ON THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In the present paper, we study three-dimensional trans-Sasakian manifolds admitting the Schouten-van Kampen connection. Also, we have proved some results on $\phi$-projectively flat, $\xi$-projectively flat and $\xi$-concircularly flat three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection. Locally $\phi$-symmetric trans-Sasakian manifolds of dimension three have been studied with respect to Schouten-van Kampen connection. Finally, we construct an example of a three-dimensional trans-Sasakian manifold admitting Schouten-van Kampen connection which verifies Theorem 4.1. and Theorem 5.2.

Key words: General geometric structures on manifolds, Schouten-van Kampen connection, Special Riemannian manifolds

1. Introduction

The Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection. Solov’ev investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection ([18], [19], [20], [21]). In 2014, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [17]. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection. Recently, G. Ghosh [10], Yildiz [26], Nagaraja [15] and D. L. Kiran Kumar [12] have studied the...
Schouten-van Kampen connection in Sasakian manifolds, $f$-Kenmotsu manifolds and Kenmotsu manifolds respectively.

A transformation of an $n$-dimensional differentiable manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle, is called a concircular transformation \[27], \[13]. A concircular transformation is always a conformal transformation \[13]. Here geodesic circle means a curve in $M$ whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor $W$ with respect to Levi-Civita connection. It is defined by \[27], \[28]

\[
W(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],
\]

where $X, Y, Z \in \chi(M)$, $R$ and $r$ are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection.

The concircular curvature tensor $\tilde{W}$ with respect to the Schouten-van Kampen connection is defined by

\[
\tilde{W}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],
\]

where $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and the scalar curvature with respect to the Schouten-van Kampen connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In 1985, a new class of $n$-dimensional almost contact manifold namely trans-Sasakian manifold was introduced by J. A. Oubina \[16\] and further study about the local structures of trans-Sasakian manifolds was carried by J. C. Marrero \[14\]. Trans-Sasakian manifolds of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are, called the cosymplectic, $\alpha$-Sasakian and $\beta$-Kenmotsu respectively (\[2\], \[11\]). In particular, if $\alpha = 0, \beta = 1; \alpha = 1, \beta = 0$; then a trans-Sasakian manifold becomes Kenmotsu and Sasakian manifolds respectively. Hence, trans-Sasakian structures give a large class of generalized Quasi-Sasakian structures. It has been proven that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha$-Sasakian and $\beta$-Kenmotsu manifold. Three-dimensional trans-Sasakian manifolds with different restrictions on curvature and smooth functions $\alpha$, $\beta$ are studied in (\[7\], \[8\], \[5\], \[6\]).

In the present paper, we have studied three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection.

The present paper is organized as follows: After the introduction in Section 1, we give some required preliminaries in Section 2. Section 3 is devoted to the study of the curvature tensor, the Ricci tensor, scalar curvature of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Section 4
is devoted to the study of $\xi$-projectively and $\phi$-projectively flat trans-Sasakian manifolds of dimension three with respect to the Schouten-van Kampen connection. In this section, we have proved that a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is $\xi$-projectively flat if and only if the scalar curvature of the manifold vanishes. In Section 5, we study $\xi$-concisurally flat trans-Sasakian manifolds of dimension three admitting Schouten-van Kampen connection. In the next section, we study locally $\phi$-symmetric trans-Sasakian manifolds of dimensional three with respect to Schouten-van Kampen connection. In Section 7, we study Weyl $\xi$-conformally flat in three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. In the last section, we construct an example of a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection to support the results obtained in Section 4 and Section 5.

