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ON NON-INVARIANT HYPERSURFACES OF AN ε-PARA SASAKIAN MANIFOLD

Shyam Kishor and Prerna Kanaujia

Abstract. Non-invariant hypersurfaces of an $\varepsilon-$ para Sasakian manifold of an induced structure (f, g, u, v, λ) have been studied in this paper. Some properties followed by this structure have ben obtained. A necessary and sufficient condition for totally umbilical non-invariant hypersurfaces equipped with $(f, g, u, v, \lambda)-$ structure of $\varepsilon-$para Sasakian manifold to be totally geodesic has also been explored.

Keywords: $\varepsilon-$ Para Sasakian Manifold, totally umbilical, totally geodesic.

1. Introduction

In 1976, Sato [1] introduced a structure (ϕ, ξ, η) satisfying $\phi^2 = I-\eta\otimes\xi$ and $\eta(\xi) = 1$ on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [2, 3] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [4] introduced almost contact manifolds equipped with associated pseudo Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo$-$ Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as $\varepsilon-$almost contact metric manifolds and $\varepsilon-$Sasakian manifolds respectively [5, 6, 7]. Also, in 1989, K. Matsumoto [8] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure, calling it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi$-$Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivated the authors in [9] to associate a semi$-$Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite

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almost paracontact metric structure an $\varepsilon-$almost paracontact structure, where the structure vector field ξ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$.

In [9] the authors studied $\varepsilon-$almost paracontact manifolds, and in particular, $\varepsilon-$para Sasakian manifolds. They gave basic definitions, some examples of $\varepsilon-$almost paracontact manifolds and introduced the notion of an $\varepsilon-$para Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the $\varepsilon-$para Sasakian manifolds were also studied in [9]. The authors in [9] proved that if a semi$-$Riemannian manifold is one of flat, proper recurrent or proper Ricci$-$recurrent, then it can not admit an $\varepsilon-$para Sasakian structure. Also. they showed that, for an $\varepsilon-$para Sasakian manifold, the conditions of being symmetric, semi$-$symmetric or of constant sectional curvature are all identical.

On the other hand In 1970, S. I. Goldberg et. al [10] introduced the notion of a non$-$invariant hypersurface of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the $(1,1)$ structure tensor field ϕ defining the almost contact structure is never tangent to the hypersurface.

The notion of $(f, g, u, v, \lambda)-$ structure was given by K.Yano [11]. It is well known [12, 13] that a hypersurface of an almost contact metric manifold always admits a $(f, g, u, v, \lambda)-$ structure. Authors [10] proved that there always exists a $(f, g, u, v, \lambda)-$ structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. R. Prasad [14] studied the non-invariant hypersurfaces of trans-Sasakian manifolds. Non-invariant hypersurfaces of nearly Trans-Sasakian manifold have been studied by S. Kishor et. al [15] . The present paper is devoted to the study of non-invariant hypersurfaces of $\varepsilon-$para Sasakian manifolds. The contents of the paper are organized as follows:

In section-2 some preliminaries are given. Section-3 deals with the study of non-invariant hypersurfaces of $\varepsilon-$para Sasakian manifolds. A necessary and sufficient condition for a totally umbilical non-invariant hypersurface of an $\varepsilon-$para Sasakian manifold to be totally geodesic is found.

2. Preliminaries

Let $\tilde{M}$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g) where ϕ is $(1,1)$ tensor field, η is $1-$ form and g is a compatible Riemannian metric such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \epsilon\eta(X) \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$.

An almost contact metric manifold is an $\varepsilon-$para Sasakian manifold if

$$(\widetilde{\nabla}_X \phi)(Y) = -g(\phi X, \phi Y) - \epsilon\eta(Y)\phi^2 X \tag{2.4}$$

for for all vector fields X, Y on $\widetilde{M}$ where $\widetilde{\nabla}$ is the operator of covariant differentiation with respect to g. From (2.4), we have

$$\widetilde{\nabla}_X \xi = \varepsilon\phi X \tag{2.5}$$

A hypersurface of an almost contact metric manifold $\widetilde{M}\,(\phi, \xi, \eta, g)$ is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of $(1,1)$ tensor field ϕ defining the contact structure is never tangent to the hypersurface. Let X be a tangent vector on a non-invariant hypersurface of an almost contact metic manifold $\widetilde{M}$, then $X\phi$ is never tangent to the hypersurface.

Let M be a non-invariant hypersurface of an almost contact metric

manifold, then defining

$$\phi X = fX + u(X)\hat{N}, \tag{2.6}$$

$$\phi\hat{N} = -U, \tag{2.7}$$

$$\xi = V + \lambda\hat{N}, \qquad \lambda = n(\hat{N}); \tag{2.8}$$

$$\eta(X) = \nu(X), \tag{2.9}$$

where f is a $(1,1)$ tensor field, u, v are $1-$forms, $\hat{N}$ is a unit normal to the hypersurface, $X \in TM$ and $u(X) \neq 0$, then we get an induced (f, g, u, v, λ) structure on M satisfying the conditions [11, 12] :

$$f^2 = -I + u \otimes U + v \otimes V, \tag{2.10}$$

$$fU = -\lambda V, \quad fV = \lambda U, \tag{2.11}$$

$$uof = \lambda v, \qquad vof = -\lambda u, \tag{2.12}$$

$$u(U) = 1 - \lambda^2, \quad u(V) = v(U) = 0, \qquad v(V) = 1 - \lambda^2, \tag{2.13}$$

$$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y), \tag{2.14}$$

$$g(X, fY) = -g(fX, Y), \qquad g(X, U) = u(X), \qquad g(X, V) = v(X), \tag{2.15}$$

for all $X, Y \in TM$, where $\lambda = n(\hat{N})$.

The Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)\hat{N}, \tag{2.16}$$

$$\tilde{\nabla}_X \hat{N} = -A_{\hat{N}} X, \tag{2.17}$$

for all $X, Y \in TM$, where $\tilde{\nabla}$ and ∇ are the Riemannian and induced Riemannian connections on $\tilde{M}$ and M respectively and $\hat{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$. The second fundamental form σ on M is related to $A_{\hat{N}}$ by

$$\sigma(X, Y) = g(A_{\hat{N}} X, Y), \quad \text{for all } X, Y \in TM. \tag{2.18}$$

3. Non-invariant Hypersurfaces

Lemma 3.1. *Let M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$structure of an $\varepsilon-$para Sasakian manifold $\tilde{M}$, then*

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X f)Y - u(Y)A_{\hat{N}} X + \sigma(X, Y)U + ((\nabla_X u)Y + \sigma(X, fY))\hat{N} \tag{3.1}$$

$$(\tilde{\nabla}_X \eta)Y = (\nabla_X u)Y - \lambda\sigma(X, Y) \tag{3.2}$$

$$\tilde{\nabla}_X \xi = \left(\nabla_X V - \lambda A_{\hat{N}} X\right) + (\sigma(X, V) + X\lambda)\hat{N} \tag{3.3}$$

for all $X, Y \in TM$.

Proof. : Consider

$$\begin{aligned}
(\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X(\phi Y) - \phi(\tilde{\nabla}_X Y) \\
&= \tilde{\nabla}_X(fY + u(Y)\hat{N}) - \phi(\nabla_X Y + \sigma(X, Y)\hat{N}) \\
&= \tilde{\nabla}_X(fY) + \tilde{\nabla}_X(u(Y)\hat{N}) - f(\nabla_X Y) - u(\nabla_X Y)\hat{N} - \sigma(X, Y)(-U) \\
&= \nabla_X(fY) + \sigma(X, fY)\hat{N} + u(Y)(-A_{\hat{N}} X) + \nabla_X(u(Y))\hat{N} - f(\nabla_X Y) \\
&\quad -u(\nabla_X Y)\hat{N} + \sigma(X, Y)U
\end{aligned}$$

which gives,

$$(\widetilde{\nabla}_X \phi)Y = (\nabla_X f)Y - u(Y)(A_{\hat{N}} X) + \sigma(X,Y)U + (\nabla_X u)Y + \sigma(X, fY)\hat{N}$$

Also,

$$\begin{aligned}(\widetilde{\nabla}_X \eta)Y &= \widetilde{\nabla}_X \eta(Y) - \eta(\widetilde{\nabla}_X Y) \\ &= \nabla_X\left(v(Y)\right) - v(\nabla_X Y) - \sigma\left(X,Y\right)\eta(\hat{N}).\end{aligned}$$

Therefore

$$(\widetilde{\nabla}_X \eta)Y = (\nabla_X u)Y - \lambda\sigma(X,Y)$$

Now consider

$$\begin{aligned}\widetilde{\nabla}_X \xi &= \nabla_X \xi + \sigma(X,\xi)\hat{N} \\ &= \nabla_X V + \nabla_X \lambda \hat{N} + \sigma(X,V)\hat{N} \\ &= \nabla_X V - \lambda \nabla_X \hat{N} + (X\lambda)\hat{N} + \sigma(X,V)\hat{N}\end{aligned}$$

which gives

$$\widetilde{\nabla}_X \xi = \left(\nabla_X V - \lambda A_{\hat{N}} X\right) + (\sigma(X,V) + X\lambda)\hat{N}.$$

□

Theorem 3.1. *Let M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$, then*

(3.4) $$\sigma(X,\xi)U = -\varepsilon f^2 X + \varepsilon u(X)U + f(\nabla_X \xi),$$

and

(3.5) $$u\left(\nabla_X \xi\right) = -\varepsilon u(fX)$$

Proof. Consider

$$\begin{aligned}\left(\widetilde{\nabla}_X \phi\right)\xi &= \widetilde{\nabla}_X\left(\phi\xi\right) - \phi\left(\widetilde{\nabla}_X \xi\right) \\ &= -\varepsilon\phi\left(\phi X\right)\end{aligned}$$

or

(3.6) $$\left(\widetilde{\nabla}_X \phi\right)\xi = -\varepsilon\phi\left(fX + u(X)\right)\hat{N}$$

Also we know that

(3.7) $$\left(\widetilde{\nabla}_X \phi\right)\xi = -\phi\left(\nabla_X \xi\right) + \sigma(X,\xi)U$$

From (3.6) and (3.7), we have

$$\begin{aligned}-\phi(\nabla_X \xi) + \sigma(X,\xi)U &= -\varepsilon\phi(fX + u(X))\hat{N}\\ &= -\varepsilon\phi(fX) + \epsilon u(X)U\end{aligned}$$

Now from (2.6) & (2.7), we have

$$-f(\nabla_X \xi) - u(\nabla_X \xi)\hat{N} + \sigma(X,\xi)U = -\varepsilon(f(fX) + u(fX)\hat{N}) + \varepsilon u(X)U$$

Equating tangential and normal parts, we get

$$\sigma(X,\xi)U = -\varepsilon f^2 X + \varepsilon u(X)U + f(\nabla_X \xi),$$

and

$$u(\nabla_X \xi) = -\varepsilon u(fX)$$

□

Theorem 3.2. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\tilde{M}$, then*

(3.8) $$(\nabla_X f)Y = -g(X,Y)V + \varepsilon v(Y)X + \sigma(X,Y)U + u(Y)A_{\hat{N}}X$$

(3.9) $$(\nabla_X u)Y = -\lambda g(X,Y) - \sigma(X,fY)$$

Proof. From equations (3.1) & (2.4), we have

$$\begin{aligned}&(\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X,Y)U + ((\nabla_X u)Y + \sigma(X,fY))\hat{N}\\ =\ &-g(X,Y)V - \lambda g(X,Y)\hat{N} + \varepsilon v(Y)X\end{aligned}$$

Equating tangential and normal parts in the above equation, we get (3.8) & (3.9) respectively. □

Theorem 3.3. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\tilde{M}$, then*

(3.10) $$\left(\tilde{\nabla}_X \phi\right)Y = -g(X,Y)V - \lambda g(X,Y)\hat{N} + \varepsilon v(Y)X + 2\sigma(X,Y)U$$

Proof. Consider

$$\begin{aligned}\left(\tilde{\nabla}_X \phi\right)Y &= \tilde{\nabla}_X(\phi Y) - \phi\left(\tilde{\nabla}_X Y\right)\\ &= \tilde{\nabla}_X(fY) + \tilde{\nabla}_X\left(u(Y)\hat{N}\right) - f(\nabla_X Y) - u(\nabla_X Y)\hat{N},\end{aligned}$$

This implies

$$(3.11)\quad \left(\widetilde{\nabla}_X\phi\right)Y = (\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X,Y)U + ((\nabla_X u)Y + \sigma(X,fY))\hat{N}$$

Using (3.8) & (3.9) ,above equation reduces to

$$\left(\widetilde{\nabla}_X\phi\right)Y = -g(X,Y)V - \lambda g(X,Y)\hat{N} + \epsilon v(Y)X + 2\sigma(X,Y)U$$

□

Furthur, we proceed for some results on totally geodesic non−invariant hypersurfaces.

Theorem 3.4. *Let M be a totally umbilical non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$, then it is totally geodesic if and only if*

$$(3.12)\quad \varepsilon u(X) - X\lambda = 0$$

Proof. Consider

$$\begin{aligned}
\widetilde{\nabla}_X\xi &= \nabla_X\xi + \sigma(X,\xi)\hat{N} \\
&= \nabla_X(V + \lambda\hat{N}) + \sigma(X,V)\hat{N} \\
&= \nabla_X V + \nabla_X\lambda\hat{N} + \sigma(X,V)\hat{N} \\
&= \nabla_X V + \lambda\nabla_X\hat{N} + (X\lambda)\hat{N} + \sigma(X,V)\hat{N}
\end{aligned}$$

or

$$(3.13)\quad \widetilde{\nabla}_X\xi = \left(\nabla_X V - \lambda A_{\hat{N}}X\right) + (\sigma(X,V) + X\lambda)\hat{N},$$

Now from (2.5) , the above equation is reduced to

$$\varepsilon(fX + u(X)\hat{N}) = \left(\nabla_X V - \lambda A_{\hat{N}}X\right) + (\sigma(X,V) + X\lambda)\hat{N},$$

Equating normal parts on both the sides, we get

$$(3.14)\quad \sigma(X,V) + X\lambda = \varepsilon u(X)$$

Now if M is totally umbilical, then $A_{\hat{N}} = \zeta I$, ζ is Kahlerian metric and equation (2.18) reduces to $\sigma(X,Y) = g\left(A_{\hat{N}}X, Y\right) = g(\zeta X, Y) = \zeta g(X,Y)$,

Therefore

$$\sigma(X,Y) = \zeta g(X,Y),$$

and equation (3.14) implies

$$\zeta g(X,Y)+X\lambda=\varepsilon u(X)$$

or

$$\varepsilon u(X)-X\lambda-\ \zeta g(X,Y)=0 \tag{3.15}$$

Now if M is totally geodesic i.e. $\zeta=0$, then (3.15) gives

$$\varepsilon u(X)-X\lambda=0.$$

□

Theorem 3.5. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$. If U is parallel, then we have*

$$\varepsilon\lambda X+f\left(A_{\hat{N}}X\right)-g(\phi X,U)V-\varepsilon\lambda v(X)V=0 \tag{3.16}$$

Proof. Consider

$$\begin{aligned}(\widetilde{\nabla}_X\phi)\hat{N} &= \widetilde{\nabla}_X\phi\hat{N}-\phi\left(\widetilde{\nabla}_X\hat{N}\right)\\ &= -\widetilde{\nabla}_XU-\phi(-A_{\hat{N}}X)\\ &= -\widetilde{\nabla}_XU-(-f(-A_{\hat{N}}X)-u(A_{\hat{N}}X)\hat{N})\end{aligned}$$

This gives

$$(\widetilde{\nabla}_X\phi)\hat{N}=-\nabla_XU+f(A_{\hat{N}}X) \tag{3.17}$$

From equation (2.4), we have

$$(\widetilde{\nabla}_X\phi)Y=-g\left(\phi X,\phi Y\right)V-\lambda g\left(\phi X,\phi Y\right)\hat{N}+\varepsilon\eta\left(Y\right)X-\varepsilon\eta\left(X\right)\eta\left(Y\right)\xi \tag{3.18}$$

Substituting $Y=\hat{N}$, we have

$$(\widetilde{\nabla}_X\phi)\hat{N}=g(\phi X,U)V+\lambda g\left(\phi X,U\right)\hat{N}-\varepsilon\lambda X+\varepsilon\lambda v(X)\xi \tag{3.19}$$

Now from (3.17) and (3.19), we have

$$-\nabla_XU+f(A_{\hat{N}}X)=g(\phi X,U)V+\lambda g\left(\phi X,U\right)\hat{N}-\varepsilon\lambda X+\varepsilon\lambda v(X)\xi$$

Equating tangential parts on both the sides, we have

$$\nabla_X U = f(A_{\hat{N}} X) - g(\phi X, U)V - \varepsilon\lambda X + \varepsilon\lambda v(X)V \tag{3.20}$$

Now if U is parellel, then

$$\varepsilon\lambda X - f(A_{\hat{N}} X) + g(\phi X, U)V - \varepsilon\lambda v(X)V = 0$$

□

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SOME CLASSES OF CONVEX FUNCTIONS ON TIME SCALES *

Fagbemigun Opeyemi Bosede and Adesanmi Mogbademu

Abstract. We have introduced diamond $\phi_{h-s,\mathbb{T}}$ derivative and diamond $\phi_{h-s,\mathbb{T}}$ integral on an arbitrary time scale. Moreover, various interconnections with the notion of classes of convex functions about these new concepts are also discussed.
Keywords: convex functions, time scale, dynamic derivatives.

1. Introduction

There has been a considerable amount of interest and publications in the theory and applications of dynamic derivatives on time scales. The study, which was initiated by Stephen Hilger (1988) in his PhD thesis, unifies traditional concepts of derivatives and differences, and it also reveals diversities in the corresponding results. The investigations are not only significant in the theoretical research of differential and difference equations, but also crucial in many computational and numerical applications (see for example [4], [14] and [25]).

The study helps to avoid proving a result twice, once for differential equations, and once for difference equations. Once we have proved a result for a general time scale, by choosing the set of real numbers $\mathbb{R}$, the derivative and integral are easily seen to be the 'usual' derivative and integral respectively. Furthermore, when we choose the time scale to be the set of integers $\mathbb{Z}$, the same general result yields a result for difference equations and integral respectively. Hence all results that are proved on a general time scale include results for both differential and difference equations.

The time scale theory has advanced fast since its introduction. Lately, the applications of its calculus in Engineering, Biology, Physics, Medical Sciences, Economics and Finance, Chemistry and Others, have come to light. For a good introduction to the theory of time scales and more details, see [1]-[6].

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2. Basic Concepts and Definitions

A *time scale*, denoted $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers. We assume that a time scale is endowed with the topology inherited from $\mathbb{R}$ with the standard topology. Thus, the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, and the non negative integers $\mathbb{N}_0$ are examples of time scales, as are $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$, and the Cantor Set, while the rational numbers $\mathbb{Q}$, irrational numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the open interval $(0,1)$, are not time scales.

Throughout this paper, we will denote a time scale by $\mathbb{T}$, and for any I, interval of $\mathbb{R}$ (open or closed, finite or infinite), $I_{\mathbb{T}} = I \cap \mathbb{T}$, a time scale interval.

Definition 2.1. Let $\mathbb{T}$ be a time scale. For all $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau \geqslant t\} \,\forall\, t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau \leqslant t\} \,\forall\, t \in \mathbb{T}.$$

We make the convention:
$\inf \emptyset = \sup \mathbb{T}$ (i.e, $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and
$\sup \emptyset = \inf \mathbb{T}$ (i.e, $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set.

The point t is said to be *right-scattered* if $\sigma(t) > t$, respectively *left-scattered* if $\rho(t) < t$. Points that are right-scattered and left-scattered at the same time are called *isolated*. The point t is called *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, respectively *left-dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty]$ is defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t) \;\forall\; t \in \mathbb{T}$$

are called, the *forward and backward graininess functions* respectively.

Suppose that $\mathbb{T}^k$, $\mathbb{T}_k$, and $\mathbb{T}^k_k$ are sets derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{M\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum point M} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

and

$$\mathbb{T}_k := \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a right-scattered minimum point m} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

We set $\mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}^k$.

Given $f : \mathbb{T} \to \mathbb{R}$, the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e $f^\sigma = f \circ \sigma$. Also, the function $f^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, i.e, $f^\rho = f \circ \rho$.

Definition 2.2. (i). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative* of f in t is the number $f^\Delta(t)$ (when it exists) with the property that for any $\epsilon > 0$, there is a neighbourhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)\,(\sigma(t) - s)\right| < \epsilon\,|\sigma(t) - s|$$

for all $s \in U$.

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

(ii). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. The *nabla derivative* of f in t is the number denoted by $f^\nabla(t)$ (when it exists), with the property that, for any $\epsilon > 0$, there is a neighbourhood V of t such that

$$\left|f(\rho(t)) - f(s) - f^\nabla(t)\,(\rho(t) - s)\right| < \epsilon\,|\rho(t) - s|\,,$$

for all $s \in V$.

(iii). The function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* on $\mathbb{T}^k$, provided $f^\Delta(t) = \lim_{s\to t} \frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$ exists where $s \to t, s \in \mathbb{T} \setminus \{\sigma(t)\}$ for all $t \in \mathbb{T}^k$.

(iv). The function $f : \mathbb{T} \to \mathbb{R}$ is *nabla differentiable* on $\mathbb{T}_k$, provided $f^\nabla(t) = \lim_{s\to t} \frac{f(s)-f(\rho(t))}{s-\rho(t)}$ exists where $s \to t, s \in \mathbb{T} \setminus \{\rho(t)\}$ for all $t \in \mathbb{T}_k$.

(v). The function $f : \mathbb{T} \to \mathbb{R}$ is said to be differentiable on $\mathbb{T}_k^k$ provided $f^\Delta(t)$ and $f^\nabla(t)$ exists for all $t \in \mathbb{T}_k^k$.

Remark 2.1. 1. When $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$ becomes the total differential operator (ordinary derivative).

2. When $\mathbb{T} = \mathbb{Z}$, then

   (i) $f^\Delta(t) = f(t+1) - f(t)$ and $f^{\Delta^r}(t) = f^{\Delta^{r-1}}(t+1) - f^{\Delta^r}(t)$ are the forward and r-th forward difference operators

   (ii) $f^\Delta(t) = f(t+\frac{1}{2}) - f(t-\frac{1}{2})$ is the central difference operator

   (iii) $f^\nabla(t) = f(t) - f(t-1)$ and $f^{\nabla^r}(t) = f^{\nabla^{r-1}}(t) - f^{\nabla^r}(t-1)$ are the backward and r-th backward difference operators

   (iv) The shift operator E, is defined as $Ef(t) = f(t+1)$ so that $E^{-1}f(t) = f(t-1)$ and $E^{-\frac{1}{2}}f(t+\frac{1}{2}) = f(t+1)$

   (v) Also, the mean operator, μ, can be given as$f^\mu(t+\frac{1}{2}) = \frac{1}{2}\{f(t) + f(t+1)\}$.

Several new important relationships may be established between these five operators $(i) - (v)$ above in terms of the shift operator, E as follows:

(i) $f^{\Delta}(t) = (E-1)f(t)$

(ii) $f^{\nabla}(t) = (1-E^{-1})f(t)$

(iii) $f^{\delta}(t) = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})f(t)$

(iv) $f^{\mu}(t) = \frac{1}{2}\{E^{\frac{1}{2}} - E^{-\frac{1}{2}}\}f(t)$.

3. Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h)-f(t)}{h}$ and $f^{\nabla}(t) = \frac{f(t)-f(t-h)}{h}$ are the h-derivatives.
4. Let $q > 1$. If $\mathbb{T} = q^{N_0}$, where $N_0 = 0, 1, 2, ...$, then $f^{\Delta}(t) = \frac{f(qt)-f(t)}{t(q-1)}$ and $f^{\nabla}(t) = \frac{q(f(t)-f(\frac{t}{q}))}{t(q-1)}$ are the q-derivatives.

The following lemma is useful in the sequel.

Lemma 2.1. *Let $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then the following holds.*

(i) If $f^{\Delta}(t)$ exists, i.e, f is delta differentiable at t, then f is continuous at t.

(ii) If f is left-continuous at t and t is right-scattered, then f is delta differentiable at t with $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If t is right-dense, then $f^{\Delta}(t)$ exists, if and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f^{Δ} exists on $\mathbb{T}^k$ and f is invertible on $\mathbb{T}$, then $(f^{-1})^{\Delta} = -(f^{\sigma})^{-1} f^{\Delta} f^{-1}$ on $\mathbb{T}^k$.

(v) If $f^{\Delta}(t)$, $g^{\Delta}(t)$ exist and $(fg)(t)$ is defined, then $(fg)^{\Delta}(t) = f(\sigma(t))g^{\Delta}(t) + f^{\Delta}(t)g(t)$.

Also, given $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$. Then the following holds.

(i) If $f^{\nabla}(t)$ exists, i.e, f is nabla differentiable at t, then f is continuous at t.

(ii) If f is right-continuous at t and t is left-scattered, then f is nabla differentiable at t with $f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}$.

(iii) If t is left-dense, then $f^{\nabla}(t)$ exists, if and only if , the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\nabla}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f^{∇} exists on $\mathbb{T}_k$ and f is invertible on $\mathbb{T}$, then $(f^{-1})^{\nabla} = -(f^{\rho})^{-1} f^{\nabla} f^{-1}$ on $\mathbb{T}_k$.

(v) If $f^{\nabla}(t)$, $g^{\nabla}(t)$ exist and $(fg)(t)$ is defined, then $(fg)^{\nabla}(t) = f(\rho(t))g^{\nabla}(t) + f^{\nabla}(t)g(t)$.

Definition 2.3. 1. A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it satisfies the following two conditions:

(i) f is continuous at all right-dense point or maximal element of $\mathbb{T}$,

(ii) the left-sided limit $\lim_{s\to t^-} f(s) = f(t^-)$ exists (finite) at each left-dense point $t \in \mathbb{T}$.

We denote by $C_{rd}(\mathbb{T}, \mathbb{R})$ the set of rd-continuous functions.

2. A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be ld-continuous if it satisfies the following two conditions:

(i) f is continuous at all left-dense point or minimal element of $\mathbb{T}$,

(ii) the right-sided limit $\lim_{s\to t^+} f(s) = f(t^+)$ exists (finite) at each right-dense point $t \in \mathbb{T}$.

We denote by $C_{ld}(\mathbb{T}, \mathbb{R})$ the set of ld-continuous function.
Obviously, the set of continuous functions on $\mathbb{T}$ contains both C_{rd} and C_{ld}.

Definition 2.4. (i). A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, the delta integral of f is defined as

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$$

for all $s, t \in \mathbb{T}$.

(ii). A function $G : \mathbb{T} \to \mathbb{R}$ is called a *nabla antiderivative* of $g : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla}(t) = g(t)$ for all $t \in \mathbb{T}_k$. In this case, the nabla integral of g is defined as

$$\int_s^t g(\tau)\nabla\tau = G(t) - G(s)$$

for all $s, t \in \mathbb{T}$.

Every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative (see [2], Theorem 1.74, [26] and [30]).

Theorem 2.1. *([2], Theorem 1.75).*

(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then $\int_t^{\sigma} tf(s)\Delta s = \mu(t)f(t)$.

(ii) If $g \in C_{ld}$ and $t \in \mathbb{T}_k$, then $\int_{\rho}^{t} tg(s)\nabla s = \nu(t)g(t)$.

The following theorem shows some basic operations with the delta integral.

Theorem 2.2. *(see [7], Theorem 2.2). If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$

(ii) $\int_a^b (\alpha f)t\Delta t = \alpha \int_a^b f(t)\Delta t$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$

(vii) $\int_a^a f(t)\Delta t = 0$

(viii) *If* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$

(ix) *If* $|f(t)| \leq g(t)$ *on* $[a,b)$, *then* $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

An analogous version of Theorem 2.2 holds for the nabla antiderivative of functions in C_{ld}.

In [7] and [9], Dinu established the diamond-α dynamic derivatives which are linear combinations of the delta and nabla dynamic derivatives on time scales. Motivated by this, we have introduced and discussed some basic properties of the diamond-$\phi_{h-s,\mathbb{T}}$ derivative.
We begin with the following new definition.

Definition 2.5. Let $\mathbb{T}$ be a time scale and $h : J_{\mathbb{T}} \to \mathbb{R}$ a given real-valued function. For $m, n \in \mathbb{T}_k^k$, set $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$. We define the diamond-$\phi_{h-s,\mathbb{T}}$ dynamic derivative of a function $f : \mathbb{T} \to \mathbb{R}$ in t to be the number denoted by $f^{\diamond_{\phi_{h-s,T}}}(t)$(when it exists), with the property that for any $\epsilon > 0$, there is a neighbourhood U of m such that, for all $n \in U,\ 0 \leq s \leq 1$ and $0 < \omega < 1$,

$$\left| \left(\frac{h(\omega)}{\omega}\right)^{-s} [f(\sigma(m)) - f(n)]\nu_{mn} + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} [f(\rho(m)) - f(n)]\mu_{mn} - f^{\diamond_{\phi_{h-s,I_{\mathbb{T}}}}}(t)\mu_{mn}\nu_{mn} \right| < \epsilon|\mu_{mn}\nu_{mn}|.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called diamond-$\phi_{h-s,\mathbb{T}}$ differentiable on $\mathbb{T}_k^k$ if $f^{\diamond_{\phi_{h-s,I_{\mathbb{T}}}}}(t)$ exists for all $t \in \mathbb{T}_k^k$. If f is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}_k^k$, and the diamond-$\phi_{h-s,\mathbb{T}}$ derivative $f^{\diamond_{\phi_{h-s,I_{\mathbb{T}}}}}(t)$ is given by

$$f^{\diamond_{\phi_{h-s,I_{\mathbb{T}}}}}(\phi(t)) = \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))$$

for all $s \in [0,1], \omega \in (0,1)$.

Remark 2.2. (i) $f^{\diamond_{\phi_{h-s,I_{\mathbb{T}}}}}(t)$ reduces to the diamond-α derivative for $\phi(t) = t, s = 1$, $h(\omega) = 1$ and $\omega = \alpha$. Thus, every diamond-α differentiable on $\mathbb{T}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable function but the converse is not true.

(ii) If f is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable for $0 \leq s \leq 1, 0 < \omega < 1$, then f is both Δ and ∇ differentiable.

(iii) Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. The diamond-$\phi_{h-s,\mathbb{T}}$ integral of f from a to b is defined by

$$\int_a^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t = \left(\frac{h(\omega)}{\omega}\right)^{-s} \int_a^b f(\phi(t))\Delta t + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} \int_a^b f(\phi(t))\nabla t$$

for all $s \in [0, 1], \omega \in (0, 1)$, provided that f has a delta and nabla integral on $[a, b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Obviously, each continuous function has a diamond-$\phi_{h-s,\mathbb{T}}$ integral. The combined derivative $\diamond_{\phi_{h-s,\mathbb{T}}}$ is not a dynamic derivative, since we do not have a $\diamond_{\phi_{h-s,\mathbb{T}}}$ antiderivative. Generally,

$$\left(\int_a^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t\right)^{\diamond_{\phi_{h-s,\mathbb{T}}}} \neq f(\phi(t)), t \in \mathbb{T}.$$

Example 2.1. Let $\mathbb{T} = [0, 1] \cup \{2, 4\}$. Define a diamond-$\phi_{h-1,\mathbb{T}}$ function $f : \mathbb{T} \to \mathbb{R}$ by $f(\phi(t)) = 1$ and $h : \mathbb{T} \to \mathbb{R}$ by $h(\omega) = 1$. For $\phi(t), \omega \in \mathbb{T}$, then

$$\left(\int_0^b f(\phi(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t\right)^{\diamond_{\phi_{h-s,\mathbb{T}}}}\Bigg|_{t=1} \neq f(\phi(t)).$$

We give some basic properties of the diamond-$\phi_{h-s,\mathbb{T}}$ integral which are similar to Theorem 2.2 of [7] and its analogue for the nabla integral.

Proposition 2.1. *Let $a, b, c \in \mathbb{T}, \beta \in \mathbb{R}$, and f, g be continuous functions on $I_{\mathbb{T}}$, then*

(i) $\int_a^b (f(t) + g(t)) \diamond_{\phi_{h-s,\mathbb{T}}} t = \int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t + \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

(ii) $\int_a^b (\beta f)t \diamond_{\phi_{h-s,\mathbb{T}}} t = \beta \int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

(iii) $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = -\int_b^a f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

(iv) $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = \int_a^c f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t + \int_c^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$

(v) $\int_a^a f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = 0.$

Lemma 2.2. *(i) If $f(t) \geq 0$ for all t, then $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \geq 0$.*

(ii) If $f(t) \leq g(t)$ for all t, then $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \leq \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$.

(iii) If $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$, then $f = 0$ if and only if $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t = 0$.

(iv) If $|f(t)| \leq g(t)$ on $[a, b)$, then $|\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t| \leq \int_a^b g(t) \diamond_{\phi_{h-s,\mathbb{T}}} t$.

(v) *If in (iv), we choose* $g(t) = |f(t)|$ *on* $[a, b]$, *we have* $|\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t| \leq \int_a^b |f(t)| \diamond_{\phi_{h-s,\mathbb{T}}} t$.

Proof. 1. Assume that f and g are continuous functions on $I_{\mathbb{T}}$, then, $\int_a^b f(t)\Delta t \geq 0$ and $\int_a^b f(t)\nabla t \geq 0$ since $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$. Since $s \in [0, 1], \omega \in (0, 1)$, the result follows.

2. Let $h(t) = g(t) - f(t)$, then $\int_a^b h(t) \diamond_{\phi_{h-s,\mathbb{T}}} t \geq 0$ and the result follows from properties (i) and (ii) above.

3. If $f(t) = 0$ for all $t \in I_{\mathbb{T}}$, the result is immediate. Now suppose that there exists $t_0 \in I_{\mathbb{T}}$ such that $f(t_0) > 0$. It is easy to see that at least one of the integrals $\int_a^b f(t)\Delta t$ or $\int_a^b f(t)\nabla t$ is strictly positive. Then we have the contradiction $\int_a^b f(t) \diamond_{\phi_{h-s,\mathbb{T}}} t > 0$.

□

Dinu [8], defined the concept of a convex function on time scales. He equally included some results connecting this concept with the idea of convex functions on a classic interval and convex sequences.

In this paper, we have given the notion of classes of convex functions on time scales, consequently established the various interconnections that exist among them, and then related their properties with the concept of classes of convex functions on classic intervals.

3. Some classes of convex functions on time scales

Previous research works have shown that the history of convex functions is tied to the concept of Jensen convex or mid-point convex functions, which deals with the arithmetic mean (see for instance [10], [15], [17] and [19]). We shall state the analogue of Jensen convexity for time scales.

Theorem 3.1. *Let $I_{\mathbb{T}} \subset \mathbb{T}$ be a time scale interval. A function $f : \mathbb{T} \to \mathbb{R}$ is called convex in the Jensen sense or J-convex or mid-point convex on $I_{\mathbb{T}}$ if for all $t_1, t_2 \in I_{\mathbb{T}}$, the inequality,*

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{f(t_1) + f(t_2)}{2} \tag{3.1}$$

holds.

Remark 3.1. (i) If $\mathbb{T} = \mathbb{R}$, then our version is the same as the classical Jensen inequality. However, if $\mathbb{T} = \mathbb{Z}$, then it reduces to the well-known arithmetic-geometric mean inequality.

(ii) The extensions of the inequality (3.1) to the convex combination of finitely many points and next to random variables associated to arbitrary probability spaces are known as the discrete Jensen and integral inequalities on time scales respectively (see [3], [15] and [16]).

Definition 3.1. We say that $f : \mathbb{T} \to \mathbb{R}$ is Godunova-Levin convex on time scales or that f belongs to the class $Q(I_\mathbb{T})$ if f is nonnegative, and that for all $t_1, t_2 \in I_\mathbb{T}$, and $\omega \in (0, 1)$,

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{1}{\omega} f(t_1) + \frac{1}{1-\omega} f(t_2). \tag{3.2}$$

Remark 3.2. (i) For $\mathbb{T} = \mathbb{R}$, the definition 3.1 above is exactly the definition of a Godunova-Levin function on a classic interval (see [12], [13], [15], and [21]).

(ii) All nonnegative monotonic and nonnegative convex functions on time scales belong to this class $Q(I_\mathbb{T})$.

(iii) If $f \in Q(I_\mathbb{T}) = SX(h_{-1}, I_\mathbb{T})$ and $t_1, t_2, t_3 \in I_\mathbb{T}$, then

$$f(t_1)(t_1-t_2)(t_1-t_3) + f(t_2)(t_2-t_1)(t_2-t_3) + f(t_3)(t_3-t_1)(t_3-t_2) \geq 0. \tag{3.3}$$

In fact, (3.3) is equivalent to (3.2) so it can alternatively be used in the definition of the class $Q(I_\mathbb{T})$.

Definition 3.2. A function $f : \mathbb{T} \to \mathbb{R}$ is a P-function on $I_\mathbb{T}$ or $f \in P(I_\mathbb{T})$ if f is nonnegative, and for all $t_1, t_2 \in I_\mathbb{T}$, and $\omega \in [0, 1]$, we have

$$f(\omega t_1 + (1-\omega)t_2) \leq f(t_1) + f(t_2). \tag{3.4}$$

Obviously, $P(I_\mathbb{T}) \subseteq Q(I_\mathbb{T})$. Also, $P(I_\mathbb{T})$ contains all nonnegative monotone, convex and quasi-convex functions on $I_\mathbb{T}$, i.e, nonnegative functions satisfying

$$f(\omega t_1 + (1-\omega)t_2) \leq max\{f(t_1) + f(t_2)\}$$

for all $t_1, t_2 \in I_\mathbb{T}$, and $\omega \in [0, 1]$, (see [24]).

Definition 3.3. A function $f : C_{I_\mathbb{T}} \subseteq \mathbb{T} \to [0, \infty)$ is of s-Godunova-Levin type on time scales, denoted $Q_s(I_\mathbb{T})$ with $s \in [0, 1]$ if

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{1}{\omega^s} f(t_1) + \frac{1}{(1-\omega)^s} f(t_2), \tag{3.5}$$

for all $\omega \in (0, 1)$ and $t_1, t_2 \subseteq C_{I_\mathbb{T}}$, where $C_{I_\mathbb{T}}$ is a convex subset of a time scale interval of a time scale linear space $\mathbb{T}$.

Remark 3.3. (i) When $s = 0$, we get a class of P-functions on $I_\mathbb{T}$.

(ii) $s = 1$ gives the class of Godunova-Levin functions on $I_\mathbb{T}$.

Definition 3.4. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \subset \mathbb{T} \to [0,\infty)$ is said to be s-convex on $I_{\mathbb{T}}$ (in the second sense on time scales) or Breckner s-convex on $I_{\mathbb{T}}$, denoted $K_2^s(I_{\mathbb{T}})$ if

$$f(\omega t_1 + (1-\omega)t_2) \leq \omega^s f(t_1) + (1-\omega)^s f(t_2), \tag{3.6}$$

for all $t_1, t_2 \in [0,\infty) \subset \mathbb{T}$ and $\omega \in [0,1]$.

Remark 3.4. Definition 3.4 is a generalization of convex functions on $I_{\mathbb{T}}$ as defined in [8]. Hence, s-convexity on $I_{\mathbb{T}}$ just means convexity when $s = 1$ on $I_{\mathbb{T}}$.

In order to unify the concepts of Definitions 3.1 – 3.4 above for functions on time scale variables we now introduce the concept of h-convex functions on time scales (see [11] and [29]).
Assume that $I_{\mathbb{T}}$ and $J_{\mathbb{T}}$ are intervals in $\mathbb{T}, [0,\infty) \subseteq J_{\mathbb{T}}$ and functions f and h are real non-negative functions defined on $I_{\mathbb{T}}$ and $J_{\mathbb{T}}$ respectively.

Definition 3.5. Let $h : J_{\mathbb{T}} \to \mathbb{R}$ with h not identical to zero. We say that $f : \mathbb{T} \to \mathbb{R}$ is an h-convex function on $I_{\mathbb{T}}$ or f belongs to the class $SX(h, I_{\mathbb{T}})$ if for all $t_1, t_2 \in I_{\mathbb{T}}, f$ is non-negative, we have

$$f(\omega t_1 + (1-\omega)t_2) \leq h(\omega) f(t_1) + h(1-\omega) f(t_2), \tag{3.7}$$

for all $\omega \in (0,1)$.

Remark 3.5. (i) If inequality (3.7) is reversed, then f is said to be h-concave on $I_{\mathbb{T}}$ i.e $f \in SV(h, I_{\mathbb{T}})$.

(ii) Obviously, if $h(\omega) = \omega$, then all non-negative functions on $I_{\mathbb{T}}$ belong to $SX(h, I_{\mathbb{T}})$ and all non-negative concave functions on $I_{\mathbb{T}}$ belong to $SV(h, I_{\mathbb{T}})$; if $h(\omega) = \frac{1}{\omega}$, then $SX(h, I_{\mathbb{T}}) = Q(I_{\mathbb{T}})$; if $h(\omega) = 1$, then $SX(h, I_{\mathbb{T}}) \supseteq P(I_{\mathbb{T}})$; and if $h(\omega) = \omega^s$, where $s \in (0,1)$, then $SX(h, I_{\mathbb{T}}) \supseteq K_s^2(I_{\mathbb{T}})$.

Definition 3.6. A function $f : I_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ is said to belong to the class $MT(I_{\mathbb{T}})$ if f is nonnegative and for all $t_1, t_2 \in I_{\mathbb{T}}$ and $\omega \in (0,1)$ satisfies the inequality:

$$f(\omega t_1 + (1-\omega)t_2) \leq \frac{\sqrt{\omega}}{2\sqrt{1-\omega}} f(t_1) + \frac{\sqrt{1-\omega}}{2\sqrt{\omega}} f(t_2). \tag{3.8}$$

Remark 3.6. (i) If $\mathbb{T} = \mathbb{R}$, and $I_{\mathbb{T}} = I$, we obtain definition 2 of Tunç and Yildirim (2012), for classical MT-convex function (see [21], [22], [23], [27] and [28]).

(ii) If we set $\omega = \frac{1}{2}$, inequality (3.8) reduces to the inequality (3.1).

(iii) Let $f, g : [1,\infty] \subset \mathbb{T} \to \mathbb{R}$, $f(t) = t^p, g(t) = (1+t)^p, p \in (0, \frac{1}{1000})$, and $h : [1, \frac{3}{2}] \subset \mathbb{T} \to \mathbb{R}$, $h(t) = (1+t_2)^m, m \in (0, \frac{1}{100})$ are MT-convex functions on $I_{\mathbb{T}}$ but they are not convex on $I_{\mathbb{T}}$.

Now, we give a variant of a new class of convex functions introduced by Olanipekun et al. in [20], but in the context of time scales.

Definition 3.7. Let $h : J_{\mathbb{T}} \to \mathbb{R}, s \in [0,1], \omega \in (0,1)$ and ϕ be a given real-valued function. Then $f : I_{\mathbb{T}} \to \mathbb{R}$ is a $\phi_{h-s,}I_{\mathbb{T}}$-convex function on time scales if for all $t_1, t_2 \in I_{\mathbb{T}}$,

$$f(\omega\phi(t_1) + (1-\omega)\phi(t_2) \leq \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f(\phi(t_2)). \quad (3.9)$$

We observe that

(i) If $s = 0$, and $\phi(t_1) = t_1$, then $f \in P(I_{\mathbb{T}})$.

(ii) If $h(\omega) = \omega^{\frac{s}{s+1}}$ and $\phi(t_1) = t_1$, , then $f \in SX(h, I_{\mathbb{T}})$.

(iii) If $s = 1, h(\omega) = 1$ and $\phi(t_1) = t_1$, then $f \in SX(I_{\mathbb{T}})$, i.e, f is convex on time scales (See, [8]).

(iv) If $h(\omega) = 1$ and $\phi(t_1) = t_1$, then $f \in K_s^2(I_{\mathbb{T}})$.

(v) If $h(\omega) = \omega^2, s = 1$ and $\phi(t_1) = t_1$, then $f \in Q(I_{\mathbb{T}})$.

(vi) If $h(\omega) = \omega^2$, and $\phi(t_1) = t_1$, then $f \in Q_s(I_{\mathbb{T}})$.

(vii) If $s = 1, h(\omega) = 2\sqrt{\omega(1-\omega)}$ and $\phi(t_1) = t_1$, then $f \in MT(I_{\mathbb{T}})$.

Moreover, suppose we denote by $Q_s(I_{\mathbb{T}})$ and $SX(\phi_{h-s}, I_{\mathbb{T}})$ the class of s-Godunova Levin and $\phi_{h-s}, I_{\mathbb{T}}$ convex functions on time scales respectively, then it is easy to see that: $P(I_{\mathbb{T}}) = Q_0(I_{\mathbb{T}}) = SX(\phi_{h-0}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-s_1}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-s_2}, I_{\mathbb{T}}) \subseteq SX(\phi_{h-1}, I_{\mathbb{T}}) = SX(\phi_h, I_{\mathbb{T}})$ for $0 \leq s_1 \leq s_2 \leq 1$ whenever ϕ is the identity function.
If inequality (3.9) is reversed, then f is $\phi_{h-s}, I_{\mathbb{T}}$ concave, that is, $f \in SV(\phi_{h-s}, I_{\mathbb{T}})$.

The permanence properties of diamond-$\phi_h, I_{\mathbb{T}}$ derivative and convexity operations on time scales constitute an important source of examples in this area.

