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APPLICATION OF FUZZY METRIC ON MANIFOLDS *

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Abstract. The relation between the fuzzy subsets and classical mathematics is fundamental to extend the new approaches in applied mathematics. This paper applies the concept of fuzzy metric on construction of fuzzy Hausdorff space and fuzzy manifold space. Based on these concepts, we have presented a concept of fuzzy metric Hausdorff spaces and fuzzy metric manifold spaces. This study extends the concept of fuzzy metric space to union and product of fuzzy metric spaces and, in this regard, investigates some producst of fuzzy metric fuzzy manifold spaces. Valued-level subsets play the main role in the connection of the notation of manifolds and fuzzy metrics.
Key words: fuzzy metric, Hausdorff spaces, manifold spaces.

1. Introduction

As a generalization of the classical set theory, fuzzy set theory was introduced by Zadeh to deal with uncertainties[14]. Fuzzy set theory is playing an important role in modeling and controlling unsure systems in nature, society and industry. Fuzzy set theory also plays a vital role in complex phenomena which is not easily characterized by classical set theory. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, progressive developments have been made in the field of fuzzy topology. Fuzzy topology is a fundamental branch of fuzzy theory which has become an area of active research because of its wide range of applications. One of the most important

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problems in fuzzy topology is to obtain the concept of properties of fuzzy metric space. This problem has been investigated by many authors from different points of view. In particular, George and Veeramani have introduced and studied a notion of fuzzy metric space with respect to the concept of t-norms. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable-topological spaces. This result permits Gregori and Romaguera to restate some classical theorems on metric completeness and metric (pre) compactness in the realm of fuzzy metric spaces[4]. Kaleva generalized the notion of the metric space by setting the distance between two points to be a non-negative fuzzy number. By defining an ordering and an addition in the set of fuzzy numbers, they obtained a triangle inequality which is analogous to the ordinary triangle inequality[9]. Historically, the notion of a differentiable manifold, that is, a set that looks locally like Euclidean Space, has been an integral part of various fields of mathematics. One may note their applications in the fields of Differential Topology [7], Algebraic Geometry, Algebraic Topology, and Lie Groups and their associated algebras. They will base our work upon the already well-established fuzzy structures via fuzzy topological spaces, fuzzy topological vector spaces, and fuzzy derivatives. However, the definition of a fuzzy derivative provided in Foster [7], is not easily generalized to a general k derivatives. Consequently, the existence of a fuzzy differentiable manifold of class greater than one has not yet been established. They shall give topological separation axioms that have not been given previously, for sake of completeness. Our principal approach is quite similar to the methods used in [7]. Namely, they shall take definitions in [3] of fuzzy continuity and fuzzy topological vector space, and use these notions to give a fuzzy topological vector space differential structure by constructing a fuzzy homeomorphism and, naturally, a fuzzy diffeomorphism of class k. To do so, they have provided a new definition of a fuzzy object, known as fuzzy vectors. They then define proper fuzzy directional derivatives along these abstract fuzzy vectors, and allude to their applications in manifold learning. For completeness, they shall define tangent vector spaces to these fuzzy manifolds. Additional materials regarding fuzzy topological spaces are available in the literature too [1, 2, 8, 9, 10, 11, 12, 13].

Motivation. The construction of manifolds based on fuzzy metrics is a novel idea via valued cuts. We try to connected topology spaces, Hausdorff spaces and manifolds to fuzzy metric spaces in this method. Thus this problem is a main motivation for the extension of topology spaces, Hausdorff spaces and manifolds to fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space.

Regarding these points, we have introduced the concept of C-graphable set and show that every non-empty set X is a C-graphable set. Also, we proved that for every given set with respect to the concept of C-graphable sets one can construct a metric space. The fuzzy metric spaces are not necessarily finite space, so one of our motivation of this work is a construction of finite fuzzy metric space. This study presents a concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space. The main focus of this work is the concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space based on t-norm such as Domby t-norm, Godel t-norm and etc.

2. Preliminaries

In this section, we will recall some definitions and results, which we will be needing in what follows.

Definition 2.1. [6] Let M be a Hausdorff topological space. We say that M is an n-dimensional topological manifold if it satisfies the following condition: for any $p \in M$, there exists

(1) an open subset U with $p \in U \subseteq M$,

(2) an open subset $E \subseteq \mathbb{R}^n$, and

(3) a homeomorphism $\psi : U \longrightarrow E$.

Such a U is called a (local) coordinate neighbourhood, and ψ is called a (local) coordinate function. We write $x = \psi(p)$ and regard $(x_1, \ldots, x_n)$ as local coordinates for the manifold M.

Definition 2.2. [6] Let S be a C^0-atlas for M. If $\psi_\alpha \circ \psi_\beta^{-1}$ is a C^∞-map for all $\alpha, \beta \in A$, we say that M is a C^∞-atlas for M. We call $\psi_\alpha \circ \psi_\beta^{-1}$ a coordinate transformation or transition function. The domain of the map $\psi_\alpha \circ \psi_\beta^{-1}$ is assumed to be $\psi_\beta(U_\alpha \cap U_\beta)$ (which could be the empty set). Thus, $\psi_\alpha \circ \psi_\beta^{-1}$ is a homeomorphism from $\psi_\beta(U_\alpha \cap U_\beta)$ to $\psi_\alpha(U_\alpha \cap U_\beta)$.

Definition 2.3. [6] Let M be a topological manifold. Let S be a C^∞-atlas for M. We say that M is a C^∞-manifold (or smooth manifold, or differentiable manifold). The concept of C^r-manifold can be defined in a similar way. However, from now on, manifold will always mean C^∞-manifold. The concept of complex manifold can be defined in a similar way, using coordinate charts $\psi : U \to \mathbb{C}^n$. However, the term complex manifold will always mean complex manifold with holomorphic (complex analytic) transition functions.

Definition 2.4. [7] A triplet (X, ρ, T) is called a KM-fuzzy metric space, if X is an arbitrary non–empty set, T is a left-continuous t-norm and $\rho : X^2 \times \mathbb{R}^{\geq 0} \to [0, 1]$ is a fuzzy set, such that for each $x, y, z, \in X$ and $t, s \geq 0$, we have:

(i) $\rho(x, y, 0) = 0$,

(ii) $\rho(x, x, t) = 1$ for all $t > 0$,

(iii) $\rho(x, y, t) = \rho(y, x, t)$(commutative property),

(iv) $T(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t + s)$(triangular inequality),

(vi) $\rho(x, y, -) : \mathbb{R}^{\geq 0} \to [0, 1]$ is a left-continuous map,

(vii) $\lim_{t\to\infty} \rho((x, y, t)) = 1$,

(viii) $\rho(x, y, t) = 1, \forall\ t > 0$ implies that $x = y$.

If (X, ρ, T) satisfies in conditions (i)–(vii), then it is called KM-fuzzy pseudometric space and ρ is called a KM-fuzzy pseudometric.

3. Fuzzy Topological Space

In this section, we recall some definitions and results, which we need in what follows.

