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ON GENERALIZED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES VIA IDEALS IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we have given $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence with regards to the intuitionistic fuzzy norm (μ, v), investigate their relationships, and make some observations about these classes. Also, we have examined the relation between these two new methods and the relation between $\mathcal{I}_2$-statistical convergence in the corresponding intuitionistic fuzzy normed space.
Key words: ideal; $I2$-lacunary statistical convergence; intuitionistic fuzzy normed space; banach space; strongly convergence.

1. Introduction

Statistical convergence of a real number sequence was firstly originated by Fast [15]. It became a notable topic in summability theory after the work of Fridy [16] and Šalát [50]. This concept was constracted to the double sequences by Mursaleen and Edely [43]. Some beneficial results on this topic can be found in [6, 22, 24, 36, 37, 38, 39, 40, 54].

Theory of $\mathcal{I}$-convergence of sequences in a metric space was given by Kostyrko et al. [32]. Other investigations and applications of ideals can be found in the study Das and Ghosal [8], Das et al. [9] and Savaş and Das [51]. Belen et al. [5] generalized the notions of statistical convergence, (λ, μ)-statistical convergence, (V, λ, μ) summability and $(C, 1, 1)$ summability for a double sequence via ideals. The other studies of this concept were examined by [21, 33, 41, 42, 44, 45, 47].

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Using lacunary sequence, Fridy and Orhan [17] examined the concept of lacunary statistical convergence. Afterwards, it was developed by Fridy and Orhan [18], Li [35], Mursaleen and Mohiuddine [46], Bakery [3]. Çakan and Altay [7] provided multidimensional analogues of the results presented by Fridy and Orhan [17]. Lacunary ideal convergence of real sequences was inquiried by Tripathy et al. [55].

Fuzzy set theory has become an important working area after the study of Zadeh [56]. Atanassov [1] investigated intuitionistic fuzzy set; this concept was utilized by Atanassov et al. [2] in the study of decision-making problems. The idea of an intuitionistic fuzzy metric space was put forward by Park [48]. In [19], it was shown that the topology generated by every IF-metric coincides with the topology generated by its F-metric. Hence, the definition of an IF-metric space needed some refinement, in the light of having independent results. In [34], motivated by Park's definition of an IF-metric, the authors defined an IF-normed spaces (IFNS for shortly) and then investigated, among other results, the fundamental theorems: open mapping, closed graph and uniform boundedness in IFNS. Several studies of the convergence of sequences in some normed linear spaces with a fuzzy setting might be revealed by the research of [10, 11, 12, 13, 14, 23, 25, 26, 27, 28, 29, 30, 31, 52, 53].

Let us start with fundamental definitions from the literature.

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|\{m \leq n : m \in K\}|,$$

where $|m \leq n : m \in K|$ denotes the number of elements of K not exceeding m.

A number sequence $x = (x_m)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq n : |x_m - L| \geq \varepsilon\}| = 0,$$

i.e.,

$$|x_m - L| < \varepsilon \quad (a.a.\ m) \tag{1.1}$$

In this case we write $st - \lim x_m = L$. For example, define $x_m = 1$ if m is a square and $x_m = 0$ otherwise. Then, $|\{m \leq n : x_m \neq 0\}| \leq \sqrt{n}$, so $st - \lim x_m = 0$. Note that we could have assigned any values whatsoever to x_m when m is a square, and we would still have $st - \lim x_m = 0$. But x is neither convergent nor bounded. It is clear that if the inequality in (1.1) holds for all but finitely many m, then $\lim x_m = L$. Statistical convergence is a natural generalization of ordinary convergence. It follows that $\lim x_m = L$ implies $st - \lim x_m = L$, so statistical convergence may be considered as a regular summability method. The sequence that converges statistically need not be convergent and also need not be bounded.

A double sequence $x = (x_{mn})$ has Pringsheim limit L (denoted by $P - \lim = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever

$m, n > N$. We shall describe such an (x_{mn}) more briefly as "P-convergent." In Pringsheim convergence the row-index m and the column-index n tend to infinity independently from each other [49].