Proposition 3.1. *Let $f, g \in SX(\phi_{h-s}, I_{\mathbb{T}})$, i.e, these are $\phi_{h-s}, I_{\mathbb{T}}$ convex, $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-$\phi_{h-s}, I_{\mathbb{T}}$ differentiable at $t \in \mathbb{T}$, and c be any constant. Then $f+g, cf, fg, \frac{1}{g}(g \neq 0), \frac{f}{g}(g \neq 0)$ are all diamond-$\phi_{h-s}, I_{\mathbb{T}}$ differentiable at $t \in \mathbb{T}$.*

Proof. Since f and g are $\diamond_{\phi_{h-s}, I_{\mathbb{T}}}$ differentiable, then let
$f^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) = \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))$
and
$g^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) = \left(\frac{h(\omega)}{\omega}\right)^{-s} g^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} g^{\nabla}(\phi(t))$, then

(i). $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-$(\phi_{h-s}, I_{\mathbb{T}})$ differentiable at $t \in \mathbb{T}$, and

$$\begin{aligned}(f,g)^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) &= \left(\frac{h(\omega)}{\omega}\right)^{-s} (f,g)^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} (f,g)^{\nabla}(\phi(t)) \\ &= f^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t) + g^{\diamond_{\phi_{h-s}, I_{\mathbb{T}}}}(t).\end{aligned}$$

(ii). For any constant $c \in \mathbb{R}, cf : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ and $(cf)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) = c\left[\left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\nabla}(\phi(t))\right]$

$$= cf^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t).$$

(iii). $fg : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ and

$$\begin{aligned}(fg)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= \left(\frac{h(\omega)}{\omega}\right)^{-s}(fg)^{\Delta}(\phi(t)) + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s}(fg)^{\nabla}(\phi(t)) \\ &= f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)(\phi(t))g(\phi(t)) + \left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\sigma(t)}(\phi(t))g^{\Delta}(\phi(t)) \\ &\quad + \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\rho(t)}(\phi(t))g^{\nabla}(\phi(t)).\end{aligned}$$

(iv). For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0, \frac{1}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ with

$$\begin{aligned}\left(\frac{1}{g}\right)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= -\frac{1}{g(\phi(t))g^{\sigma}(\phi(t))g^{\rho}(\phi(t))}\left((g^{\sigma}(\phi(t)) + g^{\rho}(\phi(t)))\, g^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)\right. \\ &= -\left(\frac{h(\omega)}{\omega}\right)^{-s} g^{\Delta}(\phi(t))g^{\sigma}(\phi(t)) \\ &\quad -\left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} g^{\nabla}(\phi(t))g^{\rho}(\phi(t)).\end{aligned}$$

(v). For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0, \frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-$\phi_{h-s,\mathbb{T}}$ differentiable at $t \in \mathbb{T}$ with

$$\begin{aligned}\left(\frac{f}{g}\right)^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t) &= \frac{1}{g(\phi(t))g^{\sigma}(\phi(t))g^{\rho}(\phi(t))}\left(f^{\diamond_{\phi_{h-s},I_{\mathbb{T}}}}(t)g^{\sigma}(\phi(t))g^{\rho}(\phi(t))\right. \\ &= -\left(\frac{h(\omega)}{\omega}\right)^{-s} f^{\sigma}(\phi(t))g^{\rho}(\phi(t))g^{\Delta}(\phi(t)) - \\ &\quad - \left.\left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f^{\rho}(\phi(t))g^{\sigma}(\phi(t))g^{\nabla}(\phi(t))\right).\end{aligned}$$

□

The following proposition easily follows and so we will omit the proof.

Proposition 3.2. *Let f be a non negative function on $I_{\mathbb{T}}$. Let h be a non negative function on $I_{\mathbb{T}}$.*

(i). If $h(\omega) \leq \omega^{1-\frac{1}{s}}$ where $s \in (0,1]$, $\omega \in (0,1)$, then $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$.

(ii). If $h(\omega) \geq \omega^{1-\frac{1}{s}}$ for any $\omega \in (0,1)$ and $s \in (0,1]$, then $f \in SV(\phi_{h-s}, I_{\mathbb{T}})$.

It is clear that Proposition 3.2 implies that all convex functions on $I_{\mathbb{T}}$ are the examples of our newly defined class of convex function on $I_{\mathbb{T}}$. An example of such particularly is $h(\omega) = \omega^k$ for $k > 1 - \frac{1}{s}$ where $s \in (0,1]$.

Remark 3.7. For $t_1, t_2 \in I_{\mathbb{T}}, p, q > 0$, the inequality (3.9) is equivalent to

$$f\left(\frac{p\phi(t_1) + q\phi(t_2)}{p+q}\right) \leq \frac{\left(\frac{h(p)}{p}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(q)}{q}\right)^{-s} f(\phi(t_2))}{p+q}.$$

The following definition is useful in defining another form of inequality (3.9) on time scales.

Definition 3.8. A function $h : J_{\mathbb{T}} \to \mathbb{R}$ is said to be a *supermultiplicative* function on $J_{\mathbb{T}} \subset \mathbb{T}$ if for all $m, n \in J_{\mathbb{T}}$,

$$h(mn) \geq h(m)h(n), \tag{3.10}$$

h is said to be a *submultiplicative* function on time scales if the inequality (3.10) is reversed and respectively a *multiplicative* function on time scales if the equality holds in (3.10).

We now prove some time scales analogue for $\phi_{h-s}, I_{\mathbb{T}}$-convex function where h is either supermultiplicative or submultiplicative. Some results in [20] are useful in the sequel.

Proposition 3.3. *Let $h : J_{\mathbb{T}} \to \mathbb{R}$ be a non negative function on $J_{\mathbb{T}} \subset \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$ be a function such that $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, where $\phi(t) = t$. Then for all $t_1, t_2, t_3 \in \mathbb{T}$ such that $t_1 < t_2 < t_3$ and $t_3 - t_1, t_3 - t_2, t_2 - t_1, \in J_{\mathbb{T}}$, the following inequality holds:*

$$[(t_3 - t_1), (t_2 - t_1), h(t_3 - t_2)]^{-s} f(t_1) - [(t_3 - t_2), (t_2 - t_1), h(t_3 - t_1)]^{-s} f(t_2)$$
$$+ [(t_3 - t_1), (t_3 - t_2), h(t_2 - t_1)]^{-s} f(t_3) \geq 0. \tag{3.11}$$

If the function h is submultiplicative and $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, then the inequality (3.11) is reversed.

Proof. Since $f \in SX(\phi_{h-s}, I_{\mathbb{T}})$, and $t_1, t_2, t_3 \in \mathbb{T}$ are points which satisfy assumptions of the proposition, then

$$\frac{t_3 - t_2}{t_3 - t_1}, \frac{t_2 - t_1}{t_3 - t_1} \in J_{\mathbb{T}} \text{ and } \frac{t_3 - t_2}{t_3 - t_1} + \frac{t_2 - t_1}{t_3 - t_1} = 1.$$

Also,

$$h(t_3 - t_2)^{-s} = \left(h\left(\frac{t_3 - t_2}{t_3 - t_1}(t_3 \dot{-} t_1)\right)\right)^{-s} \geq \left(h\left(\frac{t_3 - t_2}{t_3 - t_1}\right) h(t_3 - t_1)\right)^{-s}$$

and

$$h(t_2-t_1)^{-s} = \left(h\left(\frac{t_2-t_1}{t_3-t_1}(t_3 \dot{-} t_1)\right)\right)^{-s} \geq \left(h\left(\frac{t_2-t_1}{t_3-t_1}\right)h(t_3-t_1)\right)^{-s}.$$

Let $h(t_3-t_1)>0$. If in inequality (3.9), we set $\omega=\frac{t_3-t_2}{t_3-t_1}, 1-\omega=\frac{t_2-t_1}{t_3-t_1}$, $a=t_1, b=t_3$, we have $t_2=\omega a+(1-\omega)b$ and so

$$\begin{aligned} f(t_2) &\leq \left(\frac{h\left(\frac{t_3-t_2}{t_3-t_1}\right)}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_1) + \left(\frac{h\left(\frac{t_3-t_2}{t_3-t_1}\right)}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_3) \\ &\leq \left(\frac{\frac{h(t_3-t_2)}{h(t_3-t_1)}}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_1) + \left(\frac{\frac{h(t_3-t_2)}{h(t_3-t_1)}}{\frac{t_3-t_2}{t_3-t_1}}\right)^{-s} f(t_3). \end{aligned} \tag{3.12}$$

Multiplying inequality (3.12) by $(\frac{t_3-t_2}{t_3-t_1})^{-s}(h(t_3-t_1))^{-s}$ and further multiplication by $(t_3-t_1)^{-s}(t_2-t_1)^{-s}$ with rearrangement yields (3.11). □

Remark 3.8. (i) Inequality (3.11) can alternatively be used in Definition 3.7 since inequalities (3.9) and (3.11) are equivalent.

(ii) If we consider inequality (3.11) with $h(t)=t^2$ where $s=1$, we will obtain an alternate definition of Godunova-Levin function on time scales, that is, inequality (3.3).

(iii) Inequality (3.11) is equivalent to Definition 14 of [18] with $h(t)=1, s=1$ by considering points $t_1,t_2 \in I_{\mathbb{T}}$ with $t_1<t_2$ and $t \in I_{\mathbb{T}}$ such that $t_1<t<t_2$ and $t=\omega t_1+(1-\omega)t_2$.

(iv) Another way of writing (3.12) is:

$$\frac{f(t_1)-f(t_2)}{\left(\frac{h\left(\frac{t_1-t_2}{t_1-t_3}\right)}{\frac{t_1-t_2}{t_1-t_3}}\right)^{-s}} \leq \frac{f(t_2)-f(t_3)}{\left(\frac{h\left(\frac{t_2-t_3}{t_1-t_3}\right)}{\frac{t_2-t_3}{t_1-t_3}}\right)^{-s}},$$

where $t_1<t_3$ and $t_1,t_3 \neq t_2$.

Theorem 3.2. *Let $f: I_{\mathbb{T}} \to \mathbb{R}$ be defined and $\Delta_{\phi_{h-s,I_{\mathbb{T}}}}$ differentiable function on $I_{\mathbb{T}}^k$. If $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$ is nondecreasing (nonincreasing) on $I_{\mathbb{T}}^k$, then f is $\phi_{h-s,I_{\mathbb{T}}}$ convex (concave) on $I_{\mathbb{T}}$.*

Proof. By Remark 3.8(iv), it suffices to prove that

$$\frac{f(\phi(t))-f(\phi(t_1))}{t-t_1} \leq \frac{f(\phi(t_2))-f(\phi(t))}{t_2-t}. \tag{3.14}$$

Let $t_1 \leq \gamma_1 < \xi_2$. From the mean value Theorem (see [5]), there exists points $\xi_1,\gamma_1 \in [t_1,t)_{\mathbb{T}}$ and $\xi_2,\gamma_2 \in [t,t_2)_{\mathbb{T}}$ such that

$$f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_1) \leq \frac{f(\phi(t))-f(\phi(t_1))}{t-t_1} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1)$$

and

$$f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_2). \tag{3.15}$$

Since $t_1 < \gamma_1 < \xi_2$ and from the assumption that $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2)$, inequality (3.14) holds, then

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \leq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \leq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \tag{3.16}$$

for nondecreasing $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$ and

$$\frac{f(\phi(t)) - f(\phi(t_1))}{t - t_1} \geq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\gamma_1) \geq f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}(\xi_2) \geq \frac{f(\phi(t_2)) - f(\phi(t))}{t_2 - t} \tag{3.17}$$

for nonincreasing $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$. □

The inequality (3.16) is equivalent to the inequality (3.9) and with the $\phi_{h-s,I_{\mathbb{T}}}$-convexity of f, while the inequality (3.17) is equivalent with the $\phi_{h-s,I_{\mathbb{T}}}$-concavity of f. It is obvious that the nabla version of the Theorem (3.1) holds for nondecreasing (nonincreasing) $f^{\Delta_{\phi_{h-s,I_{\mathbb{T}}}}}$.

We now ask a question of interest: *Can the generalized class of convex function (3.19) be continuous on time scales?* The answer to this is affirmative. We shall first discuss the geometrical interpretation of $\phi_{h-s,I_{\mathbb{T}}}$ convexity on time scales in order to justify this claim.

The $\phi_{h-s,I_{\mathbb{T}}}$ convexity of a function $f : I_{\mathbb{T}} \to \mathbb{R}$ on time scales geometrically means that the points of the graph of $f(\phi(t))|[\phi(t_1), \phi(t_2)]$ are under the chord(or on the chord) joining the endpoints $(\phi(t_1), f(\phi(t_1)))$ and $(\phi(t_2), f(\phi(t_2)))$ for every $t_1, t_2 \in I_{\mathbb{T}}$. Thus

$$f(\phi(t)) \leq f(\phi(t_1)) + \frac{f(\phi(t_2)) - f(\phi(t_1))}{\phi(t_2) - \phi(t_1)}(\phi(t) - \phi(t_1))$$

for all $\phi(t) \in [\phi(t_1), \phi(t_2)]$ and all $\phi(t_1), \phi(t_2) \in I_{\mathbb{T}}$.
This shows that convex functions are majorized locally (i.e, on any compact subinterval) by affine functions.

Theorem 3.3. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous function on $I_{\mathbb{T}}$. Then f is $\phi_{h-s,I_{\mathbb{T}}}$ convex on $I_{\mathbb{T}}$ if and only if f satisfies inequality (3.1), i.e, f is midpoint convex on $I_{\mathbb{T}}$.*

Proof. Sufficiently assume for contradiction that f is not $\phi_{h-s,I_{\mathbb{T}}}$ convex on $I_{\mathbb{T}}$. Thus, there would exist subinterval $(\phi(a), \phi(b)]$ such that $f(\phi(t))|[\phi(a), \phi(b)]$ is not under the chord (or on the chord) joining $(\phi(a), f(\phi(a)))$ and $(\phi(b), f(\phi(b)))$ so that for $\phi(t) \in [\phi(a), \phi(b)]$

$$f(\psi(t)) = f(\phi(t)) - \frac{f(\phi(b)) - f(\phi(a))}{\phi(b) - \phi(a)}(\phi(t) - \phi(a)) - f(\phi(a)).$$

Thus $\xi = \sup\{f(\psi(\phi(t)))|\phi(t) \in [\phi(a), \phi(b)]\} > 0$ and $\psi(\phi(a)) = \psi(\phi(b)) = 0$ since ψ is continuous.

Also, let $K = \inf\{\phi(t) \in [\phi(a), \phi(b)]|\psi(\phi(t)) = \xi\}$, then we necessarily prove that $\phi(k) = \xi$ and $k \in (\phi(a), \phi(b))$.
For every $c > 0$ for which $k + c \in (\phi(a), \phi(b))$, we have by the definition of k, $\psi(k - c) < \psi(k)$ and $\psi(k + c) \leq \psi(k)$.
So that

$$\psi(k) > \frac{\psi(k - c) + \psi(k + c)}{2}.$$

This contradicts the fact that ψ is midpoint convex on $I_{\mathbb{T}}$. □

Remark 3.9. (i) Theorem 3.1 remains true if the condition of midpoint convexity on time scales is replaced by

$$f((1-\omega)\phi(t_1) + \omega\phi(t_2)) \leq \left(\frac{h(1-\omega)}{1-\omega}\right)^{-s} f(\phi(t_1)) + \left(\frac{h(\omega)}{\omega}\right)^{-s} f(\phi(t_2)),$$

for some $0 \leq s \leq 1, h(\omega) = 1, \phi(t_1) = t_1$ and $\omega = \frac{1}{2}$.

(ii) If we replace the condition of continuity in Theorem 3.2 by boundedness from above on every compact subinterval of time scales, Theorem 3.2 still holds.

4. Conclusion

The concepts of $\phi_{h-s,I_{\mathbb{T}}}$-convex function on time scales generalizes the time scale version of many known classes of convex functions. This implies that $\phi_{h-s,I_{\mathbb{T}}}$-convex functions on time scales readily provides many inequalities which generalize and extend the Hermite-Hadamard-type inequalities and several other inequalities for some classes of convex functions defined on time scales.

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TRIPOLAR FUZZY SOFT IDEALS AND TRIPOLAR FUZZY SOFT INTERIOR IDEALS OVER $\Gamma-$SEMIRING

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Abstract. In this paper, we have introduced the notion of tripolar fuzzy soft $\Gamma-$subsemiring, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideals over $\Gamma-$semiring and also studied some of their algebraic properties and the relations between them.
Keywords: tripolar fuzzy set, soft set, fuzzy soft set, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideal.

1. Introduction

In 1995, Murali Krishna Rao [16, 17] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [22] in 1964. As a generalization of ring, semiring was introduced by Vandiver [25] in 1934. In 1981, Sen [24] introduced the notion of $\Gamma-$semigroup as a generalization of semigroup. The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. Lister [13] introduced ternary ring.The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = Z$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. Murali Krishna Rao and Venkateswarlu [21] introduced the notion of $\Gamma-$incline and field $\Gamma-$semiring and studied properties of regular $\Gamma-$incline and field $\Gamma-$semiring.
The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty was first introduced by Zadeh [26]. Rosenfeld [23] introduced fuzzy group. There are many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, bipolar fuzzy sets, cubic sets etc. Molodtsov [15] introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Feng et al. [5] initiated the study of soft semirings. Soft rings are defined by Acar et al. [1] and Ghosh et al. [6] who initiated the study of fuzzy soft rings and fuzzy soft ideals. In 2001, Maji et al. [14] combined the concept of

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fuzzy set theory which was introduced by Zadeh [26] in 1965 and the notion of soft set theory which was introduced by Molodstov [15] in 1999. Aktas and Cagman [2] introduced the concept of fuzzy subgroup, soft sets and soft groups which was extended by Aygnnogln and Aygun [4]. Murali Krishna Rao [18] studied fuzzy soft $\Gamma-$semiring homomorphism, fuzzy soft $\Gamma-$semiring and fuzzy soft $k-$ideal over $\Gamma-$semiring, In 2000, Lee [10, 11] used the term bipolar valued fuzzy sets and applied it to algebraic structure. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In 1994, Zhang [27] initiated the concept bipolar fuzzy sets as a generalization of fuzzy sets. Jun et al. [7, 8] introduced the notion of bipolar fuzzy ideals and bipolar fuzzy filters in CI-algebras. The ideals of " an intuitionistic fuzzy set" was first introduced by Atanassor [3] as a generalization of notion of fuzzy set. Murali Krishna Rao [19] introduced the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set and introduced the notion of tripolar fuzzy ideals and tripolar fuzzy interior ideals of $\Gamma-$semigoup. In this paper, we have introduced the notion of tripolar fuzzy soft ideals and interior ideals over $\Gamma-$semiring. We have studied some of their algebraic properties and the relations between them.

2. Preliminaries

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring,if there exists a mapping $M \times \Gamma \times M \to M$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms:

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Example 2.1. Let M be a a set of all rational numbers and Γ be a set of all natural numbers are commutative semigroups with respect to usual addition. Define the mapping $M \times \Gamma \times M \to M$ by $a\alpha b$ as usual multiplication, forall $a, b \in M, \alpha \in \Gamma$. Then M is a $\Gamma-$semiring.

Definition 2.2. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0,$ forall $x \in M, \alpha \in \Gamma$.

Definition 2.3. Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.4. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.5. Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.6. Let M be a $\Gamma-$semiring. If every element of M is a regular then M is said to be regular $\Gamma-$semiring.

Definition 2.7. A non-empty subset A of $\Gamma-$semiring M is called

(i) a $\Gamma-$subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

(ii) a quasi ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.

(iii) a bi-ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.

(iv) an interior ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.

(v) a left (right) ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.

(vi) an ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

(vii) a $k-$ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

Definition 2.8. A tripolar fuzzy set A in a universe set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$, where $\mu_A : X \to [0, 1]; \lambda_A : X \to [0, 1]; \delta_A : X \to [-1, 0]$ such that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. The membership degree$\mu_A(x)$ characterizes the extent that the element x satisfies to the property corresponding to tripolar fuzzy set A,$\lambda_A(x)$ characterizes the extent that the element x satisfies to the not property(irrelevant) corresponding to tripolar fuzzy set A and $\delta_A(x)$ characterizes the extent that the element x satisfies to the implicit counter property of tripolar fuzzy set A. For simplicity $A = (\mu_A, \lambda_A, \delta_A)$ has been used for $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$.

Remark 2.1. A tripolar fuzzy set A is a generalization of a bipolar fuzzy set and an intuitionistic fuzzy set. A tripolar fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x) \mid x \in X\ )$ represents the sweet taste of food stuffs. Assuming the sweet taste of food stuff as a positive membership value $\mu_A(x)$, i.e. the element x is satisfying the sweet property. Then bitter taste of food stuff as a negative membership value $\delta_A(x,)$ i.e. the element x is satisfying the bitter property, and the remaining tastes of food stuffs like acidic, chilly etc., as a non memberships value $\lambda_A(x)$, i.e., the element is satisfying irrelevant to the sweet property.

Definition 2.9. A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is called a tripolar fuzzy $\Gamma-$subsemiring of M if A satisfies the following conditions.

(i) $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(ii) $\lambda_A(x + y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$

(iii) $\delta_A(x + y) \leq \max\{\delta_A(x), \delta_A(y)\}$

(iv) $\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(v) $\lambda_A(x\alpha y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$

(vi) $\delta_A(x\alpha y) \leq \max\{\delta_A(x), \delta_A(y)\}$, for all $x, y, z \in M, \alpha \in \Gamma$.

Definition 2.10. A tripolar fuzzy $\Gamma-$subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is called a tripolar fuzzy ideal of M if A satisfies the following conditions.

(i) $\mu_A(x\alpha y) \geq \max\{\mu_A(x), \mu_{A)}(xy\}$

(ii) $\lambda_A(x\alpha y) \leq \min\{\lambda_A(x), \lambda_A(y)\}$

(iii) $\delta_A(x\alpha y) \leq \min\{\delta_A(x), \delta_A(y)\}$, for all $x, y, z \in M, \alpha \in \Gamma$

Definition 2.11. A tripolar fuzzy $\Gamma-$subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is called a tripolar fuzzy interior ideal of M if A satisfies the following conditions.

(i) $\mu_A(x\alpha z\beta y) \geq \mu_A(z)$

(ii) $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z)$

(iii) $\delta_A(x\alpha z\beta y) \leq \delta_A(z)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.12. Let $(f, A), (g, B)$ be fuzzy soft sets over a $\Gamma-$semiring M. The intersection of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cap (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \\ f_c \cap g_c, & \text{if } c \in A \cap B. \end{cases}$$

for all $c \in A \cup B$ and $x \in M$.

Definition 2.13. Let $(f, A), (g, B)$ be fuzzy soft sets over a $\Gamma-$semiring MS. The union of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cup (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \\ f_c \cup g_c, & \text{if } c \in A \cap B. \end{cases},$$

for all $c \in A \cup B$ and $x \in M$.

Definition 2.14. Let $(f, A), (g, B)$ be fuzzy soft sets over a $\Gamma-$semiring M. "$(f, A)$and(g, B) is denoted by "$(f, A)\wedge(g, B)$" is defined by $(f, A)\wedge(g, B) = (h, C)$, where $C = A\times B$. $h_c(x) = \min\ \{f_a(x), g_b(x)\}$ for all $c = (a, b) \in A\times B$ and $x \in M$.

Definition 2.15. Let $(f, A), (g, B)$ be fuzzy soft sets over an ordered $\Gamma-$semiring M. "(f, A)or(g, B)" is denoted by $(f, A)\vee(g, B)$ is defined by $(f, A)\vee(g, B) = (h, C)$, where $C = A\times B$ and $h_c(x) = \max\ \{f_a(x), g_b(x)\}$, for all $c = (a, b) \in A\times B, x \in U$.

3. TRIPOLAR FUZZY SOFT IDEALS OVER $\Gamma-$SEMIRING

In this section, we introduce the notion of tripolar fuzzy sets to be able to deal with tripolar information as a generalization of fuzzy sets, bipolar fuzzy set and intuitionistic fuzzy sets. We introduce the notion of tripolar fuzzy soft ideals and interior ideals over $\Gamma-$semiring.

Definition 3.1. A tripolar fuzzy soft set (f, A) over $\Gamma-$semiring M is called a tripolar fuzzy soft $\Gamma-$semiring over M if $f(a) = \{\mu_{f(a)}(x), \lambda_{f(a)}(x), \delta_{f(a)}(x) \mid x \in M, a \in A\}$, where $\mu_{f(a)}(x) : M \to [0, 1]; \lambda_{f(a)}(x) : M \to [0, 1]; \delta_{f(a)}(x) : M \to [-1, 0]$ such that $0 \leq \mu_{f(a)}(x) + \lambda_{f(a)}(x) \leq 1$ and for all $x \in M$ satisfying the following conditions

(i) $\mu_{f(a)}(x + y) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$

(ii) $\lambda_{f(a)}(x + y) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$

(iii) $\delta_{f(a)}(x + y) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$

(iv) $\mu_{f(a)}(x\alpha y) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$

(v) $\lambda_{f(a)}(x\alpha y) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$

(vi) $\delta_{f(a)}(x\alpha y) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$, for all $x, y \in M, \alpha \in \Gamma$ and $a \in A$.

Definition 3.2. A tripolar fuzzy soft set (f, A) over a $\Gamma-$semiring M is called a tripolar fuzzy soft ideal over M if

(i) $\mu_{f(a)}(x + y) \geq \min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$

(ii) $\lambda_{f(a)}(x + y) \leq \max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$

(iii) $\delta_{f(a)}(x + y) \leq \max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$

(iv) $\mu_{f(a)}(x\alpha y) \geq \max\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$

(v) $\lambda_{f(a)}(x\alpha y) \leq \min\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}$

(vi) $\delta_{f(a)}(x\alpha y) \leq \min\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}$, for all $x, y \in M, \alpha \in \Gamma$ and $a \in A$.

Remark 3.1. Every tripolar fuzzy soft ideal (f, A) over a $\Gamma-$semiring M is a tripolar fuzzy soft semiring (f, A) over M but the converse is not true.

Example 3.1. Let $M = \{x_1, x_2, x_3\}, \Gamma = \{\alpha, \beta\}$. Then, we shall define the operations with the following tables:

$+$	α	β
α	α	α
β	α	β

$+$	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_2	x_3
x_3	x_3	x_3	x_2

;

α	x_1	x_2	x_3
x_1	x_1	x_3	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

and

β	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

.

Let $E = \{a, b, c\}$ and $B = \{a, b\}$. Then (ϕ, B) is tripolar fuzzy soft set defined as $(\phi, B) = \{\phi(a), \phi(b)\}$, where
$\phi(a) = \{(x_1, 0.2, 0.7, -0.2), (x_2, 0.3, 0.6, -0.3), (x_3, 0.6, 0.3, -0.3)\}$
$\phi(b) = \{(x_1, 0.4, 0.5, -0.3), (x_2, 0.6, 0.3, -0.5), (x_3, 0.5, 0.4, -0.2)\}$.
Then $(\phi, B)$is a tripolar fuzzy soft $\Gamma-$subsemiring over M and (ϕ, B) is not a tripolar fuzzy soft ideal over M.
(ϕ, B) is tripolar fuzzy soft interior ideal over M.

Definition 3.3. A tripolar fuzzy soft (f, A) over $\Gamma-$semiring M is called a tripolar fuzzy soft interior ideal of M if

(i) $\mu_{f(a)}(x\alpha z\beta y) \geq \mu_{f(a)}(z)$

(ii) $\lambda_{f(a)}(x\alpha z\beta y) \leq \lambda_{f(a)}(z)$

(iii) $\delta_{f(a)}(x\alpha z\beta y) \leq \delta_{f(a)}(z)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$ and $a \in A$.

Theorem 3.1. *Every tripolar fuzzy soft ideal over a $\Gamma-$semiring $M$is a tripolar fuzzy soft interior ideal over a $\Gamma-$semiring M.*

Proof. Let (f, A) be tripolar fuzzy soft ideal over a $\Gamma-$semiring M. Then $f(a) = \{\mu(a), \lambda(a), f(a)\}$ is a tripolar fuzzy ideal of $M, a \in A$. Then

(i) $\mu_{f(a)}(x\alpha z\beta y) \geq \mu_{f(a)}(x\alpha z) \geq \mu_{f(a)}(z)$

(ii) $\lambda_{f(a)}(x\alpha z\beta y) \leq \lambda_{f(a)}(x\alpha z) \leq \lambda_{f(a)}(z)$

(iii) $\delta_{f(a)}(x\alpha z\beta y) \leq \delta_{f(a)}(x\alpha z) \leq \delta_{f(a)}(z),$ forall $x, y, z \in M, \alpha, \beta \in \Gamma$ and $a \in A$.

Hence (f, A) is a tripolar fuzzy soft interior ideal over M. □

Theorem 3.2. *Every tripolar fuzzy soft interior ideal over a regular $\Gamma-$semiring M is a tripolar fuzzy soft ideal over M.*

Proof. Let (f, A) be tripolar fuzzy soft interior ideal over a regular $\Gamma-$semiring M. Then $f(a) = \{\mu_{f(a)}, \lambda_{f(a)}, \delta_{f(a)}\}$ is a tripolar fuzzy ideal of $M, a \in A$.

Suppose $x, y \in M, \alpha \in \Gamma$. Then $x\alpha y \in M$. Then there exist $\beta, \gamma \in \Gamma, z \in M$ such that $x\alpha y = x\alpha y\beta z\gamma x\alpha y$.

$$\begin{aligned}
\mu_{f(a)}(x\alpha y) &= \mu_{f(a)}(x\alpha y\beta z\gamma x\alpha y) \\
&= \mu_{f(a)}\left(x\alpha y\beta(z\gamma x\alpha y)\right) \\
&\geq \mu_{f(a)}(y) \\
\mu_{f(a)}(x\alpha y) &= \mu_{f(a)}\left((x\alpha y\beta z)\gamma x\alpha y\right) \\
&\geq \mu_{f(a)}(x).
\end{aligned}$$

Hence $\mu_{f(a)}$ is a fuzzy ideal of M.

$$\begin{aligned}
\lambda_{f(a)}(x\alpha y) &= \lambda_{f(a)}(x\alpha y\beta z\gamma x\alpha y) \\
&\leq \lambda_{f(a)}(y) \\
\lambda_{f(a)}(x\alpha y) &= \lambda_{f(a)}\big((x\alpha y\beta z)\gamma x\alpha y\big) \\
&\leq \lambda_{f(a)}(x).
\end{aligned}$$

Hence $\lambda_{f(a)}$ is a fuzzy ideal of M.

$$\begin{aligned}
\delta_{f(a)}(x\alpha y) &= \delta_{f(a)}(x\alpha y\beta z\gamma x\alpha y) \\
&\leq \delta_{f(a)}(y) \\
\delta_{f(a)}(x\alpha y) &\leq \delta_{f(a)}(x).
\end{aligned}$$

Hence $\delta_{f(a)}$ is a fuzzy ideal of M.
Therefore, $f(a)$ is a tripolar fuzzy ideal of $\Gamma-$semiring M.
Thus, (f, A) is a tripolar fuzzy soft ideal of $\Gamma-$semiring M. □

Theorem 3.3. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over a $\Gamma-$semiring M then $(f, A) \wedge (g, B)$ is a tripolar fuzzy soft ideal over a $\Gamma-$semiring M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft ideals over a $\Gamma-$semiring M.
Then by Definition [2.14], $(f, A) \wedge (g, B) = (f \cap g, C)$, where $C = A \times B$ and $(f \wedge g)(a, b) = f(a) \cap g(b)$, for all $(a, b) \in C$. Then

$$\begin{aligned}
\mu_{f(a)\cap g(b)}(x+y) &= \min\{\mu_{f(a)}(x+y), \mu_{g(b)}(x+y)\} \\
&\geq \min\Big\{\min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}, \min\{\mu_{g(b)}(x), \mu_{g(b)}(y)\}\Big\} \\
&= \min\Big\{\min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}, \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\}\Big\} \\
&= \min\{\mu_{f(a)\cap g(b)}(x), \mu_{f(a)\cap g(b)}(y)\}. \\
\lambda_{(f\cap g)(a,b)}(x+y) &= \min\{\lambda_{f(a)}(x+y), \lambda_{g(b)}(x+y)\} \\
&\geq \min\Big\{\max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}, \max\{\lambda_{g(b)}(x), \lambda_{g(b)}(y)\}\Big\} \\
&= \max\Big\{\min\{\lambda_{f(a)}(x), \lambda_{g(b)}(x)\}, \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\}\Big\} \\
&= \max\{\lambda_{f(a)\cap g(b)}(x), \lambda_{f(a)\cap g(b)}(y)\}.
\end{aligned}$$

$$\begin{aligned}
\delta_{(f\cap g)(a,b)}(x+y) &= \min\{\delta_{f(a)}(x+y), \delta_{g(b)}(x+y)\} \\
&\geq \min\big\{\max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}, \max\{\delta_{g(b)}(x), \delta_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\delta_{f(a)}(x), \delta_{g(b)}(x)\}, \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\}\big\} \\
&= \max\{\delta_{f(a)\cap g(b)}(x), \delta_{f(a)\cap g(b)}(y)\}.
\end{aligned}$$

$$\begin{aligned}
\mu_{f(a)\cap g(b)}(x\alpha y) &= \min\{\mu_{f(a)}(x\alpha y), \mu_{g(b)}(x\alpha y)\} \\
&\geq \min\big\{\min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}, \min\{\mu_{g(b)}(x), \mu_{g(b)}(y)\}\big\} \\
&= \min\big\{\min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}, \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\}\big\} \\
&= \min\{\mu_{f(a)\cap g(b)}(x), \mu_{f(a)\cap g(b)}(y)\}. \\
\lambda_{(f\cap g)(a,b)}(x\alpha y) &= \min\{\lambda_{f(a)}(x\alpha y), \lambda_{g(b)}(x\alpha y)\} \\
&\leq \min\big\{\max\{\lambda_{f(a)}(x), \lambda_{f(a)}(y)\}, \max\{\lambda_{g(b)}(x), \lambda_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\lambda_{f(a)}(x), \lambda_{g(b)}(x)\}, \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\}\big\} \\
&= \max\{\lambda_{f(a)\cap g(b)}(x), \lambda_{f(a)\cap g(b)}(y)\}. \\
\delta_{(f\cap g)(a,b)}(x\alpha y) &= \min\{\delta_{f(a)}(x\alpha y), \delta_{g(b)}(x\alpha y)\} \\
&\leq \min\big\{\max\{\delta_{f(a)}(x), \delta_{f(a)}(y)\}, \max\{\delta_{g(b)}(x), \delta_{g(b)}(y)\}\big\} \\
&= \max\big\{\min\{\delta_{f(a)}(x), \delta_{g(b)}(x)\}, \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\}\big\} \\
&= \max\{\delta_{f(a)\cap g(b)}(x), \delta_{f(a)\cap g(b)}(y)\}.
\end{aligned}$$

Hence $(f, A) \wedge (g, B)$ is a tripolar fuzzy soft ideal over a $\Gamma-$semiring M. $\square$

Theorem 3.4. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a $\Gamma-$semiring M then $(f, A) \wedge (g, B)$ is a tripolar fuzzy interior ideals of $\Gamma-$semiring M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a $\Gamma-$semiring M.
Then by Theorem[3.3], $(f, A) \wedge (g, B)$ is soft tripolar fuzzy $\Gamma-$subsemiring of M.
By Definition[2.14], $(f, A) \wedge (g, B) = (f \wedge g, C)$, where $C = A \times B$.
Suppose $(a, b) \in C, x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\mu_{f\wedge g(a,b)}(x\alpha y\beta z) &= \mu_{f(a)\cap g(b)}(x\alpha y\beta z) \\
&= \min\{\mu_{f(a)}(x\alpha y\beta z), \mu_{g(b)}(x\alpha y\beta z)\} \\
&\geq \min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\} \\
&= \mu_{f(a)\cap g(b)}(y) \\
&= \mu_{f\wedge g(a,b)}(y). \\
\lambda_{f\wedge g(a,b)}(x\alpha y\beta z) &= \lambda_{f(a)\cap g(b)}(x\alpha y\beta z) \\
&= \min\{\lambda_{f(a)}(x\alpha y\beta z), \lambda_{g(b)}(x\alpha y\beta z)\} \\
&\leq \min\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\} \\
&= \lambda_{f(a)\cap g(b)}(y)
\end{aligned}$$

$$
\begin{aligned}
& = \lambda_{f\wedge g(a,b)}(y). \\
\delta_{f\wedge g(a,b)}(x\alpha y\beta z) &= \delta_{f(a)\cap g(b)}(x\alpha y\beta z) \\
&= \min\{\delta_{f(a)}(x\alpha y\beta z), \delta_{g(b)}(x\alpha y\beta z)\} \\
&\leq \min\{\delta_{f(a)}(y), \delta_{g(b)}(y)\} \\
&= \delta_{f(a)\cap g(b)}(y) \\
&= \delta_{f\wedge g(a,b)}(y).
\end{aligned}
$$

Hence $(f, A) \wedge (g, B)$ is a soft tripolar fuzzy interior ideal of $\Gamma-$semiring M. □

The following theorem proofs are similar to Theorem [3.4].

Theorem 3.5. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a $\Gamma-$semiring M then $(f, A)\cup(g, B)$ is a tripolar fuzzy interior ideals of $\Gamma-$semiring M.*

Theorem 3.6. *If (f, A) and (g, B) are two tripolar fuzzy soft interior ideals over a $\Gamma-$semiring M then $(f, A)\cap(g, B)$ is a tripolar fuzzy interior ideals of $\Gamma-$semiring M.*

Theorem 3.7. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over $\Gamma-$semiring M then $(f, A) \cup (g, B)$ is a tripolar fuzzy soft ideal over M.*

Proof. Suppose (f, A) and (g, B) are two tripolar fuzzy soft ideals over $\Gamma-$semiring M. Then by Definition [2.13], we have $(f, A) \cup (g, B) = (h, C)$, where $C = A \cup B$; and

$$
h(c) = f \cup g(c) = \begin{cases} f(c) & \text{if} c \in A \setminus B; \\ g(c) & \text{if} c \in B \setminus A; \\ f(c) \cup g(c) & \text{if} c \in A \cap B, \text{ forall } c \in A \cup B. \end{cases}
$$

case(i) : If $c \in A \setminus B$ then $f \cup g(c) = f(c)$. Thus we have

$$
\begin{aligned}
\mu_{f\cup g(c)}(x+y) &= \mu_{f(c)}(x+y) \\
&\geq \min\{\mu_{f(c)}(x), \mu_{f(c)}(y)\} \\
&= \min\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}. \\
\lambda_{f\cup g(c)}(x+y) &= \lambda_{f(c)}(x+y) \\
&\leq \max\{\lambda_{f(c)}(x), \lambda_{f(c)}(y)\} \\
&= \max\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}. \\
\delta_{f\cup g(c)}(x+y) &= \delta_{f(c)}(x+y) \\
&\leq \max\{\delta_{f(c)}(x), \delta_{f(c)}(y)\} \\
&= \max\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}. \\
\mu_{f\cup g(c)}(x\alpha y) &= \mu_{f(c)}(x\alpha y)
\end{aligned}
$$

$$\begin{aligned}
&\geq \max\{\mu_{f(c)}(x\alpha y), \mu_{f(c)}(x\alpha y)\}\\
&= \max\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}.\\
\lambda_{f\cup g(c)}(x\alpha y) &= \lambda_{f(c)}(x\alpha y)\\
&\leq \min\{\lambda_{f(c)}(x), \lambda_{f(c)}(y)\}\\
&= \min\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}.\\
\delta_{f\cup g(c)}(x\alpha y) &= \delta_{f(c)}(x\alpha y)\\
&\leq \min\{\delta_{f(c)}(x), \delta_{f(c)}(y)\}\\
&= \min\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.
\end{aligned}$$

case(ii) : If $c \in B \setminus A$ then $f \cup g(c) = g(c)$.
Since $g(c)$ is a tripolar ideal of $\Gamma-$semiring M, $f\cup g(c)$ is a tripolar ideal of $\Gamma-$semiring M.
case(iii) : If $c \in A \cap B$ then $f \cup g(c) = f(c) \cup g(c)$.

$$\begin{aligned}
\mu_{f\cup g(c)}(x+y) &= \mu_{f(c)\cup g(c)}(x+y)\\
&= \max\{\mu_{f(c)}(x+y), \mu_{g(c)}(x+y)\}\\
&\geq \max\{\min\{\mu_{f(c)}(x), \mu_{f(c)}(y)\}, \min\{\mu_{g(c)}(x), \mu_{g(c)}(y)\}\}\\
&= \min\{\max\{\mu_{f(c)}(x), \mu_{g(c)}(x)\}, \max\{\mu_{f(c)}(y), \mu_{g(c)}(y)\}\}\\
&= \min\{\mu_{f(c)\cup g(c)}(x), \mu_{f(c)\cup g(c)}(y)\}\\
&= \min\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}
\end{aligned}$$

Similarly we can prove

$$\begin{aligned}
\lambda_{f\cup g(c)}(x+y) &\leq \max\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}\\
\delta_{f\cup g(c)}(x+y) &\leq \max\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.\\
\mu_{f\cup g(c)}(x\alpha y) &= \mu_{f(c)\cup g(c)}(x\alpha y)\\
&= \max\{\mu_{f(c)}(x\alpha y), \mu_{g(c)}(x\alpha y)\}\\
&\geq \max\{\max\{\mu_{f(c)}(x), \mu_{f(c)}(y)\}, \max\{\mu_{g(c)}(x), \mu_{g(c)}(y)\}\}\\
&= \max\{\max\{\mu_{f(c)}(x), \mu_{g(c)}(x)\}, \max\{\mu_{f(c)}(y), \mu_{g(c)}(y)\}\}\\
&= \max\{\mu_{f(c)\cup g(c)}(x), \mu_{f(c)\cup g(c)}(y)\}\\
&= \max\{\mu_{f\cup g(c)}(x), \mu_{f\cup g(c)}(y)\}\\
& \text{Similarly we can prove}\\
\lambda_{f\cup g(c)}(x\alpha y) &\leq \min\{\lambda_{f\cup g(c)}(x), \lambda_{f\cup g(c)}(y)\}\\
\delta_{f\cup g(c)}(x\alpha y) &\leq \min\{\delta_{f\cup g(c)}(x), \delta_{f\cup g(c)}(y)\}.
\end{aligned}$$

Therefore, $f \cup g(c)$ is a tripolar fuzzy ideal of M.
Hence $(f, A) \cup (g, B)$ is a tripolar fuzzy soft ideal over M. □

Corollary 3.1. *If (f, A) and (g, B) are two tripolar fuzzy soft ideals over $\Gamma-$semiring M then $(f, A) \cap (g, B)$ is a tripolar fuzzy soft ideal over M.*

Definition 3.4. Let (f, A) and (g, B) be two tripolar fuzzy soft sets over a $\Gamma-$semiring M. The the product (f, A) and (g, B) is defined as $(f, A) \circ (g, B) = (f \circ g, C)$, where $C = A \cup B$. Then we have $(f, A) \circ (g, B) = (h, C)$, where $C = A \cup B$; and

$$\mu_{(f \circ g)(c)}(x) = \begin{cases} \mu_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \mu_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \sup\limits_{x=y\alpha z} \{\min\{\mu_{f(c)}(y), \mu_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

$$\lambda_{(f \circ g)(c)}(x) = \begin{cases} \lambda_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \lambda_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \inf\limits_{x=y\alpha z} \{\max\{\lambda_{f(c)}(y), \lambda_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

$$\delta_{(f \circ g)(c)}(x) = \begin{cases} \delta_{f(c)}(x), & \text{if } c \in A \setminus B; \\ \delta_{g(c)}(x), & \text{if } c \in B \setminus A; \\ \inf\limits_{x=y\alpha z} \{\max\{\delta_{f(c)}(y), \delta_{g(c)}(z)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

Theorem 3.8. *If (f, A) and (g, B) are tripolar fuzzy soft interior ideals over $\Gamma-$semiring M then $(f, A) \circ (g, B)$ is a tripolar fuzzy soft interior ideal over $\Gamma-$semiring M.*

Proof. Obviously $(f, A) \circ (g, B)$ is tripolar fuzzy soft $\Gamma-$subsemiring over M. Let $x, y, z \in M, \alpha, \beta \in \Gamma$.