2. Preliminaries

Let $M$ be a connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is an $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is compatible Riemannian metric such that

\begin{align}
\phi^2 X = -X + \eta(X)\xi, & \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \\
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), & \\
g(X, \phi Y) = -g(\phi X, Y), & \quad g(X, \xi) = \eta(X),
\end{align}

for all $X, Y \in T(M)$ [1]. The fundamental 2-form $\Phi$ of the manifold is defined by

\begin{align}
\Phi(X, Y) = g(X, \phi Y),
\end{align}

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on a manifold $M$ is called trans-Sasakian structure [16] if $(M \times R, J, G)$ belongs to the class $W_4$ [9], where $J$ is the almost complex structure on $M \times R$ defined by

\begin{align}
J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),
\end{align}

for all vector fields $X$ on $M$, a smooth function $f$ on $M \times R$ and the product metric $G$ on $M \times R$. This may be expressed by the condition [3]

\begin{align}
(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),
\end{align}

for smooth functions $\alpha$ and $\beta$ on $M$. Here $\nabla$ is Levi-Civita connection on $M$. We say $M$ as the trans-Sasakian manifold of type $(\alpha, \beta)$. From (2.5) it follows that

\begin{align}
\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi),
\end{align}
(2.7) \[(\nabla_X \eta) Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).\]

In a three-dimensional trans-Sasakian manifold following relations hold [7], [8]:

(2.8) \[2\alpha \beta + \xi \alpha = 0,\]

(2.9) \[
S(X, Y) = \left\{ \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right\} g(X, Y) \\
- \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \eta(X) \eta(Y) - \{ Y \beta + (\phi X) \alpha \} \eta(Y),
\]

(2.10) \[
R(X, Y) Z = \left\{ \frac{r}{2} + 2 \xi \beta - 2(\alpha^2 - \beta^2) \right\} \left[ g(Y, Z) X - g(X, Z) Y \right] \\
- \left\{ \frac{r}{2} + 3(\alpha^2 - \beta^2) \right\} \eta(X) \xi + \left\{ \frac{r}{2} - 3(\alpha^2 - \beta^2) \right\} \left[ \eta(X) \eta(Y) - \eta(Y) \eta(X) \right],
\]

where \( S \) is the Ricci tensor of type \((0, 2)\), and \( r \) is the scalar curvature of the manifold \( M \) with respect to Levi-Civita connection.

From here after we consider \( \alpha \) and \( \beta \) are constants, then the above relations become

(2.11) \[
R(X, Y) Z = \left\{ \frac{r}{2} - (\alpha^2 - \beta^2) \right\} \left[ g(Y, Z) X - g(X, Z) Y \right] + \left\{ \frac{r}{2} - (\alpha^2 - \beta^2) \right\} \left[ g(X, Z) \eta(Y) - g(Y, Z) \eta(X) \right] \xi + \left\{ \frac{r}{2} - 3(\alpha^2 - \beta^2) \right\} \left[ \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \right],
\]

(2.12) \[
S(X, Y) = \left\{ \frac{r}{2} - (\alpha^2 - \beta^2) \right\} g(X, Y) \\
- \left\{ \frac{r}{2} - 3(\alpha^2 - \beta^2) \right\} \eta(X) \eta(Y),
\]

(2.13) \[S(X, \xi) = 2(\alpha^2 - \beta^2) \eta(X),\]
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(2.14) \[ QX = \left\{ \frac{r}{2} - (\alpha^2 - \beta^2) \right\} X - \left\{ \frac{r}{2} - 3(\alpha^2 - \beta^2) \right\} \eta(X) \xi, \]

(2.15) \[ R(X, Y) \xi = (\alpha^2 - \beta^2)(\eta(Y) X - \eta(X) Y), \]

(2.16) \[ R(\xi, X) Y = 2(\alpha^2 - \beta^2)(g(X, Y) \xi - \eta(Y) X). \]

From (2.8) it follows that if \( \alpha \) and \( \beta \) are constants, then the manifold is either \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu or cosymplectic.

