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GENERALIZED (ψ, θ, φ)–CONTRACTION WITH APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We prove a unique common fixed point theorem and some unique coupled coincidence point results satisfying generalized (ψ, θ, φ)-contraction on partially ordered metric spaces. We investigate the solution for periodic boundary value problems as an application. Our results improve, generalize and sharpen various well known results in the literature.
Key words: Fixed point, coincidence point, coupled coincidence point, generalized (ψ, θ, φ)-contraction, partially ordered metric space, ordinary differential equations.

1. Introduction

Shaddad et al. [35] studied the existence and uniqueness of fixed points on complete partially ordered metric spaces, which extends the results of Harjani and Sadarangani [18], Nieto and Rodriguez-Lopez [26] and Ran and Reurings [28]. They also established some coupled fixed point theorems, which eectionxtend and generalized the results of Harjani et al. [17], Gnana-Bhaskar and Lakshmikantham [6] and Luong and Thuan [23]. In the last s, they gave unique coupled coincidence point theorems without using compatibility, which extend and generalized the results of Alotaibi and Alsulami [3], Alsulami [4], Lakshmikantham and Ciric [22] and Razani and Parvaneh [32]. For more details one can consult ([1], [2], [5], [9]-[12], [14], [16], [19], [24], [25], [27], [29]-[35]).

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The main objective of this manuscript is to derive more general fixed point results. This manuscript is split into three sections. In the first section of our main results, we obtain the fixed point for $\beta-$non-decreasing mappings under generalized $(\psi,\,\theta,\,\varphi)$-contraction on partially ordered metric spaces and also given an example to show the usability of the obtained results. Secondly, we establish some unique coupled coincidence point results with the help of the results established in the previous section and also given an example where our results are applied, but the other existing results cannot. In the end, we investigate the solution of periodic boundary value problems to show the fruitfulness of our results. We improve and generalize the results of Alsulami [4], Ding et al. [13], Harjani et al. [17], Harjani and Sadarangani [18], Hussain et al. [19], Luong and Thuan [23], Nieto and Rodriguez-Lopez [26], Razani and Parvaneh [32], Shaddad et al. [35] and many other famous results in the literature.

2. Preliminaries

Definition 2.1. [15]. Let $F : X^2 \to X$ be a given mapping. An element $(x,\, y) \in X^2$ is called a coupled fixed point of F if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y.$$

Definition 2.2. [6]. Let $(X,\, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the mixed monotone property if for all x, $y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y), \\ y_1,\ y_2 &\in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

Definition 2.3. [22]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy.$$

Definition 2.4. [22]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\, y) \in X^2$ is called a common coupled fixed point of the mappings F and g if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy.$$

Definition 2.5. [22]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if

$$gF(x,\ y) = F(gx,\ gy),\ \text{ for all } (x,\ y) \in X^2.$$

Definition 2.6. [22]. Let $(X,\, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed $g-$monotone property if for all $x,\, y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y), \\ y_1,\ y_2 &\in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.7. [7]**.** The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x \in X, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y \in X. \end{aligned}$$

Definition 2.8. ([6]**,** [14]). A partially ordered metric space $(X,\, d,\, \preceq)$ is a metric space $(X,\, d)$ provided with a partial order $\preceq$. A partially ordered metric space $(X,\, d,\, \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $(x_n) \subseteq X$ such that $(x_n) \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X,\, d,\, \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

Definition 2.9. [14]. Let $(X,\, \preceq)$ be a partially ordered set and let $\alpha,\, \beta : X \to X$ be two mappings. We say that α is $(\beta,\, \preceq)$-non-decreasing if $\alpha x \preceq \alpha y$ for all x, $y \in X$ such that $\beta x \preceq \beta y$. If β is the identity mapping on X, we say that α is $\preceq$-non-decreasing. If α is $(\beta,\, \preceq)$-non-decreasing and $\beta x = \beta y$, then $\alpha x = \alpha y$.

Definition 2.10. [8]**.** Two self-mappings α and β of a non-empty set X are said to be commutative if $\alpha\beta x = \beta\alpha x$ for all $x \in X$.

Definition 2.11. [20]**.** Let $(X,\, d,\, \preceq)$ be a partially ordered metric space. Two mappings $\alpha,\, \beta : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(\alpha\beta x_n,\ \beta\alpha x_n) = 0,$$

provided that (x_n) is a sequence in X such that

$$\lim_{n\to\infty} \alpha x_n = \lim_{n\to\infty} \beta x_n \in X.$$

Definition 2.12. [21]**.** Two self-mappings α and β of a non-empty set X are said to be weakly compatible if they commute at their coincidence points, that is, if $\alpha x = \beta x$ for some $x \in X$, then $\alpha\beta x = \beta\alpha x$.

Definition 2.13. [8]. Let X be a non-empty set. Two mappings $g : X \to X$ and $F : X^2 \to X$ are said to be weakly compatible if they commute at their coupled coincidence points, that is, if $F(x, y) = gx$ and $F(y, x) = gy$ for some $(x, y) \in X^2$, then $gF(x, y) = F(gx, gy)$ and $gF(y, x) = F(gy, gx)$.

3. Fixed point Results

In this section, we prove a unique common fixed point theorem for mappings α, $\beta : X \to X$ in a partially ordered metric space $(X, d, \preceq)$, where X is a non-empty set. For brevity, we denote $\beta(x)$ by βx where $x \in X$. Let us start with the following definition of altering distance.

Definition 3.1. [35]. An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfied the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(x) = 0$ if and only if $x = 0$.

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and let α, $\beta : X \to X$ be two mappings such that α is $(\beta, \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(\alpha x, \ \alpha y)) \leq \theta(d(\beta x, \ \beta y)) - \varphi(d(\beta x, \ \beta y)), \tag{3.1}$$

for all x, $y \in X$ with $\beta x \preceq \beta y$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the following conditions holds.