By Definition [3.4], $(f, A) \circ (g, B) = (f \circ g, C)$, where $C = A \cup B$ and $c \in C, x \in M$.
case(i) : If $c \in A \setminus B$ then

$$\begin{aligned} \mu_{f \circ g(c)} &= \mu_{f(c)} \\ \lambda_{f \circ g(c)} &= \lambda_{f(c)} \\ \delta_{f \circ g(c)} &= \delta_{f(c)}. \end{aligned}$$

Since (f, A) is a tripolar fuzzy soft interior ideal over M, $f \circ g(c)$ is a tripolar fuzzy soft interior ideal of M.

case(ii) : If $c \in B \setminus A$ then

$$\begin{aligned} \mu_{f \circ g(c)} &= \mu_{g(c)} \\ \lambda_{f \circ g(c)} &= \lambda_{g(c)} \\ \delta_{f \circ g(c)} &= \delta_{g(c)}. \end{aligned}$$

Since (g, B) is a tripolar fuzzy soft interior ideal over M, $f \circ g(c)$ is a tripolar fuzzy soft interior ideal of M.
case(iii) : If $c \in A \cap B$ then

$$\mu_{f \circ g(c)}(x) = \sup_{x=a\alpha b} \{\min\{\mu_{f(a)}(x), \mu_{g(b)}(x)\}\}.$$

$$\begin{aligned}
\mu_{f o g(c)}(x\alpha y\beta z) &= \sup_{x=a\alpha b}\big\{\min\{\mu_{f(a)}(x\alpha y\beta z), \mu_{g(b)}(x\alpha y\beta z)\big\} \\
&\geq \sup_{x=a\alpha b}\big\{\min\{\mu_{f(a)}(y), \mu_{g(b)}(y)\big\} \\
&= \mu_{f\circ g(c)}(y). \\
\lambda_{f o g(c)}(x\alpha y\beta z) &= \inf_{x=a\alpha b}\big\{\max\{\lambda_{f(a)}(x\alpha y\beta z), \lambda_{g(b)}(x\alpha y\beta z)\big\} \\
&\leq \sup_{x=a\alpha b}\big\{\max\{\lambda_{f(a)}(y), \lambda_{g(b)}(y)\big\} \\
&= \lambda_{f\circ g(c)}(y). \\
\delta_{f o g(c)}(x\alpha y\beta z) &= \inf_{x=a\alpha b}\big\{\max\{\delta_{f(a)}(x\alpha y\beta z), \delta_{g(b)}(x\alpha y\beta z)\big\} \\
&\leq \inf_{x=a\alpha b}\big\{\max\{\delta_{f(a)}(y), \delta_{g(b)}(y)\big\} \\
&= \delta_{f o g(c)}(y).
\end{aligned}$$

Hence $f \circ g(c)$ is a tripolar fuzzy interior ideal of M.
Therefore $(f, A) \circ (g, B)$ is a tripolar fuzzy interior ideal over M. $\square$

Theorem 3.9. *Let E be a perimeter set and $\sum_E(M)$ be the set of all tripolar fuzzy soft interior ideals over $\Gamma-$semiring M. Then $(\sum_E(M), \cup, \cap)$ forms complete distributive lattices along with the relation $\subseteq$.*

Proof. Suppose (f, A) and (g, B) be soft interior ideals over M such that $A \subseteq E, B \subseteq E$.
By Theorems [3.6, 3.7], $(f, A) \cap (g, B)$ and $(f, A) \cup (g, B)$ are tripolar fuzzy soft interior ideals over M.
Obviously $(f, A) \cap (g, B)$ is lub of $\{(f, A), (g, B)\}$ and $(f, A) \cup (g, B)$ is glb of $\{(f, A)(g, B)\}$.
Every sub collection of $\sum_E(M)$ has lub and glb.
Hence $\sum_E(M)$ is a complete lattice.
We can prove $(f, A) \cap \big((g, B) \cup (h, C)\big) = \big((f, A) \cap (g, B)\big) \cup \big((f, A) \cap (h, C)\big)$.
Therefore $(\sum_E(M), \cup, \cap)$ forms a complete distributive lattice. $\square$

4. Conclusion

Murali Krishna Rao [19] introduced the notion of tripolar fuzzy set in $\Gamma-$semigroup to be able to deal with tripolar information, as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. In this paper, we have introduced the notion of tripolar fuzzy soft subsemiring, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideals over $\Gamma-$semiring and studied some of their algebraic basic properties and relations between them.The results of this paper can be extended to different algebraic structures and it can be applied to decision making problems.

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AN ANALOGUE OF COWLING-PRICE'S THEOREM FOR THE Q-FOURIER-DUNKL TRANSFORM

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Abstract. The Q-Fourier-Dunkl transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. By using the heat kernel associated to the Q-Fourier-Dunkl operator, we have established an analogue of Cowling-Price, Miyachi and Morgan theorems on $\mathbb{R}$ by using the heat kernel associated to the Q-Fourier-Dunkl transform.
Keywords: Cowling-Price's theorem; Miyachi's theorem; Uncertainty Principles; Q-Fourier-Dunkl transform.

1. Introduction

There are many theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot simultaneously be very small at infinity. This principle has several versions which were proved by M.G. Cowling and J.F. Price [3] and Miyachi [6]. In this paper, we will study an analogue of Cowling-Price's theorem and Miyachi's theorem for the Q-Fourier-Dunkl transform. Many authors have established the analogous of Cowling-Price's theorem in other various setting of harmonic analysis (see for instance [5]) The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Q-Fourier-Dunkl transform. In Section 3 we give an analogue of Cowling-Price's theorem, Miyachi's theorem, and Morgan's theorem for the Q-Fourier-Dunkl transform. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper $\alpha > \frac{-1}{2}$. Notice that if $\alpha = \frac{-1}{2}$ then the space is the classical Lebesgue one, we can follow in this case the procedures for similar transforms, such as the Fourier transform (see for example [3, 6]).

- $$Q(x) = \exp\left(-\int_0^x q(t)dt\right), \quad x \in \mathbb{R} \tag{1.1}$$

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where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$.

- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad if \quad 1 < p < \infty,$$

and $\|f\|_{\infty,\alpha} = \|f\|_\infty = esssup_{x\in\mathbb{R}}|f(x)|$.

- $L^p_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where Q is given by (1.1).

We consider the first singular differential-difference operator Λ defined on $\mathbb{R}$

$$\Lambda f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x} + q(x)f(x) \tag{1.2}$$

where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}.$$

1.1. Q-Fourier-Dunkl Transform

The following statements are proved in [1]

Lemma 1.1. *1. For each $\lambda \in \mathbb{C}$, the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1$$

admits a unique $\mathcal{C}^\infty$ solution on $\mathbb{R}$, denoted by Ψ_λ, given by

$$\Psi_\lambda(x) = Q(x)e_\alpha(i\lambda x), \tag{1.3}$$

where e_α denotes the one-dimensional Dunkl kernel defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z) \quad (z \in \mathbb{C}),$$

and j_α being the normalized spherical Bessel function of index α given by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{1.4}$$

2. For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, ...$ we have

$$\left| \frac{\partial^n}{\partial\lambda^n}\Psi_\lambda(x) \right| \le Q(x)|x|^n e^{|Im\ \lambda||x|}. \tag{1.5}$$

In particular

$$| \Psi_\lambda(x) | \le Q(x)e^{|Im\ \lambda||x|}. \tag{1.6}$$

3. *For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, we have the Laplace type integral representation*

$$\Psi_\lambda(x) = a_\alpha Q(x) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}(1+t)e^{i\lambda xt}dt, \tag{1.7}$$

where $a_\alpha = \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

Definition 1.1. The Q-Fourier-Dunkl transform associated with Λ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$\mathcal{F}_Q(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)x^{2\alpha+1}dx. \tag{1.8}$$

Theorem 1.1. 1. *Let $f \in L^1_Q(\mathbb{R})$ such that $\mathcal{F}_Q(f) \in L^1_\alpha(\mathbb{R})$. Then for allmost $x \in \mathbb{R}$ we have the inversion formula*

$$f(x)\left(Q(x)\right)^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_Q(f)(\lambda)\Psi_\lambda(x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}.$$

2. *For every $f \in L^2_Q(\mathbb{R})$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2\left(Q(x)\right)^2|x|^{2\alpha+1}dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_Q(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

3. *The Q-Fourier-Dunkl transform $\mathcal{F}_Q$ extends uniquely to an isometric isomorphism from $L^2_Q(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$.*

The heat kernel $N(x,s)$, $x \in \mathbb{R}$, $s > 0$, associated with the Q-Fourier-Dunkl transform is given by

$$N(x,s) = m_\alpha \frac{e^{-\frac{x^2}{4s}}}{(2s)^{\alpha+\frac{1}{2}}Q(x)}. \tag{1.9}$$

Some basic properties of $N(x,s)$ are the following:

- $N(x,s)Q^2(x) = m_\alpha \displaystyle\int_{\mathbb{R}} e^{-sy^2}\Psi_y(x)|y|^{2\alpha+1}dy.$
- $\mathcal{F}_Q(N(.,s))(x) = e^{-sx^2}.$

we define the heat functions W_l, $l \in \mathbb{N}$ as

$$Q^2(x)W_l(x,s) = \int_{\mathbb{R}} y^l e^{-\frac{y^2}{4s}}\Psi_y(x)|y|^{2\alpha+1}dy \tag{1.10}$$

$$\mathcal{F}_Q(W_l(.,s)) = i^l y^l e^{-sy^2}. \tag{1.11}$$

The intertwining operators associated with a Q-Fourier-Dunkl transform on the real line is given by

$$X_Q(f)(x) = a_\alpha Q(x) \int_{-1}^{1} f(tx)(1-t^2)^{\alpha-\frac{1}{2}} dt,$$

its dual is given by

$$ {}^tX_Q(f)(y)a = \quad a_\alpha \int_{|x|\geq|y|} f(x)Q(x)sgn(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)dx \tag{1.12}$$

${}^tX_Q$ can be written as

$${}^tX_Q(f)(y) = a_\alpha \int_{\mathbb{R}} f(x)Q(x)d\nu_y(x),$$

where

$$d\nu_y(x) = a_\alpha \chi_{\{|x|\geq|y|\}} sgn(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)dx$$

and $\chi_{\{|x|\geq|y|\}}$ denote the characteristic function with support in the set $\{x \in \mathbb{R} \ / \ |x| \geq |y|\}$.

Proposition 1.1. *If $f \in L^1_Q(\mathbb{R})$ then ${}^tX_Q(f) \in L^1(\mathbb{R})$ and $\|{}^tX_Q(f)\|_1 \leq \|f\|_{1,Q}$.*

For every $f \in L^1_Q(\mathbb{R})$

$$\mathcal{F}_Q = \mathcal{F} \circ^t X_Q(f), \tag{1.13}$$

where $\mathcal{F}$ is the usual Fourier transform defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x} dx.$$

2. Cowling-Price's Theorem for the Q-Fourier-Dunkl Transform

Theorem 2.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{e^{apx^2} Q^p(x)|f(x)|^p}{(1+|x|)^k} |x|^{2\alpha+1} dx < \infty \tag{2.1}$$

and

$$\int_{\mathbb{R}} \frac{e^{bq\xi^2} |\mathcal{F}_Q(f)(\xi)|^q}{(1+|\xi|)^m} d\xi < \infty, \tag{2.2}$$

for some constants $a, b > 0$, $k > 0, m > 1$ and $1 \leq p, r \leq +\infty$.

i) If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$, then $f(x) = P(x)N(x,b)$ where P is a polynomial with $degP \leq \min\{\frac{k}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{r}\}$. Especially, if

$$k \leq 2\alpha + 2 + p\min\{\frac{k}{p} + \frac{2\alpha+1}{p'}, \frac{m-1}{r}\},$$

then $f = 0$ almost everywhere. Furthermore, if $m \in]1, 1+r]$ and $k > 2\alpha + 2$, then f is a constant multiple of $N(.,b)$.

iii) If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4}a[$ all functions of the form $f(x) = P(x)N(x,\delta)$ satisfy (2.1) and (2.2).

Proof. It follows from (2.1) that $f \in L^1_Q$ and $\mathcal{F}_Q(f)(\xi)$ exists for all $\xi \in \mathbb{R}$. Moreover, it has an entire holomorphic extension on $\mathbb{C}$ satisfying for some $s > 0$,

$$|\mathcal{F}_Q(f)(z)| \leq Ce^{\frac{Imz^2}{4a}}(1 + |Imz|)^s.$$

By (1.1) we have for all $z = \xi + i\eta \in \mathbb{C}$,

$$\begin{aligned} (2.3) \quad |\mathcal{F}_Q(f)(z)| &\leq \int_{\mathbb{R}} |f(x)||\Lambda_\xi(x)||x|^{2\alpha+1}dx \\ (2.4) \quad &\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} \frac{e^{ax^2}Q(x)|f(x)|}{(1+|x|)^{\frac{k}{p}}}(1+|x|)^{\frac{k}{p}}e^{-a(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx. \end{aligned}$$

By Hölder inequality we have

$$|\mathcal{F}_Q(f)(z)| \leq e^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}} \frac{e^{pax^2}Q(x)^p|f(x)|^p}{(1+|x|)^k}|x|^{2\alpha+1}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} (1+|x|)^{\frac{kp'}{p}}e^{-ap'(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{p'}}$$

according to (2.1) we get that

$$\begin{aligned} |\mathcal{F}_Q(f)(\xi+i\eta)| &\leq Ce^{\frac{\eta^2}{4a}} \left(\int_{\mathbb{R}} (1+|x|)^{\frac{kp'}{p}}e^{-ap'(x-\frac{\eta}{2a})^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{p'}} \\ &\leq Ce^{\frac{\eta^2}{4a}} \left(\int_0^\infty (1+|x|)^{\frac{kp'}{p}+2\alpha+1}e^{-ap'(x-\frac{\eta}{2a})^2}dx\right)^{\frac{1}{p'}} \\ &\leq Ce^{\frac{\eta^2}{4a}}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}. \end{aligned}$$

If $ab = \frac{1}{4}$, then

$$|\mathcal{F}_Q(f)(\xi+i\eta)| \leq Ce^{b\eta^2}(1+|\eta|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}.$$

We put $g(z) = e^{bz^2}\mathcal{F}_Q(f)(z)$, then

$$|g(z)| \leq Ce^{b|Rez|^2}(1+|Imz|)^{\frac{k}{p}+\frac{2\alpha+1}{p'}}.$$

It follows from (2.2) that

$$\int_{\mathbb{R}} \frac{|g(z)|^r}{(1+|\xi|)^m}d\xi < \infty.$$

Lemma 2.1. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|Rez|^2}(1+|Imz|)^l$$

for some $l > 0$, $a > 0$ and

$$\int_{\mathbb{R}} \frac{|h(x)|^r}{(1+|x|)^m}|P(x)|dx < \infty$$

for some $r \geq 1$, $m > 1$ and P is a polynomial with degree m. Then h is a polynomial with $degh \leq min\{l, \frac{m-M-1}{r}\}$ and if $m \leq r+M+1$, then h is a constant.

From this Lemma, g is a polynomial, we say P_b with $degP_b \leq min\{\frac{kp^{'}}{p}+\frac{2\alpha+1}{p^{'}}, \frac{m-1}{r}\}$. Then $\mathcal{F}_Q(f)(x) = P_b(x)e^{-bx^2}$ then,

$$f(x) = Q_b(x)N(x,b)$$

where $degP_b = degQ_b$. Therefore, nonzero f satisfies (1.10) provided that

$$k > 2\alpha + 2 + p\min\left\{\frac{kp^{'}}{p} + \frac{2\alpha+1}{p^{'}}, \frac{m-1}{q}\right\}.$$

If $m < r+1$, by Lemma 1 we have g as a constant and $\mathcal{F}_Q(f)(x) = Ce^{-bx^2}$ and $f(x) = CN(x,b)$. If $m > 1$ and $k > 2\alpha+2$, these functions satisfy (2.1) and (2.2), which proves (ii).

If $ab > \frac{1}{4}$, then we can find positive constants a_1 and b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathcal{F}_Q(f)$ also satisfy (2.2) with a and b replaced by a_1 and b_1 respectively. Then $\mathcal{F}_Q(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. $\mathcal{F}_Q(f)$ cannot satisfy (2.2) unless $P_{b_1} = 0$, which implies that $f = 0$, this proves (i). If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4a}[$, the functions of the form $f(x) = P(x)N(x,\delta)$, where P is a polynomial on $\mathbb{R}$, satisfy (2.1) and (2.2). This proves (iii). □

3. Mathematical Formulas

4. Miyachi's Theorem for the Q-Fourier-Dunkl Transform

Theorem 4.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2}f \in L^p_Q(\mathbb{R}) + L^r_Q(\mathbb{R}) \tag{4.1}$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\mathcal{F}_Q(f)(\xi)e^{b\xi^2}|}{\lambda}d\xi < \infty, \tag{4.2}$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, r \leq +\infty$.

(i) if $ab > \frac{1}{4}$ *then* $f = 0$ *almost everywhere.*

(ii) if $ab = \frac{1}{4}$ *then* $f = cN(.,b)$ *with* $|c| \leq \lambda$.

(iii) if $ab > \frac{1}{4}$ *then for all* $\delta \in]b, \frac{1}{4}[$, *all functions of the form* $f(x) = P(x)N(x,\delta)$, *where* P *is a polynomial on* $\mathbb{R}$ *satisfy (2.1) and (2.2).*

To prove this result, we need the following lemmas.

Lemma 4.1. [5] *Let* h *be an entire function on* $\mathbb{C}$ *such that*

$$|h(z)| \leq Ae^{B|Rez|^2},$$

and

$$\int_{\mathbb{R}} \log^+ |h(y)|dy < \infty, \tag{4.3}$$

for some constants A *and* B. *Then* h *is a constant.*

Lemma 4.2. *Let* $r \in [1, +\infty], a > 0$. *Then for* $g \in L^r_Q(\mathbb{R})$ *there exist* $c > 0$ *such that*

$$\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r \leq c \| g \|_{r,Q} .$$

Proof. From the hypothesis, it follows that e^{-ay^2} belongs to $L^1_Q(\mathbb{R})$. Then by Proposition 1.1, ${}^tX_Q(e^{-ay^2}g)$ is defined almost everywhere on $\mathbb{R}$. Here we consider two cases:

i) If $r \in [1, +\infty[$ then

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r^r &\leq \int_{\mathbb{R}} e^{arx^2} (\int_{\mathbb{R}} Q(y)e^{-ay^2}|g(y)|d\nu_x(y))^r dx, \\
&\leq \int_{\mathbb{R}} e^{arx^2} \left(\int_{\mathbb{R}} |Q(y)g(y)|^r d\nu_x(y)\right)^{\frac{r}{r}} \left(\int_{\mathbb{R}} e^{-ar'y^2} d\nu_x(y)\right)^{\frac{r}{r'}} dx
\end{aligned}$$

where r' is the conjugate exponent for r. Since

$$\int_{\mathbb{R}} e^{-ry^2} d\nu_x(y) = Ce^{-rx^2}, \tag{4.4}$$

for $r > 0$ it follows from (4.4) that

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r^r &\leq C \int_{\mathbb{R}} {}^tX_Q(|g|^r)(x)dx, \\
&= C \int_{\mathbb{R}} |g(x)|^r |x|^{2\alpha+1} dx < \infty.
\end{aligned}$$

ii) If $r = \infty$ then it follows from (4.4) that

$$\begin{aligned}
\| e^{ax^2}\ {}^tX_Q(e^{-ay^2}g) \|_r &\leq e^{ax^2}\ {}^tX_Q(e^{-ay^2})(x)\|g\|_{Q,\infty} \\
&= C\|g\|_{Q,\infty}.
\end{aligned}$$

□

Lemma 4.3. *Let $r, p \in [1, +\infty]$ and let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2} f \in L^p_Q(\mathbb{R}) + L^r_Q(\mathbb{R}) \tag{4.5}$$

for some $a > 0$. Then for all $z \in \mathbb{C}$, the integral

$$\mathcal{F}_Q(f)(z) = \int_{\mathbb{R}} f(x)\Lambda_{z,Q}(-x)|x|^{2\alpha+1}dx$$

is well defined. $\mathcal{F}_Q(f)(z)$ is entire and there exists $C > 0$ such that for all $\xi,\ \eta \in \mathbb{R}$,

$$|\mathcal{F}_Q(f)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}. \tag{4.6}$$

Proof. From (5) and Hölder's inequality we have the first assertion. For (4.6) using (4.5) we have $f \in L^1_Q(\mathbb{R})$ and ${}^tX_Q(f) \in L^1(\mathbb{R})$. for all $\xi,\ \eta \in \mathbb{R}$,

$$\mathcal{F}_Q(f)(\xi + i\eta) = \int_{\mathbb{R}} {}^tX_Q(f)(x)e^{-ix(\xi+i\eta)}dx$$

$$\begin{aligned} |\mathcal{F}_Q\left(f\right)\left(\xi + i\eta\right)| &\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q\left(f\right)\left(x\right)\right| e^{-ax^2+x\eta-\frac{\eta^2}{4a}}dx \\ &\leq e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q\left(f\right)\left(x\right)\right| e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx. \end{aligned}$$

From (4.5) we can deduce that there exists $u \in L^p_Q(\mathbb{R})$ and $v \in L^r_Q(\mathbb{R})$ such that

$$f(x) = e^{-ax^2}u\left(x\right) + e^{-ax^2}v\left(x\right),$$

by Lemma 4 we have

$$\int_{\mathbb{R}} e^{ax^2} \left|{}^tX_Q\left(f\right)\left(x\right)\right| e^{-a\left(x-\frac{\eta}{2a}\right)^2}dx \leq C\left(\|u\|_{p,Q} + \|v\|_{r,Q}\right) < \infty,$$

which proves the Lemma. □

Proof of Theorem

- If $ab > \frac{1}{4}$. Let h be a function on $\mathbb{C}$ defined by

$$h(z) = e^{\frac{z^2}{4a}}\mathcal{F}_Q\left(f\right)\left(z\right).$$

h is entire function on $\mathbb{C}$, it follows from (4.6) that

$$\forall \xi \in \mathbb{R},\ \forall \eta \in \mathbb{R}\ \ |h\left(\xi + i\eta\right)| \leq Ce^{\frac{\xi^2}{4a}}. \tag{4.7}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}} \log^+ |h(y)|\, dy &= \int_{\mathbb{R}} \log^+ \left| e^{\frac{y^2}{4a}} \mathcal{F}_Q(f)(y) \right| dy \\ &= \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} \mathcal{F}_Q(f)(y) \right|}{\lambda} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy \\ &\leq \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} \mathcal{F}_Q(f)(y) \right|}{\lambda} dy + \int_{\mathbb{R}} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy\end{aligned}$$

because $\log_+(cd) \leq \log_+(c) + d$ for all c, $d > 0$. Since $ab > \frac{1}{4}$, (2.2) implies that

$$\int_{\mathbb{R}} \log^+ |h(y)| dy < \infty. \tag{4.8}$$

A combination of (4.7), (4.8) and Lemma 3 shows that h is a constant and

$$\mathcal{F}_Q(f)(y) = Ce^{-\frac{1}{4a}y^2}.$$

Since $ab > \frac{1}{4}$, (2.2) holds whenever $C = 0$ and the injectivity of $\mathcal{F}_Q$ implies that $f = 0$ almost everywhere.

- If $ab = \frac{1}{4}$. We deduce from previous case that $\mathcal{F}_Q(f) = Ce^{-\frac{\xi^2}{4a}}$. Then (2.2) holds whenever $|C| \leq \lambda$. Hence $f = CN(.,b)$ with $|C| \leq \lambda$.
- If $ab < \frac{1}{4}$. If f is a given form, then $\mathcal{F}_Q(f)(y) = Q(y)e^{-\delta y^2}$ for some Q.

In the contintion, we will give an analogue of Hardy's theorem [?] for the Q-Fourier-Dunkl transform.

Theorem 4.2. Hardy *Let $N \in \mathbb{N}$. Assume that $f \in L^2_Q(\mathbb{R})$ is such that*

$$|f(x)| \leq Me^{-\frac{1}{4a}x^2} \ a.e\ ,\ \forall y \in \mathbb{R}, |\mathcal{F}_Q(f)(y)| \leq M(1+|y|)^N e^{-by^2}, \tag{4.9}$$

for some constants $a > 0$, $b > 0$ and $M > 0$. Then,
i) If $ab > \frac{1}{4}$, then $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then the function f is of the form

$$f(x) = \sum_{|s| \leq N} a_s W_s\left(\frac{1}{4a}, x\right) \ a.e.\ ,\ a_s \in \mathbb{C}.$$

iii) If $ab < \frac{1}{4}$, there are infinitely many nonzero functions of f satisfying the conditions (4.9).

Proof. The first condition of (4.9) implies that $f \in L^1_Q(\mathbb{R})$. So by Proposition 1.1, the function ${}^tX_Q(f)$ is defined almost everywhere. By using the relation (1.13), we deduce that for all $x \in \mathbb{R}$,

$$|{}^tX_Q(f)(x)| \leq M_0 e^{-ax^2},$$

where M_0 is a positive constant. So

$$|{}^tX_Q(f)(x)| \leq M_0(1+|x|)^N e^{-ax^2}, \tag{4.10}$$

On the other hand from (1.13) and (4.9) we have for all $x \in \mathbb{R}$,

$$|\mathcal{F}({}^tX_Q)(f)(y)| \leq M(1+|y|)^N e^{-b|y|^2}, \tag{4.11}$$

The relations (4.10) and (4.11) show that the conditions of Proposition 3.4 of [2], p.36, are satisfied by the function ${}^tX_Q(f)$. Thus we get:
i) If $ab > \frac{1}{4}$, ${}^tX_Q(f) = 0$ a.e. Using (1.13) we deduce

$$\forall y \in \mathbb{R}, \mathcal{F}_Q(f)(y) = \mathcal{F} \circ ({}^tX_Q)(f)(y) = 0.$$

Then by the injectivity of $\mathcal{F}_Q$ we have $f = 0$ a.e.
ii) If $ab = \frac{1}{4}$, then ${}^tX_Q(f)(x) = P(x)e^{-ax^2}$, where P is a polynomial of degree lower than N. Using this relation and (1.13), we deduce that

$$\forall x \in \mathbb{R},\ \mathcal{F}_Q(f)(y) = \mathcal{F} \circ^t X_Q(f)(y) = \mathcal{F}(P(x)e^{-\delta x^2})(y).$$

but

$$\forall x \in \mathbb{R},\ \mathcal{F}(P(x)e^{-\delta x^2})(y) = S(y)e^{\frac{-y^2}{4\delta}},$$

with S a polynomial of degree lower than N.
Thus from (1.11), we obtain

$$\forall x \in \mathbb{R},\ \mathcal{F}_Q(f)(y) = \mathcal{F}_Q\left(\sum_{|s|<\frac{N-1}{2}} a_s W_s(\frac{1}{4\delta},.)\right)(y).$$

The injectivity of the transform $\mathcal{F}_Q$ implies

$$f(x) = \sum_{|s|\leq N} a_s W_s(\frac{1}{4a}, x)\ a.e.$$

iii) If $ab < \frac{1}{4}$, let $t \in]a, \frac{1}{4b}[$ and $f(x) = C\dfrac{e^{-tx^2}}{Q(x)}$ for some real constant C, these functions satisfy the conditions (4.9).
□In the next part, we will give an analogue of Morgan's theorem [7] for the Q-Fourier-Dunkl transform.

Theorem 4.3. Morgan *Let $1 < p < 2$ and r be the conjugate exponent of p. Assume that $f \in L^2_Q(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}} e^{\frac{a^p}{p}|x|^p}|f(x)||x|^{2\alpha+1}dx < +\infty,\ \text{and}\ \int_{\mathbb{R}} e^{\frac{b^r}{r}|y|^r}|\mathcal{F}_Q(f)(y)|dy < +\infty, \tag{4.12}$$

for some constants $a > 0$, $b > 0$.
Then if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$, we have $f = 0$ a.e.

Proof. The first condition of (4.12) implies that $f \in L^1_Q(\mathbb{R})$. So by Proposition 1.1, the function ${}^tX_Q(f)$ is defined almost everywhere. By using the relation (4.12) and Proposition 1.1, we deduce that:

$$\int_{\mathbb{R}} |{}^tX_Q(f)(x)| e^{\frac{a^p}{p}|x|^p} dx \leq \int_{\mathbb{R}} e^{\frac{a^p}{p}|x|^p} |f(x)||x|^{2\alpha+1} dx < +\infty.$$

So

$$\int_{\mathbb{R}} |{}^tX_Q(f)(x)| e^{\frac{a^p}{p}|x|^p} dx < +\infty \tag{4.13}$$

On the other hand, from (1.13) and (4.12) we have:

$$\int_{\mathbb{R}} e^{\frac{b^q}{q}|y|^q} |\mathcal{F}_Q(f)(y)| dy = \int_{\mathbb{R}} e^{\frac{b^q}{q}|y|^q} |\mathcal{F}({}^tX_Q)(f)(y)| dy < +\infty. \tag{4.14}$$

The relations (4.13) and (4.14) are the conditions of Theorem 1.4, p.26 of [2], which are satisfied by the function ${}^tX_Q(f)$. Thus we deduce that if $ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}}$ we have ${}^tX_Q(f) = 0$ a.e. Using the same proof as in the end of Theorem 4, we have obtained $f(y) = 0$. a.e. $y \in \mathbb{R}$. $\square$

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A SOLUTION ALGORITHM FOR p-MEDIAN LOCATION PROBLEM ON UNCERTAIN RANDOM NETWORKS

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Abstract. This paper investigates the classical p-median location problem in a network in which some of the vertex weights and the distances between vertices are uncertain and while others are random. For solving the p-median problem in an uncertain random network, an optimization model based on the chance theory is proposed first and then an algorithm is presented to find the p-median. Finally, a numerical example is given to illustrate the efficiency of the proposed method.
Keywords: Location problem; p-median; Chance theory; Uncertain random network.

1. Introduction

Location problems have received strong theoretical interest due to their relevance in practice. One of the well-known location problems which was considered in the literature is the p-median location problem which is stated as follows: Let $N = (V, E)$ be an undirected connected network with vertex set V, $\mid V \mid= n$, edge set E and let p be a constant with $p \leq n$. Every edge $e \in E$ has a positive length and each vertex $v_i \in V$ is associated with a nonnegative weight w_i that it is the demand of the client at this vertex. Let $d(x, y)$ denote the distance between x, $y \in N$ which is equal to the length of the shortest path connecting these two points. In the classical p-median problem, the aim is to locate p pairwise different facilities $m_1, \ldots, m_p$ on the network N (i.e., on vertices or edges) which minimize the sum of weighted distances from each vertex to its closest facility:

$$(\mathbf{P}_1): \quad \min_{X_p \subset N, |X_p|=p} \sum_{v_i \in V} w_i d(v_i, X_p)$$

where

$$d(v_i, X_p) = \min_{j=1,2,\ldots,p} d(v_i, m_j), \qquad X_p = \{m_1, \ldots, m_p\}.$$

An optimal solution X_p^* is called a p-median.

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Hakimi [13] showed that there exists an optimal solution among the set of vertices. This property is called vertex optimality. Later, Kariv and Hakimi [16] proved that the classical p-median problem was NP-hard, even if N was a planar graph of maximum degree 3.

Now, let $d_{ij} = d(v_i, v_j)$ be the distance from demand vertex v_i to candidate facility at vertex v_j. Also, let $w = \{w_i \mid v_i \in V\}$ and $d = \{d_{ij} \mid v_i, v_j \in V\}$ denote the set of the vertex weights and the set of distances between vertices, respectively. Then in the network N, the optimal objective value of the p-median problem is a function of w and d, which is denoted as $f(w, d)$ in this paper.

We are going to present a 0-1 linear programming formulation of the classical discrete p-median problem. Let x_{ij} be the variable that is equal to 1 if the demands of the vertex v_i are served by a facility at the vertex v_j, and 0 otherwise. Also, let the variable x_j be equal to 1 if there is an open facility at the vertex v_j, and 0 otherwise. Then, the 0-1 linear programming formulation of the classical discrete p-median problem can be stated as follows:

$$
\begin{array}{llll}
(\mathbf{P}_2): & \min & \sum_{i=1}^{n} \sum_{j=1}^{n} w_i d_{ij} x_{ij} & \\
& \text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 & \forall i = 1, \ldots, n, \\
& & x_{ij} \leq x_j & \forall i, j = 1, \ldots, n, \\
& & \sum_{j=1}^{n} x_j = p, & \\
& & x_{ij}, x_j \in \{0, 1\} & \forall i, j = 1, \ldots, n.
\end{array}
$$

This model minimizes the total weighted distance between each demand vertex and the nearest facility. The first set of constraints requires each demand vertex to be assigned to exactly one facility. The second set of constraints allows the demand of the vertex v_i to be assigned to a vertex v_j only if there is an open facility in this location. The third set of constraints states that exactly p facilities are to be located. Finally, the last constraints are the standard integrality conditions [8, 28].

The p-median location problems have been studied by many researchers. Kariv and Hakimi [16] presented an $O(p^2n^2)$ time algorithm for the p-median location problem on tree networks. Gavish and Sridhar [10] proposed an algorithm for the 2-median problem on trees which is run in $O(n \log n)$ time. Tamir [36] improved the time complexity of the p-median problem on trees to $O(pn^2)$. Benkoczi and Bhattacharya [6] presented an algorithm with time complexity of $O(pn \log n)$ for solving the p-median problem on interval networks. Also, they designed an $O(n \log^{p+2} n)$ time algorithm to solve the p-median problem on trees [1]. For the 1-median problem on wheel networks, Hatzl [14] provided an algorithm with linear running time. In addition, he showed that the 2-median problem on the cactus networks can be calculated at $O(n^2)$ time. Chang et al. [7] proved that the connected p-median problem on block graphs was $\mathcal{NP}$-hard and for the case that the lengths of the edges were equal to one, he proposed an $O(n)$ time algorithm. In 2017, Nguyen et al. [30]presented a simple algorithm with linear running time to find the 1-median location on a cactus network. In the real life, we are usually faced with various types of non-determinacy. For example, in location problems, we are usually not sure of the vertex weights and the distances between vertices of a network. For dealing with non-determinacy phenomena, probability theory was introduced by Kolmogorov [17] in 1933 for modeling frequencies, while in 2007, uncertainty theory was presented by Liu [18] for modeling belief degrees.

The early works mainly focus on handling randomness, i.e., regarding distances between vertices and vertex weights as random variables. Berman and Krass [5] investigated the p-median problem with unreliable facilities and complete information on a line. They presented an approach for solving the problem that was based on representing the stochastic problem as a linear combination of deterministic median problems for which analytical results are available. Berman and Wang [3] studied the problem of locating p facilities to serve clients residing at the vertices of a network with discrete probabilistic demand weights. Tadei et al. [37] tried to find the location of p facilities when the cost for using a facility is a stochastic variable with unknown probability distribution. Berman and Drezner [4] investigated the p-median problem with stochastic objective value. Their aim was to find the location of p facilities such that the expected value of the objective function in the future is minimized. When the stochastic network is a tree, Mirchandani and Oudjit [27] used a selective enumeration approach for solving the 2-median problem. Various results on the p-median problem on stochastic networks are given in [2, 9, 23, 24, 25, 26].

Stochastic network optimization models work well when there are enough data to estimate the probability distributions of vertex weights and distances between the vertices. When we do not have enough samples to estimate the probability distributions of the vertex weights and the distances between vertices, we have to invite experts to give the belief degrees about the vertex weights and distances between vertices. Some researchers applied the uncertainty theory to deal with the location problems: for example the uncertain models for single facility location problems were investigated by Yuan Gao [12], the hierarchical facility location for the reverse logistics network design under uncertainty was studied by Wang and Yang [38], the capacitated facility location-allocation problem under uncertain environment was investigated by Wen et al. [39], the inverse 1-median problem on a tree under uncertain cost coefficients was solved by Nguyen and Chi [29] and the classical p-center location problem on a network with the uncertain vertex weights and the uncertain distances was studied by Soltanpour et al. [35].

In many cases, uncertainty and randomness simultaneously appear in a complex system. Specifically, for some non-deterministic phenomena, we have enough observational data to obtain their probability distribution functions, while for others, we can only estimate them by expert data. In order to describe this complex system, in 2013 the chance theory was developed by Liu [21] with the concepts of uncertain random variable, chance measure and chance distribution. Liu [21] also introduced the concepts of expected value and variance of uncertain random variables. For calculating the variance of uncertain random variables, Guo and Wang [11] presented a formula based on uncertainty distribution. Sheng and Yao [34] verified a formula to calculate the variance using chance distribution and inverse chance distribution. As an important contribution to chance theory, Liu [22] presented an operational law of uncertain random variables. In addition, Hou [15] investigated the distance between uncertain random variables, and Yao and Gao [40] proved a law of large numbers for the uncertain random variables. In order to model the optimization problems with uncertainty and randomness, uncertain random programming was introduced by Liu [22] in 2013. For a survey on the uncertain random optimization problems, the reader is referred to [31, 32, 33].

In this paper, we will investigate the p-median location problem on a network with uncertain random vertex weights and distances. In the p-median problem too, there are some circumstances by which there are enough data for some of vertex weights and distances to estimate their probability distributions. On the other hand, there are no samples to estimate the probability distributions for some other vertex weights and distances so that we have to invite experts to give the belief degrees about them. Therefore, data are divided into categories; some of them have probability distributions and the others have uncertainty distributions.

The article is organized as follows: In the next section some basic concepts and properties of the uncertainty theory and chance theory will be introduced. In Section 3., we will introduce an uncertain random network and give an ideal chance distribution function of the p-median. Then, we propose an algorithm to calculate the ideal chance distribution function of the problem under investigation on uncertain random networks. Section 4. presents a model for finding the p-median and proposes an algorithm to seek the p-median of an uncertain random network. A numerical example is presented in Section 5. to illustrate the efficiency of our proposed method. Finally, Section 6. gives a brief summary to the paper.

2. Preliminary concepts and definitions

In this section, we will introduce some concepts and theorems of the uncertainty theory and chance theory.

2.1. Uncertainty theory

The uncertainty theory, introduced by Liu [18], provides a new approach to deal with non-determinacy factors. Nowadays, the uncertainty theory has become a branch of mathematics for modeling human uncertainty based on normality, duality, subadditivity, and product axioms.

In the following part, we will introduce some foundational concepts and properties of the uncertainty theory, which will be used throughout this paper [18, 19, 20].

Definition 2.1. Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in L$ (Λ^c is compliment of Λ).
Axiom 3: (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Moreover, in order to provide an operational law, Liu defined the product uncertain measure on the product σ-algebra L as follows.

Axiom 4: (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}$$

where Λ_k are arbitrary chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.2. An uncertain variable is a measurable function ξ from an uncertainty space to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined as follows.

Definition 2.3. The uncertainty distribution function ϕ of an uncertain variable ξ is defined by

$$\phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 2.4. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \{\xi_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition 2.5. Let ξ be an uncertain variable with regular uncertainty distribution function ϕ. Then the inverse function ϕ^{-1} is called the inverse uncertainty distribution function of ξ.

The distribution of a monotonous function of uncertain variables can be obtained by the following theorem.

Theorem 2.1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be the independent uncertain variables with regular uncertainty distribution functions $\phi_1, \phi_2, \ldots, \phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\nu = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution function*

$$\psi^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \ldots, \phi_n^{-1}(\alpha)).$$

2.2. Chance theory

In this subsection, we will introduce some foundational definitions and properties of the uncertain random variable, the chance measure, the chance distribution and the operational law, [21, 22].

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, Pr)$ be a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ is called a chance space.

Definition 2.6. Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of Θ is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\} dr.$$

Liu [21] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

(1) $Ch\{\Gamma \times \Omega\} = 1$.

(2) $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$ for any event Θ (Θ^c is compliment of Θ).

(3) $Ch\{\Theta_1\} \leq Ch\{\Theta_2\}$ for any real number set $\Theta_1 \subset \Theta_2$.

Moreover, Hou [15] proved the subadditivity of chance measure, that is,

$$Ch\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} Ch\{\Theta_i\}$$

for a sequence of events $\Theta_1, \Theta_2, \ldots$.

Theoretically, an uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables.

Definition 2.7. An uncertain random variable is a measurable function θ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ to the set of real numbers, i.e., $\{\theta \in B\}$ is an event for any Borel set B.

Note that random variables and uncertain variables can be regarded as special cases of uncertain random variables.

Definition 2.8. The chance distribution function of an uncertain random variable θ is defined by

$$\Phi(x) = Ch\{\theta \leq x\}$$

for any real number x.

The chance distribution function of a random variable is just its probability distribution function, and the chance distribution function of an uncertain variable is just its uncertainty distribution function.

Theorem 2.2. *Let $\eta_1, \eta_2, \ldots, \eta_m$ be the independent random variables with probability distribution functions $\Psi_1, \Psi_2, \ldots, \Psi_m$ respectively, and let $\tau_1, \tau_2, \ldots, \tau_n$ be the uncertain variables. Then the uncertain random variable*

$$\theta = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$$

has a chance distribution function

$$\Phi(x) = \int_{\mathcal{R}^m} F(x, y_1, y_2, \ldots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \ldots d\Psi_m(y_m)$$

where $F(x, y_1, y_2, \ldots, y_m)$ is the uncertainty distribution function of uncertain variable $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$ for any real numbers $y_1, y_2, \ldots, y_m$.

3. The ideal chance distribution function of the p-median

In this section, we will introduce the uncertain random network and the ideal chance distribution function of p-median. Then, we propose an algorithm for calculating the ideal chance distribution function. First, some assumptions are listed as follows.

(1) The undirected uncertain random network is connected.

(2) The weight of each vertex and the distances (shortest path length) between vertices are finite.

(3) The weight of each vertex and the distances between vertices are positive uncertain variables or positive random variables.

(4) All the uncertain variables and the random variables are independent.

Definition 3.1. The quartette $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is said to be an uncertain random network if $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and $\mathcal{U}, \mathcal{R}, \mathcal{W}$ are defined as follows.

$$\mathcal{U} = \{(v_i, v_j) |\text{ the shortest distance between vertices } v_i \textit{ and } v_j \text{ is uncertain}\}$$

$$\bigcup \{v_k |\text{ the weight of the vertex } v_k \text{ is uncertain}\},$$

$$\mathcal{R} \quad = \{(v_i, v_j)|\ \text{the shortest distance between vertices } v_i \textit{ and } v_j \text{ is random}\}$$

$$\bigcup\{v_k|\ \text{the weight of thevertex } v_k \text{ is random}\},$$

and $\mathcal{W}$ is the collection of uncertain and random vertex weights and uncertain and random vertex distances.

In this paper, all deterministic distances and weights are regarded as special uncertain distances and weights.

Let η_k and ξ_{ij} denote the weight of vertex $v_k, v_k \in \mathcal{U} \cup \mathcal{R}$, and the distance between two vertices v_i and v_j, $(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then η_k and ξ_{ij} are uncertain variables if $v_k, (v_i, v_j) \in \mathcal{U}$, and η_k and ξ_{ij} are random variables if $v_k, (v_i, v_j) \in \mathcal{R}$. Without loss of generality, we assume that the uncertain weight η_k and the distance ξ_{ij} for $v_k, (v_i, v_j) \in \mathcal{U}$ are defined on uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively, and also the random weight η_k and the distance ξ_{ij} for $v_k, (v_i, v_j) \in \mathcal{R}$ are defined on probability spaces $(\Omega_1, \mathcal{A}_1, Pr_1)$ and $(\Omega_2, \mathcal{A}_2, Pr_2)$, respectively. Since the weights and the distances are assumed to be finite, we have $a_k \leq \eta_k \leq b_k$, $a_{ij} \leq \xi_{ij} \leq b_{ij}$, where a_k, a_{ij} and b_k, b_{ij} are the lower bounds and the upper bounds, respectively.

Define $\eta = \{\eta_k|\ v_k \in \mathcal{U} \cup \mathcal{R}\}$ and $\xi = \{\xi_{ij}|\ (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$. We can denote the network with the uncertain random weights and distances as $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. The optimal value of the p-median problem is a function of weights and distances which is denoted as f in this paper. Obviously, $f(\eta, \xi)$ is an uncertain random variable. For an uncertain random network, the optimal value of the p-median problem, $f(\eta, \xi)$, is an increasing function with respect to each component of η_k and ξ_{ij}. The chance distribution function of $f(\eta, \xi)$ is called an ideal chance distribution function associated with uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Note that the ideal chance distribution function is unique for a given uncertain random network. The following theorem explains how to calculate an ideal chance distribution function.

Theorem 3.1. *Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network. Assume that the uncertain weights η_k and the uncertain distances ξ_{ij} have regular uncertainty distribution functions Υ_k and Υ_{ij} for $v_k \in \mathcal{U}$ and $(v_i, v_j) \in \mathcal{U}$ and also the random weights η_k and the random distances ξ_{ij} have probability distribution functions Ψ_k and Ψ_{ij} for $v_k \in \mathcal{R}$ and $(v_i, v_j) \in \mathcal{R}$, respectively. Then the ideal chance distribution function associated with the uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ is*

$$\Phi(z) = \int_0^\infty \cdots \int_0^\infty F(z; y_k, y_{ij}|\ v_k,\ (v_i, v_j) \in \mathcal{R}) \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i,v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij}) \tag{3.1}$$

where $F(z; y_k, y_{ij}|\ v_k,\ (v_i, v_j) \in \mathcal{R})$ is the uncertainty distribution function of the uncertain variable $f(y_k, y_{ij}|\ v_k, (v_i, v_j) \in \mathcal{R};\ \eta_k, \xi_{ij}|\ v_k, (v_i, v_j) \in \mathcal{U})$, and it is determined by its inverse uncertainty distribution function $F^{-1}(\alpha; y_k, y_{ij}|v_k, (v_i, v_j) \in \mathcal{R})$ which is equal to

$$f(y_k, y_{ij}|\ v_k, (v_i, v_j) \in \mathcal{R};\ \Upsilon_k^{-1}(\alpha), \Upsilon_{ij}^{-1}(\alpha)|\ v_k, (v_i, v_j) \in \mathcal{U}),$$

and f can be calculated by using $(\mathbf{P}_2)$.

