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TRANSLATION-FAVORABLE FLAT SURFACES IN 3-SPACES

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Abstract. In the paper, we obtain the complete classification of Translation-Factorable (TF-) surfaces with vanishing Gaussian curvature in Euclidean and Minkowski 3-spaces.
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1. Introduction

In the study of the differential geometries of surfaces in 3-spaces, it is the most popular to examine curvature properties or the relationships between the corresponding curvatures of them. Let M be a surface in 3-spaces and (x, y, z) rectangular coordinates. It is well known that M is called as translation or factorable (homothetical) surface if it is locally described as the graph of $z = f(x) + g(y)$ or $z = f(x)g(y)$, respectively. Translation surfaces having constant mean curvature (CMC) or constant Gaussian curvature (CGC) in 3-spaces have been studied in [1, 4, 15, 16, 22, 23]. Furthermore, translation surfaces in 3-spaces satisfying Weingarten condition have been studied by Dillen et. all in [10], by Sipus in [22] and also by Sipus and Dijvak in [23]. On the other hand, factorable (homothetical) surfaces whose curvatures satisfy certain conditions have been investigated in [2, 3, 17]. As an exception, surfaces with vanishing curvature have been also very much focused. It is well known that M is called as flat or minimal surface if the Gaussian curvature or the mean curvature vanishes, respectively. The study of flat or minimal surfaces have found many applications in differential geometry

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and also physics, (see in [5, 11, 24, 25]). Very recently, as a generalization of these surfaces, Difi, Ali and Zoubir described a new type surfaces called with translation-factorable (TF) surfaces in Euclidean 3-space in [9]. Moreover, author investigated these surfaces in Galilean 3-spaces, in [14]. In that paper, authors studied on the position vector of this new type surface in the 3-dimensional Euclidean space and Lorentzian-Minkowski space satisfying the special condition $\Delta r_i = \lambda_i r_i$, where Δ denotes the Laplace operator.

The main interest of this paper is to obtain the complete classification of Translation-Factorable (TF-) surfaces with vanishing Gaussian curvatures in 3-spaces, starting from this new type of surface, called as Translation-Factorable (TF-) surfaces, defined in [9]. In Sect. 2, we introduce the notations that we are going to use and give a brief summary of basic definitions in theory of surfaces in Euclidean and Minkowski 3-spaces. In Sect. 3 and 4, we give the complete classification of TF-flat surfaces in the Euclidean 3-space and Minkowski 3-space, respectively.

2. Preliminiaries

Let Euclidean and Minkowski 3-spaces denote with $\mathbb{E}^3$ and $\mathbb{E}^3_1$, respectively. One may introduce an euclidean and Lorentzian inner products between $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ as

$$\langle u, v\rangle = (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2 \quad \text{and} \quad \langle u, v\rangle_L = (d\xi_0)^2 + (d\xi_1)^2 - (d\xi_2)^2.$$

Here (ξ_0, ξ_1, ξ_2) is rectangular coordinate system of 3-spaces. These inner products induce in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ a norm in a natural way:

$$\|u\| = \sqrt{|\langle u, u\rangle|} \quad \text{and} \quad \|u\|_L = \sqrt{|\langle u, u\rangle|_L},$$

respectively. In addition, the corresponding cross products in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ shall be showed here by $\wedge$ and $\wedge_L$, respectively: notice that $\wedge_L$ should be computed as

$$u \wedge_L v = e_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - e_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} - e_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Let M^2 be a surface in $\mathbb{E}^3$ or $\mathbb{E}^3_1$. If M^2 is parameterized by an immersion

$$x(u^1, u^2) = \Big(x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\Big),$$

then M^2 is a regular surface if and only if the corresponding cross products of x_1 and x_2 don't vanish anywhere. Here, $x_k = \partial x/\partial u^k$, $k = 1, 2$. So, the normal vector field $\mathbf{N}$ of a regular surface M^2 in $\mathbb{E}^3$ or $\mathbb{E}^3_1$ is given by

$$\mathbf{N} = \frac{x_1 \wedge x_2}{\|x_1 \wedge x_2\|} \quad \text{or} \quad \mathbf{N}_L = \frac{x_1 \wedge_L x_2}{\|x_1 \wedge_L x_2\|_L}. \tag{2.1}$$