The essential deficiency of this kind of convergence is that a convergent sequence does not require to be bounded. Hardy [20] defined the concept of regular sense, does not have this hortcoming, for double sequence. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides the convergent in Pringsheim's sense.

The notion of Cesàro summable double sequences was given by [40]. Note that if a bounded sequence (x_{mn}) is statistically convergent then it is also Cesàro summable but not conversely.

Let $(x_{mn}) = (-1)^m$, $\forall n$; then $\lim_{p,r} \sum\limits_{m=1}^{p} \sum\limits_{n=1}^{r} x_{mn} = 0$, but obviously x is not statistically convergent.

The convergence of double sequences play an important role not only in pure mathematics but also in other branches of science involving computer science, biological science and dynamical systems. Also, the double sequence can be use in convergence of double trigonometric series and in the opening series of double functions and in the making differential solution.

In the wake of the study of ideal convergence defined by Kostyrko et al. [32], there has been comprehensive research to discover applications and summability studies of the classical theories.

Let $\emptyset \neq S$ be a set, and then a non empty class $\mathcal{I} \subseteq P(S)$ is said to be an *ideal* on S iff (i) $\emptyset \in \mathcal{I}$, (ii) $\mathcal{I}$ is additive under union, (iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we find $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \emptyset$ and $S \notin \mathcal{I}$. A non-empty family of sets $\mathcal{F}$ is called *filter* on S iff (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we get $A \cap B \in \mathcal{F}$, (iii) for every $A \in \mathcal{F}$ and each $B \supseteq A$, we obtain $B \in \mathcal{F}$. Relationship between ideal and filter is given as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset S : K^c \in \mathcal{I}\},$$

where $K^c = S - K$.

A non-trivial ideal $\mathcal{I}$ is (i) an *admissible ideal* on S iff it contains all singletons.

A sequence (x_m) is said to be ideal convergent to L if for every $\varepsilon > 0$, i.e.

$$A(\varepsilon) = \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}.$$

Taking $\mathcal{I} = \mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ indicates the asymptotic density of set A. If $\mathcal{I}_\delta$ is a non-trivial admissible ideal then ideal convergence coincides with statistical convergence.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$, and l_∞^2 as the space of all bounded double sequences.

A double sequence $\overline{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0,\ h_u = k_u - k_{u-1} \to \infty \ \ and \ \ l_0 = 0,\ \overline{h}_s = l_s - l_{s-1} \to \infty, \quad u, s \to \infty.$$

We will use the following notation $k_{us} := k_u l_s$, $h_{us} := h_u \overline{h}_s$ and θ_{us} is determined by

$$J_{us} := \{(k,l) : k_{u-1} < k \leq k_u \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_u := \frac{k_u}{k_{u-1}},\ \overline{q}_s := \frac{l_s}{l_{s-1}} \text{ and } q_{us} := q_u \overline{q}_s.$$

Throughout the paper, by $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

A double sequence $x = \{x_{mn}\}$ of numbers is said to be $\mathcal{I}_2$-lacunary statistical convergent or $S_{\theta_2}(\mathcal{I}_2)$-convergent to L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_u \overline{h}_s} \left|\{(m,n) \in J_{us} : |x_{mn} - L| \geq \varepsilon\}\right| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $x_{mn} \to L\,(S_{\theta_2}(\mathcal{I}_2))$ or $S_{\theta_2}(\mathcal{I}_2)\text{-}\lim_{m,n\to\infty} x_{mn} = L$.

The concept of IFNS was given by Lael and Nourouzi [34]. In order to have a different topology from the topology generated by the F-norm μ, the condition $\mu + v \leq 1$ was omitted from Park's definition.