3. Curvature tensor of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection

For an almost contact metric manifold \( M \), the Schouten-van Kampen connection \( \tilde{\nabla} \) is given by [17]

\[ \tilde{\nabla}_X Y = \nabla_X Y - \eta(Y) \nabla_X \xi + (\nabla_X \eta)(Y) \xi. \]

Let \( M \) be a three-dimensional trans-Sasakian manifold. Then from above equation we have

\[ \tilde{\nabla}_X Y = \nabla_X Y + \alpha \{ \eta(Y) \phi X - g(\phi X, Y) \xi \} + \beta \{ g(X, Y) \xi - \eta(Y) X \}. \]

We define the curvature tensor \( \tilde{R} \) of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection \( \tilde{\nabla} \) by

\[ \tilde{R}(X, Y) Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z. \]

In view of (3.2) and (3.3) we obtain

\[ \tilde{R}(X, Y) Z = R(X, Y) Z + \alpha^2 \{ g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y \]
\[ + \eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X \]
\[ - g(Y, Z) \eta(X) \xi + g(X, Z) \eta(Y) \xi \}
\[ + \beta^2 \{ g(Y, Z) X - g(X, Z) Y \}. \]

Taking inner product in both sides of (3.4) with \( W \), we have

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \alpha^2 \{ g(\phi Y, Z) g(\phi X, W) - g(\phi X, Z) g(\phi Y, W) \]
\[ + g(Y, W) \eta(X) \eta(Z) - g(X, W) \eta(Y) \eta(Z) \]
\[ - g(Y, Z) \eta(W) + g(X, Z) \eta(Y) \eta(W) \}
\[ + \beta^2 \{ g(Y, Z) g(X, W) - g(X, Z) g(Y, W) \}, \]

where \( \tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y) Z, W) \).
Taking a frame field from (3.5), we get
\[
\tilde{S}(Y, Z) = S(Y, Z) + 2\beta^2 g(Y, Z) - 2\alpha^2 \eta(Y)\eta(Z).
\]
From above equation we have
\[
\tilde{Q}_Y = Q_Y + 2\beta^2 Y - 2\alpha^2 \eta(Y)\xi.
\]
Again putting \( Y = Z = e_i \) \((i = 1, 2, 3)\) and taking summation over \( i \) in (3.6), we obtain
\[
\tilde{r} = r - 2\alpha^2 + 6\beta^2,
\]
where \( \tilde{r} \) and \( r \) are the scalar curvatures of the Schouten-van Kampen connection \( (\tilde{\nabla}) \) and Levi-Civita connection \( (\nabla) \) respectively.

Hence we have the following:

**Proposition 3.1.** A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection following statements are equivalent
(a) The curvature tensor \( \tilde{R} \) is given by (3.4),
(b) The Ricci tensor \( \tilde{S} \) is given by (3.6),
(c) \( \tilde{r} = r - 2\alpha^2 + 6\beta^2 \),
(d) The Ricci tensor \( \tilde{S} \) is symmetric,
provided \( \alpha \) and \( \beta \) are constants.

4. \( \xi \)-Projectively and \( \phi \)-projectively flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

In this section, we study projectively flat three-dimensional trans-Sasakian manifold \( M \) with respect to the Schouten-van Kampen connection. In a three-dimensional trans-Sasakian manifold, the projective curvature tensor with respect to the Schouten-van Kampen connection is given by
\[
\bar{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}.
\]

**Definition 4.1.** A three-dimensional trans-Sasakian manifold \( M \) with respect to the Schouten-van Kampen connection is said to be \( \xi \)-projectively flat if
\[
\bar{P}(X, Y) \xi = 0,
\]
for all vector fields \( X, Y \) on \( M \). This notion was first defined by Tripathi and Dwivedi [22]. If \( \bar{P}(X, Y) \xi = 0 \), just holds for \( X, Y \) orthogonal to \( \xi \), we call such a manifold a horizontal \( \xi \)-projectively flat manifold.

Using (3.4) in (4.1) we have
\[ \tilde{P}(X,Y)Z = R(X,Y)Z + \alpha^2 \{ g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y \\ + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi \} \\ + \beta^2 \{ g(Y, Z)X - g(X, Z)Y \} \\ - \frac{1}{2} \{ \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \} \tag{4.2} \]

Putting \( Z = \xi \) and using (2.1), (2.3), (2.15) and (3.6) in (4.2), we get

\[ \tilde{P}(X,Y)\xi = 0. \tag{4.3} \]

Thus we can state the following:

**Theorem 4.1.** A three-dimensional trans-Sasakian manifold is \( \xi \)-projectively flat with respect to the Schouten-van Kampen connection provided \( \alpha \) and \( \beta \) are constants.