Proof. Let $f(\eta_k, \xi_{ij}|\, v_k, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R})$ be the sum of the weighted distance of the p-median. By Definitions 2.6 and 2.8 and also Theorem 2.2, we can obtain the ideal chance distribution function as follows.

$$\begin{aligned}
\Phi(z) = &\quad Ch\{f(\eta_k, \xi_{ij}|\, v_k, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}) \leq z\} \\
= &\quad \int_0^1 Pr\{\omega_1 \in \Omega_1, \omega_2 \in \Omega_2|\, \mathcal{M}\{f(\eta_k(\omega_1), \xi_{ij}(\omega_2)|v_k, (v_i, v_j) \in \mathcal{R}, \\
&\quad \eta_k, \xi_{ij}|v_k, (v_i, v_j) \in \mathcal{U}) \leq z\} \geq r\}dr \\
= &\quad \int_0^\infty \ldots \int_0^\infty \mathcal{M}\{f(\eta_k(\omega_1), \xi_{ij}(\omega_2)|\, v_k, (v_i, v_j) \in \mathcal{R}, \eta_k, \xi_{ij}|v_k, (v_i, v_j) \in \mathcal{U}) \leq z\} \\
&\quad \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i,v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij}) \\
= &\quad \int_0^\infty \ldots \int_0^\infty F(z; y_k, y_{ij}|\, v_k,\ (v_i, v_j) \in \mathcal{R}) \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i,v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij})
\end{aligned}$$

where $F(z; y_k, y_{ij}|\, v_k,\ (v_i, v_j) \in \mathcal{R})$ is the uncertainty distribution function of uncertain variable $f(y_k, y_{ij}|\, v_k, (v_i, v_j) \in \mathcal{R};\ \eta_k, \xi_{ij}|\, v_k, (v_i, v_j) \in \mathcal{U})$ for any real numbers $y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R}$, and it is determined by its inverse uncertainty distribution $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$. By Theorem 2.1, for given $\alpha \in (0, 1)$, we have

$$\begin{aligned}
&F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R}) \\
&= f(y_k, y_{ij}|\, v_k, (v_i, v_j) \in \mathcal{R}; \Upsilon_k^{-1}(\alpha), \Upsilon_{ij}^{-1}(\alpha)|\, v_k, (v_i, v_j) \in \mathcal{U}),
\end{aligned}$$

which is just the optimal value of the p-median problem of a determinacy network and f is a strictly increasing function with respect to η_k and ξ_{ij}, where η_k and ξ_{ij} denote the weight of the vertex v_i and the shortest distance between two vertices v_i and v_j, respectively. We can calculate f by using $(\mathbf{P}_2)$. Thus the theorem is proved. $\square$

Note that, it is difficult to calculate the ideal chance distribution function by using formula (3.1). Hence, in order to calculate the ideal chance distribution function of the p-median in an uncertain random network, we propose the following algorithm:

Algorithm 1

1. For any y_k and y_{ij}, $v_k, (v_i, v_j) \in \mathcal{R}$, give partitions $\prod_k$ and $\prod_{ij}$ of intervals $[a_k, b_k]$ and $[a_{ij}, b_{ij}]$ with step $\Delta = 0.01$. Let random variables η_k and ξ_{ij} take values in $\{y_k|\, y_k = a_k + 0.01 * i \text{ for } i = 1, 2, \ldots, (b_k - a_k) * 100\}$ and $\{y_{ij}|\, y_{ij} = a_{ij} + 0.01 * i \text{ for } i = 1, 2, \ldots, (b_{ij} - a_{ij}) * 100\}$, respectively.
2. Calculate $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$ by using $(\mathbf{P}_2)$ for given y_k and y_{ij} and each $\alpha \in \{0.01, 0.02, \ldots, 0.99\}$.
3. Obtain the uncertainty distribution function of $F(z; y_k, y_{ij}|\, v_k, (v_i, v_j) \in \mathcal{R})$, from its discrete form via linear interpolation.
4. Input $F(z; y_k, y_{ij}|\, v_k, (v_i, v_j) \in \mathcal{R})$ into formula (3.1) to calculate the chance distribution function $\Phi(z)$ for each z.

4. The p-median model in uncertain random networks

In this section, we will consider the p-median problem in an uncertain random network and present an algorithm for finding the p-median.

Given a connected and undirected uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Let $\mathcal{V}_p \subseteq \mathcal{V}, |\mathcal{V}_p| = p$, and let the variable x_j be equal to 1 if $v_j \in \mathcal{V}_p$, and 0 otherwise. Also let x_{ij} be the variable that is equal to 1 if the nearest vertex to the vertex v_i in $\mathcal{V}_p$ is the vertex v_j, and 0 otherwise. A p-facility location is represented by $\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$, where

$$\begin{array}{ll} \sum_{(v_i,v_j)\in\mathcal{U}\cup\mathcal{R}} x_{ij} = 1 & \forall v_i \in \mathcal{U} \cup \mathcal{R}, \\ x_{ij} \le x_j & \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}, \\ \sum_{v_j\in\mathcal{U}\cup\mathcal{R}} x_j = p, & \\ x_{ij}, x_j \in \{0, 1\} & \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}. \end{array}$$

Therefore the sum of weighted distances of a p-facility location $\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is

$$\sum_{v_i\in\mathcal{U}\cup\mathcal{R}} \sum_{(v_i,v_j)\in\mathcal{U}\cup\mathcal{R}} \eta_i \xi_{ij} x_{ij}$$

which is obviously an uncertain random variable. Its chance distribution function is denoted by $\Psi(z)$, i.e.,

$$\Psi(z) = Ch \left\{ \sum_{v_i\in\mathcal{U}\cup\mathcal{R}} \sum_{(v_i,v_j)\in\mathcal{U}\cup\mathcal{R}} \eta_i \xi_{ij} x_{ij} \le z \right\}.$$

Based on Theorem 2.2, we suggest the following algorithm to calculate the chance distribution function which corresponds to the p-facility location

$$\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}.$$

Algorithm 2

1. Step 1 as described in Algorithm 1.
2. For given y_k and y_{ij} and each $\alpha \in \{0.01, 0.02, \ldots, 0.99\}$, calculate $F^{-1}(\alpha; y_k, y_{ij}, v_k, (v_i, v_j) \in \mathcal{R})$, which is the sum of the weighted distances of the corresponding p-facility location.
3. Use discrete form of linear interpolation to obtain the uncertainty distribution function of $F(z; y_k, y_{ij} | \, v_k, (v_i, v_j) \in \mathcal{R})$.

4. Input $F(z; y_k, y_{ij} | \, v_k, (v_i, v_j) \in \mathcal{R})$ into formula

   $$\int_0^\infty \cdots \int_0^\infty F(z; y_k, y_{ij} | \, v_k, \, (v_i, v_j) \in \mathcal{R}) \prod_{v_k \in \mathcal{R}} d\Psi_k(y_k) \prod_{(v_i, v_j) \in \mathcal{R}} d\Psi_{ij}(y_{ij})$$

   and calculate the chance distribution function $\Psi(z)$ for each z.

Theorem 4.1. *Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network and let $\Phi(z)$ be an ideal chance distribution function associated with it. Assume that $\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is a given p-facility location. Then we have*

$$\Psi(z) \le \Phi(z).$$

Proof. Obviously, by Theorem 3.1, the ideal chance distribution function of the p-median $\Phi(z)$ is the smallest sum distribution. So we can obtain the chance distribution function of any p-facility location

$$\Psi(z) \le \Phi(z).$$

The theorem is proved. □

In order to find the p-median location of an uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$, we give the following definition.

Definition 4.1. Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ be an uncertain random network and let $\Phi(z)$ be the ideal chance distribution function of the p-median. Assume that $\Psi(z)$ is the chance distribution function of a p-facility location $\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$. If $\Psi(z)$ is the closest the $\Phi(z)$, i.e.,

$$\int_0^\infty \{\Phi(z) - \Psi(z)\} dz$$

is minimum, then the p-facility location $\{x_j, x_{ij} | \, v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}\}$ is called the p-median in an uncertain random network.

Based on Definition 4.1 and the above statements, we formulate the following optimization model to determine a p-median for an uncertain random network.

$$
\begin{array}{lll}
(\mathbf{P}_3): & \min & \int_0^\infty \{\Phi(z) - \Psi(z)\} dz \\
& \text{s.t.} & \sum_{(v_i, v_j) \in \mathcal{U} \cup \mathcal{R}} x_{ij} = 1 \quad \forall v_i \in \mathcal{U} \cup \mathcal{R}, \\
& & x_{ij} \le x_j \qquad\qquad\qquad\quad \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R}, \\
& & \sum_{v_j \in \mathcal{U} \cup \mathcal{R}} x_j = p, \\
& & x_{ij}, x_j \in \{0, 1\} \qquad\quad\;\; \forall v_j, (v_i, v_j) \in \mathcal{U} \cup \mathcal{R},
\end{array}
$$

where $\Phi(z)$ is the ideal chance distribution function and

$$\Psi(z) = Ch\left\{\sum_{v_i \in \mathcal{U}\cup\mathcal{R}} \sum_{(v_i,v_j)\in\mathcal{U}\cup\mathcal{R}} \eta_i \xi_{ij} x_{ij} \le z\right\}$$

is the chance distribution function of any p-facility location.

In order to find the p-median in an uncertain random network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$, we propose the following algorithm.

Algorithm 3

1. Calculate the ideal chance distribution function for the uncertain random network by using Algorithm 1.
2. Consider all the p-facility locations in the uncertain random network.
3. Calculate the chance distribution function of the sum of weighted distances for each p-facility location by using Algorithm 2.
4. Calculate the objective function of Model $(\mathbf{P}_3)$ for given p-facility location, and choose the minimum value of objective function, which corresponds to the desired p-median.

5. An illustrative example

In this section, we give an example for the 2-median location problem in an uncertain random network to illustrate the proposed algorithms.

Consider goods distribution system. Assume that the system is given as the network N in Figure 5.1 where the vertices denote urban areas. In this system, warehouses of the distribution company and supermarkets are facilities and clients, respectively. There is a supermarket at each area. Suppose that the weight η_i of the vertex v_i is equal to the average monthly purchase of residents of this area from the supermarket located at vertex v_i. Some of the vertex weights and the distances between vertices in the network are uncertain, while others are random (see Table 5.1). Our aim is to find two vertices on the network N to locate warehouses of the distribution company which will minimize the sum of weighted distances from each supermarket to its closest warehouse.

$$\begin{aligned}
&\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\},\\
&\mathcal{U} = \{v_1, v_2, v_4, v_5, v_6\} \cup \{(v_1, v_2), (v_1, v_4), (v_2, v_5), (v_3, v_4)(v_4, v_6)\},\\
&\mathcal{R} = \{v_3\} \cup \{(v_2, v_3)\},\\
&\mathcal{W} = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\} \cup \{\xi_{12}, \xi_{14}, \xi_{23}, \xi_{25}, \xi_{34}, \xi_{46}\}.
\end{aligned}$$

Note that if the distance between some vertices or weight of some vertices are considered as constant values, then we regard them as special uncertain variables.

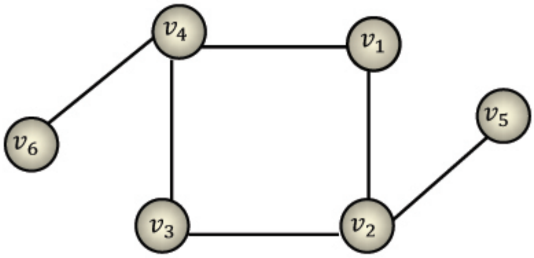

FIG. 5.1: An uncertain random network

Table 5.1: Uncertainty and probability distributions

vertex	η_i	arc	ξ_{ij}
v_1	2	(v_1, v_2)	$\mathcal{L}(2, 3)$
v_2	$\mathcal{L}(2, 4)$	(v_1, v_4)	$\mathcal{L}(3, 4)$
v_3	$U(2, 3)$	(v_2, v_3)	$U(4, 6)$
v_4	3	(v_2, v_5)	2
v_5	$\mathcal{L}(1, 2)$	(v_3, v_4)	$\mathcal{L}(2, 3)$
v_6	$\mathcal{L}(2, 3)$	(v_4, v_6)	5

From Theorem 3.1, we can obtain the ideal chance distribution function associated with the network $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ as follows:

$$\Phi(z) = \int_0^{\infty} \int_0^{\infty} F(z; y_3, y_{23}) d\Psi_3(y_3) d\Psi_{23}(y_{23})$$

where $F(z; y_3, y_{23})$ is determined by its inverse uncertainty distribution function

$$F^{-1}(\alpha; y_3, y_{23}) = f(\Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), y_3, \Upsilon_4^{-1}(\alpha), \Upsilon_5^{-1}(\alpha), \Upsilon_6^{-1}(\alpha); \Upsilon_{12}^{-1}(\alpha), \Upsilon_{14}^{-1}(\alpha), y_{23},$$
$$\Upsilon_{25}^{-1}(\alpha), \Upsilon_{34}^{-1}(\alpha), \Upsilon_{46}^{-1}(\alpha)).$$

Give a partition $\prod_3$ on interval [2, 3] with step 0.01, and let random variable η_3 take values in

$$\{y_3 \mid y_3 = 2 + 0.01i \text{ for } i = 1, \ldots, 100\}.$$

Also give a partition $\prod_{23}$ on interval [4, 6] with step 0.01, and let random variable η_{23} take values in

$$\{y_{23} \mid y_{23} = 4 + 0.01i \text{ for } i = 1, \ldots, 200\}.$$

For any $y_3 \in \prod_3$, $y_{23} \in \prod_{23}$, given $\alpha \in \{0.01, 0.02, \ldots, 0.99\}$, we calculate $F^{-1}(\alpha; y_3, y_{23})$ by using the model $(\mathbf{P}_2)$.

We may first calculate the ideal chance distribution function $\Phi(z)$ associated with the network by Algorithm 1.

Then we will calculate the chance distribution function of the total weighted distance of each 2-facility location(i.e., $\Psi(z)$) by Algorithm 2, which is shown in Figure 5.2.

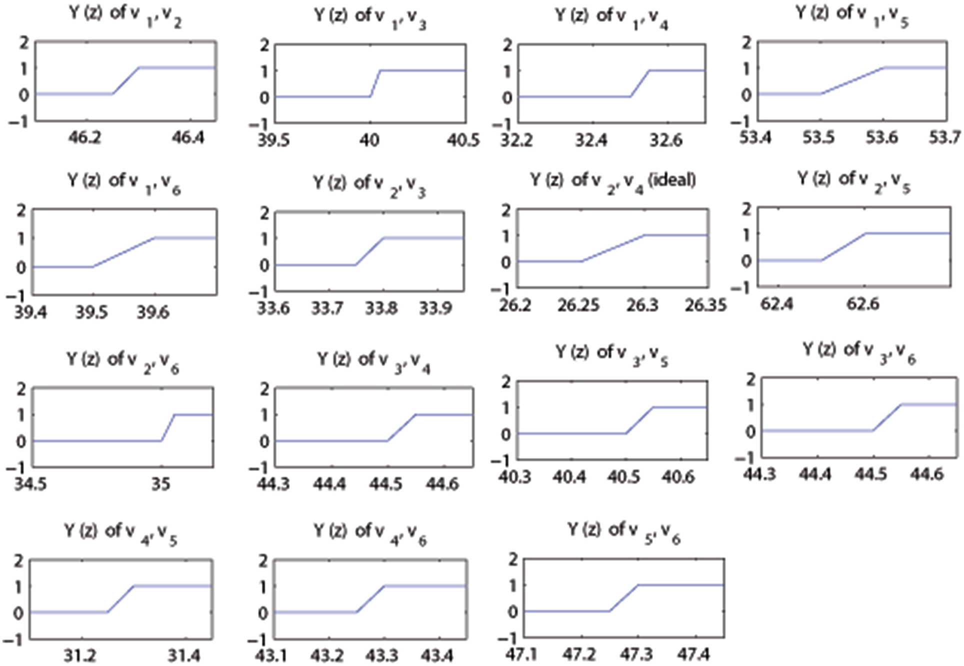

FIG. 5.2: The shapes of chance distribution functions of all 2-facilities for the example

Using Algorithm 3, we will calculate the difference between the chance distribution function of total weighted distance of each 2-facility location and the ideal one, which is given as Table 5.2.

Model $(\mathbf{P}_3)$ implies that the vertices v_2 and v_4 are the desired warehouses locations since the difference between the chance distribution function and the ideal one is minimum.

Table 5.2: Difference between the chance distribution function and the ideal chance distribution function

2-facility location	$\{v_1, v_2\}$	$\{v_1, v_3\}$	$\{v_1, v_4\}$	$\{v_1, v_5\}$	$\{v_1, v_6\}$
Difference	20.015	13.759	6.259	27.234	13.234
2-facility location	$\{v_2, v_3\}$	$\{v_2, v_4\}$	$\{v_2, v_5\}$	$\{v_2, v_6\}$	$\{v_3, v_4\}$
Difference	7.511	0	36.234	8.759	18.261
2-facility location	$\{v_3, v_5\}$	$\{v_3, v_6\}$	$\{v_4, v_5\}$	$\{v_4, v_6\}$	$\{v_5, v_6\}$
Difference	14.261	18.261	5.009	17.009	21.006

6. Conclusions

In this paper, we have investigated the p-median location problem in an uncertain random network, i.e., a network in which the weights of vertices and the distances between vertices are uncertain random variables. We first introduced the concept of the ideal chance distribution function and then presented an algorithm to calculate the ideal chance distribution function of the p-median associated with the uncertain random network. We have formulated the discrete p-median location problem in an uncertain random network and presented an algorithm to find the p-median. Finally, to illustrate the efficiency of the proposed method, we gave a numerical example.

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SOME PROPERTIES OF BIVATIATE FIBONACCI AND LUCAS QUATERNION POLYNOMIALS

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Abstract. In this work, we introduce bivariate Fibonacci quaternion polynomials and bivariate Lucas quaternion polynomials. We present generating function, Binet formula, matrix representation, binomial formulas and some basic identities for the bivariate Fibonacci and Lucas quaternion polynomial sequences. Moreover we give various kinds of sums for these quaternion polynomials.
Keywords: Bivariate Fibonacci quaternion polynomials, Bivariate Lucas quaternion polynomials, Generating function, Binet formula.

1. Introduction

In mathematics, Fibonacci and Lucas or other special numbers are investigation topic of great interest. Classical Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ is defined by a recurrence identity;

$$F_n = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ F_{n-1} + F_{n-2} & \text{if} \quad n \geq 2. \end{cases}$$

The Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$ is defined by some recurrence identity with different starting values;

$$L_n = \begin{cases} 2 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ L_{n-1} + L_{n-2} & \text{if} \quad n \geq 2. \end{cases}$$

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Let $p(x)$ and $q(x)$ be polynomials with real coefficients of the $(p,q)-$Fibonacci polynomials are defined by the recurrence relation

$$F_{p,q,n+1} = p(x)F_{p,q,n} + q(x)F_{p,q,n-1}$$

with the initial conditions $F_{p,q,0} = 0$, $F_{p,q,1} = 1$. Also for the $p(x)$ and $q(x)$ polynomials with real coefficients the $(p,q)-$Lucas polynomials are defined by the recurrence relation

$$L_{p,q,n+1} = p(x)L_{p,q,n} + q(x)L_{p,q,n-1}$$

with initial conditions $L_{p,q,0} = 2$, $L_{p,q,1} = p(x)$.

Definition 1.1. [1] For $n \geq 2$, bivariate Fibonacci polynomials are defined as recurrence relation

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y). \tag{1.1}$$

We can compute the first few bivariate Fibonacci polynomials as follow $F_0(x,y) = 0$, $F_1(x,y) = 1$, $F_2(x,y) = x$, $F_3(x,y) = x^2 + y$, $F_4(x,y) = x^3 + 2xy$. Characteristic equation of relation (1.1) is

$$h^2 - xh - y = 0 \tag{1.2}$$

and so the roots of (1.2) are $\alpha = \alpha(x,y) = \frac{x+\sqrt{x^2+4y}}{2}$ and $\beta = \beta(x,y) = \frac{x-\sqrt{x^2+4y}}{2}$. Also it has Binet's formula $F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$.

Definition 1.2. [1] For $n \geq 2$, bivariate Lucas polynomials are defined as recurrence relation

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y)$$

with initial conditionals $L_0(x,y) = 2$ and $L_1(x,y) = 1$.

Likely, let compute the first few terms of Lucas polynomials $L_0(x,y) = 2$, $L_1(x,y) = 1$, $L_2(x,y) = x + 2y$, $L_3(x,y) = x^2 + 2xy + y$, $L_4(x,y) = x^3 + 2x^2y + 2xy + 2y^2$. Also it has Binet's formula $L_n(x,y) = \alpha^n + \beta^n$ for $n \geq 0$.

Some authors considered special sequence polynomials for example generalized Fibonacci and Lucas polynomials in [7] and also bivariate Fibonacci and Lucas like polynomials in [6].

Normed division algebra, nowadays which is so important topic consists of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$ and octonions $\mathbf{O}$. Prime facie, directly we can not extend sundry results on real and complex numbers to quaternions due to quaternions are noncommutative normed division algebra over the real numbers, even it looks like things are going to be done with quaternions **H** [3]. For $a_0, a_{1,} a_{2,} a_3 \in \mathbb{R}$, a quaternion is defined by

$$e = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

where $e_0 = 1, e_{1,} e_{2,}$ and e_3 are unit vectors which verifies the following rules

$$(e_1)^2 = (e_2)^2 = (e_3)^2 = e_1 e_2 e_3 = -1. \tag{1.3}$$

From equation (1.3), we get

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_1 e_3 = -e_3 e_1 = e_2.$$

Some new quaternion and octonion polynomials are studied in [2, 4, 5, 8, 9].

2. Bivariate Fibonacci and Lucas quaternion polynomials

Now, we define new quaternion polynomials which are called bivariate Fibonacci quaternion polynomials (QBF) and bivariate Lucas quaternion polynomials (QBL).

Definition 2.1. Bivariate Fibonacci quaternion polynomials (QBF) are defined as the recurrence relation

$$\begin{aligned} QBF_n(x,y) &= \sum_{k=0}^{3} F_{n+k}(x,y) e_k \\ &= F_n(x,y)e_0 + F_{n+1}(x,y)e_1 + F_{n+2}(x,y)e_2 + F_{n+3}(x,y)e_3 \end{aligned} \tag{2.1}$$

where $F_{n+k}(x,y)$ is the $n-th$ bivariate Fibonacci polynomial with the initial conditions $QBF_0(x,y) = e_1 + xe_2 + (x^2+y)e_3$ and $QBF_1(x,y) = e_0 + xe_1 + (x^2 + y)e_2 + (x^3 + 2xy)e_3$.

Furthermore,

$$\begin{aligned} QBF_{n+1}(x,y) &= \sum_{k=0}^{3} F_{n+1+k}(x,y) e_k \\ &= x \sum_{k=0}^{3} F_{n+k}(x,y) e_k + y \sum_{k=0}^{3} F_{n+k-1}(x,y) e_k. \end{aligned}$$

So we get recurrence relation as follow

$$QBF_{n+1}(x,y) = xQBF_n(x,y) + yQBF_{n-1}(x,y). \tag{2.2}$$

Similarly, bivariate Lucas quaternion polynomials QBL are defined as the recurrence relation

$$\begin{aligned} QBL_n(x,y) &= \sum_{k=0}^{3} L_{n+k}(x,y)e_k \\ &= L_n(x,y)e_0 + L_{n+1}(x,y)e_1 + L_{n+2}(x,y)e_2 + L_{n+3}(x,y)e_3 \end{aligned} \tag{2.3}$$

where $L_{n+k}(x,y)$ is the $n-th$ bivariate Lucas polynomial and with the initial conditions $QBL_0(x,y) = 2e_0 + e_1 + (x+2y)e_2 + (x^2+2xy+y)e_3$ and $QBL_1(x,y) = e_0 + (x+y)e_1 + (x^2+2xy+y)e_2 + (x^3+2x^2y+2xy+2y^2)e_3$. Moreover, recurrence relation is

$$QBL_{n+1}(x,y) = xQBL_n(x,y) + yQBL_{n-1}(x,y). \tag{2.4}$$

Let $\alpha(x,y) = \frac{x+\sqrt{x^2+4y}}{2}$ and $\beta(x,y) = \frac{x-\sqrt{x^2+4y}}{2}$ denote the roots of the characteristic equation such that $\sqrt{x^2+4y} = \Delta$,

$$t^2 - xt - yt = 0$$

on the recurrence relation of (2.2) and (2.4).

From now on, for convenience of representation, we adopt the following notation

$$\alpha(x,y) = \alpha,\ \beta(x,y) = \beta,\ \Delta = \sqrt{x^2+4y}.$$

Equations that can be obtained with these roots are as follow

$$\begin{aligned} \alpha+\beta &= x \\ \alpha-\beta &= \Delta \\ \alpha\beta &= -y \\ \frac{\alpha}{\beta} &= -\frac{\alpha^2}{y} \\ \frac{\beta}{\alpha} &= -\frac{\beta^2}{y}. \end{aligned} \tag{2.5}$$

We continue with the generating function results.

Theorem 2.1. *The generating functions for QBF and QBL polynomials are respectively*

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{QBF_0(x,y) + [QBF_1(x,y) - xQBF_0(x,y)]\,t}{1 - xt - yt^2}$$

and

$$\sum_{n=0}^{\infty} QBL_n(x,y)t^n = \frac{QBL_0(x,y) + [QBL_1(x,y) - xQBL_0(x,y)]\, t}{1 - xt - yt^2}.$$

Proof. To compute the generating function of QBF polynomials

$$\begin{aligned} & \sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = \quad & QBF_0(x,y) + QBF_1(x,y)t + QBF_2(x,y)t^2 + \cdots + QBF_n(x,y)t^n + \cdots \end{aligned}$$

then using the equations of $-xt\left(\sum_{n=0}^{\infty} QBF_n(x,y)t^n\right)$ and $-yt^2\left(\sum_{n=0}^{\infty} QBF_n(x,y)t^n\right)$

$$\begin{aligned} & \sum_{n=0}^{\infty} QBF_n(x,y)t^n + (-xt)\sum_{n=0}^{\infty} QBF_n(x,y)t^n + (-yt^2)\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = \quad & QBF_0(x,y) + [QBF_1(x,y) - xQBF_0(x,y)]\, t \\ & + [QBF_2(x,y) - xQBF_1(x,y) - yQBF_0(x,y)]\, t^2 \\ & + \cdots + [QBF_n(x,y) - xQBF_{n-1}(x,y) - yQBF_{n-2}(x,y)]\, t^n + \cdots \end{aligned}$$

Consequently,

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n(1 - xt - yt^2) = QBF_0(x,y) + (QBF_1(x,y) - xQBF_0(x,y))t$$

is valid. Similar proof can be done for QBL polynomials. □

Now we can give the following theorems.

Lemma 2.1. *If we rearrange the Theorem 2.1, we have the generating functions as follows*

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{\frac{QBF_1(x,y) - \beta QBF_0(x,y)}{1 - \alpha t} - \frac{QBF_1(x,y) - \alpha(x,y) QBF_0(x,y)}{1 - \beta t}}{\alpha - \beta}$$

and

$$\sum_{n=0}^{\infty} QBL_n(x,y)t^n = \frac{\frac{QBL_1(x,y) - \beta QBL_0(x,y)}{1 - \alpha t} - \frac{QBL_1(x,y) - \alpha(x,y) QBL_0(x,y)}{1 - \beta t}}{\alpha - \beta}.$$

Proof. If we use Theorem 2.1 and recurrence relation (2.2), then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\
= \quad & \left(\frac{QBF_0(x,y)+(QBF_1(x,y)-(\alpha+\beta)QBF_0(x,y))t}{(1-\alpha t)(1-\beta t)}\right) \\
& \times \left(\frac{\alpha-\beta}{\alpha-\beta}\right) \\
= \quad & \frac{\left\{\begin{array}{c} QBF_1(x,y)(1-\beta t)+\beta QBF_0(x,y)(-1+\beta t) \\ +QBF_1(x,y)(-1+\alpha t)+\alpha QBF_0(x,y)(1-\alpha t) \end{array}\right\}}{(1-\alpha t)(1-\beta t)(\alpha-\beta)} \\
= \quad & \frac{\frac{QBF_1(x,y)-\beta QBF_0(x,y)}{1-\alpha t}-\frac{QBF_1(x,y)-\alpha(x,y)QBF_0(x,y)}{1-\beta t}}{\alpha-\beta}.
\end{aligned}$$

Hence the proof is completed. The other QBL polynomials can be proved similarly. □

Lemma 2.2. *For $k \geq 0$, let bivariate Fibonacci and Lucas polynomials are $F_n(x,y)$ and $L_n(x,y)$. We have*

$$\begin{aligned}
\textbf{\textit{(i)}}\ F_{k+1}(x,y)-\alpha F_k(x,y) &= \beta^k \\
\textbf{\textit{(ii)}}\ F_{k+1}(x,y)-\beta F_k(x,y) &= \alpha^k \\
\textbf{\textit{(iii)}}\ \frac{\alpha L_k(x,y)-L_{k+1}(x,y)}{\alpha-\beta} &= \beta^k \\
\textbf{\textit{(iv)}}\ \frac{L_{k+1}(x,y)-\beta L_k(x,y)}{\alpha-\beta} &= \alpha^k.
\end{aligned}$$

Proof. (i) We can prove it by induction method. Let $k=1$, then $F_2(x,y)-\alpha F_1(x,y)=\beta$.

Now let us assume that the equation is $F_n(x,y)-\alpha F_{n-1}(x,y)=\beta^{n-1}$, for $k=n-1$. For $k=n$ it becomes,

$$\begin{aligned}
\beta^n &= \beta^{n-1}\beta \\
&= ((F_n(x,y)-\alpha F_{n-1}(x,y))\beta \\
&= \beta F_n(x,y)-\alpha\beta F_{n-1}(x,y) \\
&= (\alpha+\beta-\alpha)F_n(x,y)-\alpha F_n(x,y)-\alpha\beta F_n(x,y) \\
&= xF_n(x,y)+yF_n(x,y)-\alpha F_n(x,y) \\
&= F_{n-1}(x,y)-\alpha F_n(x,y).
\end{aligned}$$

so this completes the proof. (ii), (iii) and (iv) can be done similarly. □

Now we want to derive the Binet formulas for QBF and QBL polynomials. To get this we can give the following theorem.

Theorem 2.2. *The Binet formulas of QBF and QBL polynomials are given as*

$$\begin{aligned} QBF_n(x,y) &= \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \\ QBL_n(x,y) &= \alpha^*\alpha^n + \beta^*\beta^n \end{aligned}$$

for $n \geq 0$, where $\alpha^* = \sum\limits_{k=0}^{3} \alpha^k e_k$ and $\beta^* = \sum\limits_{k=0}^{3} \beta^k e_k$.

Proof. Recall that generating function is

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{QBF_0(x,y) + (QBF_1(x,y) - xQBF_0(x,y))t}{1-xt-yt^2}.$$

So using the Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = &\sum_{k=0}^{\infty}(F_{k+1} - \beta F_{k+1})e_k \sum_{n=0}^{\infty}\alpha^n t^n - \sum_{k=0}^{\infty}(F_{k+1} - \alpha F_{k+1})e_k \sum_{n=0}^{\infty}\beta^n t^n. \end{aligned}$$

So we get,

$$\sum_{n=0}^{\infty}\left(\frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}\right)t^n$$

this is valid. Binet formula for the other QBL polynomial can be done similarly. □

We derive generating functions for the $(mk+s)-th$ order of QBF and QBL polynomials.

Theorem 2.3. *For all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{\infty} QBF_{mk+s}(x,y)x^k = \frac{QBF_s(x,y) - (-y)^m QBF_{s-m}(x,y)x}{(-y)^m - L_m(x,y) + 1}$$

and

$$\sum_{k=0}^{\infty} QBL_{mk+s}(x,y)x^k = \frac{QBL_s(x,y) - (-y)^m QBL_{s-m}(x,y)x}{(-y)^m - L_m(x,y) + 1}.$$

Proof. Using Binet formula and equation (2.5), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} QBF_{mk+s}(x,y)x^k \\
&= \sum_{k=0}^{\infty} \frac{\alpha^*\alpha^{mk+s} - \beta^*\beta^{mk+s}}{\alpha-\beta} x^k \\
&= \frac{\alpha^*\alpha^s}{\alpha-\beta} \sum_{k=0}^{\infty} \alpha^{mk}x^k - \frac{\beta^*\beta^s}{\alpha-\beta} \sum_{k=0}^{\infty} \beta^{mk}x^k \\
&= \frac{\alpha^*\alpha^s}{\alpha-\beta} \left(\frac{1}{1-\alpha^m x}\right) - \frac{\beta^*\beta^s}{\alpha-\beta} \left(\frac{1}{1-\beta^m x}\right) \\
&= \frac{\frac{\alpha^*\alpha^s-\beta^*\beta^s}{\alpha-\beta} - (\alpha\beta)^m \left(\frac{\alpha^*\alpha^{s-m}-\beta^*\beta^{s-m}}{\alpha-\beta}\right) x}{1-(\alpha^m+\beta^m)x+(\alpha\beta)^m x^2}
\end{aligned}$$

this is valid. The other result can be done similarly. □

We formulate the sum of the first n terms of these sequences of QBF and QBL polynomials.

Theorem 2.4. *The sum of the first $n-$terms of the quaternion sequences $QBF_n(x,y)$ and $QBL_n(x,y)$ is given by*

$$\sum_{k=0}^{n} QBF_k(x,y) = \frac{\left\{\begin{array}{c} -yQBF_n(x,y) - QBF_{n+1}(x,y) \\ +QBF_0(x,y) - \frac{\alpha^*\beta-\beta^*\alpha}{\alpha-\beta} \end{array}\right\}}{(\alpha-1)(\beta-1)}$$

and

$$\sum_{k=0}^{n} QBL_k(x,y) = \frac{\left\{\begin{array}{c} -yQBL_n(x,y) - QBL_{n+1}(x,y) \\ +QBL_0(x,y) + \alpha^*\beta+\beta^*\alpha \end{array}\right\}}{(\alpha-1)(\beta-1)}.$$

Proof. Using Binet formula and equation (2.5), we get

$$\begin{aligned}
&\sum_{k=0}^{n} QBF_k(x,y) \\
&= \sum_{k=0}^{n} \frac{\alpha^*\alpha^k - \beta^*\beta^k}{\alpha-\beta} \\
&= \frac{1}{\alpha-\beta}\left\{\alpha^* \sum_{k=0}^{n} \alpha^k - \beta^* \sum_{k=0}^{n} \beta^k\right\}
\end{aligned}$$

$$= \frac{1}{\alpha-\beta}\left\{\alpha^*(\frac{\alpha^{n+1}-1}{\alpha-1}) - \beta^*(\frac{\beta^{n+1}-1}{\beta-1})\right\}$$

$$= \frac{1}{(\alpha-1)(\beta-1)}\left\{\begin{array}{c}\frac{\alpha\beta(\alpha^*\alpha^n-\beta^*\beta^n)}{\alpha-\beta} - \frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta} \\ +\frac{\alpha^*-\beta^*}{\alpha-\beta} - \frac{\alpha^*\beta-\alpha\beta^*}{\alpha-\beta}\end{array}\right\}.$$

The other case can be done similarly. □

We derive summation formulas for the $(mk+s)-th$ order of QBF and QBL polynomials.

Theorem 2.5. *For all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}$,we have*

$$\sum_{k=0}^{n} QBF_{mk+s}(x,y) = \frac{\left\{\begin{array}{c}(-y)^m(QBF_{mn+s}(x,y) - QBF_{s-m}(x,y)) \\ -QBF_{mn+m+s}(x,y) + QBF_s(x,y)\end{array}\right\}}{(-y)^m - F_m(x,y) + 1}$$

and

$$\sum_{k=0}^{n} QBL_{mk+s}(x,y) = \frac{\left\{\begin{array}{c}(-y)^m(QBL_{mn+s}(x,y) - QBL_{s-m}(x,y)) \\ -QBL_{mn+m+s}(x,y) + QBL_s(x,y)\end{array}\right\}}{(-y)^m - L_m(x,y) + 1}.$$

Proof. Using Binet formula, equation (2.5), we have

$$\begin{aligned}
&\sum_{k=0}^{n} QBF_{mk+s}(x,y) \\
= &\sum_{k=0}^{n} \frac{\alpha^*\alpha^{mk+s} - \beta^*\beta^{mk+s}}{\alpha-\beta} \\
= &\frac{\alpha^*\alpha^s}{\alpha-\beta}\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right) - \frac{\alpha^*\alpha^s}{\alpha-\beta}\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right) \\
= &\frac{\left\{\begin{array}{c}\alpha^*(\alpha^{mn+s}\alpha^m\beta^m - \alpha^{mn+m+s} - \alpha^s\beta^m + \alpha^s) \\ -\beta^*(\beta^{mn+s}\alpha^m\beta^m - \beta^{mn+m+s} - \alpha^m\beta^s + \beta^s)\end{array}\right\}}{(\alpha-\beta)(\alpha^m\beta^m - \alpha^m - \beta^m + 1)} \\
= &\frac{(\alpha\beta)^m(\alpha^*\alpha^{mn+s} - \beta^*\alpha^{mn+s}) - (\alpha^*\alpha^{mn+m+s} - \beta^*\beta^{mn+m+s})}{(\alpha-\beta)(\alpha^m\beta^m - \alpha^m - \beta^m + 1)} \\
&+\frac{-(\alpha\beta)^m(\alpha^*\alpha^{s-m} - \beta^*\alpha^{s-m}) - (\alpha^*\alpha^{mn+m+s} - \beta^*\beta^{mn+m+s})}{(\alpha-\beta)(\alpha^m\beta^m - \alpha^m - \beta^m + 1)}.
\end{aligned}$$

Other case can be done similarly. □

Now, some new results for binomial summation of these sequences are derived by using their Binet forms.

Theorem 2.6. *Let n be a non-negative integer. Then we have the following binomial sum formulas for odd and even terms,*

$$\begin{aligned}
\textbf{\textit{(i)}}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}QBF_k(x,y) &= QBF_{2n}(x,y)\\
\textbf{\textit{(ii)}}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}QBF_k(x,y) &= QBF_{2n+1}(x,y)\\
\textbf{\textit{(iii)}}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}QBL_k(x,y) &= QBL_{2n}(x,y)\\
\textbf{\textit{(iv)}}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}QBL_k(x,y) &= QBL_{2n+1}(x,y).
\end{aligned}$$

Proof. (i) Let $P=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}QBF_k(x,y)$. From Binet formula, we change the right-hand side of P into:

$$P=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^{k}\left(\frac{\alpha^*\alpha^k-\beta^*\beta^k}{\alpha-\beta}\right).$$

Elementary calculations implies that

$$P=\frac{\alpha^*(y+x\alpha)^n-\beta^*(y+x\beta)^n}{\alpha-\beta}.$$

From equation (2.5), we get

$$\frac{\alpha^*\alpha^{2n}-\beta^*\beta^{2n}}{\alpha-\beta} \quad = \quad QBF_{2n}(x,y).$$

The other cases (ii),(iii) and (iv) can be done similarly. □

Now we can also formulate the Catalan's identity, Cassini's identity and d'Ocagne's identity by using Binet formulas.

Theorem 2.7. *(Catalan's Identity) For n and k non-negative integer such that $k\le n$, we have*

$$\begin{aligned}
&QBF_{n+k}(x,y)QBF_{n-k}(x,y)-QBF_n^2(x,y)\\
=\;&(-y)^{n-k}F_{n-k}(x,y)\left(\frac{\alpha^*\beta^*\beta^k-\beta^*\alpha^*\alpha^k}{(\alpha-\beta)}\right)
\end{aligned}$$

and

$$\begin{aligned}
&QBL_{n+k}(x,y)QBL_{n-k}(x,y)-QBL_n^2(x,y)\\
=\;&(-y)^{n-k}F_{n-k}(x,y)\sqrt{\triangle}(\alpha^*\beta^*\beta^k-\beta^*\alpha^*\alpha^k).
\end{aligned}$$

Proof. Using Binet formula, we obtain

$$\begin{aligned}
&QBF_{n+k}(x,y)QBF_{n-k}(x,y) - QBF_n^2(x,y)\\
=&\left(\frac{\alpha^*\alpha^{n+k}-\beta^*\beta^{n+k}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n-k}-\beta^*\beta^{n-k}}{\alpha-\beta}\right)-\left(\frac{\alpha^*\alpha^{n}-\beta^*\beta^{n}}{\alpha-\beta}\right)\\
=&\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left(\alpha^*\beta^*\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)+\beta^*\alpha^*\left(\frac{\beta^k-\alpha^k}{\alpha-\beta}\right)\right)\\
=&(\alpha\beta)^{n-k}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^*\beta^*\beta^k-\beta^*\alpha^*\alpha^k}{\alpha-\beta}\right).
\end{aligned}$$

□

Theorem 2.8. *For any natural number n, Cassini's identities for QBF and QBL polynomials are*

$$QBF_{n+1}(x,y)QBF_{n-1}(x,y) - QBF_n^2(x,y) = (-y)^{n-1}\left(\frac{\alpha^*\beta^*\beta-\beta^*\alpha^*\alpha}{\alpha-\beta}\right)$$

and

$$QBL_{n+1}(x,y)QBL_{n-1}(x,y) - QBL_n^2(x,y) = (-y)^{n-1}\sqrt{\triangle}(\alpha^*\beta^*\beta-\beta^*\alpha^*\alpha).$$

Proof. Let $k=1$ in Catalan's identity so the proof is completed for both of QBF and QBL polynomials. □

Theorem 2.9. *(d'Ocagne's Identity) Let QBF_n and QBL_n be n-th QBF and QBL polynomials. The d'Ocagne's identities are*

$$\begin{aligned}
&QBF_k(x,y)QBF_{n+1}(x,y)-QBF_{k+1}(x,y)QBF_n(x,y)\\
=&\frac{(-1)^n y^n}{\alpha-\beta}(\alpha^*\beta^*\alpha^{k-n}-\beta^*\alpha^*\beta^{k-n})
\end{aligned}$$

and

$$\begin{aligned}
&QBL_k(x,y)QBL_{n+1}(x,y)-QBL_{k+1}(x,y)QBL_n(x,y)\\
=&(\alpha-\beta)(\beta^*\alpha^*\beta^k\alpha^n-\alpha^*\beta^*\alpha^k\beta^n).
\end{aligned}$$

Proof. From Binet formula to left -hand side, we get

$$\begin{aligned}
&QBF_k(x,y)QBF_{n+1}(x,y)-QBF_{k+1}(x,y)QBF_n(x,y)\\
=&\left(\frac{\alpha^*\alpha^{k}-\beta^*\beta^{k}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}\right)-\left(\frac{\alpha^*\alpha^{k+1}-\beta^*\beta^{k+1}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n}-\beta^*\beta^{n}}{\alpha-\beta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} (\alpha^*)^2\alpha^{n+k+1} - \beta^*\alpha^*\beta^k\alpha^{n+1} - \alpha^*\beta^*\alpha^k\beta^{n+1} + (\beta^*)^2\beta^{n+k+1} \\ -(\alpha^*)^2\alpha^{n+k+1} + \beta^*\alpha^*\beta^{k+1}\alpha^n + \alpha^*\beta^*\alpha^{k+1}\beta^n - (\beta^*)^2\beta^{n+k+1} \end{array}\right\} \\
&= \frac{\beta^*\alpha^*\beta^k\alpha^n(\beta-\alpha) + \alpha^*\beta^*\alpha^k\beta^n(\alpha-\beta)}{(\alpha-\beta)^2} \\
&= \frac{(\alpha\beta)^n}{(\alpha-\beta)}(\alpha^*\beta^*\alpha^{k-n} - \beta^*\alpha^*\beta^{k-n}).
\end{aligned}
$$

The other case can be done similarly. □

The corresponding identities for QBF and QBL polynomials are contained in the next theorem.

Theorem 2.10. *For $n \geq 0$, the following statements hold:*

$$yQBF_n^2(x,y) + QBF_{n+1}^2(x,y) = \frac{(\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}}{\alpha-\beta}$$

and

$$yQBL_n^2(x,y) + QBL_{n+1}^2(x,y) = (\alpha-\beta)((\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}).$$

Proof. Using Binet formula and equation (2.5), we obtain

$$
\begin{aligned}
& yQBF_n^2(x,y) + QBF_{n+1}^2(x,y) \\
&= y\left(\frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}\right)^2 + \left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right)^2 \\
&= \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} y(\alpha^*)^2\alpha^{2n} - y\beta^*\alpha^* - y\alpha^*\beta^*(\alpha\beta)^n + y(\beta^*)^2\beta^{2n} + (\alpha^*)^2\alpha^{2n+2} \\ -\beta^*\alpha^*(\alpha\beta)^{n+1} - \alpha^*\beta^*(\alpha\beta)^{n+1} + (\beta^*)^2\beta^{2n+2} \end{array}\right\} \\
&= \frac{1}{(\alpha-\beta)^2}\left\{ y(\alpha^*)^2\alpha^{2n} + (\alpha^*)^2\alpha^{2n+2} + y(\beta^*)^2\beta^{2n} + (\beta^*)^2\beta^{2n+2} \right\} \\
&= \frac{(\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}}{(\alpha-\beta)}.
\end{aligned}
$$

Other case can be done similarly. □

Matrix method can use to get results for not only different identities but also algebraic representations in the study of recurrence relations.

In[10], the Pell quaternion matrix is defined by

$$R(n) = \begin{pmatrix} R_n & R_{n-1} \\ R_{n-1} & R_{n-2} \end{pmatrix}$$

and also was obtain equality as follow

$$\begin{pmatrix} R_n & R_{n-1} \\ R_{n-1} & R_{n-2} \end{pmatrix} = \begin{pmatrix} R_2 & R_1 \\ R_1 & R_0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{n-2}$$

where $n \geq 2$ is an integer.

Now, we define the matrix for $QBF_n(x,y)$ and $QBL_n(x,y)$. The matrix $QBF_n(x,y)(n)$ and $QBL_n(x,y)(n)$ that play role of $R(n)$. These are

$$QBF_n(x,y)(n) = \begin{pmatrix} QBF_{n+1}(x,y) & yQBF_n(x,y) \\ QBF_n(x,y) & yQBF_n(x,y) \end{pmatrix}$$

and

$$QBL_n(x,y)(n) = \begin{pmatrix} QBL_{n+1}(x,y) & yQBL_n(x,y) \\ QBL_n(x,y) & yQBL_n(x,y) \end{pmatrix}$$

for $n \geq 1$.

