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INVARIANTS FOR F-PLANAR MAPPINGS OF SYMMETRIC AFFINE CONNECTION SPACES

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Abstract. This research is motivated by similarity of basic equations of F-planar mappings of symmetric affine connection space $\mathbb{A}_N$ involved by J. Mike and N. S. Sinyukov, and which have been studied by Mikes research group (I. Hinterleitner, P. Peška, J. Stránská) and almost geodesic mappings (specially almost geodesic mappings of the second type) of the space $\mathbb{A}_N$ involved by N. S. Sinyukov and which have been studied by many authors. We used the formulas obtained by N. O. Vesic to obtain invariants for special F-planar mappings in this article. These invariants are analogous to invariants of geodesic mappings (the Thomas projective parameter and the Weyl projective tensor).
Key words: F-planar mapping, invariant, affine connection spaces

1. Introduction

In this article, we will study special F-planar mappings of a symmetric affine connection space. Our purpose is to obtain invariants for these mappings.

The F-planar mappings of symmetric affine connection spaces are involved by J. Mikeš and his research group [2–7,9,10]. This research is continued with F-planar mappings of non-symmetric affine connection spaces [11,16]. We are aimed to apply the formulas from [14] to obtain invariants for special F-planar mappings in this paper.

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1.1. Symmetric affine connection spaces

An N-dimensional differential manifold equipped with affine connection $\overset{0}{\nabla}$, whose coefficients are $L^i_{\underline{jk}}$, $L^i_{\underline{jk}} = L^i_{\underline{kj}}$, is the symmetric affine connection space $\mathbb{A}_N$ (see [10, 12]).

The affine connection coefficients $L^i_{\underline{jk}}$ are not components of a tensor. They satisfy the transformation rule [10, 12]

$$L^{i'}_{\underline{j'k'}} = x^{i'}_{\alpha} x^{\beta}_{j'} x^{\gamma}_{k'} L^{\alpha}_{\underline{\beta\gamma}} + x^{i'}_{\alpha} x^{\alpha}_{j'k'}, \tag{1.1}$$

where $x^{i'}$, $x^{j'}$, $x^{k'}$,... are components of coordinate system $O'x^{1'} \dots x^{N'}$, x^i, x^j, x^k are components of coordinate system $Ox^1 \dots x^N$, $x^{i'}_j = \partial x^{i'}/\partial x^j$, $x^i_{j'} = \partial x^i/\partial x^{j'}$, $x^i_{j'k'} = \partial^2 x^i/\partial x^{j'} \partial x^{k'}$. We will also assume $x^i_{j'k'} = x^i_{k'j'}$ below.

The next equalities hold

$$x^{\alpha'}_{\alpha} x^{\alpha}_{j'\alpha'} = x^{\alpha'}_{\alpha} x^{\beta}_{j'} x^{\gamma}_{\alpha'} x^{\alpha}_{\beta\gamma} = 0, \quad \text{and} \quad x^{\alpha}_{j'\alpha} = x^{\beta}_{j'} x^{\alpha}_{\beta\alpha} = 0. \tag{1.2}$$

Covariant derivative of a tensor $\hat{a}$ of the type $(1,1)$, whose components are a^i_j, in the direction of x^k is [10, 12]

$$a^i_{j|k} = a^i_{j,k} + L^i_{\underline{\alpha k}} a^{\alpha}_j - L^{\alpha}_{\underline{jk}} a^i_{\alpha}, \tag{1.3}$$

where partial derivative $\partial/\partial x^k$ is denoted by comma. The Einstein summation convention by mute Greek indices is used in the previous equation and will be used in the rest of paper.

One Ricci identity is founded with respect to the covariant derivative [10, 12]

$$a^i_{j|m|n} - a^i_{j|n|m} = a^{\alpha}_j \overset{0}{R}{}^i_{\alpha mn} - a^i_{\alpha} \overset{0}{R}{}^{\alpha}_{jmn}, \tag{1.4}$$

where

$$\overset{0}{R}{}^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^{\alpha}_{\underline{jm}} L^i_{\underline{\alpha n}} - L^{\alpha}_{\underline{jn}} L^i_{\underline{\alpha m}}, \tag{1.5}$$

are components of the curvature tensor $\hat{\overset{0}{R}}$ of space $\mathbb{A}_N$.