Definition 3.1. Let M be an arbitrary set and $F(M) = \{\mu : M \to [0,1]\}$. A family $\mathcal{F}_\tau$ of subset of $F(M)$ is called a fuzzy topological if satisfied in the following conditions:

(i) $\mu_0 \in \mathcal{F}_\tau$ and $\mu_1 \in \mathcal{F}_\tau$;

(ii) every $i \in I, \mu_i \in \mathcal{F}_\tau$, implies that $(\bigcup\limits_{i\in I} \mu_i) \in \mathcal{F}_\tau$;

(iii) each $1 \leq i \leq n, \mu_i \in \mathcal{F}_\tau$, implies that $(\bigcap\limits_{i=1}^{n} \mu_i) \in \mathcal{F}_\tau$.

So $(M, \mathcal{F}_\tau)$ is called a fuzzy topological space and the members of $\mathcal{F}_\tau$ are called open fuzzy subsets, where $\mu_0 \equiv 0$ and $\mu_1 \equiv 1$.

Example 3.1. Consider $M = \mathbb{R}$ and $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid \text{where } \mu_i = \dfrac{i}{i+x^2} \text{ and } i \in \mathbb{N}^*\}$. One can see that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.

Theorem 3.1. *Let $\alpha \in [0,1]$. Then $(M, \mathcal{F}_\tau)$ is a fuzzy topological space if and only if $(M, \mathcal{F}_\tau^{+\alpha})$ is a topological space.*

Proof. Since $\mu_0 \in \mathcal{F}_\tau$,we get that $\mu_0^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = \emptyset$ and so $\emptyset \in \mathcal{F}_\tau^{+\alpha}$. In addition, $\mu_1 \in \mathcal{F}_\tau$, implies that $\mu_0^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = X$, then $X \in \mathcal{F}_\tau^{\alpha^+}$. Let $\{\mu_i^{\alpha^+}\}_{i\in I} \in \mathcal{F}_\tau^{\alpha^+}$. Since $\bigvee\limits_{i\in I} \mu_i(x) = (\bigcup\limits_{i\in I} \mu_i)(x)$ and for all $i \in I$, $\mu_i(x) > \alpha$, we have $\bigvee\limits_{i\in I} \mu_i(x) > \alpha$ and so $\bigcup\limits_{i\in I} \mu_i^{\alpha^+} \in \mathcal{F}_\tau^{\alpha^+}$. If $\{\mu_i^{\alpha^+}\}_{i=1}^n \in \mathcal{F}_\tau^{\alpha^+}$,then $(\bigcap\limits_{i=1}^{n} \mu_i)(x) = (\bigwedge\limits_{i=1}^{n} \mu_i)(x) = \bigwedge\limits_{i=1}^{n} (\mu_i(x)) > \alpha$. It follows that $(X, \mathcal{F}_\tau^{\alpha^+})$ is a topological space.

The converse is similar to. $\square$

Example 3.2. Consider the fuzzy topological space $(M, \mathcal{F}_\tau)$ in Example 3.1 and $\alpha = \dfrac{1}{2}$. Then $\mu_0^{\frac{1}{2}^+} = \{x \in M \mid 0 > \frac{1}{2}\} = \emptyset$, $\mu_1^{\frac{1}{2}^+} = \{x \in M \mid 1 > \frac{1}{2}\} = \mathbb{R}$, and for $i \geq 2$, $\mu_i^{\frac{1}{2}^+} = \{x \in M \mid \dfrac{i}{i+x^2} > \dfrac{1}{2}\} = \{x \in M \mid -\sqrt{i} < x < \sqrt{i}\}$. So $(\mathbb{R}, \mathcal{F}_\tau^{\frac{1}{2}^+})$ is a topological space, where $\mathcal{F}_\tau^{\frac{1}{2}^+} = \{\emptyset, \mathbb{R}, (-\sqrt{i}, \sqrt{i}) \mid i \geq 2\}$.

Theorem 3.2. *Let M' be a set where $|M| = |M'|$ and $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then there exists a fuzzy topology $\mathcal{F}'_\tau$ on M' in such a way that $(M', \mathcal{F}'_\tau)$ is a fuzzy topological space.*

Proof. Since $|M| = |M'|$, there is a bijection $\phi : M' \to M$. Consider $\mathcal{F}'_\tau = \{\mu_i \circ \phi \mid \mu_i \in \mathcal{F}_\tau\}$, clearly $(M', \mathcal{F}'_\tau)$ is a fuzzy topological space. □

Corollary 3.1. *Let M be a set where $|M| = |\mathbb{R}|$. Then there exists a topology $\mathcal{F}_\tau$ on M in such a way that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.*

Definition 3.2. Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. A subfamily $\mathcal{FB}_\tau$ of $\mathcal{F}_\tau$ is called a base if (i), for all $x \in M$, we have $\bigvee_{\mu \in \mathcal{FB}_\tau} \mu(x) = 1$ and (ii), $\mu_1, \mu_2 \in \mathcal{FB}_\tau$ implies that $\mu_1 \cap \mu_2 \in \mathcal{FB}_\tau$.

Theorem 3.3. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Then every element of $\mathcal{F}_\tau$ is included in union of elements of $\mathcal{FB}_\tau$.*

Proof. Let $\mu \in \mathcal{F}_\tau$. Then for all $x \in M$, we have $\mu(x) = \mu(x) \bigvee 1 = \mu(x) \vee (\bigvee_{\mu_i \in \mathcal{FB}_\tau} \mu_i(x)) \leq \bigvee_{\mu_i \in \mathcal{FB}_\tau} \mu_i(x)$. It follows that $\mu \subseteq \bigcup_{\mu_i \in \mathcal{FB}_\tau} \mu_i$. □

Definition 3.3. Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Define $\langle \mathcal{FB}_\tau \rangle = \{\nu \in \mathcal{F}_\tau \mid \exists\, \mu \in \mathcal{F}_\tau, \quad \mu \subseteq \nu\}$ and it called by generated fuzzy topology by $\mathcal{FB}_\tau$.

Theorem 3.4. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then $\langle \mathcal{FB}_\tau \rangle$ is a fuzzy topology on M.*

Proof. Since for all $\mu \in \mathcal{F}_\tau, \mu \subseteq \mu_1$, we get that $\mu_1 \in \langle \mathcal{FB}_\tau \rangle$. If $\mu_0 \in \langle \mathcal{FB}_\tau \rangle$, then $\mu \subseteq \mu_0$ implies that $\mu = \mu_0$. Let $\{\nu_i\}_{i \in I}$ a family of elements $\langle \mathcal{FB}_\tau \rangle$ and $\bigcup_{i \in I} \nu_i = \nu$. Then there exist $\mu_i \in \mathcal{F}_\tau$ in such a way that $\mu_i \subseteq \nu_i$. So $\mu = \bigcup_{i \in I} \mu_i \subseteq \bigcup_{i \in I} \nu_i = \nu$. Because $\mu \in \mathcal{F}_\tau$ and $\mu \subseteq \nu$, we get that $\nu \in \langle \mathcal{FB}_\tau \rangle$. Now, if for $n \in \mathbb{N}$, $\{\nu_i\}_{i=1}^n$ is a family of elements of $\langle \mathcal{FB}_\tau \rangle$ in a similar way we have $\bigcap_{i=1}^n \nu_i \in \langle \mathcal{FB}_\tau \rangle$. □