Theorem 2.11. *For an integer $n \geq 1$, we have*

$$QBF_n(x,y)(n) = \begin{pmatrix} QBF_2(x,y) & yQBF_1(x,y) \\ QBF_1(x,y) & yQBF_0(x,y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{n-1}$$

and

$$QBL_n(x,y)(n) = \begin{pmatrix} QBL_2(x,y) & yQBL_1(x,y) \\ QBL_1(x,y) & yQBL_0(x,y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{n-1}.$$

Proof. Induction method can be used to prove it. Let $n = 1$, then basis step is clear. Now let us assume that the equation is valid for $n = k - 1$. For $n = k$, it becomes

$$\begin{aligned} & \begin{pmatrix} QBF_2(x,y) & yQBF_1(x,y) \\ QBF_1(x,y) & yQBF_0(x,y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{k-1} \\ = & \begin{pmatrix} QBF_2(x,y) & yQBF_1(x,y) \\ QBF_1(x,y) & yQBF_0(x,y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{k-2} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} QBF_k(x,y) & yQBF_{k-1}(x,y) \\ QBF_{k-1}(x,y) & yQBF_{k-2}(x,y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} QBF_{k+1}(x,y) & yQBF_k(x,y) \\ QBF_k(x,y) & yQBF_{k-1}(x,y) \end{pmatrix}. \end{aligned}$$

which completes the proof. The other case can be done similarly. □

3. Conclusion

This work studied bivariate Fibonacci and Lucas quaternion polynomials. Since bivariate Fibonacci and Lucas quaternion polynomials were not intensive studied until now, we expect to find in the future more and surprising new properties. For this purpose, Fibonacci and Lucas quaternion polynomials was used and investigated in detail particularly in the first part. Also in the other part, Binet formulas, generating functions, matrix representation and some identities of bivariate Fibonacci and Lucas quaternion polynomials were computed. Quaternions have great importance as they are used in quantum physics, applied mathematics, graph theory and differential equations. Thus, in our future studies we plan to examine bivariate Fibonacci and Lucas octonion polynomials and their key features.

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QUASI-CONFORMAL CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. The present paper deals with the study of generalized Sasakian-space-forms with the conditions $C^q(\xi, X).S = 0$, $C^q(\xi, X).R = 0$ and $C^q(\xi, X).C^q = 0$, where R, S and C^q denote Riemannian curvature tensor, Ricci tensor and quasi-conformal curvature tensor of the space-form, respectively. In the end of the paper, we have given some examples to support our results.
Keywords: Quasi-conformal curvature tensor; generalized Sasakian-space-forms; Einstein manifold; Pseudosymmetric manifold.

1. Introduction

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a generalized Sasakian-space-form if the curvature tensor of the manifold has the following form

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{(g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z)\} \\ &+ f_3\big((\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\big), \end{aligned} \tag{1.1}$$

for any vector fields X, Y, Z on M^{2n+1}. By taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature tensor, we get different kind of generalized Sasakian-space-forms. This idea was introduced by P. Alegre, D. Blair and A. Carriazo [13] in 2004. P. Alegre and Carriazo [15], A. Sarkar, S. K. Hui, etc. [19, 21, 22] studied generalized Sasakian-space-forms by considering the cosymplectic space of Kenmotsu space form as particular types of generalized Sasakian-space-forms. In 2006, U. Kim [22] studied conformally flat generalized Sasakian-space-form and locally symmetric generalized Sasakian-space-form. He proved some geometric properties of generalized Sasakian-space-forms which depends on the nature

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of the functions f_1, f_2 and f_3. Also, he proved that if a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ is locally symmetric then $(f_1 - f_3)$ is constant. In [21] De and Sarkar studied the projective curvature tensor of generalized Sasakian-space-forms and proved that generalized Sasakian-space-forms is projectively flat if and only if $f_3 = \frac{3f_2}{1-2n}$. D.G. Prakasha and H. G. Nagaraja [8] studied quasiconformally semi-symmetric generalized Sasakian-space-forms. They proved that a generalized Sasakian-space-forms is quasiconformally semi-symmetric if and only if either space form is quasiconformally flat or $f_1 = f_2$. Recently, Hui and Prakasha [17] have studied C-Bochner curvature tensor of generalized Sasakian-space-forms. S. K. Hui and D. G. Prakasha [17] studied the C-Bochner pseudosymmetric generalized Sasakian-space-forms. The generalized Sasakian-space-forms have also been studied in ([9], [18], [10], [11], [23]) and many others. Throughout their study, C-Bochner curvature tensor B satisfied the conditions $B(\xi, X).S = 0$, $B(\xi, X).R = 0$ and $B(\xi, X).B = 0$, where R and S denoted the Riemannian curvature tensor and Ricci curvature tensor of the space form respectively. After investigations of the above mentioned developments, we plan to study the quasi-conformal curvature tensor of generalized Sasakian-space-forms.

2. Preliminaries

A Riemannian manifold (M^{2n+1}, g) of dimension $(2n + 1)$ is said to be an almost contact metric manifold [7] if there exists a tensior field ϕ of type (1, 1), a vector field ξ (called the structure vector field) and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

and

$$\eta(\xi) = 1, \tag{2.3}$$

for any vector fields X, Y on M . In an almost contact metric manifold, we have $\phi\xi = 0$ and $\eta o \phi = 0$. Then such type of manifold is called a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the fundamental 2-form of $M^{(2n+1)}$. A contact metric manifold is said to be K-contact manifold if and only if the covarient derivative of ξ satisfies

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

for any vector field X on M.
The almost contact metric structure of M is said to be normal if

$$[\phi, \phi](X, Y) = -2d\eta(X, Y)\phi, \tag{2.5}$$

for any vector fields X and Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ.
A normal contact metric manifold is called Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

for any vector fields X, Y.
The generalized Sasakian-space-forms $M^{2n+1}(f_1, f_2, f_3)$ satisfies the following relations [13]

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.7}$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.8}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y). \tag{2.9}$$

Replacing Y by ξ in the equation (2.9), we get

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X). \tag{2.10}$$

Replacing X and Y by ξ in the equation (2.9), we get

$$S(\xi, \xi) = 2n(f_1 - f_3), \tag{2.11}$$

from the equation (2.9), we have

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.12}$$

Again from (2.9), we have

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi. \tag{2.13}$$

Replacing X by ξ in the above equation, we get

$$Q\xi = 2n(f_1 - f_3)\xi. \tag{2.14}$$

In a Riemannian manifold of dimension $(2n+1)$ the quasi-conformal curvature tensor is defined by [12]

$$\begin{aligned} C^q(X, Y)Z &= aR(X, Y)Z + b\big(S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY\big) \\ &- \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right]\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \tag{2.15}$$

where a and b are constants such that $a, b \neq 0$, Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$, for all X and Y and r is scalar curvature of the manifold.
Using the equations (2.7)-(2.15), we have

$$(2.16)\quad C^q(X,Y)\xi = \left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\{\eta(X)Y-\eta(Y)X\},$$

$$(2.17)\quad C^q(\xi,Y)Z = \left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\{\eta(Z)Y-g(Y,Z)\xi\},$$

and

$$\begin{aligned}\eta(C^q(X,Y)Z) &= \left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\big(g(Z,X)\eta(Y)\\ &- g(Y,Z)\eta(X)\big).\end{aligned} \tag{2.18}$$

This is required quasi-conformal curvature tensor in generalized Sasakian-space-forms.

3. Quasi-conformal Pseudosymmetric generalized Sasakian-Space-Forms

Let (M,g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M,g). A Riemannian manifold is called locally symmetric if $\nabla R=0$, where R is the Riemannian curvature tensor of (M,g). The locally symmetric manifolds have been studied by different differential geometry through various aproaches and they extended semisymmetric manifolds by [2, 3, 4, 5, 6, 24], recurrent manifolds by Walker [1], conformally recurrent manifold by Adati and Miyazawa [20].
According to Z. I. $Szab^{'}o$ [24], if the manifold M satisfies the condition

$$(R(X,Y).R)(U,V)W=0, \quad X,Y,U,V,W\in\chi(M) \tag{3.1}$$

for all vector fields X and Y, then the manifold is called semi-symmetric manifold. For a (0, k)- tensor field T on M, $k\geq 1$ and a symmetric (0, 2)-tensor field A on M the (0, k+2)-tensor fields R.T and Q(A, T) are defined by

$$\begin{aligned}(R.T)(X_1,.....X_k;X,Y) &= -T(R(X,Y)X_1,X_2,.......X_k)\\ &- .....-T(X_1,........X_{k-1},R(X,Y)X_k),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}Q(A,T)(X_1,.....X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,.......X_k)\\ &- .....-T(X_1,.....X_{k-1},(X\wedge_A Y)X_k),\end{aligned} \tag{3.3}$$

where $X\wedge_A Y$ is the endomorphism given by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{3.4}$$

According to R. Deszcz [16], a Riemannian manifold is said to be pseudosymmetric if

$$R.R = L_R\, Q(g, R), \tag{3.5}$$

holds on $U_r = \left\{x \in M | R - \frac{r}{n(n-1)} G \neq 0 \ \ at \ x\right\}$, where G is (0, 4)-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some smooth function on U_R. A Riemannian manifold M is said to be quasi-conformal pseudosymmetric if

$$R.C^q = L_{C^q}\, Q(g, C^q), \tag{3.6}$$

holds on the set $U_{C^q} = \{x \in M : C^q \neq 0 \ \ at \ x\}$, where L_{C^q} is some function on U_{C^q} and C^q is the quasi-conformal curvature tensor.
Let $M^{2n+1}(f_1, f_2, f_3)$ be quasi-conformal pseudosymmetric generalized Sasakian-space-form then from the equation(3.6), we have

$$(R(X, \xi).C^q)(U, V)W = L_{C^q}[((X \wedge_g \xi).C^q)(U, V)W]. \tag{3.7}$$

Using the equations (3.2) and (3.3) in the equation (3.7), we get

$$\begin{aligned} R(X,\xi)C^q(U,V)W &- C^q(R(X,\xi)U,V)W - C^q(U,R(X,\xi)V)W \\ &- C^q(U,V)R(X,\xi)W \\ &= L_{C^q}\big((X \wedge_g \xi)C^q(U,V)W \\ &- C^q((X \wedge_g \xi)U,V)W \\ &- C^q(U,(X \wedge_g \xi)V)W - C^q(U,V)(X \wedge_g \xi)W\big). \end{aligned} \tag{3.8}$$

Again, using the equations (2.7) and (3.4) in (3.8), we conclude the following

$$\begin{aligned} (f_1 - f_3)\big(g(\xi, C^q(U,V)W)X &- g(X, C^q(U,V)W)\xi \\ &- \eta(U)C^q(X,V)W \\ &+ g(X,U)C^q(\xi,V)W - \eta(V)\ C^q(U,X)W \\ &+ g(X,V)C^q(U,\xi)W - \eta(W)C^q(U,V)X \\ &+ g(X,W)C^q(U,V)\xi\big) \\ &= L_{C^q}\big(g(\xi, C^q(U,V)W)X - g(X, C^q(U,V)W)\xi \\ &- \eta(U)C^q(X,V)W \\ &+ g(X,U)C^q(\xi,V)W - \eta(V)C^q(U,X)W \\ &+ g(X,V)C^q(U,\xi)W - \eta(W)C^q(U,V)X \\ &+ g(X,W)C^q(U,V)\xi\big). \end{aligned} \tag{3.9}$$

The above expression can be written as

$$\begin{aligned} (f_1 - f_3 - L_{C^q})\big(g(\xi, C^q(U,V)W)X &- g(X, C^q(U,V)W)\xi \\ &- \eta(U)C^q(X,V)W + g(X,U)C^q(\xi,V)W \\ &- \eta(V)C^q(U,X)W + g(X,V)C^q(U,\xi)W \\ &- \eta(W)C^q(U,V)X + g(X,W)C^q(U,V)\xi\big) = 0, \end{aligned} \tag{3.10}$$

which implies either $L_{C^q} = f_1 - f_3$ or

$$\begin{aligned}\big(g(\xi, C^q(U,V)W)X &- g(X, C^q(U,V)W)\xi - \eta(U)C^q(X,V)W \\ &+ g(X,U)C^q(\xi,V)W - \eta(V)C^q(U,X)W \\ &+ g(X,V)C^q(U,\xi)W - \eta(W)C^q(U,V)X \\ &+ g(X,W)C^q(U,V)\xi\big) = 0. \end{aligned} \tag{3.11}$$

Putting $W = \xi$ in the equation (3.11) and using the equations (2.17) and (2.18), we have

$$\begin{aligned} C^q(U,V)X &= \left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right](g(X,U)V \\ &- g(X,V)U), \end{aligned} \tag{3.12}$$

contracting V in the above equation, we have

$$\left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]2ng(U,X) = 0, \tag{3.13}$$

this implies that

$$\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1} = 0, \tag{3.14}$$

from the above equation two conditions arise, either

$$a = -(2n-1)b \tag{3.15}$$

or

$$f_3 = \frac{3f_2}{(1-2n)}. \tag{3.16}$$

Using the equations (3.15) or (3.16) in (2.16) and (2.17), we get

$$C^q(\xi, Y)Z = 0, \tag{3.17}$$

and

$$C^q(X,Y)\xi = 0, \tag{3.18}$$

this means $M^{2n+1}(f_1, f_2, f_3)$ is quasi-conformally flat.
Thus, we conclude:

Theorem 3.1. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a (2n+1)-dimensional generalized Sasakian-space-form. If $M^{2n+1}(f_1, f_2, f_3)$ is quasi-conformal pseudosymmetric then $M^{2n+1}(f_1, f_2, f_3)$ is quasiconformally flat if at least one of the following conditions holds:*

$$(i) f_3 = \frac{3f_2}{(1-2n)} \quad (ii) a = -(2n-1)b, \quad (iii) L_{C^q} = f_1 - f_3.$$

Now we propose:

Theorem 3.2. *Let $M^{2n+1}(f_1, f_2, f_3)$ be a (2n+1)-dimensional generalized Sasakian-space-form. Then $M^{2n+1}(f_1, f_2, f_3)$ satisfies $C^q(\xi, X).S = 0$ if and only if at least one of the following conditions holds:*

$$(i)\ f_3 = \frac{3f_2}{(1-2n)}, \quad (ii)\ a = -(2n-1)b, \quad (iii)\ S(X,U) = 2n(f_1 - f_3)g(X,U).$$

Proof. If generalized Sasakian-space-form satisfies $C^q(\xi, X).S = 0$.
Then from the equation (3.2), we have

$$S(C^q(\xi, X)U, \xi) + S(U, C^q(\xi, X)\xi) = 0, \tag{3.19}$$

From the equation (2.10), we have

$$S(C^q(\xi, X)U, \xi) = 2n(f_1 - f_3)\eta(C^q(\xi, X)U). \tag{3.20}$$

Now with the help of equations (2.17) and (3.20), we can write

$$\begin{aligned} S(C^q(\xi, X)U, \xi) &= 2n(f_1 - f_3)\left[\frac{(a + (2n-1)b)((2n-1)f_3 + 3f_2)}{2n+1}\right](\eta(X)\eta(U) \\ &- g(X,U)). \end{aligned} \tag{3.21}$$

Again in view of the equation (2.17), we have

$$\begin{aligned} S(C^q(\xi, X)\xi, U) &= \left[\frac{(a + (2n-1)b)((2n-1)f_3 + 3f_2)}{2n+1}\right](S(X,U) \\ &- 2n(f_1 - f_3)\eta(X)\eta(U)). \end{aligned} \tag{3.22}$$

By using the expressions (3.21) and (3.22) in (3.19), we infer

$$\begin{aligned} &\left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right](S(X,U) \\ &-2n(f_1 - f_3)g(X,U)) = 0, \end{aligned} \tag{3.23}$$

which implies that if $C^q(\xi, X).S = 0$ then either $a = -(2n-1)b$ or $f_3 = \frac{3f_2}{(1-2n)}$ or $S(X,U) = 2n(f_1 - f_3)g(X,U)$.
Conversely, it is clear that if $a = -(2n-1)b$ or $f_3 = \frac{3f_2}{(1-2n)}$ or $S(X,U) = 2n(f_1 - f_3)g(X,U)$ then from (2.17), we have □

$$C^q(\xi, X).S = 0. \tag{3.24}$$

Now we take $C^q(\xi, U).R = 0$.
Then from the equation (3.2), we have

$$\begin{aligned} &C^q(\xi, U)R(X,Y)Z - R(C^q(\xi, U)X, Y)Z \\ &-R(X, C^q(\xi, U)Y)Z - R(X,Y)C^q(\xi, U)Z = 0, \end{aligned} \tag{3.25}$$

which in view of the equation (2.17), we have

$$\begin{aligned}\left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\big(\eta(R(X,Y)Z)U\\ -g(U,R(X,Y)Z)\xi-\eta(X)R(U,Y)Z\\ +g(U,X)R(\xi,Y)Z-\eta(Y)R(X,U)Z+g(U,Y)R(X,\xi)Z\\ -\eta(Z)R(X,Y)U+g(U,Z)R(X,Y)\xi\big)=0,\end{aligned}\tag{3.26}$$

using $Z=\xi$ and (2.2) in the above equation, we infer

$$\begin{aligned}\left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\\ \big((f_1-f_3)(g(U,Y)X-g(U,X)Y)-R(X,Y)U\big)=0,\end{aligned}\tag{3.27}$$

which implies that if $C^q(\xi,X).R=0$ then either $a=-(2n-1)b$ or $f_3=\frac{3f_2}{(1-2n)}$ or
$R(X,Y)U=(f_1-f_3)(g(U,Y)X-g(U,X)Y)$.
Thus, we conclude:

Theorem 3.3. *Let $M^{2n+1}(f_1,f_2,f_3)$ be a (2n+1)-dimensional generalized Sasakian-space-form. If $M^{2n+1}(f_1,f_2,f_3)$ satisfying $C^q(\xi,U).R=0$ then at least one of the following necessarily holds:*

$$\begin{gathered}(i)\ \ f_3=\tfrac{3f_2}{(1-2n)},\ (ii)\ \ a=-(2n-1)b,\\ (iii)\ R(X,Y)U=(f_1-f_3)(g(U,X)Y-g(U,Y)X).\end{gathered}$$

Now we propose:

Theorem 3.4. *Let $M^{2n+1}(f_1,f_2,f_3)$ be a (2n+1)-dimensional generalized Sasakian-space-form. Then $M^{2n+1}(f_1,f_2,f_3)$ satisfies $C^q(\xi,X).C^q=0$ if and only if either $f_3=\frac{3f_2}{(1-2n)}$ or $a=-(2n-1)b$.*

Proof. If generalized Sasakian-space-form satisfies $C^q(\xi,X).C^q=0$. Then, from the equation (3.2) we have

$$\begin{aligned}C^q(\xi,X)C^q(U,V)W-C^q(C^q(\xi,X)U,V)W\\ -C^q(U,C^q(\xi,X)V)W-C^q(U,V)C^q(\xi,X)W=0,\end{aligned}\tag{3.28}$$

by which in view of the equation (2.16) we get

$$\begin{aligned}\left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\big(\eta(C^q(U,V)W)X\\ -g(X,C^q(U,V)W)\xi-\eta(U)C^q(X,V)W\\ +g(X,U)C^q(V,\xi)W-\eta(V)C^q(U,X)W+g(X,V)C^q(U,\xi)W\\ +g(W,X)C^q(U,V)\xi-\eta(W)C^q(U,V)X\big)=0.\end{aligned}\tag{3.29}$$

By using $V = \xi$ in the above equation, we infer

$$\left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right]\big((C^q(U,X)W + \left[\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right](g(X,W)U) - g(U,W)X)\big) = 0, \tag{3.30}$$

which implies that either $a = -(2n-1)b$ or $f_3 = \frac{3f_2}{(1-2n)}$ or

$$C^q(U,X)W = \big(\tfrac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\big)(g(U,W)X - g(X,W)U), \tag{3.31}$$

contracting U in the above equation, we have

$$\left[\frac{(a+(2n-1)b)((2n-1)f_3+3f_2)}{2n+1}\right] 2ng(X,V) = 0, \tag{3.32}$$

this implies that either $a = -(2n-1)b$ or $f_3 = \frac{3f_2}{(1-2n)}$. Conversely, if $M^{2n+1}(f_1,f_2,f_3)$ satisfies $a = -(2n-1)b$ or $f_3 = \frac{3f_2}{(1-2n)}$, then in view of (2.17) we have $C^q(\xi,X).C^q = 0$. □

4. Examples

Example 4.1. [13] Let $N(\lambda_1,\lambda_2)$ be generalized Sasakian-space-forms of dimension 4, then by the warped product $M \times N$ endowed with the almost contact metric structure (ϕ,ξ,η,g_f), Sasakian space form $M(f_1,f_2,f_3)$ is generalized with

$$f_1 = \frac{\lambda_1 - (f^{'})^2}{f^2}, \quad f_2 = \frac{\lambda_2}{f^2}, \quad f_3 = \frac{\lambda_1 - (f^{'})^2}{f^2} + \frac{f^{''}}{f}, \tag{4.1}$$

where λ_1 and λ_2 are constants, $f = f(t)$, $t \in R$ and $f^{'}$ denotes the derivative of f with respect to t.
If we take $\lambda_1 = -\frac{3\lambda_2}{7}$ and $f(t) = e^{Kt}$, K is constant,
then $f_1 = -\frac{1}{e^{2Kt}}[\frac{3\lambda_2}{7} + K^2e^{2Kt}]$, $f_2 = \frac{\lambda_2}{e^{2Kt}}$ and $f_3 = -\frac{1}{e^{2Kt}}[\frac{3\lambda_2}{7}]$. Hence $f_3 = \frac{3f_2}{(1-2n)}$, if $n = 4$.

Example 4.2. [14] Let $N(c)$ be a complex space form, and by the warped product $M = (-\frac{\pi}{2},\frac{\pi}{2}) \times_f N$ endowed with the almost contact metric structure (ϕ,ξ,η,g_f), Sasakian space form $M(f_1,f_2,f_3)$ is generalized with functions

$$f_1 = \frac{c - 4(f^{'})^2}{4f^2}, \quad f_2 = \frac{c}{4f^2}, \quad f_3 = \frac{c - 4(f^{'})^2}{4f^2} + \frac{f^{''}}{f}, \tag{4.2}$$

where $f = f(t)$, $t \in R$ and $f^{'}$ denotes the derivative of f with respect to t.
If we take $c = 0$ and $f(t) = e^{Kt}$, K is constant,

then $f_1 = -K^2$, $f_2 = f_3 = 0$. Hence $f_3 = \frac{3f_2}{(1-2n)}$.

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ON KENMOTSU MANIFOLDS WITH A SEMI-SYMMERIC METRIC CONNECTION

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Abstract. The aim of the present paper is to study the properties of locally and globally ϕ-concircularly symmetric Kenmotsu manifolds endowed with a semi-symmetric metric connection. First, we will prove that the locally ϕ-symmetric and the globally ϕ-concircularly symmetric Kenmotsu manifolds are equivalent. Next, we will study three dimensional locally ϕ-symmetric, locally ϕ-concircularly symmetric and locally ϕ-concircularly recurrent Kenmotsu manifolds with respect to such connection and will obtain some geometrical results. In the end, we will construct a non-trivial example of Kenmotsu manifold admitting a semi-symmetric metric connection and validate our results.

Keywords: Kenmotsu manifolds, ϕ-symmetric manifolds, η-parallel Ricci tensor, semi-symmetric metric connection, concircular curvature tensor.

1. Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However, if one takes M to be an almost contact metric manifold and suppose that the product metric G on $M \times \mathbb{R}$ is Kähler, then the structure on M is cosymplectic [19] and not Sasakian. On the other hand, Oubina [25] pointed that if the conformally related metric $e^{2t}G$, t being the coordinates on $\mathbb{R}$ is Kähler, then M is Sasakian and vice versa.

In [34], Tanno classified almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such manifold M, the sectional curvature of the plane section containing ξ is constant, say c. If $c >, =, and < 0$, then M is said be a homogeneous Sasakian manifold of constant sectional curvature, product of a line or a circle with Kähler manifold of constant holomorphic sectional curvature, and warped product space $\mathbb{R} \times_f C^n$, respectively. In 1972, Kenmotsu [23] characterized the geometrical properties of the manifold when $c < 0$, called Kenmotsu manifold. The geometrical properties of this manifold have been studied

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by many geometers, for instance (see, [3], [7]-[11], [15], [16], [22], [26], [33], [36], [40], [41]).

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij} \tag{1.1}$$

of the fundamental tensor g_{ij}. A transformation which preserves the geodesic circle was first introduced by Yano [37]. The conformal transformation (1.1) satisfying the partial differential equation

$$\psi_{;i;j} = \phi g_{ij}$$

change a geodesic circle into a geodesic circle. Such transformation is known as the concircular transformation and the geometry which leads with such transformation is known as the concircular geometry [37].

A $(1,3)$ type tensor C which remains invariant under the transformation (1.1), for an n-dimensional Riemannian manifold M, given by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right] \tag{1.2}$$

for all vector fields X, Y and Z on M is known as a concircular curvature tensor [37], where R, r, and ∇ are the Riemannian curvature tensor, the scalar curvature, and the Levi-Civita connection, respectively. In view of (1.2), it is obvious that

$$(\nabla_W C)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{dr(W)}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{1.3}$$

The importance of the concircular transformation and the concircular curvature tensor are well known in the differential geometry of F-structures such as complex, almost complex, Kähler, almost Kähler, contact and almost contact structures ([37], [6], [35]). In a recent paper, Ahsan and Siddiqui [1] have studied the application of concircular curvature in general relativity and cosmology.

Let (M,g) be a Riemannian manifold of dimension n. A linear connection $\tilde{\nabla}$ on (M,g), whose torsion tensor $\tilde{T}$ of type $(1,2)$ is defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

For arbitrary vector, fields X and Y on M are said to be torsion free or symmetric if $\tilde{T}$ vanishes, otherwise it is non-symmetric. If the connection $\tilde{\nabla}$ satisfies $\tilde{\nabla}g = 0$ on (M,g), then it is called metric connection, otherwise it is non-metric. In [17], Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a differentiable manifold. Hayden [18] introduced the idea of semi-symmetric linear connection with non-zero torsion on a Riemannian manifold. The systematic study of the semi-symmetric metric connection on the Riemannian manifold was introduced by Yano [38]. He proved that a Riemannian manifold endowed with a semi-symmetric metric connection has vanishing curvature tensor with respect to the semi-symmetric metric connection if and only if it is conformally flat. This result was generalized for vanishing Ricci tensor of the semi-symmetric metric connection by T. Imai ([20], [21]). Various geometrical and physical properties of this connection have been studied by many authors among whom are ([2]-[4], [12]-[14], [27]- [31], [39]). Motivated by the above studies, the authors will continue to study the properties of the Kenmotsu manifolds equipped with a semi-symmetric metric connection. The present paper is organized in the following manner:

After the introduction in Section 1, we will notify you on the basic results of the Kenmotsu manifolds and the semi-symmetric metric connection in Section 2 and Section 3, respectively. In section 4, we will start the study of globally ϕ-concircularly symmetric Kenmotsu manifold and prove that the manifold is η-Einstein as well as locally ϕ-symmetric.The following sections deal with the study of locally ϕ-symmetric, locally ϕ-concircularly symmetric, Ricci semisymmetric, η-parallel Ricci tensor and locally ϕ-concircularly recurrent Kenmotsu manifolds equipped with a semi-symmetric metric connection. In the last section, we will construct an example of three dimensional Kenmotsu manifold admitting a semi-symmetric metric connection to verify some results of our paper.

2. Preliminaries

Let M be an $n(=2m+1)$-dimensional connected almost contact metric manifold with an almost contact structure (ϕ, ξ, η, g), that is, M admits a $(1,1)$-type tensor field ϕ, a $(1,0)$-type vector field ξ, a 1-form η, and a compatible Riemannian metric g satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \tag{2.2}$$

for all X, $Y \in T(M)$, where $T(M)$ denotes the tangent space of M [5]. If an almost contact metric manifold M satisfies

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.3}$$

for all X, $Y \in T(M)$, then M is called a Kenmotsu manifold [23]. From (2.1)-(2.3), it can be easily prove that

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.4}$$

and

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all X, $Y \in T(M)$. Let S denote the Ricci tensor of M. It is noticed that M satisfies the following relations.

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \tag{2.7}$$

and

$$S(X, \xi) = -(n-1)\eta(X) \tag{2.8}$$

for all X, $Y \in T(M)$. The curvature tensor R in a 3-dimensional Kenmotsu manifold M assumes the form

$$\begin{aligned} R(X,Y)Z = \left(\frac{r+4}{2}\right)[g(Y,Z)X - g(X,Z)Y] - \left(\frac{r+6}{2}\right)[g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{2.9}$$

After contracting X it becomes

$$S(Y,Z) = \frac{1}{2}[(r+2)g(Y,Z) - (r+6)\eta(X)\eta(Y)] \tag{2.10}$$

for all X, $Y \in T(M)$.

An n-dimensional Kenmotsu manifold (M, g) is said to be an η-Einstein manifold if its non-vanishing Ricci-tensor S takes the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.11}$$

for all X, $Y \in T(M)$, where a and b are smooth functions on (M, g). If $b = 0$ and a is constant, then η-Einstein manifold becomes Einstein manifold. Kenmotsu [23] proved that if (M, g) is an n-dimensional η-Einstein manifold, then $a + b = -(n-1)$.

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3. Semi-symmetric metric connection on Kenmotsu manifold

Let M be an n-dimensional Kenmotsu manifold endowed with a Riemannian metric g. A linear connection $\tilde{\nabla}$ on (M, g) is said to be a semi-symmetric metric connection [38] if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ and the Riemannian metric g satisfies (3.1)

$$ll.,l,kmlkvmmmmmmmmmmmmmmmmmmm\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

and

$$\tilde{\nabla} g = 0 \tag{3.2}$$

for all X, $Y \in T(M)$. The Levi-Civita connection ∇ and the semi-symmetric metric connection $\tilde{\nabla}$ on (M, g) are connected by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi \tag{3.3}$$

for all X, $Y \in T(M)$ [38]. From (2.1), (2.2) and (3.3), it follows that

$$(\tilde{\nabla}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(X)\eta(Y) + g(X,Y). \tag{3.4}$$

The curvature tensors R and $\tilde{R}$ with respect to ∇ and $\tilde{\nabla}$, respectively, are connected by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X + g(X,Z)AY - g(Y,Z)AX, \tag{3.5}$$

where α is a tensor field of type $(0,2)$ and A, a tensor field of type $(1,1)$, are related by

$$\alpha(Y,Z) = g(AY,Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z), \tag{3.6}$$

for all X, Y, $Z \in T(M)$ [38]. From (2.1), (2.5), (3.5) and (3.6), it follows that

$$\begin{aligned} \tilde{R}(X,Y)Z = R(X,Y)Z &- 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X \\ &-2\eta(X)\eta(Z)Y + 2\eta(X)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\xi. \end{aligned} \tag{3.7}$$

Contracting (3.7) along X, we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{3.8}$$

which becomes

$$\tilde{r} = r - n(3n-7) - 4. \tag{3.9}$$

Here $\tilde{S}$ and $\tilde{r}$ denote the Ricci tensor and the scalar curvature with respect to the connection $\tilde{\nabla}$. Replacing Z by ξ in (3.8) and using (2.8), we have

$$\tilde{S}(Y,\xi) = -2(n-1)g(Y,\xi). \tag{3.10}$$

Thus we can state:

Proposition 3.1. *Let M be an n-dimensional, $n \geqslant 3$, Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. Then ξ is an eigen vector of $\tilde{S}$ corresponding to the eigenvalue $-2(n-1)$.*

4. Globally ϕ-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection

In this section, we will study the properties of the globally ϕ-concircularly symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ and prove our result in the form of theorems.

Definition 4.1. A Kenmotsu manifold M of dimension n is said to be locally ϕ-symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if the non-vanishing curvature tensor $\tilde{R}$ satisfies the relation

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0$$

for all vector fields X, Y, Z and W orthogonal to ξ.

This notion was introduced by Takahashi [32] for Sasakian manifold.

Definition 4.2. An n-dimensional Kenmotsu manifold M is said to be a globally ϕ-concircularly symmetric manifold with respect to ∇ if the non-zero concircular curvature tensor C satisfies

$$\phi^2((\nabla_W C)(X, Y)Z) = 0 \tag{4.1}$$

for all vector fields X, Y, Z, $W \in T(M)$.

Definition 4.3. An n-dimensional Kenmotsu manifold M equipped with the semi-symmetric metric connection $\tilde{\nabla}$ is said to be a globally ϕ-concircularly symmetric Kenmotsu manifold with respect to $\tilde{\nabla}$ if the non-vanishing concircular curvature tensor $\tilde{C}$ with respect to $\tilde{\nabla}$ satisfies

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X, Y)Z) = 0 \tag{4.2}$$

for arbitrary vector fields X, Y, Z and W. Here $\tilde{C}$ is a concircular curvature tensor [37] with respect to $\tilde{\nabla}$ and is defined by

$$\tilde{C}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{\tilde{r}}{n(n-1)}\left[g(Y, Z)X - g(X, Z)Y\right]. \tag{4.3}$$

Theorem 4.1. *An n-dimensional, $n \geq 3$, globally ϕ-concircularly symmetric Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is an η-Einstein manifold.*

Proof. We suppose that M is a globally ϕ-concircularly symmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Then we have

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X, Y)Z) = 0.$$

In view of (2.1), the above equation becomes

$$-(\tilde{\nabla}_W \tilde{C})(X, Y)Z + \eta((\tilde{\nabla}_W \tilde{C})(X, Y)Z)\xi = 0.$$

Equation (1.3) along with above equation give

$$\begin{aligned} &-g((\tilde{\nabla}_W \tilde{R})(X, Y)Z, U) + \frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\right] \\ &+\eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z)\eta(U) - \frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\right]\eta(U) = 0. \end{aligned}$$

Replacing $X = U = e_i$, where $\{e_i, i = 1, 2, 3, ..., n\}$, be an orthonormal basis of the tangent space at each point of the manifold M and then summing over i, $1 \leq i \leq n$, we get

$$\begin{aligned} -(\tilde{\nabla}_W \tilde{S})(Y, Z) + \frac{d\tilde{r}(W)}{n} g(Y, Z) + \eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)Z) \\ - \frac{d\tilde{r}(W)}{n(n-1)}\left[g(Y, Z) - \eta(Y)\eta(Z)\right] = 0. \end{aligned}$$

Putting $Z = \xi$ in the above equation and using (2.1), we get

$$-(\tilde{\nabla}_W \tilde{S})(Y, \xi) + \frac{d\tilde{r}(W)}{n}\eta(Y) + \eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0. \tag{4.4}$$

In view of (2.1), (2.2), (2.4), (2.6), (2.7), (3.3) and (3.7), we conclude that

$$\eta((\tilde{\nabla}_W \tilde{R})(\xi, Y)\xi) = 0$$

and hence the equation (4.4) becomes

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \frac{d\tilde{r}(W)}{n}\eta(Y). \tag{4.5}$$

Substituting $Y = \xi$ in (4.5) and using (2.1) and (2.8), we get $d\tilde{r}(W) = 0$. This implies that $\tilde{r}$ is a constant. So from (4.5), we obtain

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0. \tag{4.6}$$

It is well known that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi).$$

In view of (2.1), (2.2), (2.4), (2.5), (2.8), (3.3), (3.4), (3.10) and (4.6), above equation takes the form

$$S(Y, W) = (n-3)g(Y, W) - 2(n-2)\eta(Y)\eta(W).$$

Hence the statement of the Theorem 4.1 is proved. □

From the above equation, it is clear that $r = (n-1)(n-4)$. Hence the scalar curvature under consideration is constant. Thus we have

Corollary 4.1. *An n-dimensional, $n > 3$, globally ϕ-concircularly symmetric Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.*

Theorem 4.2. *Let M be an n-dimensional, $n \geqslant 3$, Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$. Then the globally ϕ-concircularly symmetric manifold and the locally ϕ-symmetric manifold with respect to $\tilde{\nabla}$ coincide.*

Proof. We suppose that the manifold M is globally ϕ-concircularly symmetric with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Since r is constant on M and therefore $\tilde{r}$ is also constant. The covariant derivative of (4.3) gives

$$(\tilde{\nabla}_W \tilde{C})(X, Y)Z = (\tilde{\nabla}_W \tilde{R})(X, Y)Z. \tag{4.7}$$

In view of (3.3), (3.4) and (3.7), we get

$$
\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = {} & (\tilde{\nabla}_W R)(X,Y)Z + 2\{(\nabla_W \eta)(Y) - \eta(Y)\eta(W) \\
& + g(Y,W)\}\eta(Z)X + 2\{(\nabla_W \eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}\eta(Y)X \\
& - 2\{(\nabla_W \eta)(X) - \eta(X)\eta(W) + g(X,W)\}\eta(Z)Y \\
& - 2\{(\nabla_W \eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}\eta(X)Y \\
& + 2g(Y,Z)\{(\nabla_W \eta)(X) - \eta(X)\eta(W) + g(X,W)\}\xi \\
& + 2\{\nabla_W \xi + W - \eta(W)\xi\}\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z)\} \\
& - 2g(X,Z)\{(\nabla_W \eta)(Y) + \eta(Y)\eta(W) - g(Y,W)\}\xi.
\end{aligned} \tag{4.8}
$$

Using (2.4) and (2.5) in (4.8), we obtain

$$
\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = {} & (\tilde{\nabla}_W R)(X,Y)Z + 4\{-\eta(Y)\eta(W) + g(Y,W)\}\eta(Z)X \\
& + 4\{-\eta(Z)\eta(W) + g(Z,W)\}\eta(Y)X \\
& - 4\{-\eta(X)\eta(W) + g(X,W)\}\eta(Z)Y \\
& - 4\{-\eta(Z)\eta(W) + g(Z,W)\}\eta(X)Y \\
& + 4g(Y,Z)\{-\eta(X)\eta(W) + g(X,W)\}\xi \\
& + 4\{\eta(X)g(Y,Z) + \eta(Y)g(X,Z)\}\{W - \eta(W)\xi\}.
\end{aligned} \tag{4.9}
$$

If X, Y, Z and W are orthogonal to ξ then from above equation, we get

$$
(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi. \tag{4.10}
$$

In view of (4.7) and (4.10), we have

$$
(\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z + 4g(Y,Z)g(X,W)\xi.
$$

Operating ϕ^2 on either sides of the above equation and then using (2.1) we get

$$
\phi^2(\tilde{\nabla}_W \tilde{C})(X,Y)Z = \phi^2(\tilde{\nabla}_W R)(X,Y)Z \tag{4.11}
$$

for all vector fields X, Y, Z and W orthogonal to ξ. From the equations (4.7) and (4.9), it is clear that the equation (4.11) satisfies for all vector fields X, Y, Z and W on M. Hence the statement of the Theorem 4.2 is proved. □

Remark 4.1. *The last equation shows that a locally ϕ-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is always globally ϕ-concircularly symmetric manifold. Thus we conclude that on a Kenmotsu manifold locally ϕ-symmetric and globally ϕ-symmetric manifolds are equivalent corresponding to the connection $\tilde{\nabla}$.*

5. Three dimensional locally ϕ-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection

This section deals with the study of the locally ϕ-symmetric Kenmotsu manifold M with respect to a semi-symmetric metric connection $\tilde{\nabla}$. Now, we will consider a 3-dimensional locally ϕ-symmetric Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ and prove the following:

Theorem 5.1. *A 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric with respect to the connection $\tilde{\nabla}$ if and only if $dr(W) = 0$, W is an orthonormal vector field to ξ.*

Proof. From (2.9) and (3.7), we get

$$\begin{aligned}\tilde{R}(X,Y)Z = &\left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\}\\ &+\left(\frac{r+2}{2}\right)[\eta(Y)g(X,Z)\xi + \eta(X)\eta(Z)Y\\ &-\eta(X)g(Y,Z)\xi - \eta(Y)\eta(Z)X].\end{aligned} \tag{5.1}$$

Taking covariant differentiation of (5.1) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we have

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - \eta(X)\eta(Z)Y\\ &+\{-g(X,Z)\eta(Y) + g(Y,Z)\eta(X)\}\xi + \eta(Y)\eta(Z)X]\\ &+\left(\frac{r+2}{2}\right)[g(X,Z)(\tilde{\nabla}_W\eta)(Y)\xi + g(X,Z)\eta(Y)\tilde{\nabla}_W\xi\\ &-g(Y,Z)(\tilde{\nabla}_W\eta)(X)\xi - g(Y,Z)\eta(X)\tilde{\nabla}_W\xi\\ &+\eta(Z)(\tilde{\nabla}_W\eta)(X)Y + \eta(X)(\tilde{\nabla}_W\eta)(Z)Y\\ &-\eta(Z)(\tilde{\nabla}_W\eta)(Y)X - \eta(Y)(\tilde{\nabla}_W\eta)(Z)X].\end{aligned} \tag{5.2}$$

In consequence of (3.3) and (3.4), (5.2) becomes

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z = &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(X,Z)\eta(Y)\xi\\ &-\eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi + \eta(Y)\eta(Z)X]\\ &+\left(\frac{r+2}{2}\right)[-\eta(X)g(Y,Z)\{\nabla_W\xi + W - \eta(W)\xi\}\\ &-g(Y,Z)\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}\xi\\ &+\eta(Z)\{(\nabla_W\eta)(X) - \eta(X)\eta(W) + g(X,W)\}Y\\ &+\eta(X)\{(\nabla_W\eta)(Z) - \eta(W)\eta(Z) + g(Z,W)\}Y\\ &-\eta(Z)\{(\nabla_W\eta)(Y) - \eta(Y)\eta(W) + g(Y,W)\}X\\ &-\eta(Y)\{(\nabla_W\eta)(Z) - \eta(Z)\eta(W) + g(Z,W)\}X\\ &+g(X,Z)\{(\nabla_W\eta)(Y) - \eta(Y)\eta(W) + g(Y,W)\}\xi\\ &+g(X,Z)\eta(Y)\{\nabla_W\xi + W - \eta(W)\xi\}].\end{aligned} \tag{5.3}$$

Let us suppose that the vector fields X, Y, Z and W are orthogonal to ξ, therefore

(5.3) becomes

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right)[g(X,Z)\{(\nabla_W\eta)(Y) + g(Y,W)\} - g(Y,Z)\{(\nabla_W\eta)(X) + g(X,W)\}]\xi. \tag{5.4}$$

Operating ϕ^2 on both sides of (5.4) and then using (2.1) and (2.2), we obtain

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = -\frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{5.5}$$

From the equation (5.5), it is obvious that the manifold M is locally ϕ-symmetric Kenmotsu manifold with respect to $\tilde{\nabla}$ if and only if $dr(W) = 0$. Hence the statement of the Theorem 5.1 is proved. □

6. Three dimensional Locally ϕ-concircularly symmetric Kenmotsu manifold with a semi-symmetric metric connection

Definition 6.1. A Kenmotsu manifold M is said to be locally ϕ-concircularly symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if its concircular curvature tensor $\tilde{C}$ satisfies

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = 0$$

for all vector fields W, X, Y and Z orthogonal to ξ.

Theorem 6.1. *A 3-dimensional Kenmotsu manifold M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-concircularly symmetric manifold with respect to the connection $\tilde{\nabla}$ if and only if the scalar curvature r is constant.*

Proof. From (2.9) and (3.7), it follows that

$$\tilde{R}(X,Y)Z = \left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi]. \tag{6.1}$$

In view of (1.2) and (6.1), we get

$$\tilde{C}(X,Y)Z = \left(\frac{r-2}{2}\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi] + \frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.2}$$

Taking covariant derivative of (6.2) with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we have

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y \\ &+ \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi - \eta(Y)\eta(Z)X] \\ &+ \left(\frac{r+2}{2}\right)[g(X,Z)(\tilde{\nabla}_W\eta)(Y)\xi + g(X,Z)\eta(Y)\tilde{\nabla}_W\xi \\ &- g(Y,Z)(\tilde{\nabla}_W\eta)(X)\xi - g(Y,Z)\eta(X)\tilde{\nabla}_W\xi + \eta(X)(\tilde{\nabla}_W\eta)(Z)Y \\ &- \eta(Z)(\tilde{\nabla}_W\eta)(Y)X - \eta(Y)(\tilde{\nabla}_W\eta)(Z)X + \eta(Z)(\tilde{\nabla}_W\eta)(X)Y] \\ &+ \frac{dr(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.3}$$

Let us consider that the vector fields X, Y and Z are orthonormal to ξ and therefore (6.3) converts into the form

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{dr(W)}{2}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \left(\frac{r+2}{2}\right)\left\{g(X,Z)(\tilde{\nabla}_W\eta)(Y) - g(Y,Z)(\tilde{\nabla}_W\eta)(X)\right\}\xi \\ &+ \frac{dr(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{6.4}$$

Using (3.4) in (6.4), we obtain

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z &= \frac{2dr(W)}{3}\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{r+2}{2}\right)[g(X,Z)(\nabla_W\eta)(Y) \\ &- g(X,Z)\eta(Y)\eta(W) - g(Y,Z)(\nabla_W\eta)(X) + g(Y,W)g(X,Z) \\ &- g(Y,Z)g(X,W) + g(Y,Z)\eta(X)\eta(W)]\xi. \end{aligned} \tag{6.5}$$

Applying ϕ^2 on both sides of (6.5) and using (2.1), we get

$$\phi^2\left((\tilde{\nabla}_W\tilde{C})(X,Y)Z\right) = \frac{2dr(W)}{3}\{g(Y,Z)X - g(X,Z)Y\}.$$

This proved the statement of the Theorem 6.1. □

From the Theorem 5.1 and the Theorem 6.1, we can state the following:

Corollary 6.1. *A 3-dimensional Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-concircularly symmetric with respect to the connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to $\tilde{\nabla}$.*

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7. Three dimensional Ricci semisymmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

The following section delas with the study of a 3-dimensional Ricci semisymmetric Kenmotsu manifold with respect to the semi-symmetric metric connection with tha aim to prove some geometrical results.