(a) (X, d) is complete, α and β are continuous and the pair (α, β) is compatible,

(b) $(\beta(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular,

(c) (X, d) is complete, β is continuous and monotone non-decreasing, the pair (α, β) is compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then α and β have a coincidence point. Moreover, if for every x, $y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair (α, β) is weakly compatible. Then α and β have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and since $\alpha(X) \subseteq \beta(X)$, therefore there exists $x_1 \in X$ such that $\alpha x_0 = \beta x_1$. Then $\beta x_0 \preceq \alpha x_0 = \beta x_1$. Since α is $(\beta, \preceq)$-non-decreasing, $\alpha x_0 \preceq \alpha x_1$. Continuing in this manner, we get a sequence $(x_n)_{n \geq 0}$ such that (βx_n) is $\preceq$-non-decreasing, $\beta x_{n+1} = \alpha x_n \preceq \alpha x_{n+1} = \beta x_{n+2}$ and

$$\beta x_{n+1} = \alpha x_n \text{ for all } n \geq 0. \tag{3.2}$$

Let $\omega_n = d(\beta x_n, \beta x_{n+1})$ for all $n \geq 0$. Now, by using contractive condition (3.1) and the monotonicity of ψ, we have

$$\begin{aligned} \psi(d(\beta x_{n+1}, \ \beta x_{n+2})) &= \psi(d(\alpha x_n, \ \alpha x_{n+1})) \\ &\leq \theta(d(\beta x_n, \ \beta x_{n+1})) - \varphi(d(\beta x_n, \ \beta x_{n+1})). \end{aligned}$$

Thus

$$\psi(\omega_{n+1}) \leq \theta(\omega_n) - \varphi(\omega_n). \tag{3.3}$$

But we have $\psi(\omega_n) - \theta(\omega_n) + \varphi(\omega_n) > 0$. Then

$$\frac{\psi(\omega_{n+1})}{\psi(\omega_n)} \leq \frac{\theta(\omega_n) - \varphi(\omega_n)}{\psi(\omega_n)} < 1.$$

Thus $\psi(\omega_{n+1}) < \psi(\omega_n)$. It follows, from the monotonicity of ψ, that $\omega_{n+1} < \omega_n$. This shows that the sequence $(\omega_n)_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\omega \geq 0$ such that

$$\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = \omega. \tag{3.4}$$

Suppose $\omega > 0$. Taking $n \to \infty$ in (3.3), by using the property of ψ, θ, φ and (3.4), we obtain

$$\psi(\omega) \leq \theta(\omega) - \varphi(\omega),\ \text{that is, } \psi(\omega) - \theta(\omega) + \varphi(\omega) \leq 0,$$

which disagree the fact that $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Thus, by (3.4), we get

$$\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = 0. \tag{3.5}$$

We now demonstrate that $(\beta x_n)_{n\geq 0}$ is a Cauchy sequence in X. Suppose, to the contrary, that (βx_n) is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integers k, and

$$d(\beta x_{n(k)},\ \beta x_{m(k)}) \geq \varepsilon,\ \text{for } n(k) > m(k) > k.$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(\beta x_{n(k)-1},\ \beta x_{m(k)}) < \varepsilon.$$

Now, by triangle inequality, we have

$$\begin{aligned}
\varepsilon &\leq d(\beta x_{n(k)},\ \beta x_{m(k)}) \\
&\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + d(\beta x_{n(k)-1},\ \beta x_{m(k)}) \\
&\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (3.5), we have

$$\lim_{k\to\infty} d(\beta x_{n(k)},\ \beta x_{m(k)}) = \varepsilon. \tag{3.6}$$

By using triangle inequality, we have

$$\begin{aligned}
& d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) \\
\leq\ & d(\beta x_{n(k)+1},\ \beta x_{n(k)}) + d(\beta x_{n(k)},\ \beta x_{m(k)}) + d(\beta x_{m(k)},\ \beta x_{m(k)+1}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (3.5) and (3.6), we have

$$\lim_{k\to\infty} d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) = \varepsilon. \tag{3.7}$$

As $n(k) > m(k)$, $\beta x_{n(k)} \succeq \beta x_{m(k)}$ and so by using contractive condition (3.1), we have

$$\begin{aligned}\psi(d(\beta x_{n(k)+1},\ \beta x_{m(k)+1})) &= \psi(d(\alpha x_{n(k)},\ \alpha x_{m(k)}))\\ &\leq \theta(d(\beta x_{n(k)},\ \beta x_{m(k)})) - \varphi(d(\beta x_{n(k)},\ \beta x_{m(k)})).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (3.6), (3.7), we have

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction due to the fact that $\varepsilon > 0$. This claims that $(\beta x_n)_{n\geq 0}$ is a Cauchy sequence in X. Now, we claim that α and β have a coincidence point between cases $(a) - (c)$.

First suppose that (a) holds, that is, (X, d) is complete, α and β are continuous and the pair (α, β) is compatible. Since (X, d) is complete, therefore there exists $x \in X$ such that $(\beta x_n) \to x$ and (3.2) follows that $(\alpha x_n) \to x$. As α and β are continuous and so $(\alpha\beta x_n) \to \alpha x$ and $(\beta\beta x_n) \to \beta x$. Since the pair (α, β) is compatible, therefore we conclude that

$$d(\alpha x,\ \beta x) = \lim_{n\to\infty} d(\alpha\beta x_n,\ \beta\beta x_{n+1}) = \lim_{n\to\infty} d(\alpha\beta x_n,\ \beta\alpha x_n) = 0,$$

that is, x is a coincidence point of α and β.