Proposition 3.1. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $(M, \mathcal{F}_\tau)$. Then $\mathcal{FB}_\tau^{\alpha^+}$ is a base for topological space $(M, \mathcal{F}_\tau^{\alpha^+})$.*

Proof. Let $\mathcal{B} = \{\mu^{\alpha^+} \mid \mu \in \mathcal{FB}_\tau, \alpha \in [0,1]\}$, $x \in M$ and $\mu(x) = \alpha$. Then $x \in \mu^{\alpha^+} \subseteq \bigcup_{\alpha \in \mathcal{FB}_\tau} \mu^{\alpha^+}$ and so $\bigcup_{\alpha \in \mathcal{FB}_\tau} \mu^{\alpha^+} = M$. Suppose $x \in \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$. Since $\mathcal{FB}_\tau$ is a base for fuzzy topological space $(M, \mathcal{F}_\tau)$, we get $\mu_1 \cap \mu_2 \in \mathcal{FB}_\tau$. Now, $x \in (\mu_1 \cap \mu_2)^{\alpha^+} \subseteq \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$ and so $\mathcal{B}$ is a base for $\mathcal{FB}_\tau$.In the following, we will construct fuzzy topological space via topological spaces. □

Definition 3.4. Let (M, τ_M) be a topological space. For all $M_i \in \tau_M$, define $\tau_{\emptyset} = \mu_0, \tau_M = \mu_1$ and for any $i \in I, \tau_{M_i} : M \to [0,1]$ by $\tau_{M_i}(x) = \mu_i(x) = \alpha_i \in [0,1]$.

So we have the following lemma.

Theorem 3.5. *Let (M, τ_M) be a topological space. Then $(M, \mathcal{F}_\tau = \{\tau_{M_i}\}_{i\in I})$ is a fuzzy topological space.*

Proof. *Since (M, τ_M) is a topological space, we have $\emptyset, M \in \tau_M$ so by definition $\mu_0 = \tau_{\emptyset}, \mu_0 = \tau_M \in \tau$. Let for any $i \in I$, we have $\mu_i \in \mathcal{F}_\tau$. Since for all $x \in M$, $(\bigcup\limits_{i\in I} \mu_i)(x) = \bigvee\limits_{i\in I} \mu_i(x) = \bigvee\limits_{M_i\in\tau_M} \tau_{M_i}(x) = (\bigcup\limits_{M_i\in\tau_M} \tau_{M_i})(x)$ and $(M.\tau_M)$ is a topological space, we get that $\bigcup\limits_{M_i\in\tau_M} \tau_{M_i} \in \tau_M$ and so $\bigcup\limits_{i\in I} \mu_i \in \mathcal{F}_\tau$. Let $n \in \mathbb{N}$ and $\{\mu_i\}_{i=1}^n$ be a set of elements $\mathcal{F}_\tau$. In a similar way, it is shoed that $\bigcap\limits_{i=1}^{n} \mu_i \in \mathcal{F}_\tau$. therefore, $(M, \mathcal{F}_\tau$ is a fuzzy topological space.* □

Corollary 3.2. *Let M be a non-empty set. Then there exists a fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.*

Definition 3.5. Let $(M, \mathcal{F}_\tau)$ and $(M^{'}, \mathcal{F}_{\tau'})$ be fuzzy topological spaces and $f : (M, \mathcal{F}_\tau) \to (M^{'}, \mathcal{F}_{\tau'})$ be a homeomorphism, define $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M', (\mathcal{F}_{\tau'})^{\alpha^+})$ by $f^{\alpha^+}(x) = f(x)$, where $x \in M$.

Theorem 3.6. *Let $(M, \mathcal{F}_\tau)$ and $(M^{'}, \mathcal{F}_{\tau'})$ be fuzzy topological spaces. If $f : (M, \mathcal{F}_\tau) \to (M^{'}, \mathcal{F}_{\tau'})$ is a bijection and a fuzzy continuous map, then $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M^{'}, (\mathcal{F}_{\tau'})^{\alpha^+})$ is a continuous map.*

Proof. *Let $(\mu')^{\alpha^+} \in (\mathcal{F}_{\tau'})^{\alpha^+}$. Then*

$$\begin{aligned}
(f^{\alpha^+})^{-1}(\mu')^{\alpha^+} &= \{x \in M \mid f^{\alpha^+}(x) \in (\mu')^{\alpha^+}\} \\
&= \{x \in M \mid f(x) \in (\mu')^{\alpha^+}\} = \{x \in M \mid \mu'(f(x)) > \alpha\} \\
&= \{x \in M \mid \exists \mu \in \mathcal{F}_\tau \ s.t\ \mu(x) > \alpha\} \\
&= \mu^{\alpha^+} \in \mathcal{F}_\tau.
\end{aligned}$$

Since f^{α^+} is a bijection, we get f^{α^+} is a homeomorphism. □

3.1. Fuzzy manifold space

In this subsection, we introduce a concept of fuzzy Hausdorff space based on valued-cuts and in this regards, the concept of fuzzy manifold space is presented.

Definition 3.6. Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then $(M, \mathcal{F}_\tau)$ is called a fuzzy Hausdorff space if, for all $x, y \in M$ there exist $\mu_1, \mu_2 \in \mathcal{F}_\tau$ and $0 \leq \alpha < \beta \leq 1$ in such a way that $x \in \mu_1^{(\alpha,\beta)^+}, y \in \mu_2^{(\alpha,\beta)^+}$ and $\mu_1^{(\alpha,\beta)^+} \cap \mu_2^{(\alpha,\beta)^+} = \emptyset$, where for all $\mu \in \mathcal{F}_\tau$, we have $\mu^{(\alpha,\beta)^+} = \{x \in M \mid \alpha < \mu(x) < \beta\}$.

Example 3.3. Consider the fuzzy topological space, which is defined in Example 3.1. Simple computations show that for $i \neq i' \in \mathbb{N}^*$

$$\begin{aligned} J_i \quad &= \quad (-\sqrt{i(\frac{1}{\alpha}-1)}, -\sqrt{i(\frac{1}{\beta}-1)}) \cup (\sqrt{i(\frac{1}{\beta}-1)}, \sqrt{i(\frac{1}{\alpha}-1)}) \\ &J_{i'} = (-\sqrt{i'(\frac{1}{\alpha}-1)}, -\sqrt{i'(\frac{1}{\beta}-1)}) \cup (\sqrt{i'(\frac{1}{\beta}-1)}, \sqrt{i'(\frac{1}{\alpha}-1)}) \end{aligned}$$

, where $J_i = \mu_i^{(\alpha,\beta)^+}$ and $J_{i'} = \mu_{i'}^{(\alpha,\beta)^+}$. If $x, y \in \mathbb{R}$, since $\mathbb{R}$ is a Hausdorff space, there exists $i, i' \in \mathbb{N}^*$ such that $x \in J_i$, $y \in J_{i'}$, and $J_i \cap J_{i'} = \emptyset$. (For all $x \in \mathbb{R}$, consider $0 \leq \alpha < \beta \leq 1$, then $\dfrac{\alpha x^2}{1-\alpha} < i < \dfrac{\beta x^2}{1-\beta}$.)

