cosh(at)\sinh(bt)\,j + \sinh(at)\cosh(bt)\,ij \end{pmatrix}, \tag{5.15}$$

in which a and b are real numbers. From (5.10) and (5.15), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_2^* and the matrix B gives a rotation matrix in $\mathbb{R}^4_2$.

5.3. Homothetic Motion on Hypersurface M_3

Let us consider the hypersurface M_3 as follows:

$$M_3 = \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1x_4 - x_2x_3 = 0,\ x \neq 0\right\}.$$

By using generalized bicomplex numbers, M_3 can be rewritten as:

$$M_3 = \left\{x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1x_4 - x_2x_3 = 0,\ x \neq 0\right\},$$

or the hypersurface M_3 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M}_3 = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1x_4 - x_2x_3 = 0,\ x \neq 0 \right\}.$$

where M_x is the matrix representiation of generalized bicomplex number x on M_3. The metric on hypersurface M_3 is defined by $g_3(x,x) = x \cdot x^{t_3} = x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2$ and the norm of any element x on M_3 is given by $\|x\| = \sqrt{|g_3(x,x)|} = \sqrt{|x \cdot x^{t_3}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$.

Proposition 5.3. *There are following properties about the norms on the hypersurface M_3.*

i) For $x, y \in M_3$, $\|x \cdot y\| = \|x\|\,\|y\|$

ii) $\|x\|^4 = \det(M_x)$

Proof. These properties can be easily seen with direct calculations. □

Corollary 5.7. *A unit generalized bicomplex number on the hypersurface M_3 determines a rotation motion.*

Proof. It is obvious from Proposition 5.3. □

Theorem 5.13. *M_3 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex number on M_3 by M_3^*. M_3^* is given as:

$$\begin{aligned} M_3^* &= \{x \in M_3 : g_3(x,x) = 1\} \\ &= \left\{x \in M_3 : x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2 = 1\right\}. \end{aligned}$$

Theorem 5.14. *M_3^* is Lie subgroup of M_3.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_3. In that case, it can be expressed as

$$\begin{aligned} \gamma \quad &: \quad I \subset \mathbb{R} \to M_3 \\ t \quad &\to \quad \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij,\ \gamma_1(t)\gamma_4(t) - \gamma_2(t)\gamma_3(t) = 0. \end{aligned}$$

Then the matrix B corresponding to the curve γ is given as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.16}$$

Now by using this matrix B, we can define the one parameter motion on M_3 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.3. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_3 is given by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.17}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_3, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0, respectively.

Theorem 5.15. *The equation given by* (5.17) *is a homothetic motion on M_3.*

Proof. Since the curve γ is on M_3, it does not pass through the origin. So, the matrix given by (5.16) can be expressed as:

$$B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA, \tag{5.18}$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2}$. Because of $\gamma(t) \in M_3$, we have $\gamma_1(t)\gamma_4(t) - \gamma_2(t)\gamma_3(t) = 0$. By using this equality, we obtain that the matrix A in (5.18) is a real quasi-orthogonal matrix. In that case it satisfies $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_3 given by

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.17) determines a homothetic motion. □

Remark 5.3. The norm of the curve $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as $\|\gamma(t)\| = \sqrt{|\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.8. *Let $\gamma(t)$ be a curve on M_3^*. Then one-parameter motion on M_3 given by* (5.17) *is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_3^*. Then $\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2 = 1$. In that case the matrix B given by (5.16) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T \varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. □

Theorem 5.16. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_3. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve. Then $\dot{\gamma}_1^2 + \alpha\dot{\gamma}_2^2 + \beta\dot{\gamma}_3^2 + \alpha\beta\dot{\gamma}_4^2 = 1$. Also since the tangent vector of the curve γ is on M_3, we have $\dot{\gamma}_1(t)\dot{\gamma}_4(t) - \dot{\gamma}_2(t)\dot{\gamma}_3(t) = 0$. Thus $\dot{B}^T \varepsilon \dot{B} = \varepsilon$ and $\det \dot{B} = 1$. □

Theorem 5.17. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_3. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.16, $\det \dot{B} = 1$ and thus the value of $\det \dot{B}$ is independent of h. □

Theorem 5.18. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_3. Then the pole points of the motion given by* (5.17) *are* $X_0 = -\dot{B}^{-1}\dot{C}$.

Proof. Since the position vector of the curve γ is on M_3, from Theorem 5.15, the equation (5.17) is a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot{\gamma}(t) \in M_3$, from Theorem 5.16 $\det \dot{B} = 1$. Thus the equation (4.1) has only one solution. In that case the pole points of the motion are obtained as $X_0 = -\dot{B}^{-1}\dot{C}$. □

Corollary 5.9. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_3. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot{B}^{-1}$ of the speed vector of translation vector at the opposite direction $\left(-\dot{C}\right)$.*

Proof. From Theorem 5.16, the matrix $\dot{B}$ is a real quasi-orthogonal matrix. Then the matrix $\dot{B}^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_3 according to the situations of real numbers α and β.

Example 5.7. For $\alpha = \beta = 1$, M_3 becomes a hypersurface in four dimensional Euclidean space $\mathbb{R}^4$. Let $\gamma : I \subset \mathbb{R} \to M_3 \subset \mathbb{R}^4$ be a curve as:

$$\gamma(t) = h(t)\begin{pmatrix} \cos(at)\cos(bt) + \cos(at)\sin(bt)\, i \\ +\sin(at)\cos(bt)\, j + \sin(at)\sin(bt)\, ij \end{pmatrix}, \tag{5.19}$$

where a and b are real numbers. By using (5.16) and (5.19), the matrix B associated with the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ is a curve on M_3^*. In that case it becomes a spherical curve lies on M_3 and the matrix B becomes a rotation matrix in $\mathbb{R}^4$.

Example 5.8. For $\alpha = 1$, $\beta = -1$, M_3 is a hypersurface in four dimensional Euclidean space $\mathbb{R}^4_2$. Let $\gamma : I \subset \mathbb{R} \to M_3 \subset \mathbb{R}^4_2$ be a curve given by

$$\gamma(t) = h(t)\begin{pmatrix} \cosh(at)\cos(bt) + \cosh(at)\sin(bt)\, i \\ +\sinh(at)\cos(bt)\, j + \sinh(at)\sin(bt)\, ij \end{pmatrix}, \tag{5.20}$$

where a and b are real numbers. By using (5.16) and (5.20), the matrix representation of the curve γ is a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_3, that is, $\gamma(t) \in M_3^*$. The matrix representation of it is a rotation matrix in $\mathbb{R}^4_2$.

Example 5.9. For $\alpha = \beta = -1$, the hypersurface M_3 is in four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$ and the following curve lies on M_3

$$\gamma(t) = h(t)\begin{pmatrix} \cosh(at)\cosh(bt) + \cosh(at)\sinh(bt)\, i \\ +\sinh(at)\cosh(bt)\, j + \sinh(at)\sinh(bt)\, ij \end{pmatrix}, \tag{5.21}$$

in which a and b are real numbers. From (5.16) and (5.21), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_3^* and the matrix B gives a rotation matrix in $\mathbb{R}^4_2$.

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