Secondly suppose that (b) holds, that is, $(\beta(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. As (βx_n) is a Cauchy sequence in the complete space $(\beta(X), d)$, so there exists $y \in \beta(X)$ such that $(\beta x_n) \to y$. Let $x \in X$ be any point such that $y = \beta x$ and so $(\beta x_n) \to \beta x$. Since $(X, d, \preceq)$ is non-decreasing-regular and (βx_n) is $\preceq$-non-decreasing and converging to βx, therefore we have $\beta x_n \preceq \beta x$ for all $n \geq 0$. Applying the contractive condition (3.1), we have

$$\psi(d(\beta x_{n+1}, \alpha x)) = \psi(d(\alpha x_n,\ \alpha x)) \leq \theta(d(\beta x_n,\ \beta x)) - \varphi(d(\beta x_n,\ \beta x)).$$

Taking $n \to \infty$ in the above inequality, using the properties of ψ, θ, φ and the fact $(\beta x_n) \to \beta x$, we get $d(\beta x, \alpha x) = 0$, that is, x is a coincidence point of α and β.

Finally suppose that (c) holds, that is, (X, d) is complete, β is continuous and monotone non-decreasing, the pair (α, β) is compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete and so there exists $x \in X$ such that $(\beta x_n) \to x$ and (3.2) follows that $(\alpha x_n) \to x$. As β is continuous and so $(\beta\beta x_n) \to \beta x$. Moreover, the pair (α, β) is compatible, that is, $\lim_{n\to\infty} d(\beta\beta x_{n+1}, \alpha\beta x_n) = \lim_{n\to\infty} d(\beta\alpha x_n, \alpha\beta x_n) = 0$ and $(\beta\beta x_n) \to \beta x$ implies that $(\alpha\beta x_n) \to \beta x$.

Since $(X, d, \preceq)$ is non-decreasing-regular and (βx_n) is $\preceq$-non-decreasing and converging to x, therefore $\beta x_n \preceq x$ which, by the monotonicity of β, implies $\beta\beta x_n \preceq \beta x$. Using the contractive condition (3.1), we get

$$\psi(d(\alpha\beta x_n,\ \alpha x)) \leq \theta(d(\beta\beta x_n,\ \beta x)) - \varphi(d(\beta\beta x_n,\ \beta x)).$$

Taking $n \to \infty$ in the above inequality, using the properties of ψ, θ, φ and the fact that $(\beta\beta x_n) \to \beta x$, $(\alpha\beta x_n) \to \beta x$, we get $d(\beta x,\, \alpha x) = 0$, that is, x is a coincidence point of α and β.

Since the set of coincidence points of α and β is non-empty. Suppose x and y are coincidence points of α and β, that is, $\alpha x = \beta x$ and $\alpha y = \beta y$. Now, we shall show that $\beta x = \beta y$. By the assumption, there exists $z \in X$ such that αz is comparable with αx and αy. Put $z_0 = z$ and choose $z_1 \in X$ so that $\beta z_1 = \alpha z_0$. Then, we can inductively define sequences (βz_n) where $\beta z_{n+1} = \alpha z_n$ for all $n \geq 0$. Hence $\alpha x = \beta x$ and $\alpha z = \alpha z_0 = \beta z_1$ are comparable, that is, $\beta z_1 \preceq \beta x$. We will show that $\beta z_n \preceq \beta x$ for each $n \in \mathbb{N}$. In fact, we will use mathematical induction. Since $\beta z_1 \preceq \beta x$, our claim is true for $n = 1$. Suppose that $\beta z_n \preceq \beta x$ holds for some $n > 1$. Since α is $\beta-$non-decreasing with respect to $\preceq$, we get $\beta z_{n+1} = \alpha z_n \preceq \alpha x = \beta x$, and this proves our claim.

Let $\delta_n = d(\beta z_n,\ \beta x)$ for all $n \geq 0$. Since $\beta z_n \preceq \beta x$, therefore by using the contractive condition (3.1), we have

$$\psi(d(\beta z_{n+1},\ \beta x)) = \psi(d(\alpha z_n,\ \alpha x)) \leq \theta(d(\beta z_n,\ \beta x)) - \varphi(d(\beta z_n,\ \beta x)).$$

Thus

$$\psi(\delta_{n+1}) \leq \theta(\delta_n) - \varphi(\delta_n). \tag{3.8}$$

As $\psi(\delta_n) - \theta(\delta_n) + \varphi(\delta_n) > 0$ and so

$$\frac{\psi(\delta_{n+1})}{\psi(\delta_n)} \leq \frac{\theta(\delta_n) - \varphi(\delta_n)}{\psi(\delta_n)} < 1.$$

Thus $\psi(\delta_{n+1}) < \psi(\delta_n)$, which, by the monotonicity of ψ, implies $\delta_{n+1} < \delta_n$. This shows that the sequence $(\delta_n)_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(\beta z_n,\ \beta x) = \delta. \tag{3.9}$$

Now, we shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Taking $n \to \infty$ in (3.8) and by using the property of ψ, θ, φ and (3.9), we obtain

$$\psi(\delta) \leq \theta(\delta) - \varphi(\delta), \text{ that is, } \psi(\delta) - \theta(\delta) + \varphi(\delta) \leq 0,$$

which contradicts the fact that $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Thus, by (3.9), we get

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(\beta z_n,\ \beta x) = 0. \tag{3.10}$$

Similarly, we show that

$$\lim_{n\to\infty} d(\beta z_n,\ \beta y) = 0. \tag{3.11}$$

Hence, by (3.10) and (3.11), we get

$$\beta x = \beta y. \tag{3.12}$$

Since $\alpha x = \beta x$, therefore by weak compatibility of α and β, we have

$$\alpha\beta x = \beta\alpha x = \beta\beta x.$$

Let $u = \beta x$, then $\alpha u = \beta u$, that is, u is a coincidence point of α and β. Then from (3.12) with $y = u$, it follows that $\beta x = \beta u$, that is, $u = \beta u = \alpha u$, that is, u is a common fixed point of α and β. To prove the uniqueness, assume that v is another common fixed point of α and β. Then by (3.12) we have $v = \beta v = \beta u = u$.