The triplicate (X, μ, v) is said to be an IF-normed space if X is a real vector space, and μ, v are F-sets on $X \times F$ satisfying the following conditions for every $x, y \in X$ and $t, s \in \mathbb{R}$:

(a) $\mu(x,t) = 0$ for all non-positive real number t,

(b) $\mu(x,t) = 1$ for all $t \in \mathbb{R}^+$ iff $x = 0$,

(c) $\mu(cx,t) = \mu\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,

(d) $\mu(x+y, s+t) \geq \min\{\mu(x,t), \mu(y,s)\}$,

(e) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

(f) $v(x,t) = 1$ for all non-positive real number t,

(g) $v(x,t) = 0$ for all $t \in \mathbb{R}^+$ iff $x = 0$,

(h) $v(cx,t) = v\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,

$(\imath)$ $\max\{v(x,t), v(y,s)\} \geq v(x+y, t+s)$,

(i) $\lim_{t\to\infty} v(x,t) = 0$ and $\lim_{t\to 0} v(x,t) = 1$.

In this case, we will call (μ, v) an IF-norm on X. In addition, (X, μ) is called an F-normed space.

In this study, we deal with the relation between these two new methods and with relations between $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence introduced by the author in an IFNS. Also, we examine the relation between the $\mathcal{I}_2$-lacunary statistical convergence and $\mathcal{I}_2$-statistical convergence in an IFNS.

2. Main Results

Definition 2.1. Let $(X, \mu, v, *, \Theta)$ be an IFNS, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. A sequence $x=(x_{kj})$ is said to be $\mathcal{I}_2$-statistically convergent to $\xi \in X$ with regards to the IFN (μ, v), and is demonstrated by $S(\mathcal{I}_2)^{(\mu,v)} - \lim x = \xi$ or $x_{k,j} \overset{(\mu,v)}{\to} \xi\,(S(\mathcal{I}_2))$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\{k \le m, j \le n : \mu(x_{kj} - \xi, t) \le 1 - \varepsilon \text{ or } \nu(x_{kj} - \xi, t) \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

Definition 2.2. A sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-lacunary statistically convergent to $\xi \in X$ with regards to the IFN (μ, v), and is demonstrated by $S_\theta(\mathcal{I}_2)^{(\mu,v)} - \lim x = \xi$ or $x_{k,j} \overset{(\mu,v)}{\to} \xi\,(S_\theta(\mathcal{I}_2))$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\left|\{(k,j) \in J_{ru} : \mu(x_{kj} - \xi, t) \le 1 - \varepsilon \text{ or } \nu(x_{kj} - \xi, t) \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

Definition 2.3. A sequence $x = (x_{kj})$ is said to be strongly $\mathcal{I}_2$-lacunary convergent to ξ or $N_\theta(\mathcal{I}_2)$-convergent to $\xi \in X$ with regards to the IFN (μ, v) and is denoted by $x_{kj} \overset{(\mu,v)}{\to} \xi\,(N_\theta(\mathcal{I}_2))$, if for every $\delta > 0$ and $t > 0$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}} \mu(x_{kj} - \xi, t) \le 1 - \delta \text{ or } \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}} \nu(x_{kj} - \xi, t) \ge \delta\right\} \in \mathcal{I}_2.$$

Theorem 2.1. *Let $(X, \mu, v, *, \Theta)$ be an IFNS, θ be a double lacunary sequence, $\mathcal{I}_2$ be a strongly admissible ideal in $\mathbb{N}$, and $x = (x_{jk}) \in X$, then*

(i) (a) If $x_{kj} \overset{(\mu,v)}{\to} \xi\,(N_\theta(\mathcal{I}_2))$ then $x_{kj} \overset{(\mu,v)}{\to} \xi\,(S_\theta(\mathcal{I}_2))$.

(b) If $x \in l^2_\infty(X)$, the space of all bounded sequences of X and $x_{kj} \overset{(\mu,v)}{\to} \xi\,(S_\theta(\mathcal{I}_2))$ then $x_{kj} \overset{(\mu,v)}{\to} \xi\,(N_\theta(\mathcal{I}_2))$.

(ii) $S_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l^2_\infty(X) = N_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l^2_\infty(X)$.