Theorem 3.7. *Let $\alpha \in [0, 1]$. Then $(M, \mathcal{F}_\tau)$ is a fuzzy Hausdorff space if and only if $(M, \mathcal{F}_\tau^\alpha)$ is a Hausdorff space.*

Proof. Since $(M, \mathcal{F}_\tau)$ is a fuzzy topological space, by Theorem 3.1, $(M, \mathcal{F}_\tau^\alpha)$ is a topological space. Let $x, y \in M$ and $\alpha, \beta \in [0, 1]$, because of $(M, \mathcal{F}_\tau)$ is a fuzzy Hausdorff space, there exist, $\mu_1, \mu_2 \in \mathcal{F}_\tau$ and $0 \leq \alpha' < \beta' < 1$ such that $x \in \mu_1^{(\alpha',\beta')^+}$ and $y \in \mu_2^{(\alpha',\beta')^+}$. Now consider $\alpha \leq T_{min}\{\alpha', \beta'\}$. It follows that, $x \in \mu^\alpha$ and $y \in \mu^\beta$ and $\mu^\alpha \cap \mu^\beta = \emptyset$. So $(M, \mathcal{F}_\tau^\alpha)$ is a Hausdorff space.

The converse is clear by Theorem 3.1. □

Definition 3.7. Let $(M, \mathcal{F}_\tau)$ be a fuzzy Hausdorff space. Then $(M, \mathcal{F}_\tau)$ is called a fuzzy manifold if, for all $x \in M$, there exists $\mu \in \mathcal{F}_\tau$ and homeomorphism $\phi : supp(\mu) \to \mathbb{R}^n$ such that $x \in Supp(\mu)$. Each (μ, ϕ) is called a fuzzy chart and $\mathcal{A} = \{(\mu, \phi) \mid \mu \in \mathcal{F}_\tau,\ \phi : supp(\mu) \to \mathbb{R}^n\}$ is called a fuzzy atlas. Let $(\mu, \phi), (\upsilon, \psi)$ be two fuzzy charts of fuzzy atlas $\mathcal{A}$. Then $(\mu, \phi), (\upsilon, \psi)$ are called C^∞-compatible charts if $\phi : supp(\mu) \to \mathbb{R}^n$, $\psi : supp(\upsilon) \to \mathbb{R}^n$ and $\phi \circ \psi^{-1} : \psi(Supp(\mu) \cap Supp(\upsilon)) \to \phi(Supp(\mu) \cap Supp(\upsilon))$ are a C^1-fuzzy diffeomorphism.

Example 3.4. Consider the fuzzy Hausdorff space, which is defined in Example 3.3. It is clear that for all $x \in \mathbb{R}$, and for all $i \in \mathbb{N}$, we have $x \in Supp(\mu_i) = \mathbb{R}$, we get that $x \in Supp(\mu_i)$. Define $(Ln)_i : Supp(\mu_i) \longrightarrow \mathbb{R}$, so $\mathcal{A} = \{(\mu_i, (Ln)_i) \mid i \in \mathbb{N}\}$ is a fuzzy atlas.

Theorem 3.8. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy manifold and $\alpha \in [0, 1]$. Then $(M, \mathcal{F}_\tau^\alpha)$ is a manifold.*

Proof. Since $(M, \mathcal{F}_\tau)$ is a fuzzy manifold, by Theorem 3.7, $(M, \mathcal{F}_\tau^\alpha)$ is a Hausdorff topological space. In addition, for all $x \in M$, there exists $\mu \in \mathcal{F}_\tau$ and homeomorphism $\phi : supp(\mu) \to \mathbb{R}^n$ such that $x \in Supp(\mu)$. Now consider $\mathcal{A}^\alpha = \{(\mu^\alpha, \phi) \mid \mu \in$

$\mathcal{F}_\tau,\ \phi : supp(\mu) \to \mathbb{R}^n\}$. One can see that $\mathcal{A}^\alpha$ is an atlas. Thus $(M, \mathcal{F}_\tau^\alpha)$ is a manifold. $\square$

In the following, we want to extend two fuzzy topological space to a larger class of fuzzy topological spaces.

Definition 3.8. Let $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ be fuzzy topology on M and M', respectively. Define $\mathcal{F}_\tau \times \mathcal{F}_{\tau'} = \{\mu \times \mu' \mid \mu \in \mathcal{F}_\tau, \mu' \in \mathcal{F}_{\tau'}\}$, where for all $x, y \in M \times M'$ and $(\mu \times \mu')(x, y) = T_{min}(\mu(x), \mu(y))$.

Theorem 3.9. *Let $(M, \mathcal{F}_\tau)$ and $(M', \mathcal{F}_{\tau'})$ be fuzzy topological spaces. Then $(M \times M', \mathcal{F}_\tau \times \mathcal{F}_{\tau'})$ is a fuzzy topological space.*

Proof. Since $\mu_0 \in \mathcal{F}_\tau \cap \mathcal{F}_{\tau'}$ for all $(x, y) \in M \times M'$ we have $(\mu_0 \times \mu_0)(x, y) = T_{min}\{\mu_0(x), \mu_0(y)\} = 0$, and so $(\mu_0 \times \mu_0) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. In addition, $\mu_1 \in \mathcal{F}_\tau \cap \mathcal{F}_{\tau'}$, implies that $(\mu_1 \times \mu_1)(x, y) = T_{pr}\{\mu_1(x), \mu_1(y)\} = 1$ and so $\mu_1 \times \mu_1 \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Let $\{\mu_i\}_{i\in I}$ and$\{\mu'_j\}_{j\in J}$ be two families of fuzzy topologies on M and M', respectively. Then $(\bigcup\limits_{i\in I, j\in J} (\mu_i \times \mu'_j))(x, y) = (\bigvee\limits_{i\in I, j\in J} (\mu_i \times \mu'_j))(x, y) = \bigvee\limits_{i\in I, j\in J} (T_{min}(\mu_i \times \mu'_j))$ implies that $\bigcup\limits_{i\in I, j\in J} (\mu_i \times \mu'_j) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Let $\{\mu_i\}_{i=1}^n$ and $\{\mu'_j\}_{j=1}^m$ be two families of fuzzy topologies on M and M',respectively. Then $(\bigcap\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j))(x, y) = (\bigvee\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j))(x, y) = \bigvee\limits_{i=1,j=1}^{n,m} (T_{min}(\mu_i, \mu'_j))$ implies that $\bigcap\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Thus $(M \times M', \mathcal{F}_\tau \times \mathcal{F}_{\tau'})$ is a fuzzy topological space. $\square$

4. Extended fuzzy metric space

In this section, we extend the concept of fuzzy metric spaces to a larger class of fuzzy metric spaces. From now on, for all $x, y \in [0, 1]$ we consider $T_{min}(x, y) = \min\{x, y\}$, $T_{pr}(x, y) = xy, T_{lu}(x, y) = \max(0, x + y - 1), T_{do}(x, y) = \dfrac{xy}{x + y - xy}$ and $\mathcal{C}_T = \{T : [0, 1] \times [0, 1] \to [0, 1] \mid T \text{ is a continuous t-norm}\}$. In the following, we apply the concept of fuzzy metric for constructing of fuzzy metric space on any given non-empty set.