The first fundamental form of $x : U \longrightarrow M^2 \subset \mathbb{E}^3$ (or $\mathbb{E}^3_1$) is defined as:

$$I = g_{ij} du^i du^j, \quad g_{ij} = \langle x_i, x_j\rangle \quad \text{or} \quad g_{ij} = \langle x_i, x_j\rangle_L. \tag{2.2}$$

The second fundamental form II in simply and pseudo-isotropic spaces is with differentiable coefficients

$$II = h_{ij} du^i du^j, \quad h_{ij} = \langle \mathbf{N}, x_{ij} \rangle \quad \text{or} \quad h_{ij} = \langle \mathbf{N}, x_{ij} \rangle_L . \tag{2.3}$$

Therefore, the Gaussian curvature K and the mean curvature H of surface Σ are defined by, respectively,

$$K = \frac{h_{11}h_{22} - {h_{12}}^2}{W^2}, \tag{2.4}$$

$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2W^2}, \tag{2.5}$$

where $W = \sqrt{|g_{11}g_{22} - {g_{12}}^2|}$. Note that if $g_{11}g_{22} - {g_{12}}^2 < 0$ or $g_{11}g_{22} - {g_{12}}^2 > 0$, then the surface M^2 in $\mathbb{E}^3_1$ is called as time-like or space-like surface, respectively.

Now, first we would like to give the definition of the translation-factorable (TF-) surfaces in $\mathbb{E}^3$ defined in [9]. And then we would like to complete the definition of translation-factorable (TF-) surfaces in $\mathbb{E}^3_1$ given in same paper as follows:

Definition 2.1. Let M^2 be a surface in Euclidean 3-space. Then M is called a translation-factorable (TF-) surface if it can be locally written as following:

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))), \tag{2.6}$$

where f and g are some real functions and A, B are non-zero constants.

Definition 2.2. Let M^2 be a surface in Minkowski 3-space, $\mathbb{E}^3_1$. Then M is called a translation-factorable (TF-) surface if it can be locally written as one of the followings:

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))), \tag{2.7}$$

or

$$x(s,t) = (A(f(s) + g(t)) + B(f(s)g(t)), s, t), \tag{2.8}$$

which are called as first and second type and where f and g are some real functions and A, B are non-zero constants.

Remark 2.1. From Definition 2.2, one can be directly seen when taking $A = 0$ and $B \neq 0$, then surface becomes a factorable surface studied in [17]. On the other hand, if one can take $B = 0$ and $A \neq 0$, then surface is a translation surface studied in [15].

3. Classification of Translation-Factorable surfaces with vanishing Gaussian curvature in $\mathbb{E}^3$

As mentioned in the previous section, the TF-surfaces can be parametrized as in (2.6) in Euclidean 3-spaces. In this section, we calculate the Gaussian curvature for the TF-surfaces in $\mathbb{E}^3$. And then, we examine when it vanishes. Finally, we give the complete classification of of the TF-surfaces with vanishing Gaussian curvatures.

Let M^2 be a TF-surface in Euclidean 3-space, $\mathbb{E}^3$. Hence it can be parametrized as

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))). \tag{3.1}$$

Thus, the partial derivatives and $\mathbf{N}$, the unit normal vector field defined by (2.1) of this type surface are obtained by

$$x_s = (1, 0, (Bg(t) + A)f'(s)), \tag{3.2}$$
$$x_t = (0, 1, g'(t)(Bf(s) + A)), \tag{3.3}$$
$$\mathbf{N} = \frac{1}{W}(-f'(s)(Bg(t) + A), -g'(t)(Bf(s) + A), 1). \tag{3.4}$$

Here $W = \sqrt{1 + g'(t)^2(Bf(s) + A)^2 + f'(s)^2(Bg(t) + A)^2}$ and by $\prime$, we have denoted derivatives with respect to corresponding parameters. For readability, here and in the rest of the paper, we will lower the parameters of the $f(s)$ and $g(t)$ functions. Now, by considering the above into the second equalities in (2.2) and (2.3), respectively, we get

$$\begin{aligned} g_{11} &= 1 + f'^2(Bg + A)^2, \\ g_{12} &= g'f'(Bf + A)(Bg + A), \\ g_{22} &= 1 + g'^2(Bf + A)^2, \end{aligned} \tag{3.5}$$

and

$$h_{11} = \frac{f''(Bg + A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf + A)}{W}, \tag{3.6}$$

where $W^2 = 1 + g'^2(Bf + A)^2 + f'^2(Bg + A)^2$. Hence, by substituting of the last two statements into (2.4) gives

$$K = \frac{f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2}{1 + g'^2(Bf + A)^2 + f'^2(Bg + A)^2} \tag{3.7}$$

where f and g are some real functions and A, B are non-zero constants.