Proof. $(i)-(a)$. By hypothesis, for every $\varepsilon>0$, $\delta>0$ and $t>0$, let $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(N_{\theta}\left(\mathcal{I}_2\right)\right)$. Then we can write

$$\begin{aligned}
&\sum_{(k,j)\in J_{ru}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)\\
&\geq\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)\leq 1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)\geq\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)\\
&\geq\varepsilon.\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|.
\end{aligned}$$

Then observe that

$$\frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|\geq\delta$$

and

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}}\mu\left(x_{kj}-\xi,t\right)\leq(1-\varepsilon)\,\delta\text{ or }\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}}\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\delta,$$

which implies

$$\begin{aligned}
&\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\\
&\subset\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\{\sum_{(k,j)\in J_{ru}}\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\\
&\quad\text{or}\sum_{(k,j)\in J_{ru}}\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\}\geq\varepsilon\delta\}.
\end{aligned}$$

Since $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(N_{\theta}\left(\mathcal{I}_2\right)\right)$, we immediately see that $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(S_{\theta}\left(\mathcal{I}_2\right)\right)$.

$(i)-(b)$. We assume that $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(S_{\theta}\left(\mathcal{I}_2\right)\right)$ and $x\in l_{\infty}^{2}(X)$. The inequalities $\mu\left(x_{kj}-\xi,t\right)\geq 1-M$ or $\nu\left(x_{kj}-\xi,t\right)\leq M$ hold for all k,j. Let $\varepsilon>0$ be given. Then we have

$$\begin{aligned}
&\tfrac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)\\
&=\tfrac{1}{h_r\overline{h}_u}\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)\leq 1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)\geq\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)\\
&+\tfrac{1}{h_r\overline{h}_u}\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)>1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)<\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)\\
&\leq\tfrac{M}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|+\varepsilon.
\end{aligned}$$

Note that

$$A_{\mu,v}(\varepsilon,t)=\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h_u}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\ge\frac{\varepsilon}{M}\}$$

belongs to $\mathcal{I}_2$. If $r\in(A_{\mu,v}(\varepsilon,t))^c$ then we have

$$\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\mu(x_{kj}-\xi,t)>1-2\varepsilon \text{ or } \frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\nu(x_{kj}-\xi,t)<2\varepsilon.$$

Now

$$T_{\mu,v}(\varepsilon,t)=\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\mu(x_{kj}-\xi,t)\le 1-2\varepsilon \text{ or } \frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\nu(x_{kj}-\xi,t)\ge 2\varepsilon\}.$$

Hence, $T_{\mu,v}(\varepsilon,t)\subseteq A_{\mu,v}(\varepsilon,t)$ and so, by the definition of an ideal, $T_{\mu,v}(\varepsilon,t)\in\mathcal{I}_2$. Therefore, we conclude that $x_{kj}\overset{(\mu,v)}{\to}\xi(N_\theta(\mathcal{I}_2))$.

(ii) This readily follows from $(i)-(a)$ and $(i)-(b)$. □

Theorem 2.2. *Let $(X,\mu,v,*,\Theta)$ be an IFNS. If θ be a double lacunary sequence with $\liminf_r q_r>1$, $\liminf_u q_u>1$ then*

$$x_{kj}\overset{(\mu,v)}{\to}\xi(S(\mathcal{I}_2))\Rightarrow x_{kj}\overset{(\mu,v)}{\to}\xi(S_\theta(\mathcal{I}_2)).$$

Proof. Suppose first that $\liminf_r q_r>1$, $\liminf_u q_u>1$ then there exists a $\alpha,\beta>0$ such that $q_r\ge 1+\alpha$, $q_u>1+\beta$ for sufficiently large r,u, which implies that

$$\frac{h_r\overline{h_u}}{k_{ru}}\ge\frac{\alpha\beta}{(1+\alpha)(1+\beta)}.$$

If $x_{kj}\overset{(\mu,v)}{\to}\xi(S(\mathcal{I}_2))$, then for every $\varepsilon>0$ and for sufficiently large r,u, we have

$$\begin{aligned}&\frac{1}{k_{ru}}|\{k\le k_r,j\le j_u:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\\ &\ge\frac{1}{k_{ru}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\\ &\ge\tfrac{\alpha\beta}{(1+\alpha)(1+\beta)}\left(\tfrac{1}{h_r\overline{h_u}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\right)\end{aligned}$$