Theorem 4.1. *If (X, ρ, T_{min}) is a fuzzy metric space and $T \in \mathcal{C}_T$. Then (X, ρ, T) is a fuzzy metric space.*

Proof. Let $x, y, z \in X,\ r, s \in \mathbb{R}^+$ and $T \in \mathcal{C}_T$. Since for all $x, y \in [0, 1], T(x, y) \leq T_{min}(x, y)$, we get that $T(\rho(x, y, t), \rho(y, z, s)) \leq T_{min}(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t+s)$. Hence (X, ρ, T) is a fuzzy metric space. $\square$

Let X be an arbitrary set and $\alpha \in \mathbb{R}^+$. For all $x, y \in X$, define $d_\alpha : X \times X \to \mathbb{R}$ by $d_\alpha(x,y) = \begin{cases} 0 & \text{if } x = y \\ \alpha & \text{if } x \neq y \end{cases}$ as an α-discrete metric. So we have the following theorem.

Theorem 4.2. *Let X be an arbitrary set. Then there exists a fuzzy set $\rho : X^2 \times \mathbb{R}^+ \to [0,1]$, such that (X, ρ, T_{min}) is a fuzzy metric space.*

Proof. Let $|X| \geq 2$ and $\alpha \in \mathbb{R}^+$ be a fixed element. Clearly (X, d_α) is a metric space, now for all $x, y \in X, m, s, t \in \mathbb{R}^+$, define $\rho : X^2 \times \mathbb{R}^+ \to [0,1]$ by $\rho(x,y,t) = \dfrac{\varphi(t)}{\varphi(t) + md_\alpha(x,y)}$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function and for all $x, y \in X$, we have $\varphi(t) + md_\alpha(x,y) \neq 0$. Now, we show that (X, ρ, T_{min}) is a fuzzy metric space. We prove only the triangular inequality and for all $x, y, z \in X$, consider the five cases $x = y = z, x = y \neq z, x = z \neq z, x \neq y = z$ and $x \neq y \neq z$. For $x = y \neq z$, since $\varphi(t+s) \geq \varphi(s)$, we have $\varphi(t+s)(\varphi(s)+m\alpha) - \varphi(s)(\varphi(t+s)+m\alpha) \geq 0$ and so $\dfrac{\varphi(s)}{\varphi(s)+m\alpha} \leq \dfrac{\varphi(t+s)}{\varphi(t+s)+m\alpha}$. The other cases have been proven in a similar way and so (X, ρ, T_{min}) is a fuzzy metric space. □

Corollary 4.1. *Let X be an arbitrary set. Then there exists a fuzzy set $\rho : X^2 \times \mathbb{R}^+ \to [0,1]$, such that for all $T \in \mathcal{C}_T$, (X, ρ, T) is a fuzzy metric space.*

Example 4.1. Let $X = \{a, b, c\}$. For all $x, y \in X$, define $\rho(x,y,t) = \dfrac{t^5}{t^5 + d_3(x,y)}$. Then by Corollary 4.1, (X, ρ, T_{pr}) is a fuzzy metric space.

4.1. Finite fuzzy metric space based on metric

In this subsection, we apply the concept of finite metric for constructing of fuzzy metric space on any given non-empty set.

Definition 4.1. Let X be a finite set. We say that X is a C-graphable set, if $G = (X, E)$ is a connected graph, where $E \subseteq X \times X$ and $G = (X, E)$ is called an X-derived graph.

Let $\mathcal{G}_X$ be the set of all connected graphs which are constructed on X as the set of vertices, so we have the following results.

Lemma 4.1. *Let X be a non-empty set.*

(i) X is a C-graphable set.

(ii) For every graph $G = (V, E)$, there exists a non-empty set X such that G is an X-derived connected graph.

Proof. (i) Let $n \in \mathbb{N}$ and $|X| = n$. Then we construct $G = (X, E)$ as a complete graph($G = (X, E) \cong K_{|X|}$), so X is a C-graphable set.

(ii) Since every graph $G = (V, E)$ is isomorphic to a subgraph of a complete graph $K_{|X|}$, where $|X| = |V|$, so $G = (V, E)$ is an X-derived connected graph. □

Let $G = (X, E)$ be a connected graph. For all $x, y \in X$, define $d^g(x, y) = \min\{|P_{x,y}|$ where $P_{x,y}$ is a path between $x, y\}$. Obviously, d^g is a metric on X.

Theorem 4.3. *Let X be a finite set and $|X| \geq 2$. Then there exists a non-discrete metric d on X such that (X, d) is a metric space.*

Proof. Let $|X| \geq 2$. By Lemma 4.1, X is a C-graphable set and so there exists a graph $G = (X, E) \in \mathcal{G}_X$. For all $x, y \in X$, define $d(x, y) = d^g(x, y)$. Clearly (X, d^g) is a metric space. $\square$

Corollary 4.2. *Let $n \in \mathbb{N}, X$ be a set and $|X| = n$.*

(*i*) *If $G = (X, E) \cong K_n$ is the complete graph, then for metric spaces (X, d^g) and (X, d_1), we have $d^g = d_1$.*

(*ii*) *If $G = (X, E) \cong C_n$ is the cycle graph, then for metric spaces (X, d^g) and (X, d_1), we have $d_1 \leq d^g \leq d_{n-3}$.*

Theorem 4.4. *Let X be a non-empty set. Then there exists a fuzzy subset $\rho : X^2 \times \mathbb{R}^+ \to [0, 1]$, such that (X, ρ, T_{pr}) is a fuzzy metric space.*

Proof. Let $|X| \geq 2$. Then by Lemma 4.1, X is a C-graphable set and by Theorem 4.3, (X, d^g) is a metric space. For all $x, y \in X$ and for all $m, t \in \mathbb{R}^+$, define $\rho(x, y, t) = \dfrac{\varphi(t)}{\varphi(t) + md^g(x, y)}$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function($\varphi(t)+md^g(x, y) \neq 0$). Now, we show that (X, ρ, T_{pr}) is a fuzzy metric space and in this regard, only prove triangular inequality property. Let $x, y, z \in X$. Since for all $s, t, m \in \mathbb{R}^+$,

$$\begin{aligned}&\varphi(t+s)\varphi(s)md^g(x, y) + \varphi(t+s)\varphi(t)md^g(y, z)\\&\geq \varphi(t)\varphi(s)md^g(x, y) + \varphi(s)\varphi(t)md^g(y, z)\\&\geq \varphi(s)\varphi(t)md^g(x, z), \text{ and } m^2d^g(y, z)d^g(y, z)\varphi(t+s) > 0\end{aligned}$$

we get that $T_{pr}(\dfrac{\varphi(t)}{\varphi(t) + d^g(x, y)}, \dfrac{\varphi(s)}{\varphi(s) + d^g(y, z)}) \leq \dfrac{\varphi(t+s)}{\varphi(t+s) + d^g(x, z)}$. It follows that $T_{pr}(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t+s)$ and so (X, ρ, T_{pr}) is a fuzzy metric space. $\square$

Corollary 4.3. *Let X be a non-empty set. Then there exists a fuzzy subset $\rho : X^2 \times \mathbb{R}^+ \to [0, 1]$, such that for all continuous t-norm $T \leq T_{pr}$, (X, ρ, T) is a fuzzy metric space.*

4.2. Operations on fuzzy metric spaces

In this subsection, we extend fuzzy metric spaces to union and product of fuzzy metric spaces. Let (X_1, ρ_1, T) and (X_2, ρ_2, T) be fuzzy metric spaces, $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$ and $t \in \mathbb{R}^+$. For an arbitrary $T \in \mathcal{C}_T$, define $T(\rho) : (X_1 \times X_2)^2 \times \mathbb{R}^+ \to [0, 1]$ by $T(\rho)\big((x_1, y_1), (x_2, y_2), t\big) = T(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t))$. So we have the following theorem.