The components of Ricci tensor of space $\mathbb{A}_N$ are

$$\overset{0}{R}_{ij} = \overset{0}{R}{}^{\alpha}_{ij\alpha} = L^{\alpha}_{\underline{ij},\alpha} - L^{\alpha}_{\underline{i\alpha},j} + L^{\beta}_{\underline{ij}} L^{\alpha}_{\underline{\beta\alpha}} - L^{\beta}_{\underline{i\alpha}} L^{\alpha}_{\underline{j\beta}}. \tag{1.6}$$

Ricci tensor of space $\mathbb{A}_N$ is non-symmetric, i.e. it exists at least one pair of indices (i_0, j_0) such that $\overset{0}{R}_{i_0 j_0} \neq \overset{0}{R}_{j_0 i_0}$.

1.2. F-planar and second-type almost geodesic mappings

A curve ℓ, which is given by the equations (J. Mikeš, N. S. Sinyukov [9]; see [2, 4–7, 9, 10])

$$\ell = \ell(t), \quad \lambda(t) = d\ell(t)/dt (\neq 0), \quad , t \in I, \tag{1.7}$$

where t is a parameter, is called F-planar, if under a parallel translation, the tangent vector $\lambda^i = d\ell^i/dt$ remains in the small area of vectors λ^i and $\lambda^\alpha F^i_\alpha$ adjoint to it, i.e.

$$\lambda^i_{|j} = a\lambda^i + b\lambda^\alpha F^i_\alpha$$

where a and b are functions of t.

A diffeomorphism $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is called F-planar [2, 3, 6, 7, 10] if any F-planar curve in $\mathbb{A}_N$ is transformed to an $\overline{F}$-planar curve in $\overline{\mathbb{A}}_N$ by the mapping f.

The basic equation of F-planar mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is [2, 3, 6, 7, 10]

$$\overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \psi_j \delta^i_k + \psi_k \delta^i_j + 2\sigma_j F^i_k + 2\sigma_k F^i_j, \tag{1.8}$$

for 1-forms ψ_j, σ_j and an affinor F^i_j.

The F-planar mapping f transforms the affinor F^i_j to $\overline{F}^i_j$. We will stay focused on F-planar mappings which preserve the F-structure [2, 3, 6, 7, 10]. In this case, the next equation holds

$$\overline{F}^i_j = aF^i_j + b\delta^i_j, \tag{1.9}$$

for scalar functions a and b.

A mapping $f : \mathbb{A}_N \to \overline{A}_N$ whose deformation tensor is given by (1.8) is called the F-planar mapping ([10], p. 386, an alternative definition of F-planar mappings).

A mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ determined by the equations

$$\begin{cases} \overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \psi_j \delta^i_k + \psi_k \delta^i_j + 2\sigma_j F^i_k + 2\sigma_k F^i_j, \\ F^i_{j|k} + F^i_{k|j} + 2F^i_\alpha F^\alpha_j \sigma_k + 2F^i_\alpha F^\alpha_k \sigma_j = \mu_j F^i_k + \mu_k F^i_j + \nu_j \delta^i_k + \nu_k \delta^i_j, \end{cases} \tag{1.10}$$

for 1-forms ψ_j, σ_j, μ_j, ν_j and affinor structure F^i_j, is called the second type almost geodesic mapping of the space $\mathbb{A}_N$. Details about almost geodesic mappings of symmetric affine connection spaces may be found in [1, 7, 8, 10] and in many other publications.

The second type almost geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ has the property of reciprocity if it preserves the affinor structure F^i_j, $F^i_j = \overline{F}^i_j$, and the corresponding inverse mapping f^{-1} is the second type almost geodesic mapping. It is proved that the mapping f satisfies property of reciprocity if and only if $F^i_\alpha F^\alpha_j = e\delta^i_j$, $e = \pm 1, 0$.

Based on the alternative definition of F-planar mappings, we conclude that F-planar mappings of space $\mathbb{A}_N$ are subclass of the class of second type almost geodesic mappings of $\mathbb{A}_N$. The almost geodesic mappings which have the property of reciprocity are F-planar mappings which transform the F-structure by the rule (1.9) for $a = 1$ and $b = 0$. These F-planar mappings form the class $\pi^F(e)$.

1.3. Geometric mappings and their basic invariants

Infinitely many affine connections may be defined on the manifold $\mathcal{M}_N$. Let $\overset{0}{\nabla}$ and $\overset{0}{\overline{\nabla}}$ be two affine connections defined on $\mathcal{M}_N$. If $L^i_{\underline{jk}}$ and $\overline{L}^i_{\underline{jk}}$ are coefficients of these affine connections, the geometrical objects $\overset{0}{P}{}^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}}$ are components of tensor $\hat{\overset{0}{P}}$ of the type $(1,2)$. The tensor $\hat{\overset{0}{P}}$ is called the deformation tensor.