Then for any $\delta > 0$, we get

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{h_r \overline{h}_u} \left|\{(k,j) \in J_{ru} : \mu(x_{kj}-\xi,t) \le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t) \ge \varepsilon\}\right| \ge \delta\right\}$$

$$\subseteq \{(r,u) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{k_{ru}} |\{k \le k_r,\, j \le j_u : \mu(x_{kj}-\xi,t) \le 1-\varepsilon$$
$$\text{or } \nu(x_{kj}-\xi,t) \ge \varepsilon\}| \ge \tfrac{\delta\alpha\beta}{(1+\alpha)(1+\beta)}\}.$$

If $x_{kj} \overset{(\mu,v)}{\to} \xi(S(\mathcal{I}_2))$ then the set on the right-hand side belongs to $\mathcal{I}_2$ and so the set on the left-hand side belongs to $\mathcal{I}_2$. This shows that $x_{kj} \overset{(\mu,v)}{\to} \xi(S_\theta(\mathcal{I}_2))$. $\square$

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(\mathcal{I}_2)$, $\bigcup\{n : k_{r-1} < n \le k_r, r \in C\} \in F(\mathcal{I}_2)$.

Theorem 2.3. *Let* $(X,\mu,v,*,\Theta)$ *be an IFNS. If* θ *be a double lacunary sequence with* $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$ *then*

$$x_{kj} \overset{(\mu,v)}{\to} \xi(S_\theta(\mathcal{I}_2)) \Longrightarrow x_{jk} \overset{(\mu,v)}{\to} \xi(S(\mathcal{I}_2)).$$

Proof. If $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$ then without any loss of generality we can assume that there exists a M, $N > 0$ such that $q_r < M$ and $q_u < N$ for all r,u. Suppose that $x_{kj} \overset{(\mu,v)}{\to} \xi(S_\theta(\mathcal{I}_2))$, and let

$$C_{ru} := |\{(k,j) \in J_{ru} : \mu(x_{kj}-\xi,t) \le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t) \ge \varepsilon\}|.$$

Since $x_{kj} \overset{(\mu,v)}{\to} \xi(S_\theta(\mathcal{I}_2))$, it follows that for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \overline{h}_u} |\{(k,j) \in J_{ru} : \mu(x_{kj}-\xi,t) \le 1-\varepsilon$$
$$\text{or } \nu(x_{kj}-\xi,t) \ge \varepsilon\}| \ge \delta\} \in \mathcal{I}_2.$$

Hence, we can choose a positive integers $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{C_{ru}}{h_r \overline{h}_u} < \delta, \text{ for all } r > r_0, u > u_0.$$

Now let

$$K := \max\{C_{ru} : 1 \le r \le r_0, 1 \le u \le u_0\}$$

and let t and v be any integers satisfying $k_{r-1} < t \le k_r$ and $j_{u-1} < v \le j_u$. Then, we have

$$\frac{1}{tv}\left|\left\{k\leq t,\, j\leq v:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon \text{ or } \quad \nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\ \right\}\right|$$

$$\leq\frac{1}{k_{r-1}j_{u-1}}\left|\left\{k\leq k_r,\, j\leq j_u:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|$$

$$\leq\frac{1}{k_{r-1}j_{u-1}}\left(C_{11}+C_{12}+C_{21}+C_{22}+...+C_{r_0u_0}+...C_{ru}\right)$$

$$\leq\tfrac{K}{k_{r-1}j_{u-1}}.r_0u_0+\tfrac{1}{k_{r-1}j_{u-1}}\Big(h_{r_0}\overline{h}_{u_{r0}+1}\tfrac{C_{r_0,u_0+1}}{h_{r_0}\overline{h}_{u_0+1}}+h_{r_{0+1}}\overline{h}_{u_0}\tfrac{C_{r_{0+1},u_0}}{h_{r_{0+1}}\overline{h}_{u_0}}+\ldots$$
$$+h_r\overline{h}_u\tfrac{C_{ru}}{h_r\overline{h}_u}\Big)$$