Theorem 4.5. *Let (X_1, ρ_1, T) and (X_2, ρ_2, T) be fuzzy metric spaces. Then $(X_1 \times X_2, T_{min}(\rho), T)$ is a fuzzy metric space.*

Proof. Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X_1\times X_2$ and $t,s\in\mathbb{R}^+$.

(i) Since for all $x_1,x_2\in X_1, y_1,y_2\in X_2, \rho_1(x_1,x_2,t)>0$ and $\rho_2(y_1,y_2,t)>0$, we have $T_{min}(\rho)\big((x_1,y_1)$
$,(x_2,y_2),t\big)>0$.

(ii) $T_{min}(\rho)\big((x_1,y_1),(x_2,y_2),t\big)=1$ if and only if $T_{min}(\rho_1(x_1,x_2,t),\rho_2(y_1,y_2,t))=1$ if and only if $\rho_1(x_1,x_2,t)=\rho_2(y_1,y_2,t)=1$ if and only if $(x_1,y_1)=(x_2,y_2)$.

(iii) It is clear that $T_{min}(\rho)$ is a commutative map.

(iv)

$$\begin{aligned}
& T\big(T_{min}(\rho)((x_1,y_1),(x_2,y_2),t),T_{min}(\rho)((x_2,y_2),(x_3,y_3),s)\big)\\
= \;& T\big(T_{min}(\rho_1(x_1,x_2,t),\rho_2(y_1,y_2,t)),T_{min}(\rho_1(x_2,x_3,s),\rho_2(y_2,y_3,s))\big)\\
\leq \;& T_{min}(T\big(\rho_1(x_1,x_2,t),\rho_1(x_2,x_3,s)\big),T\big(\rho_2(y_1,y_2,t),\rho_2(y_2,y_3,s)\big)))\\
\leq \;& T_{min}(\rho_1(x_1,x_3,t+s),\rho_2(y_1,y_3,t+s))\\
= \;& T_{min}(\rho)((x_1,y_1),(x_3,y_3),t+s).
\end{aligned}$$

Thus $(X_1\times X_2,T_{min}(\rho),T)$ is a fuzzy metric space. □

Example 4.2. (i) Consider $X_1=X_2=[0,1]$. For all $x,y\in[0,1]$ and $t\in\mathbb{R}^+$, define $\rho_1(x,y,t)=\rho_2(x,y,t)=1-|x-y|$. It is easy to check that (X_1,ρ_1,T_{lu}) and (X_2,ρ_2,T_{lu}) are fuzzy metric spaces and by Theorem 4.5, $(X_1\times X_2,T_{min}(\rho),T_{lu})$ is a fuzzy metric space.

(ii) Let $t\in\mathbb{R}^+$. Then $(\mathbb{R}^+,\rho_1,T_{pr})$ and $(\mathbb{N},\rho_2,T_{pr})$ are fuzzy metric spaces, where $\rho_1(x,y,t)=\dfrac{\min(x,y)+t}{\max(x,y)+t}$ and $\rho_2(x,y,t)=\dfrac{\min(x,y)}{\max(x,y)}$. Applying Theorem 4.5, $(\mathbb{R}^+\times\mathbb{N},\rho,T_{pr})$ is a fuzzy metric space, where

$$\rho((x_1,y_1),(x_2,y_2),t)=\min\{\frac{\min(x_1,x_2)+t}{\max(x_1,x_2)+t},\frac{\min(y_1,y_2)}{\max(y_1,y_2)}\}.$$

Let $X_1\cap X_2=\emptyset$, (X_1,ρ_1,T) and (X_2,ρ_2,T) be fuzzy metric spaces, $x,y\in X_1\cup X_2$ and $t\in\mathbb{R}^+$. Consider $\epsilon(x,y,t)=\bigwedge_{\substack{x,u\in X_1\\ y,v\in X_2}}(\rho_1(x,u,t)\wedge\rho_2(y,v,t)))$, define $\rho_1\cup\rho_2:(X_1\cup X_2)^2\times\mathbb{R}^+\to[0,1]$ by $(\rho_1\cup\rho_2)(x,y,t)=\begin{cases}\rho_1(x,y,t), & x,y\in X_1,\\ \rho_2(x,y,t), & x,y\in X_2,\\ \epsilon(x,y,t), & x\in X_1,y\in X_2,\end{cases}$. So we have the following theorem.

Theorem 4.6. *Let (X_1,ρ_1,T) and (X_2,ρ_2,T) be fuzzy metric spaces. Then $(X_1\cup X_2,\rho_1\cup\rho_2,T)$ is a fuzzy metric space, where $X_1\cap X_2=\emptyset$.*

Proof. Let $x,y,z\in X_1\cup X_2$ and $t\in\mathbb{R}^+$. We only prove the triangular inequality property and other cases are immediate.

Let $x,y\in X_1$(for $x,y\in X_2$ is similar), then

$$T\big((\rho_1\cup\rho_2)(x,y,t),(\rho_1\cup\rho_2)(y,z,s)\big)=T\big(\rho_1(x,y,t),(\rho_1\cup\rho_2)(y,z,s)\big).$$

If $z \in X_1$, then

$$\begin{aligned} & T\big((\rho_1 \cup \rho_2)(x,y,t), (\rho_1 \cup \rho_2)(y,z,s)\big) = T\big(\rho_1(x,y,t), \rho_1(y,z,s)\big) \\ \leq \;& T\big(\rho_1(x,y,t), \rho_1(y,z,s)\big) \leq \rho_1(x,z,t+s) = (\rho_1 \cup \rho_2)(x,z,t+s). \end{aligned}$$

If $z \in X_2$, then

$$T\big((\rho_1 \cup \rho_2)(x,y,t), (\rho_1 \cup \rho_2)(y,z,s)\big) = T\big(\rho_1(x,y,t), \epsilon\big) \leq \epsilon = (\rho_1 \cup \rho_2)(x,z,t+s).$$

Let $x \in X_1, y \in X_2$. Then

$$T\big((\rho_1 \cup \rho_2)(x,y,t), (\rho_1 \cup \rho_2)(y,z,s)\big) = T\big(\epsilon, (\rho_1 \cup \rho_2)(y,z,s)\big).$$

If $z \in X_2$, since $x \in X_1$ and $y \in X_2$, we get that $(\rho_1 \cup \rho_2)(x,z,t+s) = \epsilon$ and so

$$T\big(\epsilon, (\rho_1 \cup \rho_2)(y,z,s)\big) = T\big(\epsilon, \rho_2(y,z,s)\big) \leq \epsilon = (\rho_1 \cup \rho_2)(x,z,t+s).$$

If $z \in X_1$, since $x \in X_1$ and $y \in X_2$, we get that $(\rho_1 \cup \rho_2)(x,z,t+s) \neq \epsilon$ and so

$$T\big(\epsilon, (\rho_1 \cup \rho_2)(y,z,s)\big) = T\big(\epsilon, \epsilon\big) \leq \epsilon \leq \rho_1(x,z,t+s)\big) = (\rho_1 \cup \rho_2)(x,z,t+s).$$