The affine connections $\overset{0}{\nabla}$ and $\overset{0}{\overline{\nabla}}$ defined on the manifold $\mathcal{M}_N$ generate two affine connection spaces $\mathbb{A}_N$ and $\overline{\mathbb{A}}_N$. Transformation $\overset{0}{\nabla} \xrightarrow{f} \overset{0}{\overline{\nabla}}$, i.e. $L^i_{\underline{jk}} \xrightarrow{f} \overline{L}^i_{\underline{jk}} = L^i_{\underline{jk}} + \overset{0}{P}{}^i_{\underline{jk}}$, is the mapping of space $\mathbb{A}_N$.

Different forms of deformation tensor $\overset{0}{P}{}^i_{\underline{jk}}$ generate special classes of mappings: geodesic, conformal, almost geodesic,. . .

Example 1.1. The deformation tensor of geodesic mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is

$$\overset{0}{P}{}^i_{\underline{jk}} = \psi_j \delta^i_k + \psi_k \delta^i_j, \tag{1.11}$$

for a 1-form ψ_j. After substituting the equality $\overset{0}{P}{}^i_{\underline{jk}} = \overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}}$ in the equation (1.11), one obtains

$$\overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}} = \psi_j \delta^i_k + \psi_k \delta^i_j. \tag{1.12}$$

If the equation (1.12) is contracted by i and k, one obtains

$$\psi_j = \frac{1}{N+1}\big(\overline{L}^\alpha_{\underline{j\alpha}} - L^\alpha_{\underline{j\alpha}}\big). \tag{1.13}$$

Based on the equations (1.12, 1.13), it is obtained

$$\begin{aligned}
&\overline{L}^i_{\underline{jk}} - L^i_{\underline{jk}} = \frac{1}{N+1}\delta^i_k\big(\overline{L}^\alpha_{\underline{j\alpha}} - L^\alpha_{\underline{j\alpha}}\big) + \frac{1}{N+1}\delta^i_j\big(\overline{L}^\alpha_{\underline{k\alpha}} - L^\alpha_{\underline{k\alpha}}\big) \Leftrightarrow\\
&\overline{L}^i_{\underline{jk}} - \frac{1}{N+1}\big(\delta^i_k\overline{L}^\alpha_{\underline{j\alpha}} + \delta^i_j\overline{L}^\alpha_{\underline{k\alpha}}\big) = L^i_{\underline{jk}} - \frac{1}{N+1}\big(\delta^i_k L^\alpha_{\underline{j\alpha}} + \delta^i_j L^\alpha_{\underline{k\alpha}}\big) \Leftrightarrow\\
&\overset{0}{\overline{T}}{}^i_{jk} = \overset{0}{T}{}^i_{jk},\\
&\overset{0}{T}{}^i_{jk} = L^i_{\underline{jk}} - \frac{1}{N+1}\big(\delta^i_k L^\alpha_{\underline{j\alpha}} + \delta^i_j L^\alpha_{\underline{k\alpha}}\big), \overset{0}{\overline{T}}{}^i_{jk} = \overline{L}^i_{\underline{jk}} - \frac{1}{N+1}\big(\delta^i_k \overline{L}^\alpha_{\underline{j\alpha}} + \delta^i_j \overline{L}^\alpha_{\underline{k\alpha}}\big).
\end{aligned} \tag{1.14}$$

The geometrical object $\overset{0}{T}{}^i_{jk}$ is Thomas projective parameter.

With respect to the invariance

$$\overset{0}{\overline{T}}{}^i_{jm,n} - \overset{0}{\overline{T}}{}^i_{jn,m} + \overset{0}{\overline{T}}{}^\alpha_{jm}\overset{0}{\overline{T}}{}^i_{\alpha n} - \overset{0}{\overline{T}}{}^\alpha_{jn}\overset{0}{\overline{T}}{}^i_{\alpha m} = \overset{0}{T}{}^i_{jm,n} - \overset{0}{T}{}^i_{jn,m} + \overset{0}{T}{}^\alpha_{jm}\overset{0}{T}{}^i_{\alpha n} - \overset{0}{T}{}^\alpha_{jn}\overset{0}{T}{}^i_{\alpha m},$$

the next invariant for geodesic mapping of space $\mathbb{A}_N$ is obtained

$$\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j R_{[mn]} - \frac{1}{(N+1)^2}\delta^i_m\big((N+1)L^\alpha_{\underline{j\alpha}|n} + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{n\beta}}\big) + \frac{1}{(N+1)^2}\delta^i_n\big((N+1)L^\alpha_{\underline{j\alpha}|m} + L^\alpha_{\underline{j\alpha}}L^\beta_{\underline{m\beta}}\big). \tag{1.15}$$