Again putting (3.6) in (4.2) we get

\[ \tilde{P}(X,Y)Z = P(X,Y)Z + \alpha^2 \{ g(\phi Y, Z) \phi X - g(\phi X, Z) \phi Y \\ - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi \}. \tag{4.4} \]

Putting \( Z = \xi \) in (4.4) and using (2.1) and (2.3), it follows that

\[ \tilde{P}(X,Y)\xi = P(X,Y)\xi. \tag{4.5} \]

In view of above discussion we state the following theorem:

**Theorem 4.2.** A three-dimensional trans-Sasakian manifold is \( \xi \)-projectively flat with respect to the Schouten-van Kampen connection if and only if the manifold is \( \xi \)-projectively flat with respect to the Levi-Civita connection provided \( \alpha \) and \( \beta \) are constants.

**Definition 4.2.** A trans-Sasakian manifold \( M \) with respect to the Schouten-van Kampen connection is said to be \( \phi \)-projectively flat if

\[ \phi^2 \tilde{P}(\phi X, \phi Y)\phi Z = 0. \]

It can be easily seen that \( \phi^2 \tilde{P}(\phi X, \phi Y)\phi Z = 0 \) holds if and only if

\[ g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.6} \]

for \( X, Y, Z, W \in T(M) \).
Using (4.1) and (4.6), $\phi$-projectively flat means

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2}\{\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi W) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi W)\}. \quad (4.7)$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in $\mathcal{M}$ and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (4.7) and summing up with respect to $i$, we have

$$\sum_{i=1}^{2} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2}\sum_{i=1}^{2}\{\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \quad (4.8)$$

Using (2.1), (2.2), (2.3) and (3.5) it can be easily verified that

$$\sum_{i=1}^{2} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + (\alpha^2 + \beta^2)g(Y, Z) + (\beta^2 - 3\alpha^2)e(Y)e(Z). \quad (4.9)$$

$$= S(\phi Y, \phi Z) + (\alpha^2 + \beta^2)g(Y, Z) + (\beta^2 - 3\alpha^2)e(Y)e(Z). \quad (4.10)$$

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2. \quad (4.11)$$

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = \tilde{S}(\phi Y, \phi Z). \quad (4.12)$$

Using (4.9), (4.10) and (4.11), the equation (4.8) becomes

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = 2\{S(\phi Y, \phi Z) + (\alpha^2 + \beta^2)g(Y, Z) + (\beta^2 - 3\alpha^2)e(Y)e(Z)\}. \quad (4.13)$$

Using (3.6) in (4.12), we get

$$S(\phi Y, \phi Z) = -2\alpha^2g(Y, Z) + 2(3\alpha^2 - \beta^2)e(Y)e(Z) \quad (4.14)$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (4.13) and using (2.1) (2.2) and (2.13), we obtain

$$S(Y, Z) = -2\alpha^2g(Y, Z) + 2(2\alpha^2 - \beta^2)e(Y)e(Z). \quad (4.15)$$

Conversely, let $S$ be of the form (4.14), then obviously

$$g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0.$$
Thus we can state the following:

**Theorem 4.3.** A three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is $\phi$-projectively flat if and only if the manifold is an $\eta$-Einstein manifold with respect to the Levi-Civita connection provided $\alpha, \beta$ are constants with $\beta \neq \pm \sqrt{2\alpha}, (\alpha \neq 0)$.

### 5. $\xi$-Concircularly flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

**Definition 5.1.** A trans-Sasakian manifold $M$ with respect to the Schouten-van Kampen connection is said to be $\xi$-concircularly flat if

$$\tilde{\mathcal{W}}(X,Y)\xi = 0,$$

for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on $M$.

**Theorem 5.1.** A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is horizontally $\xi$-concircularly flat if and only if the manifold with respect to the Levi-Civita connection is also $\xi$-concircular flat provided $\alpha, \beta$ are constants.