Theorem 7.1. *A 3-dimensional Ricci semisymmetric Kenmotsu manifold with respect to a semi-symmetric metric connection $\tilde{\nabla}$ possesses a constant scalar curvature.*

Proof. Let us consider a 3-dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ which satisfies $\tilde{R}(X,Y)\cdot\tilde{S}=0$, that is, M is Ricci semisymmetric with respect to $\tilde{\nabla}$ and then we have

$$\tilde{S}(\tilde{R}(X,Y)Z,W)+\tilde{S}(Z,\tilde{R}(X,Y)W)=0. \tag{7.1}$$

Replacing X by ξ in (7.1), we get

$$\tilde{S}(\tilde{R}(\xi,Y)Z,W)+\tilde{S}(Z,\tilde{R}(\xi,Y)W)=0. \tag{7.2}$$

From (5.1), it is obvious that

$$\tilde{R}(\xi,Y)Z=-2\left\{g(Y,Z)\xi-\eta(Z)Y\right\}. \tag{7.3}$$

By virtue of (3.10), (7.2) and (7.3), we obtain

$$\eta(Z)\tilde{S}(Y,W)+4\eta(W)g(Y,Z)+\eta(W)\tilde{S}(Z,Y)+4\eta(Z)g(Y,W)=0. \tag{7.4}$$

Let $\{e_i\}$, $i=1,2,3$, is an orthonormal basis of the tangent space at each point of the manifold M. Putting $Y=Z=e_i$ in (7.4) and taking summation over i, $1\leq i\leq 3$, we get

$$(\tilde{r}+12)\eta(W)=0.$$

Since $\eta(W)\neq 0$, in general, therefore $\tilde{r}=-12\,(\mathit{constant})$. This proved the statement of the Theorem 7.1. $\square$

In consequence of the Theorem 6.1 and Theorem 7.1, we state:

Corollary 7.1. *If a 3-dimensional Kenmotsu manifold M with respect to a semi-symmetric metric connection $\tilde{\nabla}$ satisfies the condition $\tilde{R}(X,Y)\cdot\tilde{S}=0$, then M is locally ϕ-symmetric as well as locally ϕ-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.*

8. η-parallel Ricci tensor with respect to the semi-symmetric metric connection

Definition 8.1. A Ricci tensor $\tilde{S}$ of a Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is called η-parallel with respect to $\tilde{\nabla}$ if it $\tilde{S}$ is non-zero and satisfies

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = 0 \tag{8.1}$$

for all vector fields X, Y and Z on M.

The notion of η-parallel Ricci tensor on a Sasakian manifold was introduced by M. Kon [24]. Since then, many authors studied the geometrical and physical properties of this tensor.

Theorem 8.1. *If a 3-dimensional Kenmotsu manifold M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ possesses an η-parallel Ricci tensor, then the scalar curvature of M is constant.*

Proof. In view of (2.2), (2.9) and (3.8), we have

$$\tilde{S}(\phi X, \phi Y) = \left(\frac{\tilde{r}+4}{2}\right)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{8.2}$$

Differentiating (8.2) covariantly with respect to the semi-symmetric metric connection $\tilde{\nabla}$ along W, we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(\phi X, \phi Y) = {} & \frac{d\tilde{r}(W)}{2}\{g(X,Y) - \eta(X)\eta(Y)\} \\ & - \left(\frac{\tilde{r}+4}{2}\right)\left\{(\tilde{\nabla}_W \eta)(X)\eta(Y) + (\tilde{\nabla}_W \eta)(Y)\eta(X)\right\} \\ & - \tilde{S}((\tilde{\nabla}_W \phi)(X), \phi Y) - \tilde{S}(\phi X, (\tilde{\nabla}_W \phi)(Y)).\end{aligned} \tag{8.3}$$

In view of (2.1), (2.3), (2.5), (3.3), (3.4), (8.1) and (8.3), it can be easily found that

$$\begin{aligned}& \frac{d\tilde{r}(W)}{2}\{g(X,Y) - \eta(X)\eta(Y)\} + 2\eta(X)\tilde{S}(\phi W, \phi Y) + 2\eta(Y)\tilde{S}(\phi W, \phi X) \\ & - (\tilde{r}+4)\{\eta(Y)g(X,W) + \eta(X)g(Y,W) - 2\eta(X)\eta(Y)\eta(W)\} = 0.\end{aligned} \tag{8.4}$$

In consequence of (8.2), (8.4) becomes

$$d\tilde{r}(W)\{g(X,Y) - \eta(X)\eta(Y)\} = 0,$$

which gives

$$d\tilde{r}(W) = 0 \Longleftrightarrow \tilde{r} \textit{ is constant}.$$

Hence the statement of the Theorem 8.1 is proved. □

In the light of the Theorem 6.1 and Theorem 8.1, we state the following corollary.

Corollary 8.1. *If a 3-dimensional Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ has η-parallel Ricci tensor, then the manifold is locally ϕ-symmetric as well as locally ϕ-concircularly symmetric with respect to $\tilde{\nabla}$, respectively.*

9. Three dimensional locally ϕ-concircularly recurrent Kenmotsu manifold with respect to the semi-symmetric metric connection

Definition 9.1. A Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is said to be ϕ-concircularly recurrent with respect to $\tilde{\nabla}$ if there exists a non-zero 1-form A on M such that

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z \tag{9.1}$$

for arbitrary vector fields X, Y, Z and W, where $\tilde{C}$ is the concircular curvature tensor with respect to the semi-symmetric metric connection $\tilde{\nabla}$. If the 1-form A vanishes identically on M, then the manifold M with $\tilde{\nabla}$ is reduced to a locally ϕ-concircularly symmetric manifold with respect to $\tilde{\nabla}$.

Theorem 9.1. *If a 3-dimensional locally ϕ-concircularly recurrent Kenmotsu manifold admits a semi-symmetric metric connection $\tilde{\nabla}$, then the curvature tensor with respect to $\tilde{\nabla}$ assumes the form (9.7).*

Proof. From (3.9) and (5.5), we have

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = -\frac{d\tilde{r}(W)}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{9.2}$$

On the other hand, from (1.3), it is seen that (for $n = 3$)

$$(\tilde{\nabla}_W \tilde{C})(X,Y)Z = (\tilde{\nabla}_W \tilde{R})(X,Y)Z - \frac{d\tilde{r}(W)}{6}\{g(Y,Z)X - g(X,Z)Y\}. \tag{9.3}$$

Applying ϕ^2 on both sides of (9.3), we get

$$\phi^2((\tilde{\nabla}_W \tilde{C})(X,Y)Z) = \phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) - \frac{d\tilde{r}(W)}{6}\{g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y\}. \tag{9.4}$$

In consequence of (2.1), (9.1) and (9.2), it is obvious that

$$\begin{aligned} A(W)\tilde{C}(X,Y)Z = &-\frac{d\tilde{r}(W)}{3}\{g(Y,Z)X - g(X,Z)Y\} \\ &-\frac{d\tilde{r}(W)}{6}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi. \end{aligned} \tag{9.5}$$

Replacing W with ξ in (9.5), we get

$$\begin{aligned} \tilde{C}(X,Y)Z = &-\frac{d\tilde{r}(\xi)}{3A(\xi)}\{g(Y,Z)X - g(X,Z)Y\} \\ &-\frac{d\tilde{r}(\xi)}{6A(\xi)}\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \end{aligned} \tag{9.6}$$

provided $A(\xi) \neq 0$. In view of (1.2) and (9.6), we have

$$\tilde{R}(X,Y)Z = a\{g(Y,Z)X - g(X,Z)Y\} - b\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \tag{9.7}$$

where $a = \left\{\frac{\tilde{r}}{6} - \frac{d\tilde{r}(\xi)}{3A(\xi)}\right\}$, $b = \frac{d\tilde{r}(\xi)}{6A(\xi)}$ and A is a non-zero 1-form. $\square$

10. Example of a Kenmotsu manifold admitting a semi-symmetric metric connection

In this section, we will construct a non-trivial example of a Kenmotsu manifold admitting the semi-symmetric metric connection and after that we will validate our results.

Example 10.1. Let

$$M = \{(x, y, z) \in \mathbb{R}^3 : x, y, z(\neq 0) \in \mathbb{R}\},$$

be a three dimensional Riemannian manifold, where (x, y, z) denotes the standard coordinates of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = z\frac{\partial}{\partial x}, \;\; e_2 = z\frac{\partial}{\partial y}, \;\; e_3 = -z\frac{\partial}{\partial z}$$

be a set of linearly independent vector fields at each point of the manifold M and therefore it forms a basis for the tangent space $T(M)$. We also define the Riemannian metric g of the manifold by $g(e_i, e_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i, j = 1, 2, 3$. Let us consider a 1-form η defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in T(M)$ and a tensor field ϕ of type $(1, 1)$ defined by

$$\phi(e_1) = -e_2, \;\; \phi(e_2) = e_1, \;\; \phi(e_3) = 0.$$

By the linearity properties of ϕ and g, we can easily verify the following relations

$$\phi^2 X = -X + \eta(X)e_3, \;\; \eta(e_3) = 1, \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for arbitrary vector fields $X, Y \in T(M)$. This shows that for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

If ∇ represents the Levi-Civita connection with respect to the Riemannian metric g, then with the help of above relations, we can easily calculate the non-vanishing components of Lie bracket as:

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = e_1, \;\; [e_2, e_3] = e_2.$$

We recall the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

for all vector fields $X, Y, Z \in T(M)$. It is obvious from Koszul's formula that

$$\begin{array}{lll} \nabla_{e_1} e_1 = -e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

From the above calculations, we can observe that $\nabla_X \xi = X - \eta(X)\xi$ for $\xi = e_3$. Thus the manifold (M, g) is a Kenmotsu manifold of dimension 3 and the structure (ϕ, η, ξ, g) denotes the Kenmotsu structure on the manifold M [16].

In consequence of (3.3) and the above results, we can find that

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_1 = -2e_3, & \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_1} e_3 = 2e_1, \\ \tilde{\nabla}_{e_2} e_1 = 0, & \tilde{\nabla}_{e_2} e_2 = -2e_3, & \tilde{\nabla}_{e_2} e_3 = 2e_2, \\ \tilde{\nabla}_{e_3} e_1 = 0, & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_3 = 0 \end{array}$$

and also the components of torsion tensor $\tilde{T}$ are

$$\tilde{T}(e_i, e_i) = \tilde{\nabla}_{e_i} e_i - \tilde{\nabla}_{e_i} e_i - [e_i, e_i] = 0, \ for \ i = 1, 2, 3$$
$$\tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = e_1, \quad \tilde{T}(e_2, e_3) = e_2.$$

This shows that $\tilde{T} \neq 0$ and, therefore, by the equation (3.1), we can say that the linear connection defined in (3.3) is a semi-symmetric connection on (M, g). By straightforward calculation, we can also find

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0$$

and other components by symmetric properties. This demonstrates that the equation (3.2) is satisfied and hence the linear connection defined by (3.3) is a semi-symmetric metric connection on M. Thus, we can say that the manifold (M, g) is a 3-dimensional Kenmotsu manifold equipped with a semi-symmetric metric connection defined by (3.3).

With the help of the above discussions, we can calculate the curvature and Ricci tensors of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ as

$$\tilde{R}(e_1, e_2)e_3 = 0, \quad \tilde{R}(e_1, e_3)e_3 = -2e_1, \quad \tilde{R}(e_3, e_2)e_2 = -2e_3,$$
$$\tilde{R}(e_3, e_1)e_1 = -2e_3, \quad \tilde{R}(e_2, e_1)e_1 = -4e_2, \quad \tilde{R}(e_2, e_3)e_3 = -2e_2,$$
$$\tilde{R}(e_1, e_2)e_2 = 0, \quad \tilde{S}(e_1, e_1) = -6, \quad \tilde{S}(e_2, e_2) = -2, \quad \tilde{S}(e_3, e_3) = -4$$

and other components can be calculated by skew-symmetric properties. We can easily observe that the equation (3.10) is verified.

Next, we have to prove that the manifold (M, g) is a Ricci semisymmetric with respect to the connection $\tilde{\nabla}$, *i.e.*, $\tilde{R} \cdot \tilde{S} = 0$. For instance,

$$(\tilde{R}(e_3, e_1) \cdot \tilde{S})(e_1, e_1) = 0, \quad (\tilde{R}(e_3, e_2) \cdot \tilde{S})(e_1, e_1) = 0, \quad (\tilde{R}(e_3, e_3) \cdot \tilde{S})(e_1, e_1) = 0.$$

In a similar way, we can verify other components. Also, we can prove that $\tilde{r} = -12$ (constant) and hence the Theorem 7.1 is verified. Moreover, it can be easily seen that the Theorem 5.1, Theorem 6.1 and the Theorem 8.1have been verified.

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CONFORMAL AND PARACONTACTLY GEODESIC TRANSFORMATIONS OF ALMOST PARACONTACT METRIC STRUCTURES

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Abstract. We give the expressions of the virtual and the structure tensor fields of an almost paracontact metric structure. We also introduce the notion of paracontactly geodesic transformation and prove that the structure tensor field is invariant under conformal and paracontactly geodesic transformations. For the particular case of para-Kenmotsu structure, we give a necessary and sufficient condition for a conformal transformation to map it to an α-para-Kenmotsu structure and show that a para-Kenmotsu manifold admits no nontrivial paracontactly geodesic transformation of the metric. In the conformal case, the virtual tensor field is invariant.
Keywords: tensor field, paracontact metric structure, geodesic transformation.

1. Introduction

Let M be a $(2n+1)$-dimensional smooth manifold, φ a $(1,1)$-tensor field called the *structure endomorphism*, ξ a vector field called the *characteristic vector field*, η a 1-form called the *contact form* and g a pseudo-Riemannian metric on M. In this case, we say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M [2] if

1. $\varphi^2 X = X - \eta(X)\xi$, for any $X \in \chi(M)$;
2. $\eta(\xi) = 1$;
3. $g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$, for any X, $Y \in \chi(M)$

and φ induces on the $2n$-dimensional distribution $\ker \eta$ an almost paracomplex structure P and the eigensubbundles corresponding to the eigenvalues 1 and -1 of P, respectively, have equal dimension n.

From the definition, it follows that $\varphi\xi = 0$, $\eta(\varphi X) = 0$, $\eta(X) = g(X, \xi)$, $g(\xi, \xi) = 1$, $g(\varphi X, Y) = -g(X, \varphi Y)$, for any X, $Y \in \chi(M)$ and $\ker \varphi^2 = \ker \varphi$.

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2. The virtual and the structure tensor fields

Consider $B_{\nabla}^{\varphi}(X,Y) := \frac{1}{2}\varphi((\nabla_{\varphi X}\varphi)\varphi Y + \varphi((\nabla_{\varphi^2 X}\varphi)\varphi Y))$ the *virtual* and $C_{\nabla}^{\varphi}(X,Y) := \frac{1}{2}\varphi((\nabla_{\varphi X}\varphi)\varphi Y - \varphi((\nabla_{\varphi^2 X}\varphi)\varphi Y))$ the *structure* tensor fields of the almost paracontact metric structure (φ,ξ,η,g) which is connected to the Nijenhuis tensor field of φ used in studying the normality of the structure.

Proposition 2.1. *The virtual and the structure tensor fields of the almost paracontact metric structure (φ,ξ,η,g) have the following properties:*

1. $\varphi(B_{\nabla}^{\varphi}(X,Y)) = B_{\nabla}^{\varphi}(\varphi X,Y) = -B_{\nabla}^{\varphi}(X,\varphi Y)$;
2. $B_{\nabla}^{\varphi}(\varphi X,\varphi Y) = -B_{\nabla}^{\varphi}(X,Y)$;
3. $g(B_{\nabla}^{\varphi}(X,Y),Z) + g(Y,B_{\nabla}^{\varphi}(X,Z)) = 0$;
4. $\varphi(C_{\nabla}^{\varphi}(X,Y)) = -C_{\nabla}^{\varphi}(\varphi X,Y) = -C_{\nabla}^{\varphi}(X,\varphi Y)$;
5. $C_{\nabla}^{\varphi}(\varphi X,\varphi Y) = C_{\nabla}^{\varphi}(X,Y)$;
6. $g(C_{\nabla}^{\varphi}(X,Y),Z) + g(Y,C_{\nabla}^{\varphi}(X,Z)) = 0$,

for any X, Y, $Z \in \chi(M)$.

Proof. Notice that

$$B_{\nabla}^{\varphi}(X,Y) := \frac{1}{2}\varphi(\nabla_{\varphi X}\varphi^2 Y - \varphi(\nabla_{\varphi X}\varphi Y) + \varphi(\nabla_{\varphi^2 X}\varphi^2 Y - \varphi(\nabla_{\varphi^2 X}\varphi Y))) =$$

$$= \frac{1}{2}[\varphi(\nabla_{\varphi X}\varphi^2 Y) - \varphi^2(\nabla_{\varphi X}\varphi Y) + \varphi^2(\nabla_{\varphi^2 X}\varphi^2 Y) - \varphi^3(\nabla_{\varphi^2 X}\varphi Y)].$$

We have:

$$\varphi(B_{\nabla}^{\varphi}(X,Y)) = \frac{1}{2}[\varphi^2(\nabla_{\varphi X}\varphi^2 Y) - \varphi^3(\nabla_{\varphi X}\varphi Y) + \varphi^3(\nabla_{\varphi^2 X}\varphi^2 Y) - \varphi^4(\nabla_{\varphi^2 X}\varphi Y)],$$

$$B_{\nabla}^{\varphi}(\varphi X,Y) = \frac{1}{2}[\varphi(\nabla_{\varphi^2 X}\varphi^2 Y) - \varphi^2(\nabla_{\varphi^2 X}\varphi Y) + \varphi^2(\nabla_{\varphi^3 X}\varphi^2 Y) - \varphi^3(\nabla_{\varphi^3 X}\varphi Y)],$$

$$B_{\nabla}^{\varphi}(X,\varphi Y) = \frac{1}{2}[\varphi(\nabla_{\varphi X}\varphi^3 Y) - \varphi^2(\nabla_{\varphi X}\varphi^2 Y) + \varphi^2(\nabla_{\varphi^2 X}\varphi^3 Y) - \varphi^3(\nabla_{\varphi^2 X}\varphi^2 Y)],$$

$$B_{\nabla}^{\varphi}(\varphi X,\varphi Y) = \frac{1}{2}[\varphi(\nabla_{\varphi^2 X}\varphi^3 Y) - \varphi^2(\nabla_{\varphi^2 X}\varphi^2 Y) + \varphi^2(\nabla_{\varphi^3 X}\varphi^3 Y) - \varphi^3(\nabla_{\varphi^3 X}\varphi^2 Y)].$$

Because $\varphi^3 = \varphi$ and $\varphi^4 = \varphi^2$ it follows $\varphi(B_{\nabla}^{\varphi}(X,Y)) = B_{\nabla}^{\varphi}(\varphi X,Y) = -B_{\nabla}^{\varphi}(X,\varphi Y)$ and $B_{\nabla}^{\varphi}(\varphi X,\varphi Y) = -B_{\nabla}^{\varphi}(X,Y)$.

Similarly we can show $\varphi(C_{\nabla}^{\varphi}(X,Y)) = -C_{\nabla}^{\varphi}(\varphi X,Y) = -C_{\nabla}^{\varphi}(X,\varphi Y)$ and $C_{\nabla}^{\varphi}(\varphi X,\varphi Y) = C_{\nabla}^{\varphi}(X,Y)$.

Taking into account that $g(\varphi X, Y) = -g(X, \varphi Y)$ and $g(\varphi^2 X, \varphi^2 Y) = -g(\varphi X, \varphi Y)$, for any X, $Y \in \chi(M)$, we get:

$$g(B^{\varphi}_{\nabla}(X,Y),Z) + g(Y, B^{\varphi}_{\nabla}(X,Z)) =$$

$$= \frac{1}{2}[-g(\nabla_{\varphi X}\varphi^2 Y, \varphi Z) - g(\nabla_{\varphi X}\varphi Y, \varphi^2 Z) + g(\nabla_{\varphi^2 X}\varphi^2 Y, \varphi^2 Z) + g(\nabla_{\varphi^2 X}\varphi Y, \varphi Z) -$$

$$-g(\nabla_{\varphi X}\varphi^2 Z, \varphi Y) - g(\nabla_{\varphi X}\varphi Z, \varphi^2 Y) + g(\nabla_{\varphi^2 X}\varphi^2 Z, \varphi^2 Y) + g(\nabla_{\varphi^2 X}\varphi Z, \varphi Y)] =$$

$$= \frac{1}{2}[-\varphi X(g(\varphi^2 Y, \varphi Z)) - \varphi X(g(\varphi Y, \varphi^2 Z)) + \varphi^2 X(g(\varphi^2 Y, \varphi^2 Z)) + \varphi^2 X(g(\varphi Y, \varphi Z))] = 0.$$

Similarly we can show $g(C^{\varphi}_{\nabla}(X,Y),Z) + g(Y, C^{\varphi}_{\nabla}(X,Z)) = 0$. □

Let m and l be the complementary projectors on the tangent bundle of M, defined by:

$$m := \eta \otimes \xi, \quad l := I - \eta \otimes \xi$$

and denoted by $\mathfrak{M} := Im(m)$ and $\mathfrak{L} := Im(l)$ (obviously, $l = \varphi^2$). Then $TM = \mathfrak{M} \oplus \mathfrak{L}$, and from the properties of the almost paracontact metric structure it follows that $\mathfrak{M} = \ker \varphi$ and $\mathfrak{L} = \ker \eta$.

By N_φ, we denoted the Nijenhuis tensor field of φ:

$$N_\varphi(X,Y) := \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] =$$

$$= \varphi^2(\nabla_X Y) - \varphi^2(\nabla_Y X) + \nabla_{\varphi X}\varphi Y - \nabla_{\varphi Y}\varphi X -$$

$$-\varphi(\nabla_{\varphi X} Y) + \varphi(\nabla_Y \varphi X) - \varphi(\nabla_X \varphi Y) + \varphi(\nabla_{\varphi Y} X).$$

Proposition 2.2. *If (φ, ξ, η, g) is an almost paracontact metric structure on M, then $\varphi^2(N_\varphi(\varphi X, \varphi Y)) = 2(C^{\varphi}_{\nabla}(Y,X) - C^{\varphi}_{\nabla}(X,Y))$, for any X, $Y \in \chi(M)$.*

Proof. We have:

$$\varphi^2(N_\varphi(\varphi X, \varphi Y)) = \varphi^2(\varphi^2(\nabla_{\varphi X}\varphi Y) - \varphi^2(\nabla_{\varphi Y}\varphi X) + \nabla_{\varphi^2 X}\varphi^2 Y - \nabla_{\varphi^2 Y}\varphi^2 X -$$

$$-\varphi(\nabla_{\varphi^2 X}\varphi Y) + \varphi(\nabla_{\varphi Y}\varphi^2 X) - \varphi(\nabla_{\varphi X}\varphi^2 Y) + \varphi(\nabla_{\varphi^2 Y}\varphi X)) =$$

$$= \varphi^4(\nabla_{\varphi X}\varphi Y) - \varphi^4(\nabla_{\varphi Y}\varphi X) + \varphi^2(\nabla_{\varphi^2 X}\varphi^2 Y) - \varphi^2(\nabla_{\varphi^2 Y}\varphi^2 X) -$$

$$-\varphi^3(\nabla_{\varphi^2 X}\varphi Y) + \varphi^3(\nabla_{\varphi Y}\varphi^2 X) - \varphi^3(\nabla_{\varphi X}\varphi^2 Y) + \varphi^3(\nabla_{\varphi^2 Y}\varphi X).$$

Because $\varphi^3 = \varphi$ and $\varphi^4 = \varphi^2$ it follows

$$\varphi^2(N_\varphi(\varphi X, \varphi Y)) = \varphi^2(\nabla_{\varphi X}\varphi Y) - \varphi^2(\nabla_{\varphi Y}\varphi X) + \varphi^2(\nabla_{\varphi^2 X}\varphi^2 Y) - \varphi^2(\nabla_{\varphi^2 Y}\varphi^2 X) -$$

$$-\varphi(\nabla_{\varphi^2 X}\varphi Y) + \varphi(\nabla_{\varphi Y}\varphi^2 X) - \varphi(\nabla_{\varphi X}\varphi^2 Y) + \varphi(\nabla_{\varphi^2 Y}\varphi X) =$$

$$= -[\varphi(\nabla_{\varphi X}\varphi^2 Y) - \varphi^2(\nabla_{\varphi X}\varphi Y) - \varphi^2(\nabla_{\varphi^2 X}\varphi^2 Y) + \varphi(\nabla_{\varphi^2 X}\varphi Y)] +$$

$$+[\varphi(\nabla_{\varphi Y}\varphi^2 X) - \varphi^2(\nabla_{\varphi Y}\varphi X) - \varphi^2(\nabla_{\varphi^2 Y}\varphi^2 X) + \varphi(\nabla_{\varphi^2 Y}\varphi X)] =$$

$$= -2C^{\varphi}_{\nabla}(X,Y) + 2C^{\varphi}_{\nabla}(Y,X).$$

□

Corollary 2.1. *Let (φ, ξ, η, g) be an almost paracontact metric structure on M.*

1. *If the Nijenhuis tensor field of φ vanishes identically, then the structure tensor field is symmetric.*
2. *If the structure tensor field is symmetric, then $N_\varphi(\varphi X, \varphi Y) \in \mathfrak{M}$, for any X, $Y \in \chi(M)$.*

Proof. From Proposition 2.2 we have for any X, $Y \in \chi(M)$:

$$\varphi^2(N_\varphi(\varphi X, \varphi Y)) = 2(C^\varphi_\nabla(Y, X) - C^\varphi_\nabla(X, Y)).$$

1. If $N_\varphi = 0$ follows $C^\varphi_\nabla(Y, X) - C^\varphi_\nabla(X, Y) = 0$, for any X, $Y \in \chi(M)$.
2. If $C^\varphi_\nabla(Y, X) = C^\varphi_\nabla(X, Y)$, for any X, $Y \in \chi(M)$ follows $\varphi^2(N_\varphi(\varphi X, \varphi Y)) = 0$ i.e. $N_\varphi(\varphi X, \varphi Y) \in \ker \varphi^2 = \ker \varphi = \mathfrak{M}$.

□

3. Paracontactly geodesic transformations

We will introduce the notion of paracontactly geodesic transformation of an almost paracontact metric structure and study the invariance of the virtual and structure tensor fields under paracontactly geodesic transformations.

Recall that a diffeomorphism between two pseudo-Riemannian manifolds $\Phi : (M, g) \to (\bar{M}, \bar{g})$ is called *geodesic map*, if it takes each geodesic of (M, g) to a geodesic of $(\bar{M}, \bar{g})$. In this case, the pseudo-Riemannian metric $\tilde{g} := \Phi^* \bar{g}$ on M is called *geodesic transformation* of g. Note that the metrics g and $\tilde{g}$ have common geodesics.

Let (φ, ξ, η, g) be an almost paracontact metric structure on the smooth manifold M. Then:

Definition 3.1. A geodesic transformation $g \to \tilde{g}$ of the pseudo-Riemannian metric g on M is called paracontactly geodesic transformation if $(\varphi, \xi, \eta, \tilde{g})$ is also an almost paracontact metric structure on M.

A simple example similar like in the almost contact case [3] is the following.

Example 3.1. Let $\Phi : (M, \varphi, \xi, \eta, g) \to (\bar{M}, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a geodesic map preserving the almost paracontact structure, that is, $\bar{\varphi} = \Phi_* \circ \varphi \circ (\Phi_*)^{-1}$, $\bar{\xi} = \Phi_*(\xi)$, $\bar{\eta} = (\Phi^*)^{-1}\eta$. Then $g \to \tilde{g} := \Phi^* \bar{g}$ is a paracontactly geodesic transformation of g on M.

Let (φ, ξ, η, g) be an almost paracontact metric structure on the smooth manifold M and $g \to \tilde{g}$ a paracontactly geodesic transformation of g.

It was proved [5] that the tensor T of the affine deformation from the Levi-Civita connection ∇ of g to the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ has the form $T(X, Y) := \tilde{\nabla}_X Y - \nabla_X Y = \psi(X)Y + \psi(Y)X$, X, $Y \in \chi(M)$, for ψ an exact 1-form on M called the 1-*form of geodesic distortion*. In this case, the Levi-Civita connections associated to g and $\tilde{g}$ satisfy $(\tilde{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y + \psi(\varphi Y)X - \psi(Y)\varphi X$, for any X, $Y \in \chi(M)$.

Proposition 3.1. *The virtual and the structure tensor fields of the transformed structure have the following properties:*

1. $B^{\varphi}_{\tilde{\nabla}}(X,Y) = B^{\varphi}_{\nabla}(X,Y) + \psi(\varphi^2 Y)\varphi^2 X - \psi(\varphi Y)\varphi X$;
2. $C^{\varphi}_{\tilde{\nabla}}(X,Y) = C^{\varphi}_{\nabla}(X,Y)$,

for any X, $Y \in \chi(M)$.

Proof.

$$B^{\varphi}_{\tilde{\nabla}}(X,Y) := \frac{1}{2}\varphi((\tilde{\nabla}_{\varphi X}\varphi)\varphi Y + \varphi((\tilde{\nabla}_{\varphi^2 X}\varphi)\varphi Y)) =$$
$$= \frac{1}{2}\varphi((\nabla_{\varphi X}\varphi)\varphi Y + \psi(\varphi^2 Y)\varphi X - \psi(\varphi Y)\varphi^2 X +$$
$$+\varphi((\nabla_{\varphi^2 X}\varphi)\varphi Y + \psi(\varphi^2 Y)\varphi^2 X - \psi(\varphi Y)\varphi^3 X)) :=$$
$$:= B^{\varphi}_{\nabla}(X,Y) + \frac{1}{2}[\psi(\varphi^2 Y)\varphi^2 X - \psi(\varphi Y)\varphi^3 X + \psi(\varphi^2 Y)\varphi^4 X - \psi(\varphi Y)\varphi^5 X].$$

Because $\varphi^3 = \varphi$, $\varphi^4 = \varphi^2$ and $\varphi^5 = \varphi$, we get the first relation. The second one can be similarly obtained. □

We can therefore state the theorem:

Theorem 3.1. *The structure tensor field of an almost paracontact metric structure is invariant under paracontactly geodesic transformations.*

Concerning the virtual tensor, we give necessary and sufficient conditions for it to be invariant.

Theorem 3.2. *The virtual tensor field of the almost paracontact metric structure* (φ,ξ,η,g) *is invariant under paracontactly geodesic transformations* $g \to \tilde{g}$ *(i.e.* $B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla}$*) if and only if* $\psi(\varphi^2 Y)\varphi X - \psi(\varphi Y)X \in \mathfrak{M}$, *for any* X, $Y \in \chi(M)$.

Proof. From Proposition 3.1, the condition $B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla}$ is equivalent to $0 = \psi(\varphi^2 Y)\varphi^2 X - \psi(\varphi Y)\varphi X = \varphi[\psi(\varphi^2 Y)\varphi X - \psi(\varphi Y)X]$, for any $X, Y \in \chi(M)$. □

4. Conformal transformations

By a *conformal transformation* of the almost paracontact metric structure (φ,ξ,η,g) we understand the passage to the almost paracontact metric structure $(\varphi,\tilde{\xi},\tilde{\eta},\tilde{g})$, where $\tilde{\xi} := e^f\xi$, $\tilde{\eta} := e^{-f}\eta$, $\tilde{g} := e^{-2f}g$, for f a smooth function on the manifold M. In the part to follow, we shall study the invariance of the virtual and the structure tensor fields under conformal transformations.

Because the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ and ∇ of g satisfy $\tilde{\nabla}_X Y = \nabla_X Y - [X(f)Y + Y(f)X - g(X,Y)grad_g(f)]$ it follows $(\tilde{\nabla}_X\varphi)Y = (\nabla_X\varphi)Y - df(\varphi Y)X + df(Y)\varphi X + g(X,\varphi Y)grad_g(f) - g(X,Y)\varphi(grad_g(f))$, for any X, $Y \in \chi(M)$.

Proposition 4.1. *The virtual and the structure tensor fields of the transformed structure have the following properties:*

1. $B_{\tilde{\nabla}}^{\varphi}(X,Y) = B_{\nabla}^{\varphi}(X,Y) - df(\varphi^2 Y)\varphi^2 X + df(\varphi Y)\varphi X - g(\varphi X, \varphi Y)\varphi^2(grad_g(f)) + g(\varphi X, Y)\varphi(grad_g(f))$;
2. $C_{\tilde{\nabla}}^{\varphi}(X,Y) = C_{\nabla}^{\varphi}(X,Y)$,

for any X, $Y \in \chi(M)$.

Proof.

$$B_{\tilde{\nabla}}^{\varphi}(X,Y) := \frac{1}{2}\varphi((\tilde{\nabla}_{\varphi X}\varphi)\varphi Y + \varphi((\tilde{\nabla}_{\varphi^2 X}\varphi)\varphi Y)) =$$

$$= \frac{1}{2}\varphi((\nabla_{\varphi X}\varphi)\varphi Y - df(\varphi^2 Y)\varphi X + df(\varphi Y)\varphi^2 X + g(\varphi X, \varphi^2 Y)grad_g(f) -$$

$$-g(\varphi X, \varphi Y)\varphi(grad_g(f)) + \varphi((\nabla_{\varphi^2 X}\varphi)\varphi Y - df(\varphi^2 Y)\varphi^2 X + df(\varphi Y)\varphi^3 X +$$

$$+g(\varphi^2 X, \varphi^2 Y)grad_g(f) - g(\varphi^2 X, \varphi Y)\varphi(grad_g(f)))) :=$$

$$:= B_{\nabla}^{\varphi}(X,Y) + \frac{1}{2}[-df(\varphi^2 Y)\varphi^2 X + df(\varphi Y)\varphi^3 X + g(\varphi X, \varphi^2 Y)\varphi(grad_g(f)) -$$

$$-g(\varphi X, \varphi Y)\varphi^2(grad_g(f)) - df(\varphi^2 Y)\varphi^4 X + df(\varphi Y)\varphi^5 X +$$

$$+g(\varphi^2 X, \varphi^2 Y)\varphi^2(grad_g(f)) - g(\varphi^2 X, \varphi Y)\varphi^3(grad_g(f))].$$

Because $\varphi^3 = \varphi$, $\varphi^4 = \varphi^2$ and $\varphi^5 = \varphi$, $g(\varphi X, Y) = -g(X, \varphi Y)$ and $g(\varphi^2 X, \varphi^2 Y) = -g(\varphi X, \varphi Y)$, for any X, $Y \in \chi(M)$, we get the first relation. The second one can be similarly obtained. □

We can therefore state the theorem:

Theorem 4.1. *The structure tensor field of an almost paracontact metric structure is invariant under conformal transformations.*

Concerning the virtual tensor, we shall find a necessary and sufficient condition for it to be also invariant. Note that in the particular case when $\tilde{g} = \lambda g$, for λ a constant positive function, $B_{\tilde{\nabla}}^{\varphi} = B_{\nabla}^{\varphi}$.

Theorem 4.2. *The virtual tensor field of the almost paracontact metric structure* (φ, ξ, η, g) *is invariant under conformal transformations* $g \to \tilde{g} := e^{-2f}g$ *(i.e.* $B_{\tilde{\nabla}}^{\varphi} = B_{\nabla}^{\varphi}$*) if and only if* $grad_g(f) \in \mathfrak{M}$.

Proof. We follow the steps used in proving an analogue result for the almost contact metric case [4]. Therefore, $B_{\tilde{\nabla}}^{\varphi} = B_{\nabla}^{\varphi}$ if and only if for any X, $Y \in \chi(M)$

$$df(\varphi^2 Y)\varphi^2 X - df(\varphi Y)\varphi X = -g(\varphi X, \varphi Y)\varphi^2(grad_g(f)) + g(\varphi X, Y)\varphi(grad_g(f)).$$

Changing the role of X and Y in the previous relation, we obtain:

$$df(\varphi^2X)\varphi^2Y - df(\varphi X)\varphi Y = -g(\varphi Y, \varphi X)\varphi^2(grad_g(f)) + g(\varphi Y, X)\varphi(grad_g(f)) =$$

$$= -g(\varphi Y, \varphi X)\varphi^2(grad_g(f)) - g(Y, \varphi X)\varphi(grad_g(f))$$

and adding the two relations we get:

$$df(\varphi^2Y)\varphi^2X - df(\varphi Y)\varphi X + df(\varphi^2X)\varphi^2Y - df(\varphi X)\varphi Y = -2g(\varphi X, \varphi Y)\varphi^2(grad_g(f)).$$

Then for $Y := X$ we obtain:

$$df(\varphi^2X)\varphi^2X - df(\varphi X)\varphi X = -g(\varphi X, \varphi X)\varphi^2(grad_g(f)).$$

If we assume that $g(X, X) = 1$ it follows that $g(\varphi X, \varphi X) = -1 + [\eta(X)]^2$ and if $X \in \mathfrak{L}$ we get $\eta(X) = 0$ and $\varphi^2X = X$. Therefore, for $X \in \mathfrak{L}$ such that $g(X, X) = 1$ we obtain $g(\varphi X, \varphi X) = -1$ and $\varphi^2X = X$. Then the equality becomes

$$df(X)X - df(\varphi X)\varphi X = \varphi^2(grad_g(f)),$$

which means that $\varphi^2(grad_g(f))$ is a linear combination of the vector fields X and φX. Writing this for $X := Y$ we get:

$$df(Y)Y - df(\varphi Y)\varphi Y = \varphi^2(grad_g(f))$$

and substracting the two relations

$$df(X)X - df(\varphi X)\varphi X - df(Y)Y + df(\varphi Y)\varphi Y = 0.$$

Since the four vector fields $\{X, \varphi X, Y, \varphi Y\}$ are linearly independent, we deduce that $df(X) = 0$ equivalent to $g(grad_g(f), X) = 0$, for $X \in \mathfrak{L}$, which means that $grad_g(f) \in \mathfrak{L}^\perp = \mathfrak{M}$.

Conversely, if $grad_g(f) \in \mathfrak{M} = \ker\varphi$ we get $\varphi^2(grad_g(f)) = \varphi(grad_g(f)) = 0$. Then

$$df(\varphi^2Y)\varphi^2X - df(\varphi Y)\varphi X = g(grad_g(f), \varphi^2Y)\varphi^2X - g(grad_g(f), \varphi Y)\varphi X =$$

$$= g(\varphi^2(grad_g(f)), Y)\varphi^2X + g(\varphi(grad_g(f)), Y)\varphi X = 0$$

and so $B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla}$. $\square$

5. Applications

We shall consider the particular case of a para-Kenmotsu manifold. Let α be a smooth real function on the smooth manifold M.

Definition 5.1. [6] We say that the almost paracontact metric structure (φ, ξ, η, g) is α-para-Kenmotsu if the Levi-Civita connection ∇ of g satisfies $(\nabla_X\varphi)Y = \alpha[g(\varphi X, Y)\xi - \eta(Y)\varphi X]$, for any X, $Y \in \chi(M)$.

For $\alpha \equiv 1$, we call (φ, ξ, η, g) *para-Kenmotsu structure.*

Theorem 5.1. *A conformal transformation with defining function f maps a para-Kenmotsu structure to an e^f-para-Kenmotsu structure if and only if f is locally constant.*

Proof. Let (φ, ξ, η, g) be a para-Kenmotsu structure on M and $(\varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be a conformal transform of (φ, ξ, η, g) with $\tilde{\xi} = e^f \xi$, $\tilde{\eta} = e^{-f}\eta$, $\tilde{g} = e^{-2f} g$.

We know that:

$$(\tilde{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y - df(\varphi Y)X + df(Y)\varphi X + g(X, \varphi Y) grad_g(f) - g(X, Y)\varphi(grad_g(f)),$$

for any X, $Y \in \chi(M)$.

If (φ, ξ, η, g) is para-Kenmotsu structure, we have:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

for any X, $Y \in \chi(M)$. It follows:

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X - df(\varphi Y)X + df(Y)\varphi X +$$

$$+ g(X, \varphi Y) grad_g(f) - g(X, Y)\varphi(grad_g(f)),$$

for any X, $Y \in \chi(M)$. Replacing $\xi = e^{-f}\tilde{\xi}$, $\eta = e^f \tilde{\eta}$, $g = e^{2f}\tilde{g}$, we obtain:

$$(\tilde{\nabla}_X \varphi)Y = e^f[\tilde{g}(\varphi X, Y)\tilde{\xi} - \tilde{\eta}(Y)\varphi X] - df(\varphi Y)X + df(Y)\varphi X +$$

$$+ e^{2f}[\tilde{g}(X, \varphi Y) grad_g(f) - \tilde{g}(X, Y)\varphi(grad_g(f))],$$

for any X, $Y \in \chi(M)$.

Then $(\varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is e^f-para-Kenmotsu structure, i.e.

$$(\tilde{\nabla}_X \varphi)Y = e^f[\tilde{g}(\varphi X, Y)\tilde{\xi} - \tilde{\eta}(Y)\varphi X],$$

for any X, $Y \in \chi(M)$ if and only if

$$df(\varphi Y)X - df(Y)\varphi X = e^{2f}[\tilde{g}(X, \varphi Y) grad_g(f) - \tilde{g}(X, Y)\varphi(grad_g(f))],$$

for any X, $Y \in \chi(M)$. For $X := \xi$ and $Y := \xi$, because $\varphi\xi = 0$, we get $\varphi(grad_g(f)) = 0$, i.e. $grad_g(f) \in \ker \varphi = \mathfrak{M}$. Replacing this in the previous relation, we have:

$$df(\varphi Y)X - df(Y)\varphi X = e^{2f}\tilde{g}(X, \varphi Y) grad_g(f),$$

for any X, $Y \in \chi(M)$. For $Y := \xi$, because $\varphi\xi = 0$, we get $df(\xi) = 0$ which is equivalent with $g(grad_g(f), \xi) = 0$ and with $grad_g(f) \perp \xi$, i.e. $grad_g(f) \in \ker \eta = \mathfrak{L}$.

Therefore, $grad_g(f) \in \mathfrak{M} \cap \mathfrak{L} = \{0\}$, which means that f is locally constant.

Conversely, if f is locally constant, then $df = 0$, $grad_g(f) = 0$ and

$$(\tilde{\nabla}_X \varphi)Y = e^f[\tilde{g}(\varphi X, Y)\tilde{\xi} - \tilde{\eta}(Y)\varphi X] - df(\varphi Y)X + df(Y)\varphi X+$$
$$+e^{2f}[\tilde{g}(X, \varphi Y)grad_g(f) - \tilde{g}(X, Y)\varphi(grad_g(f))] =$$
$$= e^f[\tilde{g}(\varphi X, Y)\tilde{\xi} - \tilde{\eta}(Y)\varphi X],$$

for any X, $Y \in \chi(M)$, i.e. $(\varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is e^f-para-Kenmotsu structure. □

Corollary 5.1. *If a conformal transformation with defining function f maps a para-Kenmotsu structure to an e^f-para-Kenmotsu one, then the virtual tensor field is invariant.*

Proof. From Theorem 5.1 we have that f is locally constant and from Proposition 4.1 we obtain $B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla}$. □

Theorem 5.2. *A para-Kenmotsu manifold admits no nontrivial paracontactly geodesic transformation of the metric.*

Proof. Let (φ, ξ, η, g) be a para-Kenmotsu structure on M and $g \mapsto \tilde{g}$ be a paracontactly geodesic transformation with $\tilde{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X$, for X, $Y \in \chi(M)$.

We know that:

$$(\tilde{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y + \psi(\varphi Y)X - \psi(Y)\varphi X,$$

for any X, $Y \in \chi(M)$.

If (φ, ξ, η, g) is para-Kenmotsu structure, we have:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

for any X, $Y \in \chi(M)$. It follows:

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X + \psi(\varphi Y)X - \psi(Y)\varphi X,$$

for any X, $Y \in \chi(M)$.

If the transformed structure $(\varphi, \xi, \eta, \tilde{g})$ would be para-Kenmotsu, too, then:

$$(\tilde{\nabla}_X \varphi)Y = \tilde{g}(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

for any X, $Y \in \chi(M)$ and replacing this in the previous relation, we obtain:

$$[\tilde{g}(\varphi X, Y) - g(\varphi X, Y)]\xi = \psi(\varphi Y)X - \psi(Y)\varphi X,$$

for any X, $Y \in \chi(M)$. For $X := \xi$, because $\varphi\xi = 0$, we get $\psi(\varphi Y) = 0$, for any $Y \in \chi(M)$. It follows:

$$[\tilde{g}(\varphi X, Y) - g(\varphi X, Y)]\xi = -\psi(Y)\varphi X,$$

for any X, $Y \in \chi(M)$ and applying φ, we get:

$$\psi(Y)\varphi X = 0,$$

for any X, $Y \in \chi(M)$, which implies $\psi = 0$ or $\varphi = 0$, that is impossible. Therefore, the paracontactly geodesic transformation $g \mapsto \tilde{g}$ can not map the para-Kenmotsu structure (φ, ξ, η, g) to a para-Kenmotsu one. □

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PROPERTIES OF T-SPREAD PRINCIPAL BOREL IDEALS GENERATED IN DEGREE TWO *

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Abstract. In this paper, we have studied the stability of t-spread principal Borel ideals in degree two. We have proved that $\mathrm{Ass}^{\infty}(I) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}$, where $I = B_t(u) \subset S$ is a t-spread Borel ideal generated in degree 2 with $u = x_i x_n, t+1 \leq i \leq n-t$. Indeed, I has the property that $\mathrm{Ass}(I^m) = \mathrm{Ass}(I)$ for all $m \geq 1$ and $i \leq t$, in other words, I is normally torsion free. Moreover, we have shown that I is a set theoretic complete intersection if and only if $u = x_{n-t}x_n$. Also, we have derived some results on the vanishing of Lyubeznik numbers of these ideals.
Keywords: Monomial ideals, t-spread principal Borel ideals, Arithmetical rank, Complete intersection.