After contracting the difference $\overset{0}{\widetilde{\widetilde{\mathcal{W}}}}{}^i_{jmn} - \overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn} = 0$, the Weyl projective tensor as invariant for the geodesic mapping of $\mathbb{A}_N$ is obtained

$$\overset{0}{W}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} + \frac{1}{N+1}\delta^i_j \overset{0}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j}. \tag{1.16}$$

The process for obtaining the Thomas projective parameter and the Weyl projective tensor motivated N. O. Vesić to obtain general formulae of invariants for different geometric mappings [14].

If the deformation tensor of mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is $P^i_{\underline{jk}} = \overset{0}{\overline{\omega}}{}^i_{jk} - \overset{0}{\omega}{}^i_{jk}$, the basic associated invariant of the Thomas type for this mapping is

$$\overset{0}{\widetilde{T}}{}^i_{jk} = L^i_{\underline{jk}} - \omega^i_{jk}, \tag{1.17}$$

and the basic associated invariant of the Weyl type for this mapping is

$$\overset{0}{\widetilde{W}}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} - \omega^i_{jm|n} + \omega^i_{jn|m} + \omega^\alpha_{jm}\omega^i_{\alpha n} - \omega^\alpha_{jn}\omega^i_{\alpha m}. \tag{1.18}$$

The formulas (1.17, 1.18) were applied in [15, 16] for obtaining invariants of mappings of non-symmetric affine connection spaces. In this paper, we will use these formulas to obtain invariants for F-planar mappings of the type $\pi^F(e)$.

1.4. Motivation

Symmetric affine connection spaces and mappings between them have been studied by many authors. Some of these authors are J. Mikeš, N. S. Sinyukov, I. Hinterleitner, and many others.

The theory of invariants for mappings between non-symmetric affine connection spaces has been developed in last decades. Some of results from this subject research are obtained by the following authors (M. S. Stanković [11, 13, 15]; M. Lj. Zlatanović [13, 16–18]) and many others. Numerous authors have studied this subject of differential geometry.

Almost geodesic mappings of second type which have the property of reciprocity are the special class of F-planar mappings. The researches about invariants for almost geodesic mappings of the second type [15] motivated the research presented below.

The main purpose of this paper is to obtain basic associated invariants of Thomas and Weyl type for F-planar mappings of space $\mathbb{A}_N$ of the type $\pi^F(e)$.

The next aim is to examine how many invariants for F-planar mappings of the type $\pi^F(e)$ may be obtained with respect to the corresponding basic invariants.

The last goal of this paper is to examine these invariants tensors or parameters.

2. Invariants for F-planar mappings of space $\mathbb{A}_N$

Let $f:\mathbb{A}_N\to\overline{\mathbb{A}}_N$ be an F-planar mapping of the class $\pi^F(e)$. Because $\overline{F}^i_j=F^i_j$ and the deformation tensors $P^i_{\underline{jk}}$ and $\overline{P}^i_{\underline{jk}}$ of the mapping f and its inverse mapping f^{-1} satisfy the equality $\overline{P}^i_{\underline{jk}}=-P^i_{\underline{jk}}$, it exists the 1-form $\overline{\sigma}_j$ such that $\overline{\sigma}_j=-\sigma_j$.

For this reason, the deformation tensor $P^i_{\underline{jk}}$ of mapping $f:\mathbb{A}_N\to\overline{\mathbb{A}}_N$ from the class $\pi^F(e)$ is

$$P^i_{\underline{jk}}=\overline{L}^i_{\underline{jk}}-L^i_{\underline{jk}}=\psi_j\delta^i_k+\psi_k\delta^i_j-\overline{\sigma}_j\overline{F}^i_k-\overline{\sigma}_k\overline{F}^i_j+\sigma_jF^i_k+\sigma_kF^i_j. \tag{2.1}$$

After contracting the equation (2.1) by i and k, one obtains

$$\psi_j=\frac{1}{N+1}\left(\overline{L}^\alpha_{\underline{j\alpha}}+\overline{\sigma}_j\overline{F}+\overline{\sigma}_\alpha\overline{F}^\alpha_j\right)-\frac{1}{N+1}\left(L^\alpha_{\underline{j\alpha}}+\sigma_jF+\sigma_\alpha F^\alpha_j\right) \tag{2.2}$$

for $F=F^\alpha_\alpha$ and $\overline{F}=\overline{F}^\alpha_\alpha$.