$$\leq\tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\tfrac{1}{k_{r-1}j_{u-1}}\left(\sup_{r>r_0,u>u_0}\tfrac{C_{ru}}{h_r\overline{h}_u}\right)\left(h_{r_0}\overline{h}_{u_0+1}+h_{r_{0+1}}\overline{h}_{u_0}+...+h_r\overline{h}_u\right)$$

$$\leq\tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\varepsilon.\tfrac{(k_r-k_{r_0})(j_u-j_{u-1})}{k_{u-1}j_{s-1}}$$

$$\leq\tfrac{u_0s_0.K}{k_{r-1}j_{u-1}}+\varepsilon.q_u.q_s\leq\tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\varepsilon.M.N$$

Since $k_{r-1}j_u\to\infty$ as $t,v\to\infty$, it follows that

$$\frac{1}{tv}\left|\left\{k\leq t,\, j\leq v:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\ \right\}\right|\to 0$$

and consequently for any $\delta_1>0$, the set

$$\left\{(t,v)\in\mathbb{N}\times\mathbb{N}:\frac{1}{tv}\left|\left\{k\leq t,j\leq v:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|\right\}\in\mathcal{I}_2.$$

This shows that $x_{jk}\overset{(\mu,v)}{\to}\xi\left(S\left(\mathcal{I}_2\right)\right)$. $\square$

Combining Theorem 2.2 and Theorem 2.3 we have

Theorem 2.4. *Let θ be a strongly lacunary sequence. IFNS. If $1<\liminf_r q_r\leq\limsup_r q_r<\infty,$ and $1<\liminf_u q_u\leq\limsup_u q_u<\infty$ then*

$$x_{kj}\overset{(\mu,v)}{\to}\xi\left(S_\theta\left(\mathcal{I}_2\right)\right)\Leftrightarrow x_{kj}\overset{(\mu,v)}{\to}\xi\left(S\left(\mathcal{I}_2\right)\right).$$

Proof. This readily follows from Theorem 2.2 and Theorem 2.3. $\square$

Theorem 2.5. *Let $(X,\mu,v,*,\Theta)$ be an IFNS such that $\frac{1}{4}\varepsilon_{mn}\Theta\frac{1}{4}\varepsilon_{mn}<\frac{1}{2}\varepsilon_{mn}$ and $\left(1-\frac{1}{4}\varepsilon_{mn}\right)*\left(1-\frac{1}{4}\varepsilon_{mn}\right)>1-\frac{1}{2}\varepsilon_{mn}$. If X is a Banach space then $S_\theta\left(\mathcal{I}_2\right)^{(\mu,v)}\cap l^2_\infty\left(X\right)$ is a closed subset of $l^2_\infty\left(X\right)$.*

Proof. We first assume that $(x^{mn}) = \left(x_{kj}^{mn}\right)$ be a convergent sequence in $S_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l_\infty^2(X)$. Suppose $x^{(mn)}$ convergent to x. It is clear $x \in l_\infty^2(X)$. We need to show that $x \in S_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l_\infty^2(X)$. Since $x^{mn} \in S_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l_\infty^2(X)$ there exists real numbers L_{mn} such that

$$x_{kj}^{mn} \overset{(\mu,v)}{\to} L_{mn}(S_\theta(\mathcal{I}_2)) \text{ for } m,n = 1,2,3,...$$