1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $I \subset S$ a graded ideal. By a well-known result of Brodmann [4], there exists an integer $k \geq 1$ such that $\mathrm{Ass}(I^m) = \mathrm{Ass}(I^k)$ for all $m \geq k$. A prime ideal $P \in \mathrm{Ass}^{\infty}(I) = \bigcup_{m\geq 1} \mathrm{Ass}(I^m)$ is called *persistent* with respect to I, and whenever $P \in \mathrm{Ass}(I^k)$ we have $P \in \mathrm{Ass}(I^{k+1})$. The ideal I has the *persistence property* if all the prime ideals $P \in \mathrm{Ass}^{\infty}(I)$ are persistent, that is, if $\mathrm{Ass}(I) \subseteq \mathrm{Ass}(I^2) \subseteq \cdots \subseteq \mathrm{Ass}(I^m) \subseteq \cdots$.

The persistence property for monomial ideals has been intensively studied in the last years; see for example, [10] and the references therein. Recently, it has been proved in [1] that t-spread principal Borel ideals have the persistence property. The so-called t-spread ideals were introduced in [7].

Let $t \geq 1$ be an integer. A monomial $x_{i_1} \cdots x_{i_d} \in S$ with $i_1 \leq \cdots \leq i_d$ is called *t-spread* if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. We recall from [7] that a monomial ideal $I \subset S$ with the minimal system of monomial generators $G(I)$ is called *t-spread principal Borel* if there exists a monomial $u \in G(I)$ such that $I = B_t(u)$, where $B_t(u)$ denotes the smallest t-spread strongly stable ideal which contains u. A monomial ideal I is

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called *t-spread strongly stable* if it satisfies the following condition: for all $u \in G(I)$ and $j \in \operatorname{supp}(u)$, if $i < j$ and $x_i(u/x_j)$ is t-spread, then $x_i(u/x_j) \in I$.

In this paper, we will study several properties of t-spread principal Borel ideals $B_t(u)$ generated in small degree. Most part of the paper is devoted to the study of $\operatorname{Ass}^\infty(B_t(u))$. In the second part of the paper we will study the arithmetical rank of $B_t(u)$. In the last part, we will derive some results on the vanishing of Lyubeznik numbers of $B_t(u)$.

The main result of the first section shows that if $I = B_t(u) \subset S$ is a t-spread Borel ideal generated in degree 2 with $u = x_i x_n, t+1 \leq i \leq n-t$, then $\operatorname{Ass}(I^m)$ is already stabilized at $m = 2$ and $\operatorname{Ass}^\infty(I) = \operatorname{Min}(I) \cup \{\mathfrak{m}\}$, where $\operatorname{Min}(I)$ denotes the set of minimal prime ideals of I and $\mathfrak{m}$ is the maximal graded ideal of S. The hypothesis $i \geq t+1$ might look restrictive, but as we explain in Remark 2.4, this is the only case when $\operatorname{Ass}^\infty(I) \supsetneq \operatorname{Min}(I)$.

For the proof, one has to consider monomial localization of a monomial ideal. Let $P = P_A = (x_j : j \notin A)$ be a monomial prime ideal and $I \subset S$ a monomial ideal. Then the localization of I with respect to P is $I(P) \subset S(P) = K[\{x_j : j \notin A\}]$ which is obtained from I by applying the K-algebra homomorphism $S \to S(P)$ induced by $x_j \mapsto 1$ for $j \notin A$. Moreover, by [11, Lemma 2.3], we have $P \in \operatorname{Ass}(I)$ if and only if $\operatorname{depth} S(P)/I(P) = 0$.

It was observed in [1] that all the powers of a t-spread principal Borel ideal have linear quotients with respect to the decreasing lexicographic order. By monomial localization of a t-spread principal Borel ideal generated in degree 2, we can get monomial ideals which still have linear quotients though they are not generated in a single degree. Therefore, we can compute the depth of their powers by using the projective dimension formula given in [9, Chapter 8]. Namely, let $I \subset S$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. We say that I has linear quotients with respect to the order $u_1, \ldots, u_m$ of its minimal monomial generators if for every $j \geq 1$, the ideal quotient $L_j = (u_1, \ldots, u_{j-1}) : u_j$ is generated by variables. If r_j is the number of variables which generate L_j for every j, then $\operatorname{proj\,dim} S/I = \max\{r_1, \ldots, r_m\} + 1$, hence

$$\operatorname{depth} S/I = n - 1 - \max\{r_1, \ldots, r_m\}. \tag{1.1}$$

We should note that the persistence property of every t-spread principal Borel ideal $B_t(u)$ generated in degree 2 may be derived by using [6, Theorem 2.15] since $B_t(u)$ can be viewed as the edge ideal of a graph.

Let $I \subset S$ be a homogeneous ideal and $\sqrt{I}$ the radical of I. Then the *arithmetical rank* of I is defined as

$$\operatorname{ara}(I) = \min\{r \geq 1 : \text{ there exists } f_1, \ldots, f_r \in I \text{ such that } \sqrt{I} = \sqrt{(f_1, \ldots, f_r)}\}.$$

It is known that for every squarefree monomial ideal $I \subset S$, we have

$$\operatorname{ara}(I) \geq \operatorname{cd}(I) = \operatorname{proj\,dim}(S/I), \tag{1.2}$$

where $\mathrm{cd}(I)$ denotes the cohomological dimension of I [14].

If $\mathrm{height}(I) = \mathrm{ara}(I)$, the ideal I is called a set-theoretic complete intersection. An ideal I is called cohomologically complete intersection if $ht(I) = cd(I)$.

There are several classes of squarefree monomial ideals for which equality holds in inequality (1.2); see, for example, [3, 5, 8, 12]. In [12] and [5] it was shown that if $I \subset S$ is a squarefree monomial ideal with a 2-linear resolution, then $\mathrm{ara}(I) = \mathrm{proj\,dim}(S/I)$. As a consequence of [7, Theorem 1.4], it follows that every t-spread principal Borel ideal has a 2-linear resolution, thus if $I = B_t(u)$ where u is a t-spread monomial of degree 2, then we have $\mathrm{ara}(I) = \mathrm{proj\,dim}(S/I)$. In Section 3. we give a direct proof of this equality by using the Schmitt-Vogel Lemma (see [15]) which might be interesting for the reader. In particular, we derive that $I = B_t(u)$ is a set theoretic complete intersection ideal if and only if $u = x_{n-t}x_n.$.

Finally, in Section 4., we derive some results on the vanishing of Lyubeznik numbers of t-spread principal Borel ideals in degree two.

2. Stability for the associated primes

In this section, we aim at proving the following:

Theorem 2.1. *Let I be a t-spread principal Borel ideal, where $u = x_i x_n$, $t+1 \leq i \leq n-t$. Then*

$$\mathrm{Ass}(I^m) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}, \text{ for } m \geq 2.$$

In particular,

$$\mathrm{Ass}^\infty(I) = \mathrm{Min}(I) \cup \{\mathfrak{m}\}.$$

In order to prove this theorem, we need some preparation.
Let $u = x_i x_n$ with $i \leq t$ and $I = B_t(u)$. We set $\mathcal{S}(I) = \bigcup_{v \in G(I)} \mathrm{supp}(v)$. If $i < t$, then $\mathcal{S}(I) \subsetneq [n]$. Then, as it was observed in the proof of [1, Theorem 3.1], since I satisfies the l- exchange property, it follows that I^m has linear quotients with respect to $>_{lex}$ for every $m \geq 1$. This means that if $G(I^m) = \{u_1 >_{lex} u_2 >_{lex} \ldots u_q >_{lex}\}$ then for every $j \geq 1$, the ideal quotient $(u_1, \ldots, u_{j-1}) : u_j$ is generated by variables.

Lemma 2.2. *In the above settings, for every $j \geq 1$, $x_n, x_i \notin (u_1, ..., u_{j-1}) : u_j$.*

Proof. Clearly $x_n \notin (u_1, ..., u_{j-1}) : u_j$ since we cannot write $x_n u_j$ as a multiple of u_l with $l \leq j-1$.

As $i \leq t$, the generators of I are the form of $x_{i_l} x_{j_l}$ with $1 \leq i_l \leq i \leq t$, $j_l > t$. Assume that there exists $j \geq 2$ such that $x_i u_j \in (u_1, ..., u_{j-1})$. Let $u_j = (x_{i_1} x_{j_1}) \ldots (x_{i_m} x_{j_m})$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq i \leq t$ and $t < j_1, \ldots, j_m \leq n$. Then $u_j = (x_{i_1} \ldots x_{i_m})(x_{j_1} \ldots x_{j_m})$. If $x_i u_j \in (u_1, ..., u_{j-1})$, then there exists some monomial $u_l \in G(I^m)$ with $l \leq j-1$ such that $x_i u_j = u_l x_s$, for some $s > i$. Let $u_l = (x_{i'_1} \ldots x_{i'_m})(x_{j'_1} \ldots x_{j'_m})$ with $1 \leq i'_1 \leq i'_2 \leq \ldots \leq i'_m \leq i \leq t$ and $t < j'_1, \ldots, j'_m \leq n$.

We have $x_i(x_{i_1}\dots x_{i_m})(x_{j_1}\dots x_{j_m}) = (x'_{i_1}\dots x'_{i_m})(x'_{j_1}\dots x'_{j_m})x_s$ with $s > i$. But then,

$$\sum_{j=1}^{i}\deg_{x_j}(x_iu_j) = m+1 > m = \sum_{j=1}^{i}\deg_{x_j}(u_lx_s)$$

which is contradiction. $\square$

In particular, by (1.1), the above lemma shows that

$$\operatorname{depth}(K[\{x_j : j \in \mathcal{S}(I)\}]/I^m) > 0, \text{ for every } m \geq 1.$$

First, we will identify the minimal prime ideals of $I = B_t(u)$, where $u = x_ix_n$ and $t+1 \leq i \leq n-t$. By applying [1, Theorem 1.1], it follows that

$$\text{(2.1)} \qquad \operatorname{Min}(I) = \{(x_1, ..., x_i)\} \cup \{(x_1, ..., x_{j_1-1}, x_{j_1+t}, ..., x_n) : \ 1 \leq j_1 \leq i\}.$$

Let Q be a monomial prime ideal associated to I^m for some $m \geq 2$. Then $Q = Q_A = (x_j : j \notin A)$ for some set $A \subset [n]$ and $\operatorname{depth} S(Q)/I(Q)^m = 0$, where $S(Q) = K[\{x_j : j \notin A\}]$ and $I(Q)$ is the localization of the ideal I with respect to Q, that is, $I(Q)$ is obtained from I by mapping the variables $x_j \to 1$ for $j \in A$. Therefore, in order to find all the associated monomial prime ideals of I^m for $m \geq 2$, we need to consider the localization of I with respect to some variable.

Lemma 2.3. *Let k be a positive integer and $P_{\{k\}} = (x_j : j \in [n]\backslash\{k\})$. Let $I = B_t(u)$ with $u = x_ix_n$, $t+1 \leq i \leq n-t$, and let $k \in [n]$. Then*

(1) *If $k = 1$, then $I(P_{\{k\}}) = (x_{1+t}, \dots, x_n)$.*

(2) *If $1 < k \leq t$, then*

$$I(P_{\{k\}}) = (x_{k+t}, \dots, x_n) + \bar{B}_{t-1}(x_{k-1}x_{k+t-1})S(P_{\{k\}})$$

where $\bar{B}_{t-1}(x_{k-1}x_{k+t-1})$ is the $(t-1)$-spread principal Borel ideal generated by $x_{k-1}x_{k+t-1}$ in the polynomial ring $K[\{x_1, \dots, x_{k+t-1}\} \setminus \{x_k\}]$.

(3) *If $t < k \leq i$, then*

$$I(P_{\{k\}}) = (x_1, \dots, x_{k-t}, x_{k+t}, \dots, x_n) + \bar{B}_{t-1}(x_{k-1}x_{k+t-1})S(P_{\{k\}})$$

where $\bar{B}_{t-1}(x_{k-1}x_{k+t-1})$ is the $(t-1)$-spread principal Borel ideal in the polynomial ring $K[\{x_{k-1}, \dots, x_{k+t-1}\} \setminus \{x_k\}]$.

(4) *If $i < k < i+t$, then*

$$I(P_{\{k\}}) = (x_1, \dots, x_{k-t}) + \bar{B}_{t-1}(x_ix_n)S(P_{\{k\}})$$

where $\bar{B}_{t-1}(x_ix_n)$ is the $(t-1)$-spread principal Borel ideal in the polynomial ring $K[\{x_{k-t+1}, \dots, x_n\} \setminus \{x_k\}]$.

(5) *If* $k \geq i+t$, *then* $I(P_{\{k\}}) = (x_1, \ldots, x_i)$.

Proof. Assumptions and definition of monomial localization imply that $I(P_{\{k\}})$ for all cases, as desired. □

Proof of Theorem 1.1 In order to prove the statement of the theorem, we have to show that for $m \geq 2$, I^m there is no other associated prime ideal except the minimal prime ideals of I and the maximal ideal. Notice that $\mathfrak{m} \in \operatorname{Ass}(I^m)$ for every $m \geq 2$ by [1, Theorem 3.1].

Let $Q = Q_A = (x_j : j \notin A)$ be a monomial prime ideal which contains I^m, $Q \neq \mathfrak{m}$. Then, $Q \in \operatorname{Ass}(I^m)$ if and only if depth $\frac{S(Q)}{I(Q)^m} = 0$ where $S(Q) = K[\{x_j : j \notin A\}]$ and $I(Q)$ is the localization of I with respect to Q. Thus, in order to prove the desired statement, we have to show that if $Q \notin \operatorname{Min}(I)$, then depth $S(Q)/I(Q)^m > 0$.

We will distinguish the following cases.

Case (i). $Q = Q_A \supset (x_1, \ldots, x_i)$. Let $k = \max A$. If $k \geq i+t$, then $I(Q) = I(P_{\{k\}}) = (x_1, \ldots, x_i)$. Since $Q \neq (x_1, \ldots, x_i)$, there exists $x_l \in Q$ with $l > i$. Thus, depth $S(Q)/I(Q)^m > 0$ since x_l is regular on $S(Q)/I(Q)^m$. Thus Q is not an associated prime of I^m.

Now we assume that $k = \max A < i+t$. Obviously, we have $k \geq \min A > i$. Then $Q = Q_A \supset (x_1, \ldots, x_i, x_{i+t}, \ldots, x_n)$. Then by using Lemma 2.3, we get $I(Q) = (x_1, \ldots, x_{k-t}) + \bar{B}_{t-1}(x_i x_n)S(Q)$, where $\bar{B}_{t-1}(x_i x_n)$ is the $(t-1)$-spread principal Borel ideal in the polynomial ring $K[\{x_{k-t+1}, \ldots, x_n\} \setminus \{x_k\}]$. Then

$$I(Q)^m = \sum_{l=0}^{m} (x_1, \ldots, x_{k-t})^{m-l} \left(\bar{B}_{t-1}(x_i x_n)\right)^l.$$

It is easily seen that $I(Q)^m$ has linear quotients with respect to decreasing pure lexicographic order. Let $G(I(Q)^m) = \{w_1 >_{\text{lex}} \ldots >_{\text{lex}} w_q\}$ be the minimal set of generators of $I(Q)^m$ ordered with respect to the pure lexicographic order. Clearly, the smallest monomials in $G(I(Q)^m)$ are the minimal generators of $(B_{t-1}(x_i x_n))^m$ ordered decreasingly with respect to the lexicographic order. By Lemma 2.2, since $i-(k-t+1) = (i-k)+(t-1) < t$, no ideal quotient of $G((B_{t-1}(x_i x_n))^m)$ contains x_i and x_n. Therefore, by using formula (1.1) we get depth $S(Q)/I(Q)^m > 0$. This shows that $Q = Q_A$ is not an associated prime of $I(Q)^m$.

Case (ii). $Q = Q_A \supset (x_1, \ldots, x_{j_1-1}, x_{j_1+t}, \ldots, x_n)$ for some $j_1 \leq i$. Then $A \subset [j_1, j_1+t]$, thus $k = \max A < i+t$ and $l = \min A \geq j_1$. If $l = 1$, that is, $j_1 = 1$, then $I(Q) = I(P_{\{1\}}) = (x_{1+t}, \ldots, x_n)$, by Lemma 2.3. In this case depth $S(Q)/I(Q)^m > 0$ since $Q \supset (x_{1+t}, \ldots, x_n)$, thus there exists $x_l \in S(Q)$ which is regular on $S(Q)/I(Q)^m$. Let now $j_1 \geq 2$. Then $l \geq 2$. We consider the following subcases:

(a) $i < l \leq k < i+t$;

(b) $l \leq i < k < i+t$;

(c) $l \leq k \leq i$.

In subcase (a), we get $I(Q) = I(P_{\{k\}})$ and we derive that depth $S(Q)/I(Q)^m > 0$ as in case (i). For (b) and (c), we observe that $I(Q)$ is of the form $I(Q) = (x_1, \ldots, x_{s-t}, x_{s+t}, \ldots, x_n, \bar{B}_{t-1}(x_{s-1}x_{s+t-1}))$ for some s , where $\bar{B}_{t-1}(x_{s-1}x_{s+t-1}) \subset K[\{x_{s-1}, \ldots, x_{s+t-1}\} \backslash \{x_s\}]$. Then, we order the minimal generators of $(I(Q))^m$ decreasingly with respect to the pure lexicographic order induced by

$$x_1 > \cdots > x_{s-t} > x_{s+t} > \cdots > x_n > x_{s-t+1} > x_{s-t+2} > \cdots > x_{s+t-1}.$$

By a similar argument to the one used in case (i), we get depth $S(Q)/I(Q)^m > 0$ since $\bar{B}_{t-1}(x_{s-1}x_{s+t-1})$ is a $(t-1)$-spread principal Borel ideal of the form given in Lemma 2.2. Therefore, no monomial as in Case (ii) is an associated prime of I^m. □

Remark 2.4. *Of course, we may consider the behavior of* $\mathrm{Ass}(I^m)$ *when* $I = B_t(u)$ *is a t-spread principal Borel ideal generated by* $u = x_i x_n$ *with* $i \leq t$. *To begin with, we consider* $i < t$. *In this case,* $\mathcal{S}(I) = \bigcup_{v \in G(I)} \mathrm{supp}(v) = [n] \setminus \{i+1, i+2, \ldots, t\}$ *and* $I = B_t(u)$ *is in fact an i-spread ideal in the polynomial ring* $K[\{x_j : j \notin \{i+1, i+2, \ldots, t\}\}]$. *Therefore, we are reduced to considering a t-spread principal Borel ideal* $I = B_t(u)$ *where* $u = x_t x_n$. *Then we see that I is the edge ideal of a bipartite graph on the vertex set* $\{1, 2, \ldots, t\} \cup \{t+1, t+2, \ldots, n\}$. *Consequently, by [16, Theorem 5.9], I has the property that* $\mathrm{Ass}(I^m) = \mathrm{Ass}(I)$ *for all* $m \geq 1$, *in other words, I is normally torsion free.*

3. Arithmetical rank of principal Borel ideals generated in degree two

In this section, we will give a direct proof of Theorem 3.2 on the arithmetic rank of a principal Borel ideals of degree 2. As we have mentioned in Introduction, we can get this result by using [12, Corollary 5.3]. A useful tool in our proof is the Schmitt-Vogel Lemma (see [15])

Lemma 3.1. *Let* $I \subset S$ *be a squarefree monomial and* $A_1, \ldots, A_r$ *be some subsets of the set of monomials of I. Suppose that the following conditions hold:*

(SV1) $|A_1| = 1$ *and* A_i *is a finite set for any* $2 \leq i \leq r$;

(SV2) *The union of all the sets* A_i, $i = 1, \ldots, r$, *contains the set of the minimal monomial generators of I.*

(SV3) *For any* $i \geq 2$ *and for any two different monomials* $m_1, m_2 \in A_i$ *there exists* $j < i$ *and a monomial* $m' \in A_j$ *such that* $m' | m_1 m_2$.

Let $g_i = \sum_{m_i \in A_i} m_i$ *for* $1 \leq i \leq r$. *Then* $\sqrt{(g_1, ..., g_r)} = I$. *In particular,* $\mathrm{ara}(I) \leq r$.

Theorem 3.2. *Let I be a t-spread principal Borel ideal, where $u = x_i x_n$, $i \leq n-t$. Then*

$$\operatorname{ara}(I) = \operatorname{proj}\dim_S(S/I) = n-t.$$

Proof. By [7, Theorem 2.3] we have $\operatorname{proj}\dim_S(S/I) = n-t$. We show that $\operatorname{ara}(I) = n-t$ by using the Schmitt-Vogel Lemma. We will display the minimal generators of I in an upper triangular tableau as follows. In the first row, we will put the generators divisible by x_1 order decreasingly with respect to the lexicographic order. In the same manner, in the second row, we will order the monomials divisible by x_2. We shall continue this way up to the row containing the monomial divisible by x_i where we shall put the generators $x_i x_{n-t} \ldots x_i x_n$. Then our tableau looks as follows.

$$\begin{array}{cccccccccc}
x_1x_{t+1} & x_1x_{t+2} & x_1x_{t+3} & \cdots & x_1x_{i+t} & \cdots & x_1x_{n-2} & x_1x_{n-1} & x_1x_n \\
 & x_2x_{t+2} & x_2x_{t+3} & \cdots & x_2x_{i+t} & \cdots & x_2x_{n-2} & x_2x_{n-1} & x_2x_n \\
 & & \ddots & & \vdots & & \vdots & \vdots & \vdots \\
 & & & & x_ix_{t+i} & \cdots & x_ix_{n-2} & x_ix_{n-1} & x_ix_n
\end{array}$$

Next we define the sets $A_1, A_2, \ldots, A_{n-t}$ in the following way. In the first set, we will put the monomial from the right up corner of the tableau. In the second set, we will put the two monomials from the right up parallel to the diagonal of triangular tableau. In the third set, we will collect the three monomials from the next parallel to the diagonal, and so on. Explicitly, the sets are the following ones.

$$\begin{aligned}
&A_1 = \{x_1x_n\} \\
&A_2 = \{x_1x_{n-1}, x_2x_n\} \\
&A_3 = \{x_1x_{n-2}, x_2x_{n-1}, x_3x_n\} \\
&\vdots \\
&A_j = \{x_1x_{n-j+1}, x_2x_{n-j+2}, ..., x_jx_n\}, \text{ for } i \geq j \\
&\vdots \\
&A_j = \{x_1x_{n-j+1}, x_2x_{n-j+2}, ..., x_ix_{n-j+i}\}, \text{ for } i < j \\
&\vdots \\
&A_{n-t-1} = \{x_1x_{t+2}, x_2x_{t+3}, ..., x_ix_{i+t+1}\} \\
&A_{n-t} = \{x_1x_{t+1}, x_2x_{t+2}, ..., x_ix_{i+t}.\}
\end{aligned}$$

One may easily check that the sets $A_1, \ldots, A_{n-t}$ verify all conditions of the Schmitt-Vogel Lemma. The first two conditions of Lemma 3.1 are clearly fulfilled. We hall sgive an explanation for the third condition only. If we pick up two different monomials in the set A_j for some $j \geq 2$, let us say m_1 from the k-th row and m_2 from the l-th row of the tableau with $k < l$, then we put the monomial which is the intersection element of the k-th row and the column of m_2 as m' which divides the product m_1m_2 and $m' \in A_r$ for some $r < j$. For instance, if $m_1 = x_kx_{n-j+k}, m_2 = x_lx_{n-j+l} \in A_j$ for some $k < l$ then we choose $m' = x_kx_{n-j+l} \in A_{k+j-l}$ which divides $m_1m_2 = x_kx_lx_{n-j+k}x_{n-j+l}$. $\square$

We recall from [2] that the ideal I is called a set-theoretic complete intersection if $\operatorname{height}(I) = \operatorname{ara}(I)$. An ideal I is called cohomologically complete intersection if $ht(I) = cd(I)$.

Proposition 3.3. *Let $I = B_t(u)$ be a t-spread principal Borel ideal generated in degree 2. Then I is a set theoretic compete intersection if and only if $u = x_{n-t}x_n$.*

Proof. Let $u = x_i x_n$. By Theorem 3.2, we have $\operatorname{ara}(I) = \operatorname{proj}\dim(S/I) = n - t$. By [1, Theorem 1.1], we know that $\operatorname{height}(I) = i$. Thus $\operatorname{height}(I) = \operatorname{ara}(I)$ if and only if $i = n - t$. □

Proposition 3.4. *Let $t \geq 1$ be an integer and $I_{n,d,t} \subset S$ the t-spread Veronese ideal generated in degree d. Then I is a cohomologically complete intersection ideal. In particular,* $\operatorname{cd}(I_{n,d,t}) = n - t(d-1)$.

Proof. By [7, Theorem 2.3], I is Cohen-Macaulay and $\operatorname{cd}(R, I_{n,d,t}) = \operatorname{height}(I_{n,d,t}) = n - t(d-1)$. So $I_{n,d,t}$ is cohomologically intersection. □

4. Lyubeznik numbers

Suppose that (R, m, K) is a local ring admitting a surjection from an n-dimensional regular local ring (S, n, K) containing a field, and let I denote the kernel of the surjection. Given $i, j \in \mathbb{N}$, the Lyubeznik number of R with respect to $i, j \in \mathbb{N}$, is defined as

$$\lambda_{i,j}(R) = \dim_K \operatorname{Ext}^i_S(K, H^{n-j}_I(S))$$

and is denoted $\lambda_{i,j}(R)$. Put $d = \dim R$, Lyubeznik numbers satisfy the following properties:

(a) $\lambda_{i,j}(R) = 0 \;\; for \;\; j > d \;\; or \;\; i > j.$

(b) $\lambda_{d,d}(R) \neq 0.$

(c) If R is Cohen-Macaulay, then $\lambda_{d,d}(R) = 1$.

(d) Euler characteristic,

$$\sum_{0 \leq i,j \leq d} (-1)^{i-j} \lambda_{i,j}(R) = 1.$$

Therefore, we can record all nonzero Lyubeznik numbers in the so-called *Lyubeznik table*:

$$\begin{bmatrix} \lambda_{0,0} & . & . & . & \lambda_{0,d} \\ 0 & . & & & . \\ 0 & 0 & . & & . \\ 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \lambda_{d,d} \end{bmatrix}$$

where $\lambda_{i,j} := \lambda_{i,j}(R)$ for every $0 \leq i, j \leq d$, see for example [2].

Corollary 4.1. *Lyubeznik table of* $I_{n,d,t} = J \subset S$ *is*

$$\lambda_{i,j}(S/J) = 0 \ \textit{for all} \ 0 \leq i, j < d \ \textit{and} \ \lambda_{d,d} = 1,$$

where $\dim(S/J) = d$.

Proof. [7, Theorem 2.3]. □

Lemma 4.2. *Let* $S = k[x_1, \ldots, x_n]$ *be a polynomial ring over a field* k, m *which denotes its homogeneous maximal ideal* $(x_1, \ldots, x_n)$ *and* $I = B_t(u)$ *where* $u = x_{n-t}x_n$. *Then*

$$\lambda_{i,j}(S/I) = 0 \ \textit{for all} \ 0 \leq i, j < d \ \textit{and} \ \lambda_{d,d} = 1.$$

Proof. As I is cohomologically complete intersection,

$$\dim(S/I) = \text{fgrade}(I, S).$$

So

$$\text{depth}(S/I) \leq \text{fgrade}(I, S).$$

By [2, lemma 3.2] we conclude that

$$\lambda_{i,j}(S/I) = 0 \ \text{for all} \ 0 \leq i, j < d \ \text{and} \ \lambda_{d,d} = 1.$$

□

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$\alpha\beta-$STATISTICAL CONVERGENCE ON TIME SCALES

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Abstract. In this paper, we introduce the concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$Cesàro summability of delta measurable functions on an arbitrary time scale. Then some inclusion relations and results about these new concepts are presented. We will also investigate the relationship between statistical convergence and $\alpha\beta-$statistical convergence on a time scale.
Keywords: statistical convergence, time scale, delta measurable functions, Cesàro summable.

1. Introduction

The idea of statistical convergence for sequences of real and complex numbers was introduced by Fast [14] and Steinhaus [15] independently in the same year (1951) as follows. Let $K \subseteq \mathbb{N}$, the set of natural numbers and $K_n = \{k \leqslant n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n. A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| = 0.$$

In this case, we write $st - \lim x = L$. It is known that every convergent sequence is statistically convergent, but not conversely. For example, suppose that the sequence $x = (x_k)$ defined by $x_k = \sqrt{k}$ if k is square and $x_k = 0$ otherwise. It is clear that the sequence $x = (x_k)$ is statistically convergent to 0 but it is not convergent. Over the years, generalizations and applications of this notion have been investigated by various researchers [2, 8, 10, 11, 13, 16, 17, 18, 19, 20, 21, 24, 26, 29].

Aktuglu [13] introduced $\alpha\beta-$statistical convergence as follows. Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

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$P_1 : \alpha$ and β are both non $-$ decreasing,
$P_2 : \beta(n) \geqslant \alpha(n)$,
$P_3 : \beta(n) - \alpha(n) \to \infty$ as $n \to \infty$.
Let Λ denote the set of pairs (α, β) satisfying P_1, P_2 and P_3.

For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leqslant 1$ and $K \subset \mathbb{N}$, we define

$$\delta^{\alpha,\beta}(K, \gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}}$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S.

Definition 1.1. [13] A sequence $x = (x_n)$ is said to be $\alpha\beta-$statistically convergent of order γ to L, if for every $\varepsilon > 0$

$$\delta^{\alpha,\beta}(\{k : |x_k - L| \geqslant \varepsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geqslant \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0,$$

which is denoted by $st_{\alpha\beta}^{\gamma} - \lim x_n = L$. For $\gamma = 1$, we say that x is $\alpha\beta-$statistically convergent to L, and this is denoted by $st_{\alpha\beta} - \lim x_n = L$.

The purpose of our study is to introduce the concept of $\alpha\beta-$statistical convergence on an arbitrary time scale.

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was introduced by Hilger in his Ph. D. thesis supervised by Auldbach in 1988 (see [3, 27]), in order to unify continuous and discrete analysis. Since this theory is applicable to any field in which dynamic processes can be described with discrete or continuous models and is also effective in modeling some real life problems, it has a tremendous potential for applications and has recently received much attention, see [1, 12, 22, 23, 28]. In addition, statistical convergence is applied to time scales by various researchers in literature. For instance, Seyyidoglu and Tan [25] defined some new notions such as $\Delta-$convergence and $\Delta-$Cauchy, by using $\Delta-$density. Turan and Duman introduced the concepts of density, statistical convergence and lacunary statistical convergence of delta measurable real-valued functions defined on time scales in [5] and [6], respectively. Also, in [7], they obtained a Tauberian condition for statistical convergence, and established a relationship between statistical convergence and lacunary statistical convergence on time scales. Altin, Koyunbakan and Yilmaz [30] gave the notions of $m-$ and $(\lambda, m)-$uniform density of a set and $m-$ and $(\lambda, m)-$uniform statistical convergence on an arbitrary time scales. Furthermore, $\lambda-$statistical convergence on time scales was defined by Yilmaz, Altin and Koyunbakan [9]. Recently, Sozbir and Altundag [4] introduced the concepts of weighted statistical convergence and $\left[\bar{N}, p\right]_{\mathbb{T}}-$summability of delta measurable functions on time scales, and investigated their relations. We here recall some concepts and notations about the theory of time scales.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ can be defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

for $t \in \mathbb{T}$. And the graininess function $\mu : \mathbb{T} \to [0, \infty)$ can be defined by $\mu(t) = \sigma(t) - t$. In this definition we put $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ is an empty set. A closed interval in a time scale $\mathbb{T}$ is given by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Open intervals or half-open intervals are defined accordingly.

Let F_1 denote the family of all left closed and right open intervals of $\mathbb{T}$ of the form $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t < b\}$ with $a, b \in \mathbb{T}$ and $a \leqslant b$. The interval $[a, a)$ is understood as the empty set. F_1 is a semiring of subsets of $\mathbb{T}$. Let $m_1 : F_1 \to [0, \infty)$ be a set function on F_1 such that $m_1\left([a, b)_{\mathbb{T}}\right) = b - a$. Then, it is known that m_1 is a countably additive measure on F_1. Now, the Caratheodory extension of the set function m_1 associated with family F_1 is said to be the Lebesgue $\Delta-$measure on $\mathbb{T}$ is denoted by μ_Δ. In this case, it is known that if $a \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta\left(\{a\}\right) = \sigma\left(a\right) - a$. If $a, b \in \mathbb{T}$ and $a \leqslant b$, then $\mu_\Delta\left([a, b)_{\mathbb{T}}\right) = b - a$ and $\mu_\Delta\left((a, b)_{\mathbb{T}}\right) = b - \sigma\left(a\right)$. If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \leqslant b$, then $\mu_\Delta\left((a, b]_{\mathbb{T}}\right) = \sigma\left(b\right) - \sigma\left(a\right)$ and $\mu_\Delta\left([a, b]_{\mathbb{T}}\right) = \sigma\left(b\right) - a$ (see [12]).

We should note that throughout the paper, we consider that $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$. Turan and Duman [5] introduced the concepts of density, statistical convergence and strong $p-$Cesàro summability of measurable real valued functions defined on time scales in the following way.

Definition 1.2. [5] Let Ω be a $\Delta-$measurable subset of $\mathbb{T}$. Then, for $t \in \mathbb{T}$, we define the set $\Omega\left(t\right)$ by

$$\Omega\left(t\right) = \left\{s \in [t_0, t]_{\mathbb{T}} : s \in \Omega\right\}.$$

In this case, we define the density of Ω on $\mathbb{T}$, denoted by $\delta_{\mathbb{T}}\left(\Omega\right)$, as follows:

$$\delta_{\mathbb{T}}\left(\Omega\right) = \lim_{t \to \infty} \frac{\mu_\Delta\left(\Omega\left(t\right)\right)}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)}$$

provided that the above limit exists.

Definition 1.3. [5] Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta-$measurable function. We say that f is statistically convergent on $\mathbb{T}$ to a number L, if for every $\varepsilon > 0$

$$\delta_{\mathbb{T}}\left(\left\{t \in \mathbb{T} : \left|f\left(t\right) - L\right| \geqslant \varepsilon\right\}\right) = 0$$

holds, i.e., for every $\varepsilon > 0$,

$$\lim_{t \to \infty} \frac{\mu_\Delta\left(\left\{s \in [t_0, t]_{\mathbb{T}} : \left|f\left(s\right) - L\right| \geqslant \varepsilon\right\}\right)}{\mu_\Delta\left([t_0, t]_{\mathbb{T}}\right)} = 0,$$

which is denoted by $st_{\mathbb{T}} - \lim\limits_{t \to \infty} f\left(t\right) = L$.

Definition 1.4. [5] Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta-$measurable function and $0<p<\infty$. We say that f is strongly $p-$Cesàro summable on the time scale $\mathbb{T}$ to a number L, if there exists some $L\in\mathbb{R}$ such that

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta\left([t_0,t]_\mathbb{T}\right)}\int\limits_{[t_0,t]_\mathbb{T}}|f(s)-L|^p\Delta s=0.$$

2. Main Results

In this section, we will begin by introducing the new concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$Cesàro summability on an arbitrary time scale, which are our main definitions, and we establish some relations about these notions. We also examine the relationship between statistical convergence and $\alpha\beta-$statistical convergence on a time scale.

Now let $\alpha,\beta:\mathbb{T}\to\mathbb{R}^+$ be two functions satisfying the following conditions:

T_1 : α and β are both non $-$ decreasing,

T_2 : $\sigma(\beta(t))>\alpha(t)\geqslant t_0$ for all $t\in\mathbb{T}$,

T_3 : $\sigma(\beta(t))-\alpha(t)\to\infty$ as $t\to\infty$.

And let $\Lambda_\mathbb{T}$ denote the set of pairs (α,β) satisfying T_1, T_2 and T_3.

Definition 2.1. Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta-$measurable function and $(\alpha,\beta)\in\Lambda_\mathbb{T}$. Then, f is said to be $\alpha\beta-$statistically convergent to $L\in\mathbb{R}$ on a time scale $\mathbb{T}$, if for every $\varepsilon>0$

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_\mathbb{T}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_\mathbb{T}\right)}=0,$$

which is denoted by $st_{\mathbb{T}-\alpha\beta}-\lim f(t)=L$.

This definition includes the following special cases:

i) If we take $\alpha(t)=t_0$ and $\beta(t)=t$ for all $t\in\mathbb{T}$, then $\alpha\beta-$statistical convergence is reduced to statistical convergence on a time scale introduced in [5].

ii) Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1}\leqslant\lambda_n+1$ and $\lambda_1=1$. For $\mathbb{T}=\mathbb{N}$, if we choose $\alpha(t)=t-\lambda_t+t_0$ and $\beta(t)=t$, then $\alpha\beta-$statistical convergence on a time scale is reduced to $\lambda-$statistical convergence introduced in [24].

Remark 2.1. Let $\theta=(k_r)$ be an increasing sequence of non-negative integers with $k_0=0$ and $\sigma(k_r)-\sigma(k_{r-1})\to\infty$ as $r\to\infty$, which means that θ is a lacunary sequence with respect to $\mathbb{T}$. If we take $\mathbb{T}=\mathbb{N}$, $\alpha(t)=k_{t-1}+1$ and $\beta(t)=k_t$, then $\alpha\beta-$statistical convergence on $\mathbb{T}$ gives us the concept of lacunary statistical convergence introduced in [19]. However, for an arbitrary time scale $\mathbb{T}$, this is not clear, and we leave it as an open problem.

Proposition 2.1. *If $f:\mathbb{T}\to\mathbb{R}$ is $\alpha\beta-$statistically convergent, then its limit is unique.*

Proposition 2.2. *If $f, g : \mathbb{T} \to \mathbb{R}$ with $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L_1$ and $st_{\mathbb{T}-\alpha\beta} - \lim g(t) = L_2$, then we have the following:*

i) $st_{\mathbb{T}-\alpha\beta} - \lim (f(t) + g(t)) = L_1 + L_2$,

ii) $st_{\mathbb{T}-\alpha\beta} - \lim (cf(t)) = cL_1$ *for any* $c \in \mathbb{R}$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta-$measurable function and $(\alpha, \beta) \in \Lambda_{\mathbb{T}}$. Then, one says f is said to be strongly $\alpha\beta-$Cesàro summable on a time scale $\mathbb{T}$, if there exists some $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t),\beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s = 0.$$

Theorem 2.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be $\Delta-$measurable function and $L \in \mathbb{R}$. Then we have the following:*

i) *If f is strongly $\alpha\beta-$Cesàro summable to L, then $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$, but not conversely.*

ii) *If $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$ and f is a bounded function, then f is strongly $\alpha\beta-$Cesàro summable to L.*

Proof. ***i)*** *Let f is strongly $\alpha\beta-$Cesàro summable to L. Then, for every $\varepsilon > 0$, we can write that*

$$\int_{[\alpha(t),\beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s \geqslant \int_{[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon} |f(s) - L| \Delta s$$

$$\geqslant \varepsilon\mu_\Delta(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geqslant \varepsilon\}),$$

which implies that $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$.

To prove the converse, define a function f in each intervals $[\alpha(t), \beta(t)]_{\mathbb{T}}$ by

$$f(s) = \begin{cases} 1, & \text{if } s \in [\alpha(t), \alpha(t) + 1)_{\mathbb{T}}, \\ 2, & \text{if } s \in [\alpha(t) + 1, \alpha(t) + 2)_{\mathbb{T}}, \\ \vdots & \\ \left[\left|\sqrt{u_t}\right|\right], & \text{if } s \in \left[\alpha(t) + \left[\left|\sqrt{u_t}\right|\right] - 1, \alpha(t) + \left[\left|\sqrt{u_t}\right|\right]\right)_{\mathbb{T}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $u(t) = \sigma(\beta(t)) - \alpha(t)$.

Then, for every $\varepsilon > 0$, we observe that

$$\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)} = \frac{\mu_\Delta\left([\alpha(t),\alpha(t)+[|\sqrt{u_t}|])_{\mathbb{T}}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}$$

$$= \frac{[|\sqrt{u_t}|]}{u_t} \to 0 \quad (\text{as } t \to \infty).$$

Thus, $st_{\mathbb{T}-\alpha\beta} - \lim\limits_{t\to\infty} f(t) = 0$.

On the other hand,

$$\frac{1}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}}|f(s)|\,\Delta s$$

$$=\frac{1}{\sigma(\beta(t))-\alpha(t)}\sum_{m=1}^{[|\sqrt{u_t}|]} m\mu_\Delta\left([\alpha(t)+m-1,\alpha(t)+m)_{\mathbb{T}}\right)$$

$$=\frac{1}{u_t}\sum_{m=1}^{[|\sqrt{u_t}|]} m$$

$$=\frac{1+2+\ldots+[|\sqrt{u_t}|]}{u_t}$$

$$=\frac{[|\sqrt{u_t}|]([|\sqrt{u_t}|]+1)/2}{u_t}\to\frac{1}{2}\neq 0 \quad (\text{as } t\to\infty).$$

Hence, we obtain that f is not strongly $\alpha\beta-$Cesàro summable to 0*. This completes the proof.*

ii) *Let f be bounded and $st_{\mathbb{T}-\alpha\beta}-\lim f(t)=L$. Then, there exists a positive number M such that $|f(t)|\leqslant M$ for all $t\in\mathbb{T}$, and for a given $\varepsilon>0$, we also have*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}=0.$$

Then, we can easily see that

$$\frac{1}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}}|f(s)-L|\,\Delta s$$

$$=\frac{1}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon}|f(s)-L|\,\Delta s$$

$$+\frac{1}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|<\varepsilon}|f(s)-L|\,\Delta s$$

$$\leqslant\frac{M+|L|}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon}\Delta s+\frac{\varepsilon}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}}\Delta s$$

$$=(M+|L|)\,\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}+\varepsilon.$$

Letting $t\to\infty$ on the both sides of the last inequality, since $\varepsilon>0$ is arbitrary, we have

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\int\limits_{[\alpha(t),\beta(t)]_{\mathbb{T}}}|f(s)-L|\,\Delta s=0.$$

So, the proof is completed. □

Theorem 2.2. *If* $\liminf\limits_{t\to\infty}\frac{\sigma(\beta(t))}{\alpha(t)} > 1$, *then* $st_{\mathbb{T}} - \lim f(t) = L$ *implies* $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$.

Proof. *Suppose that* $\liminf\limits_{t\to\infty}\frac{\sigma(\beta(t))}{\alpha(t)} > 1$. *Then, for sufficiently large* t, *there exists* $\delta > 0$ *such that* $\frac{\sigma(\beta(t))}{\alpha(t)} \geqslant 1+\delta$, *and hence* $\frac{\sigma(\beta(t))-\alpha(t)}{\sigma(\beta(t))} \geqslant \frac{\delta}{1+\delta}$. *For a given* $\varepsilon > 0$, *we have*

$$\begin{aligned}
&\frac{\mu_\Delta\left(\left\{s\in[t_0,\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,\beta(t)]_{\mathbb{T}}\right)}\\
&\qquad\geqslant \frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-t_0}\\
&\qquad\geqslant \frac{\sigma(\beta(t))-\alpha(t)}{\sigma(\beta(t))}\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-\alpha(t)}\\
&\qquad\geqslant \frac{\delta}{1+\delta}\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-\alpha(t)}.
\end{aligned}$$

Letting $t\to\infty$ *on the both sides of the last inequality and also using the* $st_{\mathbb{T}} - \lim f(t) = L$, *we get*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)} = 0.$$

This completes the proof of the theorem. □

Theorem 2.3. *If* $\lim\limits_{t\to\infty}\frac{\alpha(t)-t_0}{\sigma(\beta(t))-t_0} = 0$, *then* $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$ *implies* $st_{\mathbb{T}} - \lim f(t) = L$.

Proof. *Assume that* $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$. *Then, for every* $\varepsilon > 0$, *we may write*

$$\begin{aligned}
&\frac{\mu_\Delta\left(\left\{s\in[t_0,\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,\beta(t)]_{\mathbb{T}}\right)}\\
&\qquad= \frac{\mu_\Delta\left(\left\{s\in[t_0,\alpha(t))_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,\beta(t)]_{\mathbb{T}}\right)} + \frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,\beta(t)]_{\mathbb{T}}\right)}\\
&\qquad= \frac{\mu_\Delta\left(\left\{s\in[t_0,\alpha(t))_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-t_0} + \frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-t_0}\\
&\qquad\leqslant \frac{\alpha(t)-t_0}{\sigma(\beta(t))-t_0} + \frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta(t))-\alpha(t)}.
\end{aligned}$$

Taking limit as $t\to\infty$ *on the both sides of last inequality and using the condition of* $\lim\limits_{t\to\infty}\frac{\alpha(t)-t_0}{\sigma(\beta(t))-t_0} = 0$, *we have*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[t_0,\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left([t_0,\beta(t)]_{\mathbb{T}}\right)} = 0,$$

which completes the proof. $\square$

Now, let $(\alpha,\beta)\in\Lambda_{\mathbb{T}}$ and $(\alpha',\beta')\in\Lambda_{\mathbb{T}}$. In the following theorem $\alpha\beta-$statistical convergence and $\alpha'\beta'-$statistical convergence are compared under the restriction

$$\alpha(t)\leqslant\alpha'(t)<\beta'(t)\leqslant\beta(t)$$

for all $t\in\mathbb{T}$. Under these conditions above,we have the following theorem:

Theorem 2.4. *If* $\lim\limits_{t\to\infty}\frac{\sigma(\beta'(t))-\alpha'(t)}{\sigma(\beta(t))-\alpha(t)}>0$, *then* $st_{\mathbb{T}-\alpha\beta}-\lim f(t)=L$ *implies* $st_{\mathbb{T}-\alpha'\beta'}-\lim f(t)=L$.