Hence, it holds

$$\begin{aligned}P^i_{\underline{jk}}=&\frac{1}{N+1}\delta^i_j\left(\overline{L}^\alpha_{\underline{k\alpha}}+\overline{\sigma}_k\overline{F}+\overline{\sigma}_\alpha\overline{F}^\alpha_k\right)+\frac{1}{N+1}\delta^i_k\left(\overline{L}^\alpha_{\underline{j\alpha}}+\overline{\sigma}_j\overline{F}+\overline{\sigma}_\alpha\overline{F}^\alpha_j\right)\\&-\frac{1}{N+1}\delta^i_j\left(L^\alpha_{\underline{k\alpha}}+\sigma_kF+\sigma_\alpha F^\alpha_k\right)-\frac{1}{N+1}\delta^i_k\left(L^\alpha_{\underline{j\alpha}}+\sigma_jF+\sigma_\alpha F^\alpha_j\right)\\&-\overline{\sigma}_j\overline{F}^i_j-\overline{\sigma}_k\overline{F}^i_k+\sigma_jF^i_k+\sigma_kF^i_j.\end{aligned}$$

Based on this equation, we get

$$\begin{aligned}\omega^i_{jk}=&\frac{1}{N+1}\delta^i_j\left(L^\alpha_{\underline{k\alpha}}+\sigma_kF+\sigma_\alpha F^\alpha_k\right)+\frac{1}{N+1}\delta^i_k\left(L^\alpha_{\underline{j\alpha}}+\sigma_jF+\sigma_\alpha F^\alpha_j\right)\\&-\sigma_jF^i_k-\sigma_kF^i_j,\end{aligned}$$

and the corresponding $\overline{\omega}^i_{jk}$.

With respect to this ω^i_{jk}, the basic invariant of Thomas type for F-planar mapping $f:\mathbb{A}_N\to\overline{\mathbb{A}}_N$ is

$$\begin{aligned}\mathcal{F}\overset{0}{\widetilde{\mathcal{T}}}{}^i_{jk}=&L^i_{\underline{jk}}+\sigma_jF^i_k+\sigma_kF^i_j\\&-\frac{1}{N+1}\left(\delta^i_j\left(L^\alpha_{\underline{k\alpha}}+\sigma_kF+\sigma_\alpha F^\alpha_k\right)+\delta^i_k\left(L^\alpha_{\underline{j\alpha}}+\sigma_jF+\sigma_\alpha F^\alpha_j\right)\right).\end{aligned} \tag{2.3}$$

The basic invariant of Weyl type for F-planar mapping $f:\mathbb{A}_N\to\overline{\mathbb{A}}_N$ is

$$\begin{aligned}\widetilde{\mathcal{F}\mathbf{W}}{}^{\overset{0}{}i}_{jmn} &= \overset{0}{R}{}^i_{jmn}+\frac{1}{N+1}\delta^i_j\overset{0}{R}_{[mn]}-\frac{1}{N+1}\delta^i_j\overset{0}{U}_{[mn]}-\frac{1}{N+1}\delta_{[m}L^\alpha_{\underline{j\alpha}|n]}\\ &-\frac{1}{N+1}\delta^i_{[m}\overset{0}{U}_{jn]}+\frac{1}{N+1}\delta^i_n\overset{0}{V}_{jm}-\frac{1}{N+1}\delta^i_m\overset{0}{V}_{jn}\\ &-\sigma_{j|[m}F^i_{n]}+\sigma_jF^i_{[m|n]}-\sigma_{[m|n]}F^i_j+\sigma_{[m}F^i_{j|n]}\\ &+\sigma_\alpha\sigma_j(F^\alpha_mF^i_n-F^\alpha_nF^i_m)+\sigma_\alpha F^\alpha_j(\sigma_mF^i_n-\sigma_nF^i_m)\\ &+\sigma_je(\sigma_n\delta^i_m-\sigma_m\delta^i_n),\end{aligned}\tag{2.4}$$

for $L^i_{\underline{jm}|n}=L^i_{\underline{jm},n}+L^i_{\underline{\alpha n}}L^\alpha_{\underline{jm}}-L^\alpha_{\underline{jn}}L^i_{\underline{\alpha m}}-L^\alpha_{\underline{mn}}L^i_{\underline{j\alpha}}$ and