**Proof.** Combining (1.1),(1.2) and using (3.4), (3.6) (3.8), we get

$$\tilde{\mathcal{W}}(X,Y)Z = \mathcal{W}(X,Y)Z + \alpha^2\{g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Y)\xi + 2\alpha^2(\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y)\}. $$

(5.2)

Putting $Z = \xi$ in (5.2) we get

$$\tilde{\mathcal{W}}(X,Y)\xi = \mathcal{W}(X,Y)\xi + \frac{2\alpha^2}{3}\{\eta(X)Y - \eta(Y)X\}. $$

(5.3)

From (5.3), implies that

$$\tilde{\mathcal{W}}(X,Y)\xi = \mathcal{W}(X,Y)\xi; \text{ for all } X, Y \text{ orthogonal to } \xi. $$

Hence the proof of theorem is complete.

**Theorem 5.2.** A three-dimensional trans-Sasakian manifold is $\xi$-concircularly flat with respect to the Schouten-van Kampen connection if and only if the scalar curvature $\tilde{r}$ is zero, provided $\alpha$ and $\beta$ are constants.

**Proof.** Putting $Z = \xi$ in (1.2) and using (2.1), (2.3), (2.3), (2.15) and (3.4), we have

$$\tilde{\mathcal{W}}(X,Y)\xi = -\frac{\tilde{r}}{6}\{\eta(Y)X - \eta(X)Y\}. $$

(5.5)

Thus the theorem is proved.
6. Locally $\phi$-symmetric trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

**Definition 6.1.** A trans-Sasakian manifold $M$ with respect to the Schouten-van Kampen connection is called to be locally $\phi$-symmetric if

\[
\phi^2(\bar{\nabla}_W \bar{R})(X,Y)Z = 0,
\]

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$ on $M$. This notion was introduced by Takahashi [24], for Sasakian manifolds.

We know that

\[
(\bar{\nabla}_W \bar{R})(X,Y)Z = \bar{\nabla}_W(\bar{R}(X,Y)Z) - \bar{R}(\bar{\nabla}_W X,Y)Z - \bar{R}(X,\bar{\nabla}_W Y)Z - \bar{R}(X,Y)\bar{\nabla}_W Z.
\]

(6.2)

By virtue of (3.1), above equation is reduced to

\[
(\bar{\nabla}_W \bar{R})(X,Y)Z = (\nabla_W \bar{R})(X,Y)Z + \eta(X)\bar{R}(\nabla_W \xi, Y)Z + (\nabla_W \eta)(X)\bar{R}(\xi, Y)Z + \eta(Y)\bar{R}(X,\nabla_W \xi)Z + (\nabla_W \eta)(Y)\bar{R}(X, \xi)Z + \eta(Z)\bar{R}(X, Y)\nabla_W Z + (\nabla_W \eta)(Z)\bar{R}(X,Y)\xi.
\]

(6.3)

Now differentiating (3.4) with respect to $W$, using (2.1), (2.2), (2.3), (2.5) and (2.7) we obtain

\[
(\nabla_W \bar{R})(X,Y)Z = (\nabla_W \bar{R})(X,Y)Z + \alpha^2\{g(X,Y)g(\phi Y, Z) - g(W,Y)g(\phi X, Z)\}\xi + \{g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)\}W + \alpha^2\beta\{g(\phi W, X)g(\phi Y, Z) - g(\phi W, Y)g(\phi X, Z)\}\xi + \{g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)\}z + (\alpha^2 - \beta^2)\{\alpha(g(\phi W, Y)X - g(\phi W, X)Y) - \beta^2g(\phi W, \phi Y)X + g(\phi W, \phi X)Y\}\eta(Z) + (\beta g(\phi W, \phi Z) - \alpha g(\phi W, X))(\eta(X)Y - \eta(Y)X) + \alpha^2\{g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)\}(\alpha g(\phi W, Y)X + \beta (W - \eta(W)\xi)) - \alpha^2\{\alpha (g(\phi Y, Z)g(\phi W, X) + g(\phi X, Z)g(\phi W, Y)) + \beta (g(Y, Z)g(\phi W, \phi X) + g(X, Z)g(\phi W, \phi Y))\}\xi + \alpha^2\{\alpha (g(W, \phi Z)\eta(Y) + g(W, \phi Y)\eta(Z)) - \beta (g(\phi W, \phi Z)\eta(Y) + g(\phi W, \phi Y)\eta(Z))\}X + \{\alpha (g(W, \phi Z)\eta(X) + g(W, \phi X)\eta(Z))\}Y + \beta (g(\phi W, \phi Z)\eta(X) + g(\phi W, \phi X)\eta(Z))Y.
\]