Hence the common fixed point of α and β is unique. $\square$

Put $\psi(t) = t$ and $\varphi(t) = 0$ for all $t \geq 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$d(\alpha x,\ \alpha y) \leq \theta(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$, where $\theta(0) = 0$ and $t - \theta(t) > 0$ for all $t > 0$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions (a) − (c) of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Put $\theta(t) = k\psi(t)$ with $0 \leq k < 1$ and $\varphi(t) = 0$, for all $t \geq 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists an altering distance function ψ such that*

$$\psi(d(\alpha x,\ \alpha y)) \leq k\psi(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$ where $k < 1$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions (a) − (c) of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Put $\psi(t) = \theta(t)$ for all $t \geq 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\psi(d(\alpha x,\ \alpha y)) \leq \psi(d(\beta x,\ \beta y)) - \varphi(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$, where $\varphi(0) = 0$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Put $\psi(t) = \theta(t) = t$ for all $t \geq 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.4. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and there exists a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$d(\alpha x,\ \alpha y) \leq d(\beta x,\ \beta y) - \varphi(d(\beta x,\ \beta y)),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$, where $\varphi(0) = 0$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $0 \leq k < 1$ for all $t \geq 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.5. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings such that α is $(\beta,\ \preceq)$-non-decreasing, $\alpha(X) \subseteq \beta(X)$ and*

$$d(\alpha x,\ \alpha y) \leq kd(\beta x,\ \beta y),$$

for all $x,\ y \in X$ such that $\beta x \preceq \beta y$, where $k < 1$. Suppose that there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Also assume that, at least, one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $z \in X$ such that αz is comparable to αx and αy and also the pair $(\alpha,\ \beta)$ is weakly compatible. Then α and β have a unique common fixed point.

Put $\beta = I$ (the identity mapping) in Theorem 3.1, we get the following corollary.

Corollary 3.6. *Let $(X,\, d,\, \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping for which there exists an altering distance function ψ, an upper semi-continuous function $\theta : [0,\, +\infty) \to [0,\, +\infty)$ and a lower semi-continuous function $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ such that*

$$\psi(d(\alpha x,\ \alpha y)) \le \theta(d(x,\ y)) - \varphi(d(x,\ y)),$$

for all $x,\, y \in X$ with $x \preceq y$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\, d,\, \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Put $\beta = I$ (the identity mapping) in Corollary 3.1, we get the following corollary.

Corollary 3.7. *Let $(X,\, d,\, \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping for which there exists an upper semi-continuous function $\theta : [0,\, +\infty) \to [0,\, +\infty)$ such that*

$$d(\alpha x,\ \alpha y) \le \theta(d(x,\ y)),$$

for all $x,\, y \in X$ such that $x \preceq y$, where $\theta(0) = 0$ and $t - \theta(t) > 0$ for all $t > 0$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\, d,\, \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Put $\beta = I$ (the identity mapping) in Corollary 3.2, we get the following corollary.

Corollary 3.8. *Let $(X,\, d,\, \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping for which there exists an altering distance function ψ such that*

$$\psi(d(\alpha x,\ \alpha y)) \le k\psi(d(x,\ y)),$$

for all $x,\, y \in X$ such that $x \preceq y$ where $k < 1$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\, d,\, \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Put $\beta = I$ (the identity mapping) in Corollary 3.3, we get the following corollary.

Corollary 3.9. *Let $(X,\, d,\, \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping for which there exists an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ such that*

$$\psi(d(\alpha x,\ \alpha y)) \le \psi(d(x,\ y)) - \varphi(d(x,\ y)),$$

for all $x,\ y \in X$ such that $x \preceq y$, where $\varphi(0) = 0$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\ d,\ \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\ y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Put $\beta = I$ (the identity mapping) in Corollary 3.4, we get the following corollary.

Corollary 3.10. *Let $(X,\ d,\ \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping and there exists a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$d(\alpha x,\ \alpha y) \leq d(x,\ y) - \varphi(d(x,\ y)),$$

for all $x,\ y \in X$ such that $x \preceq y$, where $\varphi(0) = 0$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\ d,\ \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\ y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Put $\beta = I$ (the identity mapping) in Corollary 3.5, we get the following corollary.

Corollary 3.11. *Let $(X,\ d,\ \preceq)$ be a complete partially ordered metric space and let $\alpha : X \to X$ be a non-decreasing mapping satisfying*

$$d(\alpha x,\ \alpha y) \leq kd(x,\ y),$$

for all $x,\ y \in X$ such that $x \preceq y$, where $k < 1$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$. Also suppose that α is continuous or $(X,\ d,\ \preceq)$ is regular. Then α has a fixed point. Moreover, if for each $x,\ y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

Example 3.1. Let $X = \mathbb{R}$ be a metric space with the usual metric $d : X \times X \to [0, +\infty)$ equipped with the natural ordering of real numbers $\leq$. Let $\alpha,\ \beta : X \to X$ be defined as

$$\alpha x = \frac{x^2}{3} \text{ and } \beta x = x^2 \text{ for all } x \in X.$$

Clearly, α and β satisfied the contractive condition of Theorem 3.1 with $\psi(t) = \theta(t) = t$ and $\varphi(t) = 2t/3$ for $t \geq 0$. Moreover, all the other conditions of Theorem 3.1 are satisfied and $u = 0$ is a unique common fixed point of α and β.

4. Coupled coincidence point results

In this section, we derive some unique coupled coincidence point results with the help of the results established in the previous section. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the n^{th} Cartesian product $X \times X \times ... \times X$ (n times). Let $(X,\ \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order.

$$W \sqsubseteq V \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } W = (u,\ v),\ V = (x,\ y) \in X^2.$$

Definition 4.1. ([14], [34]). Let $(X,\ d)$ be a metric space. Define $\delta : X^2 \times X^2 \to [0,\ +\infty)$ by

$$\delta(V,\ W) = \max\{d(x,\ u),\ d(y,\ v)\},\ \text{for all } V = (x,\ y),\ W = (u,\ v) \in X^2.$$

Then δ is metric on X^2 and $(X,\ d)$ is complete and regular if and only if $(X^2,\ \delta)$ is complete and regular.