It follows that $(X_1 \cup X_2, \rho_1 \cup \rho_2, T)$ is a fuzzy metric space. $\square$

Example 4.3. Consider $X_1 = \{a,b,c\}$ and $X_2 = \{d,e\}$. Then (X_1, ρ_1, T_{min}) and (X_2, ρ_2, T_{min}) are fuzzy metric spaces as follows:

$$\begin{aligned} &\rho_1(a,a,t) = \rho_1(b,b,t) = \rho_1(c,c,t) = 1, \\ &\rho_1(a,b,t) = \rho_1(b,c,t) = \frac{t}{t+4}, \text{ and } \rho_1(a,c,t) = \frac{t}{t+2}, \\ &\rho_2(d,d,t) = \rho_2(e,e,t) = 1 \text{ and } \rho_2(d,e,t) = \frac{t}{t+1}. \end{aligned}$$

Using Theorem 4.6, $(X_1 \cup X_2, \rho_1 \cup \rho_2, T_{min})$ is a fuzzy metric space as follows:

$$\begin{aligned} &(\rho_1 \cup \rho_2)(a,a,t) = (\rho_1 \cup \rho_2)(b,b,t) = (\rho_1 \cup \rho_2)(c,c,t) = (\rho_1 \cup \rho_2)(d,d,t) = \\ &(\rho_1 \cup \rho_2)(e,e,t) = 1, \\ &(\rho_1 \cup \rho_2)(a,b,t) = (\rho_1 \cup \rho_2)(b,c,t) = (\rho_1 \cup \rho_2)(b,d,t) = (\rho_1 \cup \rho_2)(d,c,t) = \\ &(\rho_1 \cup \rho_2)(a,e,t) = (\rho_1 \cup \rho_2)(e,c,t) = (\rho_1 \cup \rho_2)(b,e,t) = (\rho_1 \cup \rho_2)(a,d,t) = \frac{t}{t+4} \end{aligned}$$

and $(\rho_1 \cup \rho_2)(a,c,t) = \dfrac{t}{t+2}$.

Corollary 4.4. *Let (X_1, ρ, T) and (X_2, ρ, T) be fuzzy metric spaces, where $X_1 \cap X_2 = \emptyset$. Then $(X_1 \cup X_2, \rho, T)$ is a fuzzy metric space.*

4.3. Fuzzy metric manifolds

In this subsection, we introduce the concept of fuzzy metric topological space and investigate its properties.

Definition 4.2. Let (M, ρ, T) be a fuzzy metric space and $F(M) = \{\mu : M \to [0, 1]\}$. Then $(M, \rho, T, \mathcal{F}_\tau)$ is called a fuzzy metric topological space, if

(i) $\mathcal{F}_\tau$ is a fuzzy topology on M;

(ii) for all $x, y \in M$, $t \in \mathbb{R}^+$ and $\mu_1 \neq \mu \in \mathcal{F}_\tau$, we have $T(\mu(x), \mu(y)) \leq \rho(x, y, t)$.

Example 4.4. Let $M = \mathbb{R}$. Then (M, T_{pr}, ρ) is a fuzzy metric space, where for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}^+, \rho(x, y, t) = |\dfrac{\min\{x, y\} + t}{\max\{x, y\} + t}|$. It is easy to see that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space, where $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid \text{where } \mu_i = \dfrac{i}{i + x^2} \text{ and } i \in \mathbb{N}^*\}$.

Theorem 4.7. *Let (M, τ_M) be a topological space and T be a t-norm on M. Then there exists a fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$ such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.*

Proof. *Let $\mu \in \mathcal{F}_\tau, t \in \mathbb{R}^+$ and $x, y \in M$. Define*

$$\rho(x, y, t) = \begin{cases} T(\mu(x), \mu(y)) & \text{if } x \neq y, \\ 1 & \text{otherwise.} \end{cases}$$

Now, we show that $H = (M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space. By definition, for all $x \in M$, $\rho(x, x, t) = 1$ and for all $y \in M$, $\rho(x, y, t) > 0$. Let $x, y, z \in M$. Then for all $t, s \in \mathbb{R}^+$

$$\begin{aligned} T(\rho(x, y, t), \rho(x, y, s)) &= T(T(\mu(x), \mu(y), T(\mu(y), \mu(z))) \\ &\leq T(\mu(x), \mu(z)) = \rho(x, y, t + s). \end{aligned}$$

So (M, ρ, T) is a fuzzy metric space. Let $x, y \in M, t \in \mathbb{R}^+$ and $\mu_1 \neq \mu_2 \in \mathcal{F}_\tau$. Hence $T(\mu(x), \mu(y)) = \rho(x, y, t) \leq \rho(x, y, t)$. Therefore, $(M, \rho, T, \mathcal{F}_\tau$ is a fuzzy metric topological space. $\square$

Theorem 4.8. *Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space, $x, y \in M$ and $\alpha \in [0, 1]$. Then*

(i) If $\mu = \mu_1$, then $T(\mu(x), \mu(y)) \leq \rho(x, y, t)$ implies that $x = y$.

(ii) For any $i \in I, \mu_i \in \mathcal{F}_\tau$, we have $T((\bigcup_{i \in I} \mu_i)(x), (\bigcup_{i \in I} \mu_i)(y)) \leq \rho(x, y, t)$,

(iii) For any $i \in \mathbb{N}, \mu_i \in \mathcal{F}_\tau$, we get $T((\bigcap_{i=1}^{n} \mu_i)(x), (\bigcap_{i=1}^{n} \mu_i)(y)) \leq \rho(x, y, t)$.

Proof. (i) since for all $x \in M$, $\mu(x) = 1$, we get $T(\mu(x), \mu(x)) = T(1, 1) = 1 \leq \rho(x, y, t)$. Thus $\rho(x, y, t) = 1$ and so $x = y$.

(ii) Let $x, y \in M$. Then $T((\bigcap_{i=1}^{n} \mu_i)(x), (\bigcap_{i=1}^{n} \mu_i)(y)) = T(\bigwedge_{i=1}^{n} \mu_i(x), \bigwedge_{i=1}^{n} \mu_i(y) \leq T(\mu_i(x), \mu_i(y)) \leq \rho(x, y, t)$.