(6.4)
with respect to the Schouten-van Kampen connection ˜\nabla curvature is constant, provided α,β are constants. 

A three-dimensional trans-Sasakian manifold is locally -symmetry with respect to the Levi-Civita connection ˜\nabla if and only if the scalar curvature is constant provided α,β are constants.

Using (6.4) in (6.3) we have

\[
(\nabla_W \hat{R})(X, Y)Z = (\nabla_W R)(X, Y)Z
\]

\[
+ \alpha^2\beta[(g(X, Y)g(\phi Y, Z) - g(W, Y)g(\phi X, Z))\xi
\]

\[
+ \{g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)\}W
\]

\[
+ \alpha^2\beta[\{g(\phi W, X)g(\phi Y, Z) - g(\phi W, Y)g(\phi X, Z)\}\xi
\]

\[
+ \{g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)\}W]
\]

\[
+(\alpha^2 - \beta^2)\{\alpha(g(W, Y)X - g(W, X)Y)
\]

\[
- \beta^2(g(\phi W, \phi Y)X + g(\phi W, \phi X)Y)\}\eta(Z)
\]

\[
+(\beta g(\phi W, \phi Z) - \alpha g(\phi W, Z)(\eta(X)Y - \eta(Y)X)]
\]

\[
+ \alpha^2\beta(g(Y, Z)\eta(Y) - g(Y, Z)\eta(X))(-\alpha gW + \beta(W - \eta(W))\xi
\]

\[
- \alpha^2\beta[-\alpha(g(Y, Z)g(\phi W, X) + g(X, Z)g(\phi W, Y))
\]

\[
+ \beta g(Y, Z)g(\phi W, \phi Y) + g(X, Z)g(\phi W, \phi Y)\}\eta(Z)
\]

\[
+ \beta^2\beta[-\alpha(g(W, \phi Z)\eta(Y) + g(W, \phi Y)\eta(Z)]
\]

\[
- \beta g(\phi W, \phi Z)\eta(Y) + g(\phi W, \phi Y)\eta(Z))\}X
\]

\[
+ \{\alpha g(W, \phi Z)\eta(X) + g(W, \phi X)\eta(Z)
\]

\[
- \beta g(\phi W, \phi Z)\eta(X) + g(\phi W, \phi X)\eta(Z))\}Y]
\]

\[
+ \eta(X)\hat{R}(\nabla_W \xi, Y)Z + (\nabla_W \eta)(X)\hat{R}(\xi, Y)Z
\]

\[
+ \eta(Y)\hat{R}(X, \nabla_W \xi)Z + (\nabla_W \eta)(Y)\hat{R}(X, \xi)Z
\]

\[
+ \eta(Z)\hat{R}(X, Y)\nabla_W Z + (\nabla_W \eta)(Z)\hat{R}(X, Y)\xi.
\]

Now applying \phi^2 on both sides of (6.5) and taking X, Y, Z, W are orthogonal to \xi and using (2.1), (2.3) we get from above equation

\[
(\phi^2(\nabla_W \hat{R}))(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z.
\]

Hence we can state the following:

**Theorem 6.1.** A three-dimensional trans-Sasakian manifold is locally \phi-symmetry with respect to the Schouten-van Kampen connection ˜\nabla if and only if the manifold is also locally \phi-symmetry with respect to the Levi-Civita connection ˜\nabla provided α, β are constants.

U. C. De and Avijit Sarkar [7] have proved that a trans-Sasakian manifold is locally \phi-symmetry if and only if the scalar curvature is constant provided α, β are constants.