Definition 4.2. [2]. Let $(X,\ d)$ be a metric space. Define $\Delta_n : X^n \times X^n \to [0,\ +\infty)$, for $A = (a_1,\ a_2,\ ...,\ a_n)$, $B = (b_1,\ b_2,\ ...,\ b_n) \in X^n$, by

$$\Delta_n(A,\ B) = \frac{1}{n}\sum_{i=1}^{n} d(a_i,\ b_i).$$

Then Δ_n is metric on X^n and $(X,\ d)$ is complete and regular if and only if $(X^n,\ \Delta_n)$ is complete and regular.

Theorem 4.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space. Suppose $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F has the mixed $g-$monotone property with respect to $\preceq$ on X for which there exists an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} (4.1) \qquad & \psi(d(F(x,\ y),\ F(u,\ v))) \\ \leq\ & \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}), \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Suppose that $F(X^2) \subseteq g(X)$, $g(X)$ is complete, g is continuous and monotone non-decreasing. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exists two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point. Furthermore, suppose that for every $(x,\ y),\ (u,\ v) \in X^2$, there exists a point $(z,\ w) \in X^2$ such that $(z,\ w)$ is comparable to $(x,\ y)$ and $(u,\ v)$. Then F and g have a unique coupled coincidence point.

Proof. Define $S : g(X) \times g(X) \to g(X)$ by

$$(4.2) \qquad S(V) = (F(x,\ y),\ F(y,\ x)),$$

for all $V = (gx,\ gy) \in g(X) \times g(X)$. Clearly S is well defined as g is monotone non-decreasing. It is noticeable that $(g(X),\ d,\ \preceq)$ is a complete regular partially ordered metric space. Also S is continuous since F is continuous.

Let $V=(gx,\,gy)$ and $W=(gu,\,gv)\in g(X)\times g(X)$ be such that $V\sqsubseteq W$. Then $gx\preceq gu$ and $gy\succeq gv$. As F has the mixed $g-$monotone property with respect to $\preceq$ and so $F(x,\,y)\preceq F(u,\,v)$ and $F(y,\,x)\succeq F(v,\,u)$. Thus $S(V)\sqsubseteq S(W)$. Thus S is $\sqsubseteq$-non-decreasing.

Now, there exist two elements $x_0,\,y_0\in X$ such that $gx_0\preceq F(x_0,\,y_0)$ and $gy_0\succeq F(y_0,\,x_0)$. It follows that, there exists $V_0=(gx_0,\,gy_0)\in g(X)\times g(X)$ such that $V_0\sqsubseteq S(V_0)$.

Again, suppose that $V=(gx,\,gy)$ and $W=(gu,\,gv)\in g(X)\times g(X)$ such that $V\sqsubseteq W$. Then $gx\preceq gu$ and $gy\succeq gv$, by using (4.1) and (4.2), we have

$$\begin{aligned}&\psi(d(F(x,\ y),\ F(u,\ v)))\\ \leq\ &\theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\})-\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).\end{aligned}$$

Furthermore $gy\succeq gv$ and $gx\preceq gu$, the contractive condition (4.1) and (4.2) gives

$$\begin{aligned}&\psi(d(F(y,\ x),\ F(v,\ u)))\\ \leq\ &\theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\})-\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).\end{aligned}$$

Combining them, we get

$$\begin{aligned}&\max\{\psi(d(F(x,\ y),\ F(u,\ v))),\ \psi(d(F(y,\ x),\ F(v,\ u)))\}\\ \leq\ &\theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\})-\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).\end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned}&\psi(\max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\})\\ \leq\ &\theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\})-\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).\end{aligned}\tag{4.3}$$

Thus, by using (4.3), we get

$$\begin{aligned}&\psi(\delta(S(V),\ S(W)))\\ =\ &\psi(\max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\})\\ \leq\ &\theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\})-\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\})\\ \leq\ &\theta(\delta(V,\ W)-\varphi(\delta(V,\ W)).\end{aligned}$$

Thus

$$\psi(\delta(S(V),\ S(W)))\leq\theta(\delta(V,\ W)-\varphi(\delta(V,\ W)),$$

for all $V,W\in g(X)\times g(X)$ with $V\sqsubseteq W$. Consequently, S satisfies all the conditions of Corollary 3.6 in the complete partially ordered metric space $(g(X)\times g(X),\,\delta,\,\sqsubseteq)$. Thus S has a fixed point, which leads that F and g have a coupled coincidence point.

Let us now show the uniqueness of coupled coincidence point. We suppose that $(x,\,y)\in X^2$ is a coupled coincidence point of F and g. Now, we take $(u,\,v)\in X^2$ is another coupled coincidence point of F and g, then there exists $(z,\,w)\in X^2$ such that $(z,\ w)$ is comparable to $(x,\ y)$ and $(u,\ v)$. Therefore, $(gx,\ gy)$ and $(gu,\ gv)$ are fixed points of S with $(gx,\ gy) \sqsubseteq (gz,\ gw)$ and $(gu,\ gv) \sqsubseteq (gz,\ gw)$. Hence by Corollary 3.6, we find that S has a unique fixed point, which leads to the uniqueness of the coupled coincidence point of F and g. □

Put $g = I$ (the identity mapping) in Theorem 4.1, we get the following Corollary:

Corollary 4.1. *Let $(X,\ d,\ \preceq)$ be a complete partially ordered metric space. Suppose $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exists an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying*

$$\begin{aligned} &\psi(d(F(x,\ y),\ F(u,\ v))) \\ \leq\ &\theta(\max\{d(x,\ u),\ d(y,\ v)\}) - \varphi(\max\{d(x,\ u),\ d(y,\ v)\}), \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, with $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point. Furthermore, suppose that for every $(x,\ y)$, $(u,\ v) \in X^2$, there exists a point $(z,\ w) \in X^2$ such that $(z,\ w)$ is comparable to $(x,\ y)$ and $(u,\ v)$. Then F has a unique coupled fixed point.