Now, we would like to investigate the vanishing Gaussian curvature problem for TF-surfaces in $\mathbb{E}^3$. As well known, the surfaces with vanishing Gaussian curvature are called flat. Now, we examine TF- flat surface in Euclidean 3-space, whose Gaussian curvature is identically zero. Then the following classification theorem is valid.

Theorem 3.1. *Let M^2 be a TF-surface defined by* (3.1) *in the Euclidean 3-space. Then, M^2 is a flat surface if and only if it can be parametrized as one of the followings:*

1. *M^2 is a part of a plane,*
2. *M^2 is a regular surface in $\mathbb{E}^3$ parametrized by*

$$x(s,t) = (s, t, g(t)(Bc + A) + Ac)\,, \tag{3.8}$$

where $f = c$ is a constant function or

$$x(s,t) = (s, t, f(s)(Bc + A) + Ac) \tag{3.9}$$

where $g = c$ is a constant function.

3. *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{3.10}$$

4. *f and g are given by*

$$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}}. \end{aligned} \tag{3.11}$$

Proof. Let M^2 be the TF- flat surface. Thus, from (3.7), it is clear that is sufficient that

$$f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2 = 0. \tag{3.12}$$

Let us consider on the following possibilities:

Case (1): $f' = 0$ and $g' = 0$. Then, the equation (3.12) is trivially satisfied. By considering these assumptions in (3.1), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1) of Theorem 3.1.

Case (2): $f' = 0$ or $g' = 0$. First, assume that $f' = 0$, i.e., f be constant. In case, the equation (3.12) is trivially satisfied. But, in case g is a arbitrary smooth function. Thus, we get (3.8). Similarly, by considering the assumption of g as $g' = 0$, we can get (3.9) in Theorem 3.1.

Case (3): Let $f'' = 0$ or $g'' = 0$, but not both. First, assume that $f'' = 0$, i.e., f be a linear function. In this case, one get $g' = 0$ to provide the equation (3.12). Second, let $g'' = 0$. Then by the similar way, $f' = 0$ must be. Note that one can easily see that these cases are covered by Case (2).

Case (4): Let f', g', f'' and g'' be non-zero. Then, the equation (3.12) can be rewritten as

$$\frac{f''(A + Bf)}{B(f')^2} = \frac{B(g')^2}{g''(A + Bg)} = C, \tag{3.13}$$

for non-zero constant C. We are going to consider the following cases seperately:

Case (4a): $C = 1$. In this case (3.13) implies that

$$f''(A + Bf) = B(f')^2 \quad \text{and} \quad B(g')^2 = g''(A + Bg), \tag{3.14}$$

from which, we get (3.10) in Case (3) in Theorem 3.1.

Case (4b): $C \neq 1$. In this case we solve (3.13) to obtain (3.11).

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 3.1 vanishes identically. □

4. Classification of Translation-Factorable surfaces with vanishing Gaussian curvature in $\mathbb{E}^3_1$

In this section, we study two types of TF-surfaces in the 3-dimensional Minkowski space. Let M^2 be a TF-surface parametrized in (2.7) or (2.8) in Minkowski 3-spaces. Namely, M^2 can be parametrized as

$$x(s,t) = (s, t, A(f(s) + g(t)) + Bf(s)g(t)), \tag{4.1}$$

or

$$x(s,t) = (A(f(s) + g(t)) + Bf(s)g(t), s, t), \tag{4.2}$$

which are called as first and second type TF-surfaces .

First, we would like to consider on the type I TF-surface parametrized as in (4.1). Thus, we have,

$$x_s = (1, 0, f'(A + Bg)), \tag{4.3}$$

$$x_t = (0, 1, g'(A + Bf)). \tag{4.4}$$

Also, $\mathbf{N}_L$ the unit normal vector field of M^2 defined by (2.1) is given by

$$\mathbf{N}_L = \frac{1}{W}(f'(A + Bg), -g'(A + Bf), 1). \tag{4.5}$$

Here with ′, we have denoted derivatives with respect to corresponding parameters and

$$W = \sqrt{\left|1 - g'^2(A + Bf)^2 - f'^2(A + Bg)^2\right|}. \tag{4.6}$$

By considering (4.3), (4.4) and (4.5) into the third equalities in (2.2) and (2.3), respectively, we obtain

$$g_{11} = 1 - f'^2(A + Bg)^2, \quad g_{12} = -f'g'(A+Bf)(A+Bg), \quad g_{22} = 1 - g'^2(A + Bf)^2, \tag{4.7}$$

and

$$h_{11} = \frac{f''(Bg + A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf + A)}{W}. \tag{4.8}$$