In view of above result we can state the following:

**Theorem 6.2.** A three-dimensional trans-Sasakian manifold is locally \phi-symmetric with respect to the Schouten-van Kampen connection ˜\nabla if and only if the scalar curvature is constant, provided α, β are constants.
7. Weyl conformally flat trans-Sasakian manifold with respect to Schouten-van Kampen connection

The Weyl conformal curvature tensor $\tilde{C}$ of type $(1,3)$ of $M$, an $n$-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is given by [23]

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-2}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y] - \frac{\tilde{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y],$$

(7.1)

where $\tilde{Q}$ is the Ricci operator with respect to the Schouten-van Kampen connection.

Let us consider that a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is Weyl conformally flat, that is $\tilde{C} = 0$. Then from (7.1), we get

$$\tilde{R}(X,Y)Z = [\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y] - \frac{\tilde{r}}{2}[g(Y, Z)X - g(X, Z)Y].$$

(7.2)

Let us take inner product of the equation (7.2) with $W$. Then we get

$$g(\tilde{R}(X,Y)Z, W) = [\tilde{S}(Y,Z)g(X, W) - \tilde{S}(X,Z)g(Y, W) + g(Y, Z)g(\tilde{Q}X, W) - g(X, Z)g(\tilde{Q}Y, W)] - \frac{\tilde{r}}{2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

(7.3)

Using (2.1), (2.3), (3.5)-(3.8), we get

$$g(\tilde{R}(X,Y)Z, W) = [S(Y,Z)g(X, W) - S(X,Z)g(Y, W) + g(Y, Z)g(QX, W) - g(X, Z)g(QY, W)] - \frac{r - 2\alpha^2}{2}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

(7.4)

Putting $X = W = \xi$ in (7.4) and using (2.1) and (2.3), we get

$$g(\tilde{R}(\xi,Y)Z, \xi) = [S(Y,Z) - S(\xi,Z)\eta(Y) + g(Y, Z)S(\xi, \xi) - \eta(\xi)S(Y, \xi)] - \frac{r}{2}[g(Y, Z) - \eta(Z)\eta(Y)].$$

(7.6)
where \( g(QY, Z) = S(Y, Z) \).

Now, using (2.13) and (2.16), we get

\[
S(Y, Z) = \frac{r}{2} g(Y, Z) + [6(\alpha^2 - \beta^2) - \frac{r}{2}] \eta(Y) \eta(Z).
\]

Therefore

\[
S(Y, Z) = ag(Y, Z) + b \eta(Y) \eta(Z),
\]

where \( a = \frac{r}{2} \) and \( b = [6(\alpha^2 - \beta^2) - \frac{r}{2}] \).

This shows that the manifold \( M \) is an \( \eta \)-Einstein manifold.

Thus we can state the following:

**Theorem 7.1.** A three-dimensional Weyl conformally flat trans-Sasakian manifold with respect to the Schouten-van Kampen connection \( \tilde{\nabla} \) is an \( \eta \)-Einstein manifold provided \( \alpha, \beta \) are constants with \( \alpha \neq \beta \).

8. **Example of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen Connection**

In this section, we wanted to construct an example of a three-dimensional trans-Sasakian manifold with respect to Schouten-van Kampen connection.

We have considered the three-dimensional manifold \( M = \{ (x, y, z) \in \mathbb{R}^3, z \neq 0 \} \), where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). The vector fields

\[
e_1 = e^{-z} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_2 = e^{-z} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z},
\]

are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined by

\[
g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.
\]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1,1)\) tensor field defined by \( \phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0 \). Then using the linearity of \( \phi \) and \( g \) we have

\[
\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z) \eta(W),
\]

for any \( Z, W \in \chi(M) \). Thus for \( e_3 = \xi, (\phi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \). Now, by direct computations we obtain

\[
[e_1, e_2] = 0, \quad [e_2, e_3] = e_2, \quad [e_1, e_3] = e_1.
\]
The Riemannian connection $\nabla$ of the metric tensor $g$ is given by the Koszul’s formula which is

$$2g(\nabla X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$  (8.1)