Theorem 4.2. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F has the mixed $g-$monotone property with respect to $\preceq$ on X for which there exists an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} &\psi(d(F(x,\ y),\ F(u,\ v))) \\ \leq\ &\theta\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - \varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right), \end{aligned} \tag{4.4}$$

for all $x,\ y,\ u,\ v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Moreover $\psi(\frac{a+b}{2}) \leq \frac{\psi(a)+\psi(b)}{2}$, for all $a,\ b \in (0,\ \infty)$. Suppose that $F(X^2) \subseteq g(X)$, $g(X)$ is complete, g is continuous and monotone non-decreasing. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point. Furthermore, suppose that for every $(x,\ y),\ (u,\ v) \in X^2$, there exists a point $(z,\ w) \in X^2$ such that $(z,\ w)$ is comparable to $(x,\ y)$ and $(u,\ v)$. Then F and g have a unique coupled coincidence point.

Proof. Let $S : g(X) \times g(X) \to g(X)$ be a mapping defined by (4.2). From condition (4.4), we have

$$\psi(d(F(x,\ y),\ F(u,\ v))) \leq \theta\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - \varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right),$$

and

$$\psi(d(F(y,\ x),\ F(v,\ u))) \leq \theta\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - \varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right).$$

By summing the above inequalities, we get

$$\begin{aligned} &\psi(d(F(x,\ y),\ F(u,\ v))) + \psi(d(F(y,\ x),\ F(v,\ u))) \\ \leq\ & 2\theta\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - 2\varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) \end{aligned}$$

Utilizing the condition $\psi(\frac{a+b}{2}) \leq \frac{\psi(a) + \psi(b)}{2}$, for all $a,\ b \in (0,\ \infty)$, we obtain

$$\begin{aligned} &\psi(\frac{d(F(x,\ y),\ F(u,\ v))) + d(F(y,\ x),\ F(v,\ u))}{2}) \\ \leq\ & \theta\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - \varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) \end{aligned}$$

It means that

$$\psi(\Delta_2(S(V),\ S(W))) \leq \theta(\Delta_2(V,\ W)) - \varphi(\Delta_2(V,\ W)),$$

$V = (gx,\ gy)$ and $W = (gu,\ gv) \in g(X) \times g(X)$ such that $V \sqsubseteq W$. Consequently, S satisfies all the conditions of Corollary 3.6 in the complete partially ordered metric space $(g(X) \times g(X),\ \Delta_2,\ \sqsubseteq)$. Thus S has a fixed point, which leads that F and g have a coupled coincidence point. The rest of the proof is similar to the proof of Theorem 4.1. □

Put $g = I$ (the identity mapping) in Theorem 4.2, we get the following Corollary:

Corollary 4.2. *Let $(X,\ d,\ \preceq)$ be a complete partially ordered metric space. Suppose $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exists an* *altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying*

$$\psi(d(F(x,\ y),\ F(u,\ v))) \leq \theta\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right) - \varphi\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right),$$

for all $x,\ y,\ u,\ v \in X$, with $x \preceq u$ and $y \succeq v$. Moreover $\psi(\frac{a+b}{2}) \leq \frac{\psi(a)+\psi(b)}{2}$, for all $a,\ b \in (0,\ \infty)$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point. Furthermore, suppose that for every $(x,\ y),\ (u,\ v) \in X^2$, there exists a point $(z,\ w) \in X^2$ such that $(z,\ w)$ is comparable to $(x,\ y)$ and $(u,\ v)$. Then F has a unique coupled fixed point.

In a similar way, we may state the results analog of Corollary 3.1, Corollary 3.2, Corollary 3.3, Corollary 3.4 and Corollary 3.5 for Theorem 4.1, Corollary 4.1, Theorem 4.2 and Corollary 4.2.

Example 4.1. Let $X = [0,\ 1]$ be a metric space with the metric $d : X^2 \to [0,\ +\infty)$ defined by $d(x,\ y) = |x-y|$, for all $x,\ y \in X$, with the natural ordering of real numbers $\leq$. Let $F : X^2 \to X$ be defined by

$$F(x,\ y) = \frac{1}{8}\left(x - y + \frac{3}{2}\right), \text{ for all } x,\ y \in X,$$

and $g : X \to X$ be defined as

$$gx = \frac{x}{2}, \text{ for all } x \in X.$$

Thus, F has the mixed $g-$monotone property. First we shall show that F and g are not compatible. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n,\ y_n) = \lim_{n\to\infty} gx_n = a \text{ and } \lim_{n\to\infty} F(y_n,\ x_n) = \lim_{n\to\infty} gy_n = b.$$

Thus $a = b = 3/32$ and then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{3}{16}.$$

Also

$$\begin{aligned} &\lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) \\ = &\lim_{n\to\infty}\left|\frac{1}{16}\left(x_n - y_n + \frac{3}{2}\right) - \frac{1}{8}\left(\frac{x_n}{2} - \frac{y_n}{2} + \frac{3}{2}\right)\right| = \frac{3}{32} \neq 0. \end{aligned}$$

Now, we shall show that the contractive condition of Theorem 4.1 should satisfy by the mappings F and g. Let $\psi(t) = \theta(t) = t$ and $\varphi(t) = \dfrac{t}{2}$ for $t \geq 0$. Now, for all $x,\ y,\ u,\ v \in X$ such that $gx \preceq gu$ and $gy \succeq gv$, we have

$$\begin{aligned}
& \psi(d(F(x,\ y),\ F(u,\ v))) \\
= \ & d(F(x,\ y),\ F(u,\ v)) \\
= \ & \left| \frac{1}{8}\left(x - y + \frac{3}{2}\right) - \frac{1}{8}\left(u - v + \frac{3}{2}\right) \right| \\
\leq \ & \frac{1}{4}\left(\left| \frac{x}{2} - \frac{u}{2} \right| + \left| \frac{y}{2} - \frac{v}{2} \right| \right) \\
\leq \ & \frac{1}{4}(d(gx,\ gu) + d(gy,\ gv)) \\
\leq \ & \frac{1}{2}\max\{d(gx,\ gu),\ d(gy,\ gv)\} \\
\leq \ & \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).
\end{aligned}$$

Thus the contractive condition of Theorem 4.1 is satisfied for all $x,\ y,\ u,\ v \in X$. Hence all the other conditions of Theorem 4.1 are satisfied and $z = (3/16,\ 3/16)$ is a unique coupled coincidence point of F and g.