Thus, by substituting of these above statements into (2.4) gives

$$K_L = \frac{f''g''(Bf+A)(Bg+A) - B^2(f')^2(g')^2}{W^4} \tag{4.9}$$

where f and g are some real functions, A, B are non-zero constants and W is given as in (4.6).

Now, we would like to give the following theorem being the classification of type I TF-surfaces with vanishing Gaussian curvature in $\mathbb{E}^3_1$.

Theorem 4.1. *Let M^2 be a type I TF-surface defined by* (4.1) *in the Minkowski 3-space. Then,*

1. *M^2 is a type I space-like flat surface if and only if it can be parametrized as one of the followings:*
   - (a) *M^2 is a part of a plane,*
   - (b) *M^2 is a space-like surface in $\mathbb{E}^3_1$ parametrized by*

     $$x(s,t) = (s, t, g(t)(A+Bc) + Ac)\,, \tag{4.10}$$

     where $f = c$ is a constant function and $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$ or

     $$x(s,t) = (s, t, f(s)(A+Bc) + Ac) \tag{4.11}$$

     where $g = c$ is a constant function and $\frac{-1}{A+Bc} < f' < \frac{1}{A+Bc}$.
   - (c) *f and g are given by*

     $$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}, \tag{4.12}$$

     such that satisfy the condition (4.18).
   - (d) *f and g are given by*

     $$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \end{aligned} \tag{4.13}$$

     such that satisfy the condition (4.18).
2. *M^2 is a type I time-like flat surface if and only if it can be parametrized as one of the followings:*
   - (a) *M^2 is a time-like surface in $\mathbb{E}^3_1$ parametrized by*

     $$x(s,t) = (s, t, g(t)(Bc+A) + Ac)\,, \tag{4.14}$$

     where $f = c$ is a constant function or

     $$x(s,t) = (s, t, f(s)(Bc+A) + Ac) \tag{4.15}$$

     where $g = c$ is a constant function.

(b) f and g are given by

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{4.16}$$

(c) f and g are given by

$$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}}. \end{aligned} \tag{4.17}$$

Proof. Let M^2 be a type I TF- flat surface. First, let M^2 be a type I space-like surface. Then from (4.6), we have

$$g'^2(A + Bf)^2 + f'^2(A + Bg)^2 < 1. \tag{4.18}$$

Since M^2 is a flat surface, then from (4.9), it is clear that is sufficient that

$$f''g''(A + Bf)(A + Bg) - B^2(f')^2(g')^2 = 0. \tag{4.19}$$

Let us consider on the following possibilities:

Case (1): $f' = 0$ and $g' = 0$. Then, the equation (4.18) and (4.19) are trivially satisfied. By considering these assumptions in (4.1), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1a) of Theorem 4.1.

Case (2): $f' = 0$ or $g' = 0$. First, assume that $f' = 0$, i.e., f be a constant. In case, the equation (4.19) is trivially satisfied and also from (4.18) yields g is satisfied $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$. Thus, we get (4.10). Similarly, by considering the assumption of g as $g' = 0$, we can get (4.11) in Theorem 4.1.

Case (3): Let $f'' = 0$ or $g'' = 0$, but not both. First, assume that $f'' = 0$, i.e., $f' = c_1$ and $f = c_1 s + c_2$ be a linear function. In this case, one get $g' = 0$, namely $g = C_1$, to provide the equation (4.19). Thus, from (4.18), we get the condition $1 < c_1^2 C_1^2$. Second, let $g'' = 0$. Then by the similar way, $f' = 0$ must be. Note that one can easily see that these cases are covered by Case (1b).

Case (4): Let f', g', f'' and g'' be non-zero. Then, the equation (4.19) can be rewritten as

$$\frac{f''(A + Bf)}{B(f')^2} = \frac{B(g')^2}{g''(A + Bg)} = C, \tag{4.20}$$

for non-zero constant C. We are going to consider the following cases seperately:

Case (4a): $C = 1$. In this case (4.20) implies that

$$f''(A + Bf) = B(f')^2 \quad \text{and} \quad B(g')^2 = g''(A + Bg), \tag{4.21}$$

from which, we get (4.12) in Case (1c) in Theorem 4.1.