Take a double sequence $\{\varepsilon_{mn}\}$ of strictly decreasing positive numbers converging to zero. Then for every $m,n = 1,2,3,...$ there is positive N_{mn} such that if $m,n \geq N_{mn}$ then $\sup_{m,n} \nu(x - x^{mn}, t) \leq \frac{\varepsilon_{mn}}{4}$. Without loss of generality assume that $N_{mn} = mn$ and choose a $\delta > 0$ such that $\delta < \frac{1}{5}$. Now set

$$A_{\mu,v}(\varepsilon_{mn}, t) = \left\{ \begin{array}{c} (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h_u}} \left|\{(k,j) \in J_{ru} : \right. \\ \mu\left(x_{kj}^{mn} - L_{mn}, t\right) \leq 1 - \frac{\varepsilon_{mn}}{4} \text{ or} \\ \left. \nu\left(x_{kj}^{mn} - L_{mn}, t\right) \geq \frac{\varepsilon_{mn}}{4}\}\right| < \delta \end{array} \right\}$$

belongs to $F(\mathcal{I}_2)$ and

$$B_{\mu,v}(\varepsilon_{m+1,n+1}, t) = \left\{ \begin{array}{c} (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h_u}} \left|\{(k,j) \in J_{ru} : \right. \\ \mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \leq 1 - \frac{\varepsilon_{m+1,n+1}}{4} \text{ or} \\ \left. \nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \geq \frac{\varepsilon_{m+1,n+1}}{4}\}\right| < \delta \end{array} \right\}$$

belongs to $F(\mathcal{I}_2)$. Since $A_{\mu,v}(\varepsilon_{mn}, t) \cap B_{\mu,v}(\varepsilon_{m+1,n+1}, t) \in F(\mathcal{I}_2)$ and $\emptyset \notin F(\mathcal{I}_2)$, we can choose $(r,u) \in A_{\mu,v}(\varepsilon_{mn}, t) \cap B_{\mu,v}(\varepsilon_{m+1,n+1}, t)$. Then

$$\begin{aligned} &\tfrac{1}{h_r\overline{h_u}} \left|\{(k,j) \in J_{ru} : \mu\left(x_{kj}^{mn} - L_{mn}, t\right) \leq 1 - \tfrac{\varepsilon_{mn}}{4} \text{ or } \nu\left(x_{kj}^{mn} - L_{mn}, t\right) \geq \tfrac{\varepsilon_{mn}}{4}\right. \\ &\vee \mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \leq 1 - \tfrac{\varepsilon_{m+1,n+1}}{4} \text{ or } \nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \\ &\left.\geq \tfrac{\varepsilon_{m+1,n+1}}{4}\}\right| \leq 2\delta < 1. \end{aligned}$$

Since $h_r\overline{h_u} \to \infty$ and $A_{\mu,v}(\varepsilon_{mn}, t) \cap B_{\mu,v}(\varepsilon_{m+1,n+1}, t) \in F(\mathcal{I}_2)$ is finite, we can choose the above r,u so that $h_r\overline{h_u} > 5$. Hence there must exist a $(k,j) \in J_{ru}$ for which we have simultaneously, $\mu\left(x_{kj}^{mn} - L_{mn}, t\right) > 1 - \frac{\varepsilon_{mn}}{4}$ or $\nu\left(x_{kj}^{mn} - L_{mn}, t\right) < \frac{\varepsilon_{mn}}{4}$ and $\mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) > 1 - \frac{\varepsilon_{mn}}{4}$ or $\nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) < \frac{\varepsilon_{mn}}{4}$. For a given $\varepsilon_{mn} > 0$ choose $\frac{\varepsilon_{mn}}{2}$ such that $\left(1 - \frac{1}{2}\varepsilon_{mn}\right) * \left(1 - \frac{1}{2}\varepsilon_{mn}\right) > 1 - \varepsilon_{mn}$ and $\frac{1}{2}\varepsilon_{mn}\Theta\frac{1}{2}\varepsilon_{mn} < \varepsilon_{mn}$. Then it follows that

$$\nu\left(L_{mn} - x_{kj}^{mn}, \frac{t}{2}\right)\Theta\nu\left(L_{m+1,n+1} - x_{kj}^{m+1,n+1}, \frac{t}{2}\right) \leq \frac{\varepsilon_{mn}}{4}\Theta\frac{\varepsilon_{mn}}{4} < \frac{\varepsilon_{mn}}{2}$$

and

$$\begin{aligned} \nu\left(x_{kj}^{mn} - x_{kj}^{m+1,n+1}, t\right) &\leq \sup_{m,n} \nu\left(x - x^{mn}, \tfrac{t}{2}\right)\Theta \sup_{m,n} \nu\left(x - x^{m+1,n+1}, \tfrac{t}{2}\right) \\ &\leq \tfrac{\varepsilon_{mn}}{4}\Theta\tfrac{\varepsilon_{mn}}{4} < \tfrac{\varepsilon_{mn}}{2}. \end{aligned}$$