$$\overset{0}{U}_{ij}=\sigma_{i|j}F+\sigma_iF_{,j}-\sigma_{\alpha|i}F^\alpha_j+\sigma_\alpha F^\alpha_{i|j},\tag{2.5}$$

$$\begin{aligned}\overset{0}{V}_{ij} &= L^\beta_{\underline{\alpha\beta}}F^\alpha_j\sigma_i+FF^\alpha_j\sigma_i\sigma_\alpha+L^\beta_{\underline{\alpha\beta}}F^\alpha_i\sigma_j+FF^\alpha_i\sigma_j\sigma_\alpha+2e\sigma_i\sigma_j\\ &-\frac{1}{(N+1)}(L^\alpha_{\underline{i\alpha}}+\sigma_iF+\sigma_\alpha F^\alpha_i)(L^\beta_{\underline{j\beta}}+\sigma_jF+\sigma_\beta F^\beta_j).\end{aligned}\tag{2.6}$$

Because $\overline{\sigma}_j=-\sigma_j$ and $\overline{F}^i_k=F^i_k$, we get $\overline{F}^i_\alpha\overline{F}^\alpha_j=\overline{e}\delta^i_j=F^i_\alpha F^\alpha_j=e\delta^i_j$, which gives $\overline{e}=e$, $\overline{\sigma}_i\overline{\sigma}_j=\sigma_i\sigma_j$, $\overline{\sigma}_i\overline{\sigma}_j\overline{F}^p_q\overline{F}^r_s=\sigma_i\sigma_jF^p_qF^r_s$. Moreover, it holds $\overline{F}_{,i}=F_{,i}$. Hence, we get $\overline{\sigma}_i\overline{F}_{,j}=\sigma_iF_{,j}$. It also holds

$$\begin{aligned}(L^\alpha_{\underline{i\alpha}}+\sigma_iF+\sigma_\alpha F^\alpha_i)(L^\beta_{\underline{j\beta}}+\sigma_jF+\sigma_\beta F^\beta_j) &= L^\alpha_{\underline{i\alpha}}L^\beta_{\underline{j\beta}}+L^\alpha_{\underline{i\alpha}}\big(\sigma_jF+\sigma_\beta F^\beta_j\big)\\ &+L^\alpha_{\underline{j\alpha}}\big(\sigma_iF+\sigma_\beta F^\beta_i\big)\\ &+\big(\sigma_iF+\sigma_\alpha F^\alpha_i\big)\big(\sigma_jF+\sigma_\beta F^\beta_j\big).\end{aligned}\tag{2.7}$$

Based on invariants from the previous paragraph, and with respect to the form of the geometrical object $\widetilde{\mathcal{F}\mathcal{W}}{}^{\overset{0}{}i}_{jmn}$ given by (2.4), such as the equation (2.7), we simplify the expressions (2.4, 2.5, 2.6) to

$$\begin{aligned}\widetilde{\mathcal{F}\mathcal{W}}{}^{\overset{0}{}i}_{jmn} &= \overset{0}{R}{}^i_{jmn}+\frac{1}{N+1}\delta^i_j\big(\overset{0}{R}_{[mn]}-\overset{0}{U}_{[mn]}\big)\\ &-\frac{1}{N+1}\big(\delta^i_{[m}L^\alpha_{\underline{j\alpha}|n]}+\delta^i_{[m}\overset{0}{U}_{jn]}+\delta^i_{[m}\overset{0}{V}_{jn]}\big)\\ &-\sigma_{j|[m}F^i_{n]}+\sigma_jF^i_{[m|n]}-\sigma_{[m|n]}F^i_j+\sigma_{[m}F^i_{j|n]},\end{aligned}\tag{2.4'}$$

$$\overset{0}{U}_{ij}=\sigma_{i|j}F-\sigma_{\alpha|i}F^\alpha_j+\sigma_\alpha F^\alpha_{i|j},\tag{2.5'}$$

$$\overset{0}{V}_{ij} = L^{\beta}_{\underline{\alpha\beta}} F^{\alpha}_{j} \sigma_i + L^{\beta}_{\underline{\alpha\beta}} F^{\alpha}_{i} \sigma_j - \frac{1}{N+1} L^{\alpha}_{\underline{i\alpha}} L^{\beta}_{\underline{j\beta}} - \frac{1}{N+1} L^{\alpha}_{(\underline{i\alpha}} \delta^{\gamma}_{j)} \big(\sigma_\gamma F + \sigma_\beta F^{\beta}_{\gamma}\big). \tag{2.6'}$$

The next lemma holds.