5. Application to ordinary differential equations

In this section, first we investigate the solution of the following first-order periodic problem:

$$(5.1) \qquad \begin{cases} u'(t) = f(t,\ u(t)),\ t \in [0,\ T], \\ u(0) = u(T), \end{cases}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Taking into account of the space $X = C(I,\ \mathbb{R})$ $(I = [0,\ T])$ of all continuous functions from I to $\mathbb{R}$, which is a regular complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{t \in I} |x(t) - y(t)|\,, \text{ for all } x,\ y \in X,$$

with a partial order, for all $x,\ y \in X$, given by

$$x \preceq y \Longleftrightarrow x(t) \leq y(t), \text{ for all } t \in I.$$

Definition 5.1. A lower solution of (5.1) is a function $\zeta \in C^1(I,\ \mathbb{R})$ such that

$$\begin{aligned}
\zeta'(t) \ &\leq \ f(t,\ \zeta(t)) \text{ for } t \in I, \\
\zeta(0) \ &= \ \zeta(T) = 0.
\end{aligned}$$

Theorem 5.1. *Consider the problem (5.1) with continuous function $f : I \times \mathbb{R} \to \mathbb{R}$ and suppose there exists $\lambda > 0$ such that for $x,\ y \in \mathbb{R}$ with $x \geq y$,*

$$0 \leq f(t,\ x) + \lambda x - f(t,\ y) - \lambda y \leq \lambda \ln\left[\frac{1}{2}(x - y) + 1\right].$$

Then the existence of a lower solution of (5.1) deliver us the existence of a solution of (5.1).

Proof. It is noticeable that, problem (5.1) is equivalent to the following integral equation

$$u(t) = \int_0^T G(t,\ s)[f(s,\ u(s)) + \lambda u(s)]ds,$$

where $G(t,\ s)$ is the Green function given by

$$G(t,\ s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, \ 0 \leq s < t \leq T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, \ 0 \leq t < s \leq T. \end{cases}$$

Define $\psi,\ \varphi : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\psi(t) = \theta(t) = t \text{ and } \varphi(t) = t - \ln(\frac{t}{2} + 1).$$

Obviously ψ and θ is continuous, increasing, positive in $(0,\ \infty)$ and $\psi(0) = 0$. Also, φ is continuous, positive in $(0,\ \infty)$ and $\varphi(0) = 0$. Thus ψ, θ and φ satisfy all the mentioned properties.

Now define the mapping $\alpha : X \to X$ as follows:

$$\alpha(x)(t) = \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s)]ds.$$

If $x_1 \geq x_2$, then by using our assumption, we have $f(t,\ x_1(t)) + \lambda x_1(t) \geq f(t,\ x_2(t)) + \lambda x_2(t)$ for all $t \in I$. Since $G(t,\ s) > 0$, for $t \in I$, therefore one can obtain

$$\begin{aligned} \alpha(x_1)(t) &= \int_0^T G(t,\ s)[f(s,\ x_1(s)) + \lambda x_1(s)]ds \\ &\geq \int_0^T G(t,\ s)[f(s,\ x_2(s)) + \lambda x_2(s)]ds \\ &= \alpha(x_2)(t). \end{aligned}$$

Consequently α is a non-decreasing mapping. Now, for $x \geq y$, we have

$$\begin{aligned}
& \psi(d(\alpha x,\ \alpha y)) \\
= \ & d(\alpha x,\ \alpha y) \\
= \ & \sup_{t\in I} |\alpha(x)(t) - \alpha(y)(t)| \\
= \ & \sup_{t\in I} \left| \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s) - f(s,\ y(s)) - \lambda y(s)]ds \right| \\
\leq \ & \sup_{t\in I} \left| \int_0^T G(t,\ s) \cdot \lambda \ln \left[\frac{1}{2}(x(s) - y(s)) + 1 \right] ds \right| \\
\leq \ & \lambda \ln \left[\frac{1}{2} d(x,\ y) + 1 \right] \sup_{t\in I} \left| \int_0^T G(t,\ s)ds \right| \\
\leq \ & \lambda \ln \left[\frac{1}{2} d(x,\ y) + 1 \right] \sup_{t\in I} \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right| \\
\leq \ & \ln \left[\frac{1}{2} d(x,\ y) + 1 \right] \\
\leq \ & d(x,\ y) - \left\{ d(x,\ y) - \ln \left[\frac{1}{2} d(x,\ y) + 1 \right] \right\} \\
\leq \ & \theta(d(x,\ y)) - \varphi(d(x,\ y)).
\end{aligned}$$

Hence

$$\psi(d(\alpha x, \alpha y)) \leq \theta(d(x, y)) - \varphi(d(x, y)).$$

Thus the contractive condition of Corollary 3.6 is satisfied. Finally, suppose that $\zeta \in X$ is a lower solution of (5.1), then

$$\zeta'(s) + \lambda \zeta(s) \leq f(s,\ \zeta(s)) + \lambda \zeta(s), \text{ for } t \in I.$$

Multiplying by $G(t,\ s)$ and then integrating, we get

$$\int_0^T \zeta'(s) G(t,\ s)ds + \lambda \int_0^T \zeta(s) G(t,\ s)ds \leq \alpha(\zeta)(t), \text{ for } t \in I.$$

Then, for all $t \in I$, we have

$$\int_0^t \zeta'(s) \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \zeta'(s) \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds + \lambda \int_0^T \zeta(s) G(t,\ s)ds \leq \alpha(\zeta)(t).$$