By Koszul formula

$$\begin{align*}
\nabla_{e_1} e_3 &= e_1, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_1 &= -e_3, \\
\nabla_{e_2} e_3 &= e_2, & \nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_1 &= 0, \\
\nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_1 &= 0.
\end{align*}$$

From above we see that the manifold satisfies (2.6) for $\alpha = 0$, $\beta = 1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, 1)$. With the help of the above results it can be verified that

$$\begin{align*}
R(e_1, e_2) e_3 &= 0, & R(e_2, e_3) e_3 &= -e_2, & R(e_1, e_3) e_2 &= -e_1, \\
R(e_1, e_2) e_2 &= e_3, & R(e_2, e_3) e_2 &= 0, & R(e_1, e_3) e_2 &= 0, \\
R(e_1, e_2) e_1 &= 0, & R(e_2, e_3) e_1 &= 0, & R(e_1, e_3) e_1 &= e_3.
\end{align*}$$

Now we consider the Schouten-Van Kampen connection to this example.

Using (3.2) and above result we have

$$\begin{align*}
\tilde{\nabla}_{e_1} e_3 &= (1 - \beta)e_1 + \alpha e_2, & \tilde{\nabla}_{e_1} e_2 &= -\alpha e_3, & \tilde{\nabla}_{e_1} e_1 &= (\beta - 1)e_3, \\
\tilde{\nabla}_{e_2} e_3 &= -\alpha e_1 + (1 - \beta)e_2, & \tilde{\nabla}_{e_2} e_2 &= (\beta - 1)e_3, & \tilde{\nabla}_{e_2} e_1 &= 0, \\
\tilde{\nabla}_{e_3} e_3 &= 0 & \tilde{\nabla}_{e_3} e_2 &= -\beta e_2, & \tilde{\nabla}_{e_3} e_1 &= -\beta e_1.
\end{align*}$$

Using (3.4) we get

$$\begin{align*}
\tilde{R}(e_1, e_2) e_3 &= 0, & \tilde{R}(e_2, e_3) e_3 &= (\beta^2 - \alpha^2 - 1)e_2, \\
\tilde{R}(e_1, e_3) e_3 &= (\beta^2 - \alpha^2 - 1)e_1, & \tilde{R}(e_1, e_2) e_2 &= \alpha^2 e_2 + (\beta^2 + \alpha^2 - 1)e_1, \\
\tilde{R}(e_2, e_3) e_2 &= (\beta^2 + \alpha^2 + 1)e_3, & \tilde{R}(e_1, e_3) e_2 &= 0, \\
\tilde{R}(e_1, e_2) e_1 &= (1 - \beta^2 - \alpha^2)e_2, & \tilde{R}(e_2, e_3) e_1 &= 0, \\
\tilde{R}(e_1, e_3) e_1 &= (1 + \alpha^2 - \beta^2)e_3.
\end{align*}$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = \sum_{i=1}^{3} g(R(e_i, e_1) e_1, e_i) = -2.$$

Similarly, we have

$$S(e_2, e_2) = -2 \quad \text{and} \quad S(e_3, e_3) = -2.$$
\[ \tilde{S}(e_1, e_2) = \tilde{S}(e_2, e_2) = 2(\beta^2 - 1) \quad \tilde{S}(e_3, e_3) = 2(\beta^2 - \alpha^2 - 1). \]

\[ r = -6 \quad \tilde{r} = 6\beta^2 - 2\alpha^2 - 6. \]

From above we see that \( \tilde{r} = 0 \) for \( \alpha = 0, \beta = 1 \). Therefore, the manifold under consideration satisfies the Theorem 5.2.

Using (4.1) and above relations, we get

\[ P(e_1, e_2)e_3 = P(e_1, e_3)e_3 = P(e_2, e_3)e_3 = 0, \]

\[ \tilde{P}(e_1, e_2)e_3 = \tilde{P}(e_1, e_3)e_3 = \tilde{P}(e_2, e_3)e_3 = 0. \]

Therefore, the manifold will be \( \xi \)-projectively flat on a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection which verifies the Theorem 4.1.

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References