Lemma 2.1. *Let $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ be an F-planar mapping of the type $\pi^F(e)$. The geometrical objects $\mathcal{F}\overset{0}{\mathcal{T}}{}^{i}_{jk}$ and $\mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^{i}_{jmn}$, respectively given by equations* (2.3, 2.4'), *for $\overset{0}{U}_{ij}$ and $\overset{0}{V}_{ij}$ given by* (2.5', 2.6'), *are invariants for the mapping f.* □

Corollary 2.1. *The invariants $\mathcal{F}\overset{0}{\mathcal{T}}{}^{i}_{jk}$ and $\mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^{i}_{jmn}$ for F-planar mapping f : $\mathbb{A}_N \to \overline{\mathbb{A}}_N$ given by* (2.3, 2.4') *are parameters expected in the special case.*

Proof. The invariant $\mathcal{F}\overset{0}{\mathcal{T}}{}^{i}_{jk}$ for mapping f is the sum of the Thomas projective parameter and tensor of the type $(1,2)$. For this reason, this invariant is parameter.

Based on equations (2.5', 2.6'), the invariant $\mathcal{F}\overset{0}{\widetilde{\mathcal{F}}}$ given by (2.4') is expressed as sum of tensor and the geometrical object

$$\mathcal{M}^{i}_{jmn} = \frac{1}{N+1} \delta^{i}_{[m} L^{\alpha}_{\underline{j\alpha}|n]} + \frac{1}{(N+1)^2} \delta^{i}_{[m} L^{\alpha}_{\underline{j\alpha}} L^{\beta}_{\underline{n]\beta}}. \tag{2.8}$$

It holds the equality $L^{\alpha}_{\underline{j\alpha}|k} = L^{\alpha}_{\underline{j\alpha},k} - L^{\alpha}_{\underline{jk}} L^{\beta}_{\underline{\alpha\beta}}$.

Based on the equation (1.1), one obtains

$$L^{\alpha'}_{\underline{i'\alpha'}|'j'} = x^{\beta}_{i'} x^{\gamma}_{j'} L^{\alpha}_{\underline{\beta\alpha}|\gamma} - x^{\beta}_{j'k'} L^{\alpha}_{\underline{\beta\alpha}}, \tag{2.9}$$

$$L^{\alpha'}_{\underline{i'\alpha'}} L^{\beta'}_{\underline{j'\beta'}} = x^{\beta}_{i'} x^{\gamma}_{j'} L^{\beta}_{\underline{\alpha\beta}} L^{\delta}_{\underline{\gamma\delta}}. \tag{2.10}$$

From (2.9, 2.10), we obtain

$$M^{i'}_{j'm'n'} = x^{i'}_{\alpha} x^{\beta}_{j'} x^{\gamma}_{m'} x^{\delta}_{n'} M^{\alpha}_{\beta\gamma\delta} - \frac{1}{N+1} \delta^{i'}_{[m'} x^{\beta}_{j'n']} L^{\alpha}_{\underline{\beta\alpha}}. \tag{2.11}$$

That means that the invariants $\mathcal{F}\overset{0}{\widetilde{\mathcal{T}}}{}^{i}_{jmn}$ and $\mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^{i}_{jmn}$ are tensors if and only if $x^{i'}_{\alpha} x^{\alpha}_{j'k'} - \frac{1}{N+1}\big(\delta^{i'}_{k'} x^{\alpha}_{j'\alpha} + \delta^{i'}_{j'} x^{\alpha}_{k'\alpha}\big) = 0$ and $\delta^{i'}_{[m'} x^{\beta}_{j'n']} L^{\alpha}_{\underline{\beta\alpha}} = 0$, respectively. □