(iii) Since $(M, \mathcal{F}_\tau)$ is a fuzzy topological space, for all $i \in I$ and $\mu_i \in \mathcal{F}_\tau, \mu = \bigcup_{i \in I} \mu_i \in \mathcal{F}_\tau$. So

$$T((\bigcup_{i \in I} \mu_i)(x), (\bigcup_{i \in I} \mu_i)(y)) = T(\mu(x), \mu(y)) \leq \rho(x, y, t).$$

□

Theorem 4.9. *Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space and $\alpha \in [0, 1]$. Then $(M, \rho, T, \mathcal{F}_\tau^{+\alpha})$ is a topological space.*

Proof. It is similar to Theorem 3.1. □

Theorem 4.10. *Let M be a non-empty set. Then there exists a t-norm T, a fuzzy metric subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$, and fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.*

Proof. Let M be a non-empty set. Then there exists a topology τ on M such that (M, τ) is a topological space. Using Theorem 3.2, there exists a fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space. We apply Theorem 4.4, so there exists a fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$, such that (M, ρ, T_{pr}) is a fuzzy metric space. Thus there exists a t-norm T, a fuzzy metric subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$, and fuzzy topology $\mathcal{F}_\tau$ such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space. □

Theorem 4.11. *Let M' be a set where $|M| = |M'|$ and $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space. Then there exists a topology $\mathcal{F'}_\tau$ on M' and fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$ in such a way that $(M', \rho, T, \mathcal{F'}_\tau)$ is a fuzzy metric topological space.*

Proof. Since $|M| = |M'|$, there is a bijection $\phi : M' \to M$. Consider $\mathcal{F'}_\tau = \{\mu_i \circ \phi \mid \mu_i \in \mathcal{F}_\tau\}$, clearly $(M', \mathcal{F'}_\tau)$ is a fuzzy topological space. Based on Theorem 4.7, there exists a fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$ and a t-norm T such that $(M', \rho, T, \mathcal{F'}_\tau)$ is a fuzzy metric topological space. □

Corollary 4.5. *Let M be a set where $|M| = |\mathbb{R}|$. Then there exists a fuzzy topology $\mathcal{F}_\tau$ on M, a fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0, 1]$ and a t-norm T such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.*

In the following, results are similar to previous section.

Definition 4.3. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space. A subfamily $\mathcal{FB}_\tau$ of $\mathcal{F}_\tau$ is called a base if (i), for all $x \in M$, we have $\bigvee_{\mu \in \mathcal{FB}_\tau} \mu(x) = 1$ and $(ii), \mu_1, \mu_2 \in \mathcal{FB}_\tau$ implies that $\mu_1 \cap \mu_2 \in \mathcal{FB}_\tau$.

Theorem 4.12. *Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topology space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Then every element of $\mathcal{F}_\tau$ is inclosed in union of elements of $\mathcal{FB}_\tau$.*

Definition 4.4. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Define $\langle \mathcal{FB}_\tau \rangle = \{\nu \in \mathcal{F}_\tau | \ \exists \ \mu \in \mathcal{F}_\tau, \quad \mu \subseteq \nu\}$ and it called by generated topology by $\mathcal{FB}_\tau$.

Theorem 4.13. *Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space. Then $\langle \mathcal{FB}_\tau \rangle$ is a fuzzy topology on M.*

Proposition 4.1. *Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space and $\mathcal{FB}_\tau$ be a base for $(M, \rho, T, \mathcal{F}_\tau)$. Then $\mathcal{FB}_\tau^{\alpha^+}$ is a base for topological space $(M, \mathcal{F}_\tau^{\alpha^+})$.*

Definition 4.5. Let $(M, \rho, T, \mathcal{F}_\tau)$ and $(M', \rho, T, \mathcal{F}_{\tau'})$ be fuzzy metric topological spaces and $f : (M, \mathcal{F}_\tau) \to (M', \mathcal{F}_{\tau'})$ be a homeomorphism, define $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M', (\mathcal{F}_{\tau'})^{\alpha^+})$ by $f^{\alpha^+}(x) = f(x)$, where $x \in M$.

Theorem 4.14. *Let $(M, \rho, T, \mathcal{F}_\tau)$ and $(M', \rho, T, \mathcal{F}_{\tau'})$ be fuzzy metric topological spaces. If $f : (M, \rho, T, \mathcal{F}_\tau) \to (M', \rho, T, \mathcal{F}_{\tau'})$ be a fuzzy continuous map, then $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M', (\mathcal{F}_{\tau'})^{\alpha^+})$ is a continuous map.*

Definition 4.6. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space. Then $(M, \rho, T, \mathcal{F}_\tau)$ is called a fuzzy metric Hausdorff space if, for all $x, y \in M$ there exist $\mu_1, \mu_2 \in \mathcal{F}_\tau$ in such a way that $x \in supp(\mu_1), y \in supp(\mu_2)$ and $\mu_1 \cap \mu_2 = \emptyset$ where $Supp(\mu) = \{x \mid \mu(x) \neq 0\}$ and $\mu_1 \cap \mu_2 = \emptyset$ means that $(\mu_1 \cap \mu_2)(x) = 0$.

Definition 4.7. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric Hausdorff space. Then $(M, \rho, T, \mathcal{F}_\tau)$ is called a fuzzy metric manifold if, for all $x \in M$, there exists $\mu \in \mathcal{F}_\tau$ and homeomorphism $\phi : supp(\mu) \to \mathbb{R}^n$ such that $x \in Supp(\mu)$.Each (μ, ϕ) is called a fuzzy chart and $\mathcal{A} = \{(\mu, \phi) \mid \mu \in \mathcal{F}_\tau, \ \phi : supp(\mu) \to \mathbb{R}^n\}$ is called a fuzzy atlas.Let $(\mu, \phi), (\upsilon, \psi)$ be two fuzzy chart of fuzzy atlas $\mathcal{A}$. Then $(\mu, \phi), (\upsilon, \psi)$ are called C^∞-compatible charts if $phi : supp(\mu) \to \mathbb{R}^n$, $\psi : supp(\upsilon) \to \mathbb{R}^n$ and $\phi \circ \psi^{-1} : \psi(Supp(\mu) \cap Supp(\upsilon)) \to \phi(Supp(\mu) \cap Supp(\upsilon))$ are a C^1-fuzzy diffeomorphism.

Example 4.5. Consider the fuzzy Hausdorff space, which is defined in Example 4.4. It is clear that for all $x \in \mathbb{R}$, and for all $i \in \mathbb{N}$, we have $x \in Supp(\mu_i) = \mathbb{R}$, we get that $x \in Supp(\mu_i)$. Define $(Ln)_i : Supp(\mu_i) \longrightarrow \mathbb{R}$, so $\mathcal{A} = \{(\mu_i, (Ln)_i) \mid i \in \mathbb{N}\}$ is a fuzzy atlas. Now, define for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}^+, \rho(x, y, t) = |\dfrac{\min\{x, y\} + t}{\max\{x, y\} + t}|$. It is easy to see that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space, where $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid$ where $\mu_i = \dfrac{i}{i + x^2}$ and $i \in \mathbb{N}^*\}$ and consequently $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric manifold.

5. Conclusion

The current paper has introduced a novel concept of fuzzy Hausdorff space, fuzzy manifold space. Also:

(i) Based on fuzzy metric spaces, every non empty set converted to a fuzzy metric space.

(ii) It is showed that the product and union of fuzzy metric spaces is a fuzzy metric space.

(iii) The extended fuzzy metric spaces are constructed using the some algebraic operations on fuzzy metric spaces.

(iv) The concept of fuzzy Hausdorff space and fuzzy manifold space has been defined and some of its properties have been investigated.

One of advantage of this work is approaches of manifolds based on fuzzy subsets via fuzzy metric that any one based any welldefined fuzzy metric present a new manifold. We hope that these results are helpful for further studies in theory of fuzzy metric Hausdorff space fuzzy metric manifold space. In our future studies, we hope to obtain more results regarding instuitic metric Hausdorff spaces, neutrosophic metric manifold spaces and their applications.

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