Using integration by parts and $\zeta(0) = \zeta(T) = 0$, we get

$$\zeta(t) \leq \alpha(\zeta)(t) \text{ for all } t \in I.$$

This proves that $\zeta \preceq \alpha(\zeta)$. Thus all the hypothesis of Corollary 3.6 are satisfied. Consequently, α has a fixed point $x \in X$ which is the solution of (5.1) in $X = C(I,\ \mathbb{R})$. □

Now, we investigate the solution of the following two-point boundary value problem.

$$\text{(5.2)} \qquad \begin{cases} -x''(t) = h(t,\ x(t),\ x(t)),\ x \in (0,\ +\infty),\ t \in [0,\ 1], \\ x(0) = x(1) = 0. \end{cases}$$

Theorem 5.2. *Under the following assumptions*

(i) $h : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) *Suppose that there exists* $0 \leq \gamma \leq 8$ *such that for all* $t \in I$, $x \geq u$ *and* $y \leq v$,

$$0 \leq h(t,\ x,\ y) - h(t,\ u,\ v) \leq \frac{\gamma}{4}((x-u)+(y-v)).$$

(iii) *There exists* $(a,\ b) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ *such that*

$$\begin{cases} -a''(t) \leq h(t,\ a(t),\ b(t)),\ t \in [0,\ 1], \\ -b''(t) \geq h(t,\ b(t),\ a(t)),\ t \in [0,\ 1], \\ a(0) = a(1) = b(0) = b(1) = 0. \end{cases}$$

Then (5.2) has unique solution in $C^2(I,\ \mathbb{R})$.

Proof. Notice that (5.2) is equivalent to the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t,\ s)h(s,\ x(s),\ x(s))ds \text{ for } t \in [0,\ 1],$$

where $G(t,\ s)$ is the Green function of differential operator $-\dfrac{d^2}{dt^2}$ with Dirichlet boundary condition $x(0) = x(1) = 0$, that is,

$$G(t,\ s) = \begin{cases} t(1-s),\ 0 \leq t \leq s \leq 1, \\ s(1-t),\ 0 \leq s \leq t \leq 1. \end{cases}$$

Define now the mapping $F : X^2 \to X$ by

$$F(x,\ y)(t) = \int_0^1 G(t,\ s)h(s,\ x(s),\ y(s))ds,\ t \in [0,\ 1] \text{ and } x,\ y \in X.$$

From (ii), it is clear that F has the mixed monotone property with respect to the partial order $\preceq$ in X. Let $x,\ y,\ u,\ v \in X$ such that $x \geq u$ and $y \leq v$, then by (ii), we have

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ = \ & \sup_{t\in I} |F(x,\ y)(t) - F(u,\ v)(t)| \\ = \ & \sup_{t\in I} \int_0^1 G(t,\ s)[h(s,\ x(s),\ y(s)) - h(s,\ u(s),\ v(s))]ds \\ \leq \ & \sup_{t\in I} \int_0^1 G(t,\ s) \cdot \frac{\gamma}{4}((x(s)-u(s)) + (y(s)-v(s)))ds \\ \leq \ & \frac{\gamma}{2}\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right) \sup_{t\in I} \int_0^1 G(t,\ s)ds. \end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{\gamma}{2}\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right)\sup_{t\in I}\int_0^1 G(t,\ s)ds. \tag{5.3}$$

It is noticeable that

$$\int_0^1 G(t,\ s)ds = -\frac{t^2}{2}+\frac{t}{2} \text{ and } \sup_{t\in[0,\ 1]}\int_0^1 G(t,\ s)ds = \frac{1}{8}.$$

These facts, the inequality (5.3) and the hypothesis $0 < \gamma \leq 8$ give us

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &\leq \frac{\gamma}{16}\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right) \\ &\leq \frac{1}{2}\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right). \end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{1}{2}\left(\frac{d(x,\ u)+d(y,\ v)}{2}\right).$$

Thus, the contractive condition of Corollary 4.2 is satisfied with $\psi(t) = \theta(t) = t$ and $\varphi(t) = t/2$ for $t \geq 0$. By *(iii)*, there exists $(a,\ b) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ such that

$$-a''(s) \leq h(s,\ a(s),\ b(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t,\ s)$, we get

$$\int_0^1 -a''(s)G(t,\ s)ds \leq F(a,\ b)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0,\ 1]$, we have

$$-(1-t)\int_0^t sa''(s)ds - t\int_t^1 (1-s)a''(s)ds \leq F(a,\ b)(t).$$

Using integration by parts and $a(0) = a(1) = 0$, for all $t \in [0,\ 1]$, we get

$$-(1-t)(ta'(t)-a(t)) - t(-(1-t)a'(t)-a(t)) \leq F(a,\ b)(t).$$

Thus, we have

$$a(t) \leq F(a,\ b)(t), \text{ for } t \in [0,\ 1].$$

This implies that $a \leq F(a,\ b)$. Similarly, one can show that $b \geq F(b,\ a)$. Thus all the hypothesis of Corollary 4.2 are satisfied. Consequently, F has a coupled fixed point $(x,\ y) \in X^2$ which is the solution of (5.2) in $X = C(I,\ \mathbb{R})$. $\square$

Conclusion 5.1. (1) *Using the same technique, one can easily obtain tripled, quadruple and in general, multidimensional version of our results.*

(2) *Theorem 3.1 generalized the results of Harjani and Sadarangani [18].*

(3) *Corollary 3.11 generalized the results of Ran and Reurings* [28] *and Nieto and Rodríguez-López* [26].

(4) *The results of Harjani et al.* [17] *and Luong and Thuan* [23] *are extended and generalized by Corollary* 4.1 *and Corollary* 4.2, *respectively.*

(5) *Corollary* 4.1 *and Corollary* 4.2 *generalized the results of Gnana-Bhaskar and Lakshmikantham* [6] *and Ding et al.* [13].

(6) *Theorem* 4.2 *is an extension of the main result of Alotaibi and Alsulami* [3].

(7) *Theorem* 4.1 *and Theorem* 4.2 *is generalize the results of Razani and Parvaneh* [32] *and Alsulami* [4].

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