Now, using $\mathcal{F}\overset{0}{\widetilde{\overline{\mathcal{W}}}}{}^i_{jmn} - \mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn} = 0$, where $\mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn}$ is basic invariant of the F-planar mapping is given by (2.4') and $\mathcal{F}\overset{0}{\widetilde{\overline{W}}}{}^i_{jmn}$ is its image, one obtains

$$
\begin{aligned}
0 &= \mathcal{F}\overset{0}{\widetilde{\overline{\mathcal{W}}}}{}^i_{jmn} - \mathcal{F}\overset{0}{\widetilde{\mathcal{W}}}{}^i_{jmn}\\
&= \overline{R}^i_{jmn} - R^i_{jmn} + \frac{1}{N+1}\delta^i_j\big(\overline{R}_{[mn]} - R_{[mn]} - \overline{U}_{[mn]} + U_{[mn]}\big)\\
&- \delta^i_{[m}A_{jn]} - \frac{1}{N+1}\big(\delta^i_{[m}\overline{L}^{\alpha}_{\underline{j\alpha}\|n]} - \delta^i_{[m}L^{\alpha}_{\underline{j\alpha}|n]}\big)\\
&- \overline{\sigma}_{j\|[m}\overline{F}^i_{n]} + \overline{\sigma}_j\overline{F}^i_{[m\|n]} - \overline{\sigma}_{[m\|n]}\overline{F}^i_j + \overline{\sigma}_{[m}\overline{F}^i_{j\|n]}\\
&+ \sigma_{j|[m}F^i_{n]} - \sigma_j F^i_{[m|n]} + \sigma_{[m|n]}F^i_j - \sigma_{[m}F^i_{j|n]}
\end{aligned}
\tag{2.12}
$$

where is $A_{ij} = \dfrac{1}{N+1}\big(\overline{U}_{ij} - U_{ij} + \overline{V}_{ij} + V_{ij}\big)$.

After contracting the previous equation by i and n, we get

$$
\begin{aligned}
A_{jm} = &-\frac{N}{N^2-1}\big(\overline{R}_{jm} - R_{jm}\big) - \frac{1}{N^2-1}\big(\overline{R}_{mj} - R_{mj}\big)\\
&+ \frac{N}{N^2-1}\big(\overline{U}_{jm} - U_{jm}\big) + \frac{1}{N^2-1}\big(\overline{U}_{mj} - U_{mj}\big)\\
&- \frac{1}{N-1}\big(\overline{L}^{\alpha}_{\underline{j\alpha}\|m} - L^{\alpha}_{\underline{j\alpha}|m}\big) + \frac{1}{N-1}\big(\overline{B}_{jm} - B_{jm}\big),
\end{aligned}
$$

where

$$
B_{ij} = \big(\sigma_{j|\alpha} - \sigma_{\alpha|j}\big)F^{\alpha}_i - \big(\sigma_{i|\alpha} - \sigma_{\alpha|i}\big)F^{\alpha}_j - \sigma_i F^{\alpha}_{j|\alpha} - \sigma_j F^{\alpha}_{i|\alpha},
\tag{2.13}
$$

and the corresponding $\overline{B}_{ij}$.

If involves the expression of A_{ij} in the equation (2.12), we get

$$
\mathcal{F}\overset{0}{\overline{W}}{}^i_{jmn} = \mathcal{F}\overset{0}{W}{}^i_{jmn},
$$

where

$$
\begin{aligned}
\mathcal{F}\overset{0}{W}{}^i_{jmn} = \overset{0}{R}{}^i_{jmn} &+ \frac{1}{N+1}\delta^i_j\overset{0}{R}_{[mn]} + \frac{N}{N^2-1}\delta^i_{[m}\overset{0}{R}_{jn]} + \frac{1}{N^2-1}\delta^i_{[m}\overset{0}{R}_{n]j}\\
&- \frac{1}{N+1}\delta^i_j U_{[mn]} - \frac{N}{N^2-1}\delta^i_{[m}U_{jn]} - \frac{1}{N^2-1}\delta^i_{[m}U_{n]j} - \frac{1}{N-1}\delta^i_{[m}B_{jn]}\\
&+ \sigma_{j|[m}F^i_{n]} + \sigma_{[m|n]}F^i_j - \sigma_j F^i_{[m|n]} - \sigma_{[m}F^i_{j|n]}.
\end{aligned}
\tag{2.14}
$$

The next theorem holds.

Theorem 2.1. *Let $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ be an F-planar mapping of symmetric affine connection space $\mathbb{A}_N$. The geometrical object $\mathcal{F}\overset{0}{W}{}^i_{jmn}$ given by (2.14) is an invariant for the mapping f. This invariant is tensor.* □

3. Conclusion

In this paper, we studied F-planar mappings of the type $\pi^F(e)$. The basic invariants of Thomas and Weyl type for these mappings are obtained in Lemma 2.1. The derived invariants of Weyl type for these mappings is obtained in Theorem 2.1. In Corollary 2.1 and in the second part of Theorem 2.1, we examined tensor characters of the obtained invariants.

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