Case (4b): $C \neq 1$. In this case we solve (4.20) to obtain (4.13).

Secondly, let M^2 be a type I time-like surface in $\mathbb{E}^3_1$. Then from (4.6), we have

$$g'^2(A+Bf)^2 + f'^2(A+Bg)^2 > 1. \tag{4.22}$$

In view of this condition, the proof of the second case can be made similar to the previous case.

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 4.1 vanishes identically. □

Now, secondly let M^2 be a type II TF-surfaces given as in (4.2). Thus, we have,

$$x_s = (f'(A+Bg), 1, 0), \tag{4.23}$$

$$x_t = (g'(A+Bf), 0, 1). \tag{4.24}$$

Also, $\mathbf{N}_L$ the unit normal vector field of M^2 defined by (2.1) is given by

$$\mathbf{N}_L = \frac{1}{W}(1, -f'(A+Bg), g'(A+Bf)). \tag{4.25}$$

Here with $'$, we have denoted derivatives with respect to corresponding parameters and

$$W = \sqrt{\left|1 + f'^2(A+Bg)^2 - g'^2(A+Bf)^2\right|}. \tag{4.26}$$

By considering (4.23), (4.24) and (4.25) into the third equalities in (2.2) and (2.3), respectively, we obtain

$$g_{11} = 1 + f'^2(A+Bg)^2, \quad g_{12} = f'g'(A+Bf)(A+Bg), \quad g_{22} = g'^2(A+Bf)^2 - 1, \tag{4.27}$$

and

$$h_{11} = \frac{f''(Bg+A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf+A)}{W}. \tag{4.28}$$

Thus, by substituting of these above statements into (2.4) gives

$$K_L = \frac{f''g''(Bf+A)(Bg+A) - B^2(f')^2(g')^2}{W^4} \tag{4.29}$$

where f and g are some real functions, A, B are non-zero constants and W is given as in (4.26). As well knowing that if M^2 is a space-like surface then, from (4.26) yields

$$g'^2(A+Bf)^2 - f'^2(A+Bg)^2 < 1. \tag{4.30}$$

On the other hand, if M^2 is a time-like surface then, from (4.26) yields

$$g'^2(A+Bf)^2 - f'^2(A+Bg)^2 > 1. \tag{4.31}$$

Now we would like to give the following theorem being the classification of type II TF-flat surfaces in $\mathbb{E}^3_1$.

Theorem 4.2. *Let M^2 be a type II TF-surface defined by (4.2) in the Minkowski 3-space. Then,*

1. *M^2 is a type II space-like flat surface if and only if it can be parametrized as one of the followings:*

   (a) *M^2 is a part of a plane,*

   (b) *M^2 is a space-like surface in $\mathbb{E}_1^3$ parametrized by*

   $$x(s,t) = (s, t, g(t)(A + Bc) + Ac)\,, \tag{4.32}$$

   where $f = c$ is a constant function and $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$ or

   $$x(s,t) = (s, t, f(s)(A + Bc) + Ac) \tag{4.33}$$

   where $g = c$ is a constant function and $0 < f'^2(A + Bc)^2 + 1$.

   (c) *f and g are given by*

   $$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}, \tag{4.34}$$

   such that satisfy the condition (4.30).

   (d) *f and g are given by*

   $$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \end{aligned} \tag{4.35}$$

   such that satisfy the condition (4.30).

2. *M^2 is a type I time-like flat surface if and only if it can be parametrized as one of the followings:*

   (a) *M^2 is a time-like surface in $\mathbb{E}_1^3$ parametrized by*

   $$x(s,t) = (s, t, g(t)(Bc + A) + Ac)\,, \tag{4.36}$$

   where $f = c$ is a constant function or

   $$x(s,t) = (s, t, f(s)(Bc + A) + Ac) \tag{4.37}$$

   where $g = c$ is a constant function.

   (b) *f and g are given by*

   $$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}, \tag{4.38}$$

   such that satisfy the condition (4.31).

(c) f and g are given by

$$f(s) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}},$$
$$g(t) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \tag{4.39}$$

such that satisfy the condition (4.31).

Proof. In view of the condition (4.6), the proof of this theorem can be made similar to the previous Theorem 4.1. □

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