Hence, we have

$$\begin{aligned}\nu\left(L_{mn}-L_{m+1,n+1},t\right) &\leq \Big[\nu\left(L_{mn}-x_{kj}^{mn},\tfrac{t}{3}\right)\Theta\nu\left(x_{kj}^{m+1,n+1}-L_{m+1,n+1},\tfrac{t}{3}\right)\\ &\quad \Theta\nu\left(x_{kj}^{mn}-x_{kj}^{m+1,n+1},\tfrac{t}{3}\right)\Big]\\ &\leq \tfrac{\varepsilon_{mn}}{2}\Theta\tfrac{\varepsilon_{mn}}{2}<\varepsilon_{mn}\end{aligned}$$

and similarly $\mu\left(L_{mn}-L_{m+1,n+1},t\right)>1-\varepsilon_{mn}$. This implies that $\{L_{mn}\}_{m,n\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_{mn}\to L\in X$ as $m,n\to\infty$. We shall prove that $x\overset{(\mu,v)}{\to}L_{mn}\left(S_\theta\left(\mathcal{I}_2\right)\right)$. For anay $\varepsilon>0$ and $t>0$, choose $(m,n)\in\mathbb{N}\times\mathbb{N}$ such that $\varepsilon_{mn}<\frac{1}{4}\varepsilon$, $\sup_{m,n}\nu\left(x-x^{mn},t\right)<\frac{1}{4}\varepsilon$, $\nu\left(L_{mn}-L,t\right)>1-\frac{1}{4}\varepsilon$ or $\nu\left(L_{mn}-L,t\right)<\frac{1}{4}\varepsilon$. Now since

$$\begin{aligned}&\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\nu\left(x_{kj}-L,t\right)\geq\varepsilon\right\}\right|\\ &\quad\leq\tfrac{1}{h_r\overline{h}_u}\Big|\Big\{(k,j)\in J_{ru}:\nu\left(x_{kj}-x_{kj}^{mn},\tfrac{t}{3}\right)\Theta\\ &\quad\Big[\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\Theta\nu\left(L_{mn}-L,\tfrac{t}{3}\right)\Big]\geq\varepsilon\Big\}\Big|\\ &\quad\leq\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\geq\tfrac{\varepsilon}{2}\right\}\right|\end{aligned}$$

and similarly

$$\begin{aligned}&\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-L,t\right)\leq1-\varepsilon\right\}\right|\\ &\quad>\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\leq1-\tfrac{\varepsilon}{2}\right\}\right|.\end{aligned}$$

It follows that

$$\begin{aligned}&\Big\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-L,t\right)\leq1-\varepsilon\right.\right.\\ &\qquad\left.\left.\text{or }\nu\left(x_{kj}-L,t\right)\geq\varepsilon\right\}\right|\geq\delta\}\\ &\quad\subset\Big\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\Big|\Big\{(k,j)\in J_{ru}:\mu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\leq\\ &\qquad1-\tfrac{\varepsilon}{2}\text{ or }\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\geq\tfrac{\varepsilon}{2}\Big\}\Big|\geq\delta\Big\}\end{aligned}$$

for any given $\delta>0$. Hence we have $x\overset{(\mu,v)}{\to}L_{mn}\left(S_\theta\left(\mathcal{I}_2\right)\right)$. $\square$

3. Conclusion

In this paper we introduce the notions of $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence with respect to the IFN (μ,v), investigate their relationship, and make some observations about these classes. Our study of $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence of sequences in IFN spaces also provides a tool to deal with convergence problems of sequences of fuzzy real numbers. These results can be used to study the convergence problems of sequences of fuzzy numbers having a chaotic pattern in IFN spaces.

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