Proof. Suppose that $\lim\limits_{t\to\infty}\frac{\sigma(\beta'(t))-\alpha'(t)}{\sigma(\beta(t))-\alpha(t)}>0$ *and* $st_{\mathbb{T}-\alpha\beta}-\lim f(t)=L$. *We have the inclusion*

$$\{s\in[\alpha'(t),\beta'(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}\subseteq\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}$$

for every $\varepsilon>0$, *and hence*

$$\begin{aligned}&\mu_\Delta(\{s\in[\alpha'(t),\beta'(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\})\\&\qquad\leqslant\mu_\Delta(\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\}).\end{aligned}$$

So, we may write that

$$\begin{aligned}&\frac{\mu_\Delta\left(\left\{s\in[\alpha(t),\beta(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\\&\qquad\geqslant\frac{\mu_\Delta\left(\left\{s\in\left[\alpha'(t),\beta'(t)\right]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\mu_\Delta\left([\alpha(t),\beta(t)]_{\mathbb{T}}\right)}\\&\qquad=\frac{\sigma\left(\beta'(t)\right)-\alpha'(t)}{\sigma(\beta(t))-\alpha(t)}\frac{\mu_\Delta\left(\left\{s\in\left[\alpha'(t),\beta'(t)\right]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\right\}\right)}{\sigma(\beta'(t))-\alpha'(t)}.\end{aligned}$$

Since $st_{\mathbb{T}-\alpha\beta}-\lim f(t)=L$, *taking limit as* $t\to\infty$ *on the both sides of last inequality, we get*

$$\lim_{t\to\infty}\frac{\mu_\Delta(\{s\in[\alpha'(t),\beta'(t)]_{\mathbb{T}}:|f(s)-L|\geqslant\varepsilon\})}{\sigma(\beta'(t))-\alpha'(t)}=0,$$

which means $st_{\mathbb{T}-\alpha'\beta'}-\lim f(t)=L$. *Hence, the proof is completed.* $\square$

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THE MOSTAR INDEX OF FULLERENES IN TERMS OF AUTOMORPHISM GROUP

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Abstract. Let G be a connected graph. For an edge $e = uv \in E(G)$, suppose $n(u)$ and $n(v)$ are respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. The Mostar index is a topological index which is defined as $Mo(G) = \sum_{e\in E(G)} f(e)$, where $f(e) = |n(u) - n(v)|$. In this paper, we will compute the Mostar index of a family of fullerene graphs in terms of the automorphism group.
Keywords: Automorphism group, Mostar index, group action.

1. Introduction

For arbitrary vertices u and v of a graph G, the distance $d(u, v)$ is defined as the length of a shortest path connecting u and v. For the edge $e = uv \in E(G)$, suppose $n(u)$ and $n(v)$ are respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u. The Mostar index is defined as $Mo(G) = \sum_{e\in E(G)} f(e)$, where $f(e) = |n(u) - n(v)|$, see [10].

Let G be a group which acts on the non-empty set Ω. The left action G on Ω induces a group homomorphism φ from G into the symmetric group S_Ω, that satisfies the following two axioms (where we denote $\varphi(g, \alpha)$ as α^g): $\alpha^e = \alpha$ for all $\alpha \in \Omega$ (e denotes the identity element of group G) and $\alpha^{(gh)} = (\alpha^g)^h$ for all $g, h \in G$ and all $\alpha \in \Omega$. The orbit of an element $\alpha \in \Omega$ is denoted by α^G and it is defined as the set of all α^g's, where $g \in G$. The size of Ω is called the degree of this action. The stabilizer of an element $\alpha \in \Omega$ is defined as $G_\alpha = \{g \in G : \ \alpha^g = \alpha\}$. Let $H = G_\alpha$, then for $\beta \in \Omega$ ($\alpha \neq \beta$), H_β is denoted by $G_{\alpha,\beta}$. On the other hand, the orbit-stabilizer theorem implies that $|\alpha^G|.|G_\alpha| = |G|$, see [9].

A bijection σ on the vertex set of graph G is called an automorphism of G if it preserves the edge set. In other words, if α is an automorphism of G, then $e = uv$

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is an edge if and only if $\sigma(e) = \sigma(u)\sigma(v)$ is an edge of G. Let $Aut(G)$ be the set of all automorphisms of G. Then $Aut(G)$ under the composition of mappings forms a group. The graph G is called vertex-transitive if its automorphism group has one orbit. This means that for two arbitrary vertices $x, y \in V(G)$, there is an autoorphism $\varphi \in Aut(G)$ such that $\varphi(x) = y$. We can similarly define an edge-transitive graph.

The aim of this paper is to compute the Mostar index of an infinite family of fullerene graphs. To do this, we shall first compute the automorphism group of the fullerene graph, and afterwards we shall compute all edge-orbits. Finally, we shall determine the contribution of each edge in the formula of Mostar index.

2. Mostar index of fullerenes

If G is a vertex-transitive graph, for every edge $e = uv \in E(G)$, we have $n(u) = n(v)$ and thus $\mathrm{Mo}(G) = 0$. Here, by relyinh on this and knowing that the action of a group on its orbits is transitive, we will compute the Mostar index of a benzenoid graph by means of orthogonal cuts. Let F be an orthogonal cut. For the arbitrary edge $e \in E(G)$, all vertices in one shore of F are closer to the end-vertex of e belonging to the same shore than to the other one. Hence, all edges of the same orthogonal cut contribute equally to $\mathrm{Mo}(B)$. It is proved in [10] that if $F \subset E(B)$ is an orthogonal cut of a benzenoid graph B of size p and if the shores of F have n_1 and n_2 vertices, respectively, then the total contribution of edges from F to $\mathrm{Mo}(B)$ is equal to $p|n_1 - n_2|$. Hence, we have the following theorem.

Theorem 2.1. *Let B be a benzenoid graph on n vertices and let $F_1, \ldots, F_q$ be all its orthogonal cuts. Let p_i denotes the size of F_i, and n_{i_1} and n_{i_2} be the number of vertices in its shores. Then*

$$\mathrm{Mo}(B) = \sum_{i=1}^{q} p_i |n_{i_1} - n_{i_2}|.$$

Došlić et al. in [10] proved that since the dodecahedron and the Buckminster fullerene are the only two vertex-transitive fullerene graphs, we have $\mathrm{Mo}(C_{20}) = \mathrm{Mo}(C_{60} : I_h) = 0$. They also introduced the following open problem:

Problem [10]. Are there other fullerene graphs G such that $\mathrm{Mo}(G) = 0$?

Here, we will compute the Mostar index of an infinite family of fullerenes. This is the first attempt to give some new results about the above problem. We conjecture that if F is a fullerene (except dodecahedron and the buckminsterfullerene) then $\mathrm{Mo}(F) \neq 0$. Let F be a fullerene with $\mathrm{Mo}(F) = 0$. Then for all edges $e = uv$, we have $n(u) = n(v)$ and thus F is a distance balanced graph. In other words, we conjectured that a fullerene is distance-balance if and only if F is vertex-transitive.

Theorem 2.2. *Let $E_1, \cdots, E_r$ be the orbits of graph G under the action of $Aut(G)$ on the set $E(G)$. Then*

$$\text{Mo}(G) = \sum_{i=1}^{r} \sum_{e_i \in E_i} |E_i| \times |n(u_i) - n(v_i)|. \tag{2.1}$$

Proof. Let $E_1, \ldots, E_r$ be the orbits of graph G under the action of $Aut(G)$ on the set of edges. For two edges $e = uv$ and $f = ab$ in the same edge-orbit of G, one can prove that $\{n(u), n(v)\}$=$\{n(a), n(b)\}$. This completes the proof. □

Fullerenes are polyhedral molecules made entirely of carbon atoms. The most symmetric fullerene is the famous buckminster fullerene, C_{60}, whose discovery in 1985 marked the birth of fullerene chemistry [23]. In 1991, the buckminster fullerene was declared "The Molecule of the Year" by Science magazine, and since then, these new graphs have been attracting attention of various research communities. Many methods of graph theory have been applied to investigate the mathematical models of fullerene molecules called fullerene graphs. M. Ghorbani and A. R. Ashrafi, in a series of papers [1–8, 12–17, 21], introduced some infinite classes of fullerene graphs. At first, they tried to classify fullerenes with respect to their automorphism group. However, this problem is still open, although Fowler and his co-authors in [11] showed that fullerenes are realizable within 28 point groups. Recently, Ghorbani et al. have computed the automorphism group of some classes of polyhedral graphs, see [18–20].

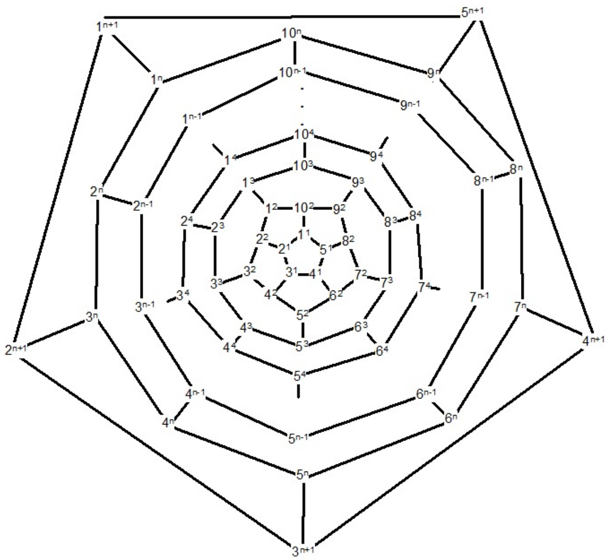

FIG. 2.1: C_{10n}, n is even.

Example 2.1. Consider the fullerene graph C_{10n} (n is even) as depicted in Figure 2.1. The vertices of central pentagon are labeled by $\{1^1, 2^1, 3^1, 4^1, 5^1\}$. These vertices compose the first layer of fullerene graph C_{10n}. The vertices of the second layer are the boundary vertices of five pentagons adjacent to the central pentagon and so on. In [22], it is shown that the following elements are in the automorphism group of fullerene graph C_{10n}. Let α be a symmetry element that fixes the vertices $1^1, 10^2, 10^3, \cdots, 10^n, 5^2, 5^3, \cdots, 5^n$ and 3^{n+1} and $\sigma = (1^1, 1^{n+1}, 2^1, 2^{n+1}, 3^1, 3^{n+1}, 4^1, 4^{n+1}, 5^1, 5^{n+1})(1^2, 2^n, 3^2, 4^n, 5^2, 6^n, 7^2, 8^n, 9^2, 10^n)(2^2, 3^n, 4^2, 5^n, 6^2, 7^n, 8^2, 9^n, 10^2, 1^n)\ (1^3, 2^{n-1}, 3^3, 4^{n-1}, 5^3, 6^{n-1}, 7^3, 8^{n-1}, 9^3, 10^{n-1})\ (2^3, 3^{n-1}, 4^3, 5^{n-1}, 6^3, 7^{n-1}, 8^3, 9^{n-1}, 10^3, 1^{n-1}) \cdots (1^{n/2}, 2^{(n+4)/2}, 3^{n/2}, 4^{(n+4)/2}, 5^{n/2}, 6^{(n+4)/2}, 7^{n/2}, 8^{(n+4)/2}, 9^{n/2}, 10^{(n+4)/2})\ (2^{n/2}, 3^{(n+4)/2}, 4^{n/2}, 5^{(n+4)/2}, 6^{n/2}, 7^{(n+4)/2}, 8^{n/2}, 9^{(n+4)/2}, 10^{n/2}, 1^{(n+4)/2})\ (1^{(n+2)/2}, 2^{(n+2)/2}, 3^{(n+2)/2}, 4^{(n+2)/2}, 5^{(n+2)/2}, 6^{(n+2)/2}, 7^{(n+2)/2}, 8^{(n+2)/2}, 9^{(n+2)/2}, 10^{(n+2)/2})$.

It is clear that $\alpha^2 = \sigma^{10} = 1$, $\alpha\sigma\alpha = \sigma^{-1}$ and $G = \langle\alpha, \sigma\rangle \leqslant A = Aut(\mathrm{C}_{10n})$. On the other hand, every symmetry element which fixes 1^1, must also fix $10^2, 10^3, \cdots, 10^n, 5^2, 5^3, \cdots, 5^n$ and 3^{n+1}. The identity element and the symmetry element α do this, too. Hence, the orbit-stabilizer property ensures that $|A| = |\ {1^1}^{A}|.|A_{1^1}|$ and thus $|A| = 10 \times 2 = 20$ which implies that $A \cong D_{20}$. All orbits of the automorphism group C_{10n} are given in Table 1.

Example 2.2. Consider the fullerene graph C_{10n} (n is odd) as depicted in Figure 2.2. Assume that α is a symmetry element which fixes the points 1^1, 10^2, 10^3, $\cdots$, 10^n, 1^{n+1}, 5^2, 5^3, $\cdots$, 5^{n-1} and 5^n and σ is a symmetry element by the following permutation presentation:
$\sigma = (1^1, 4^{n+1}, 2^1, 5^{n+1}, 3^1, 1^{n+1}, 4^1, 2^{n+1}, 5^1, 3^{n+1})\ (1^2, 7^n, 3^2, 9^n, 5^2, 1^n, 7^2, 3^n, 9^2, 5^n)\ (2^2, 8^n, 4^2, 10^n, 6^2, 2^n, 8^2, 4^n, 10^2, 6^n)\ (1^3, 7^{n-1}, 3^3, 9^{n-1}, 5^3, 1^{n-1}, 7^3, 3^{n-1}, 9^3, 5^{n-1})\ (2^3, 8^{n-1}, 4^3, 10^{n-1}, 6^3, 2^{n-1}, 8^3, 4^{n-1}, 10^3, 6^{n-1}) \cdots (1^{(n+1)/2}, 7^{(n+3)/2}, 3^{(n+1)/2}, 9^{(n+3)/2}, 5^{(n+1)/2}, 1^{(n+3)/2}, 7^{(n+1)/2}, 3^{(n+3)/2}, 9^{(n+1)/2}, 5^{(n+3)/2})\ (2^{(n+1)/2}, 8^{(n+3)/2}, 4^{(n+1)/2}, 10^{(n+3)/2}, 6^{(n+1)/2}, 2^{(n+3)/2}, 8^{(n+1)/2}, 4^{(n+3)/2}, 10^{(n+1)/2}, 6^{(n+3)/2})$.

Similar to the last case, one can see $G = \langle\alpha, \sigma\rangle = Aut(\mathrm{C}_{10n})$ is isomorphic with the dihedral group D_{20}. The orbits of the automorphism group are given in Table 4.

In the following part, we will count all orbits of fullerene C_{10n}. To do this, let fix(g) be the set of elements of X fixed by g. By applying Burnside's Lemma, if group G acts on the set X, then for $g \in G$, the number of orbits is

$$\#O = \frac{1}{|G|} \sum_{g \in G} |fix(g)|. \tag{2.2}$$

For every edge $e = uv$ and each automorphism $\alpha \in Aut(G)$, define $\bar{\alpha}(e) = \{\alpha(u), \alpha(v)\}$. Thus, Aut$(G)$ acts on the set of edges by the above rule and the Burnside's Lemma for the set of edges can be rewritten as follows:

$$\#\bar{O} = \frac{1}{|\bar{G}|} \sum_{\bar{g} \in \bar{G}} |fix(\bar{g})|. \tag{2.3}$$

Again, consider the fullerene graph C_{10n}, where n is even, as depicted in Figure 2.1. In this part, we will find the permutation presentation of elements of $Aut(C_{10n})$.

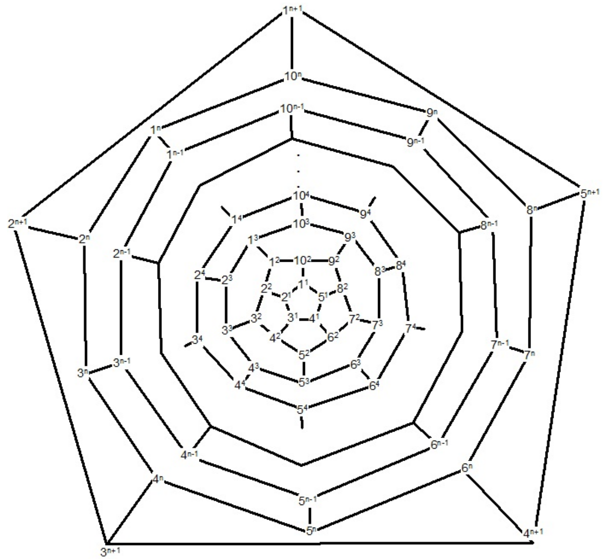

FIG. 2.2: C_{10n}, n is odd.

It is not difficult to see that there are five symmetry elements of order two in $Aut(C_{10n})$ denoted by α_i, $1 \le i \le 5$. One can easily check that
$fix(\alpha_1) = \{1^1, 10^2, 10^3, \cdots, 10^n, 5^2, 5^3, \cdots, 5^n, 3^{n+1}\}$,
$fix(\alpha_2) = \{2^1, 2^2, 2^3, \cdots, 2^n, 7^2, 7^3, \cdots, 7^n, 4^{n+1}\}$,
$fix(\alpha_3) = \{3^1, 4^2, 4^3, \cdots, 4^n, 9^2, 9^3, \cdots, 9^n, 5^{n+1}\}$,
$fix(\alpha_4) = \{4^1, 6^2, 6^3, \cdots, 6^n, 1^2, 1^3, \cdots, 1^n, 1^{n+1}\}$,
$fix(\alpha_5) = \{5^1, 8^2, 8^3, \cdots, 8^n, 3^2, 3^3, \cdots, 3^n, 2^{n+1}\}$.

This means that $|fix(\alpha_i)| = 2n$, $(1 \leqslant i \leqslant 5)$. Suppose β_1 is an involution that maps 1^1 to 2^{n+1}, 2^1 to 1^{n+1}, 3^1 to 5^{n+1}, 4^1 to 4^{n+1}, 5^1 to 3^{n+1}, 1^2 to 2^n, 2^2 to 1^n, 3^2 to 10^n, 4^2 to 9^n, 5^2 to 8^n, 6^2 to 7^n, 7^2 to 6^n, 8^2 to 5^n, 9^2 to 5^n, 10^2 to 3^n and so on. It is clear that fix $\beta_1 = \phi$. If we continue with this method, all permutation presentations of β_i's are as follows:
$\beta_1 = (1^1, 2^{n+1})(2^1, 1^{n+1})(3^1, 5^{n+1})(4^1, 4^{n+1})(5^1, 3^{n+1})(1^2, 2^n)(2^2, 1^n)(3^2, 10^n)(4^2, 9^n)(5^2, 8^n)(6^2, 7^n)(7^2, 6^n)(8^2, 5^n)(9^2, 4^n)(10^2, 3^n)(1^3, 2^{n-1})(2^3, 1^{n-1})(3^3, 10^{n-1})(4^3, 9^{n-1})(5^3, 8^{n-1})(6^3, 7^{n-1})(7^3, 6^{n-1})(8^3, 5^{n-1})(9^3, 4^{n-1})(10^3, 3^{n-1}) \cdots (1^{n/2}, 2^{(n+4)/2})(2^{n/2}, 1^{(n+4)/2})(3^{n/2}, 10^{(n+4)/2})(4^{n/2}, 9^{(n+4)/2})(5^{n/2}, 8^{(n+4)/2})(6^{n/2}, 7^{(n+4)/2})(7^{n/2}, 6^{(n+4)/2})(8^{n/2}, 5^{(n+4)/2})(9^{n/2}, 4^{(n+4)/2})(10^{n/2}, 3^{(n+4)/2})(1^{(n+2)/2}, 2^{(n+2)/2})(3^{(n+2)/2}, 10^{(n+2)/2})(4^{(n+2)/2}, 9^{(n+2)/2})(5^{(n+2)/2}, 8^{(n+2)/2})(6^{(n+2)/2}, 7^{(n+2)/2})$,

$\beta_2 = (1^1, 3^{n+1})(2^1, 2^{n+1})(3^1, 1^{n+1})(4^1, 5^{n+1})(5^1, 4^{n+1})(1^2, 4^n)(2^2, 3^n)(3^2, 2^n)(4^2, 1^n)(5^2, 10^n)(6^2, 9^n)(7^2, 8^n)(8^2, 7^n)(9^2, 6^n)(10^2, 5^n)(1^3, 4^{n-1})(2^3, 3^{n-1})(3^3, 2^{n-1})(4^3, 1^{n-1})(5^3, 10^{n-1})(6^3, 9^{n-1})(7^3, 8^{n-1})(8^3, 7^{n-1})(9^3, 6^{n-1})(10^3, 5^{n-1}) \cdots (1^{n/2}, 4^{(n+4)/2})(2^{n/2}, 3^{(n+4)/2})(3^{n/2}, 2^{(n+4)/2})(4^{n/2}, 1^{(n+4)/2})(5^{n/2}, 10^{(n+4)/2})(6^{n/2}, 9^{(n+4)/2})(7^{n/2}, 8^{(n+4)/2})(8^{n/2}, 7^{(n+4)/2})(9^{n/2}, 6^{(n+4)/2})(10^{n/2}, 5^{(n+4)/2})(1^{(n+2)/2},$

$4^{(n+2)/2})(2^{(n+2)/2},3^{(n+2)/2})(5^{(n+2)/2},10^{(n+2)/2})(6^{(n+2)/2},9^{(n+2)/2})(7^{(n+2)/2},$ $8^{(n+2)/2})$,

$\beta_{\ 3}=(1^1,4^{n+1})(2^1,3^{n+1})(3^1,2^{n+1})(4^1,1^{n+1})(5^1,5^{n+1})(1^2,6^n)(2^2,5^n)(3^2,4^n)(4^2,$ $3^n)(5^2,2^n)(6^2,1^n)(7^2,10^n)(8^2,9^n)(9^2,8^n)(10^2,7^n)(1^3,6^{n-1})(2^3,5^{n-1})(3^3,4^{n-1})$ $(4^3,3^{n-1})(5^3,2^{n-1})(6^3,1^{n-1})(7^3,10^{n-1})(8^3,9^{n-1})(9^3,8^{n-1})(10^3,7^{n-1})\cdots(1^{n/2},$ $6^{(n+4)/2})(2^{n/2},5^{(n+4)/2})(3^{n/2},4^{(n+4)/2})(4^{n/2},3^{(n+4)/2})(5^{n/2},2^{(n+4)/2})(6^{n/2},$ $1^{(n+4)/2})(7^{n/2},10^{(n+4)/2})(8^{n/2},9^{(n+4)/2})(9^{n/2},8^{(n+4)/2})(10^{n/2},7^{(n+4)/2})(1^{(n+2)/2},$ $6^{(n+2)/2})(2^{(n+2)/2},5^{(n+2)/2})(3^{(n+2)/2},4^{(n+2)/2})(7^{(n+2)/2},10^{(n+2)/2})(8^{(n+2)/2},$ $9^{(n+2)/2})$,

$\beta_{\ 4}=(1^1,5^{n+1})(2^1,4^{n+1})(3^1,3^{n+1})(4^1,2^{n+1})(5^1,1^{n+1})(1^2,8^n)(2^2,7^n)(3^2,6^n)(4^2,$ $5^n)(5^2,4^n)(6^2,3^n)(7^2,2^n)(8^2,1^n)(9^2,10^n)(10^2,9^n)(1^3,8^{n-1})(2^3,7^{n-1})(3^3,6^{n-1})$ $(4^3,5^{n-1})(5^3,4^{n-1})(6^3,3^{n-1})(7^3,2^{n-1})(8^3,1^{n-1})(9^3,10^{n-1})(10^3,9^{n-1})\cdots(1^{n/2},$ $8^{(n+4)/2})(2^{n/2},7^{(n+4)/2})(3^{n/2},6^{(n+4)/2})(4^{n/2},5^{(n+4)/2})(5^{n/2},4^{(n+4)/2})(6^{n/2},$ $3^{(n+4)/2})(7^{n/2},2^{(n+4)/2})(8^{n/2},1^{(n+4)/2})(9^{n/2},10^{(n+4)/2})(10^{n/2},9^{(n+4)/2})(1^{(n+2)/2},$ $8^{(n+2)/2})(2^{(n+2)/2},7^{(n+2)/2})(3^{(n+2)/2},6^{(n+2)/2})(4^{(n+2)/2},5^{(n+2)/2})(9^{(n+2)/2},$ $10^{(n+2)/2})$,

$\beta_{\ 5}=(1^1,1^{n+1})(2^1,5^{n+1})(3^1,4^{n+1})(4^1,3^{n+1})(5^1,2^{n+1})(1^2,10^n)(2^2,9^n)(3^2,8^n)(4^2,$ $7^n)(5^2,6^n)(6^2,5^n)(7^2,4^n)(8^2,3^n)(9^2,2^n)(10^2,1^n)(1^3,10^{n-1})(2^3,9^{n-1})(3^3,8^{n-1})$ $(4^3,7^{n-1})(5^3,6^{n-1})(6^3,5^{n-1})(7^3,4^{n-1})(8^3,3^{n-1})(9^3,2^{n-1})(10^3,1^{n-1}\cdots(1^{n/2},$ $10^{(n+4)/2}(2^{n/2},9^{(n+4)/2})(3^{n/2},8^{(n+4)/2})(4^{n/2},7^{(n+4)/2})(5^{n/2},6^{(n+4)/2})(6^{n/2},$ $5^{(n+4)/2})(7^{n/2},4^{(n+4)/2})(8^{n/2},3^{(n+4)/2})(9^{n/2},2^{(n+4)/2})(10^{n/2},1^{(n+4)/2})(1^{(n+2)/2},$ $10^{(n+2)/2})(2^{(n+2)/2},9^{(n+2)/2})(3^{(n+2)/2},8^{(n+2)/2})(4^{(n+2)/2},7^{(n+2)/2})(5^{(n+2)/2},$ $6^{(n+2)/2})$,

$\beta_{\ 6}=(1^1,3^{n+1})(2^1,4^{n+1})(3^1,5^{n+1})(4^1,1^{n+1})(5^1,2^{n+1})(1^2,6^n)(2^2,7^n)(3^2,8^n)(4^2,$ $9^n)(5^2,10^n)(6^2,1^n)(7^2,2^n)(8^2,3^n)(9^2,4^n)(10^2,5^n)(1^3,6^{n-1})(2^3,7^{n-1})(3^3,8^{n-1})$ $(4^3,9^{n-1})(5^3,10^{n-1})(6^3,1^{n-1})(7^3,2^{n-1})(8^3,3^{n-1})(9^3,4^{n-1})(10^3,5^{n-1}\cdots(1^{n/2},$ $6^{(n+4)/2})(2^{n/2},7^{(n+4)/2})(3^{n/2},8^{(n+4)/2})(4^{n/2},9^{(n+4)/2})(5^{n/2},10^{(n+4)/2})(6^{n/2},$ $1^{(n+4)/2})(7^{n/2},2^{(n+4)/2})(8^{n/2},3^{(n+4)/2})(9^{n/2},4^{(n+4)/2})(10^{n/2},5^{(n+4)/2})(1^{(n+2)/2},$ $6^{(n+2)/2})(2^{(n+2)/2},7^{(n+2)/2})(3^{(n+2)/2},8^{(n+2)/2})(4^{(n+2)/2},9^{(n+2)/2})(5^{(n+2)/2},$ $10^{(n+2)/2})$.

This yields that $Aut(C_{10n})$ includes four rotational elements γ_i $(1\le i\le 4)$ and four permutations σ_i $(1\le i\le 4)$ of order 10 with the following permutation presentation:

$\gamma_{\ 1}=(1^1,2^1,3^1,4^1,5^1)(1^2,3^2,5^2,7^2,9^2)(2^2,4^2,6^2,8^2,10^2)(1^3,3^3,5^3,7^3,9^3)(2^3,4^3,$ $6^3,8^3,10^3)\cdots(1^{n-1},3^{n-1},5^{n-1},7^{n-1},9^{n-1})(2^{n-1},4^{n-1},6^{n-1},8^{n-1},10^{n-1})(1^n,3^n,$ $5^n,7^n,9^n)(2^n,4^n,6^n,8^n,10^n)(1^{n+1},2^{n+1},3^{n+1},4^{n+1},5^{n+1})$,

$\gamma_{\ 2}=(1^1,3^1,5^1,2^1,4^1)(1^2,5^2,9^2,3^2,7^2)(2^2,6^2,10^2,4^2,8^2)(1^3,5^3,9^3,3^3,7^3)(2^3,6^3,$ $10^3,4^3,8^3)\cdots(1^{n-1},5^{n-1},9^{n-1},3^{n-1},7^{n-1})(2^{n-1},6^{n-1},10^{n-1},4^{n-1},8^{n-1})(1^n,5^n,$ $9^n,3^n,7^n)(2^n,6^n,10^n,4^n,8^n)(1^{n+1},3^{n+1},5^{n+1},2^{n+1},4^{n+1})$,

$\gamma_{\ 3}=(1^1,5^1,4^1,3^1,2^1)(1^2,9^2,7^2,5^2,3^2)(2^2,10^2,8^2,6^2,4^2)(1^3,9^3,7^3,5^3,3^3)(2^3,10^3,$ $8^3,6^3,4^3)\cdots(1^{n-1},9^{n-1},7^{n-1},5^{n-1},3^{n-1})(2^{n-1},10^{n-1},8^{n-1},6^{n-1},4^{n-1})(1^n,9^n,$ $7^n,5^n,3^n)(2^n,10^n,8^n,6^n,4^n)(1^{n+1},5^{n+1},4^{n+1},3^{n+1},2^{n+1})$,

$\gamma_{\,4} = (1^1, 4^1, 2^1, 5^1, 3^1)(1^2, 7^2, 3^2, 9^2, 5^2)(2^2, 8^2, 4^2, 10^2, 6^2)(1^3, 7^3, 3^3, 9^3, 5^3)(2^3, 8^3, 4^3, 10^3, 6^3) \cdots (1^{n-1}, 7^{n-1}, 3^{n-1}, 9^{n-1}, 5^{n-1})(2^{n-1}, 8^{n-1}, 4^{n-1}, 10^{n-1}, 6^{n-1})(1^n, 7^n, 3^n, 9^n, 5^n)(2^n, 8^n, 4^n, 10^n, 6^n)(1^{n+1}, 4^{n+1}, 2^{n+1}, 5^{n+1}, 3^{n+1})$.

$\sigma_{\,1} = (1^1, 5^{n+1}, 5^1, 4^{n+1}, 4^1, 3^{n+1}, 3^1, 2^{n+1}, 2^1, 1^{n+1})(1^2, 10^n, 9^2, 8^n, 7^2, 6^n, 5^2, 4^n, 3^2, 2^n)(2^2, 1^n, 10^2, 9^n, 8^2, 7^n, 6^2, 5^n, 4^2, 3^n)(1^3, 10^{n-1}, 9^3, 8^{n-1}, 7^3, 6^{n-1}, 5^3, 4^{n-1}, 3^3, 2^{n-1})(2^3, 1^{n-1}, 10^3, 9^{n-1}, 8^3, 7^{n-1}, 6^3, 5^{n-1}, 4^3, 3^{n-1}) \cdots (1^{n/2}, 10^{(n+4)/2}, 9^{n/2}, 8^{(n+4)/2}, 7^{n/2}, 6^{(n+4)/2}, 5^{n/2}, 4^{(n+4)/2}, 3^{n/2}, 2^{(n+4)/2})(2^{n/2}, 1^{(n+4)/2}, 10^{n/2}, 9^{(n+4)/2}, 8^{n/2}, 7^{(n+4)/2}, 6^{n/2}, 5^{(n+4)/2}, 4^{n/2}, 3^{(n+4)/2})(1^{(n+2)/2}, 10^{(n+2)/2}, 9^{(n+2)/2}, 8^{(n+2)/2}, 7^{(n+2)/2}, 6^{(n+2)/2}, 5^{(n+2)/2}, 4^{(n+2)/2}, 3^{(n+2)/2}, 2^{(n+2)/2})$,

$\sigma_{\,2} = (1^1, 1^{n+1}, 2^1, 2^{n+1}, 3^1, 3^{n+1}, 4^1, 4^{n+1}, 5^1, 5^{n+1})(1^2, 2^n, 3^2, 4^n, 5^2, 6^n, 7^2, 8^n, 9^2, 10^n)(2^2, 3^n, 4^2, 5^n, 6^2, 7^n, 8^2, 9^n, 10^2, 1^n)(1^3, 2^{n-1}, 3^3, 4^{n-1}, 5^3, 6^{n-1}, 7^3, 8^{n-1}, 9^3, 10^{n-1})(2^3, 3^{n-1}, 4^3, 5^{n-1}, 6^3, 7^{n-1}, 8^3, 9^{n-1}, 10^3, 1^{n-1} \cdots (1^{n/2}, 2^{(n+4)/2}, 3^{n/2}, 4^{(n+4)/2}, 5^{n/2}, 6^{(n+4)/2}, 7^{n/2}, 8^{(n+4)/2}, 9^{n/2}, 10^{(n+4)/2})(2^{n/2}, 3^{(n+4)/2}, 4^{n/2}, 5^{(n+4)/2}, 6^{n/2}, 7^{(n+4)/2}, 8^{n/2}, 9^{(n+4)/2}, 10^{n/2}, 1^{(n+4)/2})(1^{(n+2)/2}, 2^{(n+2)/2}, 3^{(n+2)/2}, 4^{(n+2)/2}, 5^{(n+2)/2}, 6^{(n+2)/2}, 7^{(n+2)/2}, 8^{(n+2)/2}, 9^{(n+2)/2}, 10^{(n+2)/2})$,

$\sigma_{\,3} = (1^1, 2^{n+1}, 4^1, 5^{n+1}, 2^1, 3^{n+1}, 5^1, 1^{n+1}, 3^1, 4^{n+1})(1^2, 4^n, 7^2, 10^n, 3^2, 6^n, 9^2, 2^n, 5^2, 8^n)(2^2, 5^n, 8^2, 1^n, 4^2, 7^n, 10^2, 3^n, 6^2, 9^n)(1^3, 4^{n-1}, 7^3, 10^{n-1}, 3^3, 6^{n-1}, 9^3, 2^{n-1}, 5^3, 8^{n-1})(2^3, 5^{n-1}, 8^3, 1^{n-1}, 4^3, 7^{n-1}, 10^3, 3^{n-1}, 6^3, 9^{n-1}) \cdots (1^{n/2}, 4^{(n+4)/2}, 7^{n/2}, 10^{(n+4)/2}, 3^{n/2}, 6^{(n+4)/2}, 9^{n/2}, 2^{(n+4)/2}, 5^{n/2}, 8^{(n+4)/2})(2^{n/2}, 5^{(n+4)/2}, 8^{n/2}, 1^{(n+4)/2}, 4^{n/2}, 7^{(n+4)/2}, 10^{n/2}, 3^{(n+4)/2}, 6^{n/2}, 9^{(n+4)/2})(1^{(n+2)/2}, 4^{(n+2)/2}, 7^{(n+2)/2}, 10^{(n+2)/2}, 3^{(n+2)/2}, 6^{(n+2)/2}, 9^{(n+2)/2}, 2^{(n+2)/2}, 5^{(n+2)/2}, 8^{(n+2)/2})$,

$\sigma_{\,4} = (1^1, 4^{n+1}, 3^1, 1^{n+1}, 5^1, 3^{n+1}, 2^1, 5^{n+1}, 4^1, 2^{n+1})(1^2, 8^n, 5^2, 2^n, 9^2, 6^n, 3^2, 10^n, 7^2, 4^n)(2^2, 9^n, 6^2, 3^n, 10^2, 7^n, 4^2, 1^n, 8^2, 5^n)(1^3, 8^{n-1}, 5^3, 2^{n-1}, 9^3, 6^{n-1}, 3^3, 10^{n-1}, 7^3, 4^{n-1})(2^3, 9^{n-1}, 6^3, 3^{n-1}, 10^3, 7^{n-1}, 4^3, 1^{n-1}, 8^3, 5^{n-1}) \cdots (1^{n/2}, 8^{(n+4)/2}, 5^{n/2}, 2^{(n+4)/2}, 9^{n/2}, 6^{(n+4)/2}, 3^{n/2}, 10^{(n+4)/2}, 7^{n/2}, 4^{(n+4)/2})(2^{n/2}, 9^{(n+4)/2}, 6^{n/2}, 3^{(n+4)/2}, 10^{n/2}, 7^{(n+4)/2}, 4^{n/2}, 1^{(n+4)/2}, 8^{n/2}, 5^{(n+4)/2})(1^{(n+2)/2}, 8^{(n+2)/2}, 5^{(n+2)/2}, 2^{(n+2)/2}, 9^{(n+2)/2}, 6^{(n+2)/2}, 3^{(n+2)/2}, 10^{(n+2)/2}, 7^{(n+2)/2}, 4^{(n+2)/2})$.

Hence, C_{10n} (n is even) has $\frac{n-2}{2} \times 2 + 2 = n$ orbits each of them has 10 vertices, see Table 1.

Vertex	**Orbit members**
1^1	$1^1, 2^1, 3^1, 4^1, 5^1, 1^{n+1}, 2^{n+1}, 3^{n+1}, 4^{n+1}, 5^{n+1}$
1^2	$1^2, 3^2, 5^2, 7^2, 9^2, 2^n, 4^n, 6^n, 8^n, 10^n$
2^2	$2^2, 4^2, 6^2, 8^2, 10^2, 1^n, 3^n, 5^n, 7^n, 9^n$
$\vdots$	$\vdots$
$1^{n/2}$	$1^{n/2}, 3^{n/2}, 5^{n/2}, 7^{n/2}, 9^{n/2},$ $2^{(n+4)/2}, 4^{(n+4)/2}, 6^{(n+4)/2}, 8^{(n+4)/2}, 10^{(n+4)/2}$
$2^{n/2}$	$2^{n/2}, 4^{n/2}, 6^{n/2}, 8^{n/2}, 10^{n/2},$ $1^{(n+4)/2}, 3^{(n+4)/2}, 5^{(n+4)/2}, 7^{(n+4)/2}, 9^{(n+4)/2}$
$1^{(n+2)/2}$	$1^{(n+2)/2}, 2^{(n+2)/2}, 3^{(n+2)/2}, 4^{(n+2)/2}, 5^{(n+2)/2},$ $6^{(n+2)/2}, 7^{(n+2)/2}, 8^{(n+2)/2}, 9^{(n+2)/2}, 10^{(n+2)/2}$

Table 1. Members of orbits of C_{10n}, n is even.

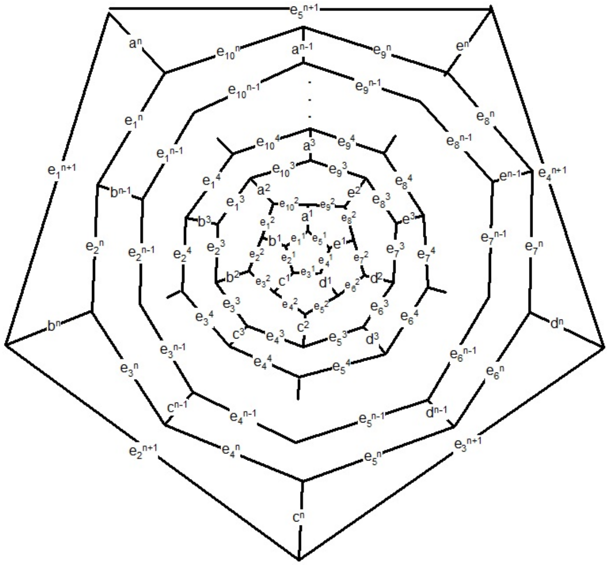

FIG. 2.3: C_{10n}, n is even.

It is not difficult to see that $|fix(\overline{\alpha_i})| = n + 2$ $(1 \le i \le 5)$, $|fix(\overline{\beta_j})| = 2$ $(1 \le j \le 5)$ and $|fix(\overline{\beta_6})| = |fix(\overline{\gamma_k})| = |fix(\overline{\sigma_l})| = 0$ $(1 \le k \le 4), (1 \le l \le 4)$. By considering the action of $Aut(C_{10n})$ on the set of edges and using Eq. 2.3, one can prove that the number of orbits is $n+1$ for which n is even. They are $O(e_1^1)$, $O(e^1)$, $O(e_1^2)$, $O(e^2)$, $\cdots$, $O(e_1^{(n-2)/2})$, $O(e^{(n-2)/2})$, $O(e_1^{n/2})$, $O(e^{n/2})$ and $O(e_1^{(n+2)/2})$. Hence, we proved the following theorem.

Theorem 2.3. *Consider the fullerene graph C_{10n}, n is even. Then there are $n+1$ orbits under the action of automorphism group on the set of edges.*

Theorem 2.4. *Consider the fullerene graph C_{10n}, where n is even and $n \ge 10$. Then*

$$Mo(C_{10n}) = 75n^2 - 100n + 3980.$$

Proof. Suppose $e_1^1 = \{1^1, 2^1\}$ and $e^1 = \{5^1, 8^2\}$. Then

$$\begin{aligned}
N_{1^1} &= \{1^1, 5^1, 7^2, 8^2, 9^2, 10^2, 7^3, 8^3, 9^3, 7^4, 8^4, 7^5\},\\
N_{1^1} &= \{1^1, 5^1, 7^2, 8^2, 9^2, 10^2, 7^3, 8^3, 9^3, 7^4, 8^4, 7^5\},\\
N_{2^1} &= \{2^1, 3^1, 2^2, 3^2, 4^2, 5^2, 3^3, 4^3, 5^3, 4^4, 5^4, 5^5\},\\
N_{1^1,2^1} &= V(C_{10n}) - N_{1^1} - N_{2^1},\\
N_{5^1} &= \{1^1, 2^1, 3^1, 4^1, 5^1, 2^2, 3^2, 4^2, 3^3\},\\
N_{8^2} &= V(C_{10n}) - N_{1^1} - N_{5^1,8^2},\\
N_{5^1,8^2} &= \{1^2, 5^2, 6^2, 10^2, 1^3, 2^3, 4^3, 5^3, 2^4, 3^4, 4^4, 3^5\}.
\end{aligned}$$

This means that $n(1^1) = 12$, $n(2^1) = 12$, $n(1^1, 2^1) = 10n - 24$, $n(5^1) = 9$, $n(8^2) = 10n - 21$ and $n(5^1, 8^2) = 12$. Also if $e_1^2 = \{2^2, 1^2\}$, $e^2 = \{9^2, 9^3\}$. Then

$$\begin{aligned}
N_{2^2} &= \{2^1, 3^1, 4^1, 2^2, 3^2, 4^2, 5^2, 6^2, 3^3, 4^3, 5^3, 6^3, 4^4, 5^4, 6^4, 5^5, 6^5, 6^6\},\\
N_{1^2} &= V(C_{10n}) - N_{2^2} - N_{2^2,1^2},\\
N_{2^2,1^2} &= \{1^1, 5^1\},\\
N_{9^2} &= \{1^1, 2^1, 3^1, 4^1, 5^1, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2\},\\
N_{9^3} &= V(C_{10n}) - N_{9^2} - N_{9^2,9^3},\\
N_{9^2,9^3} &= \varnothing,
\end{aligned}$$

and thus $n(2^2) = 18$, $n(1^2) = 10n - 20$, $n(2^2, 1^2) = 2$, $n(9^2) = 15$, $n(9^3) = 10n - 15$, $n(9^2, 9^3) = 0$, and so on, see Table 2. By using Theorem 2.2, for every edge $e = \{u, v\}$, one can determine the contributions of $n(u)$ and $n(v)$ of edge $e = \{u, v\}$ as reported in Table 2. The summation of these integers yields that

$$\begin{aligned}
&Mo(C_{10n}) = 10 \times (12 - 12) + 10 \times (10n - 30) + 20 \times (10n - 38)\\
&+10 \times (10n - 30) + 20 \times (10n - 50) + 10 \times (10n - 50)\\
&+20 \times (10n - 65) + 10 \times (10n - 70) + 20 \times (10n - 81)\\
&+10 \times \textstyle\sum_{i=0}^{n/2-5} 10n - 2(45 + 10i) + 20 \times \sum_{i=0}^{n/2-6} 10n - 2(50 + 10i)\\
&+10 \times (5n - 5n) = 75n^2 - 100n + 3980.
\end{aligned}$$

□

Type of edge	$\boldsymbol{n(u)}$**,** $\boldsymbol{n(v)}$**,** ***equidistant***	**Number**
e_1^1	12,12,10n-24	10
e^1	9,10n-21,12	10
e_1^2	18,10n–20,2	20
e^2	15,10n-15,0	10
e_1^3	24,20n-26,2	20
e^3	25,10n–25,0	10
e_1^4	32,10n-33,1	20
e^4	35,10n-35,0	10
e_1^5	40,10n-41,1	20
e^5	45,10n-45,0	10
e_1^6	50,10n-50,0	20
e^6	55,10n-55,0	10
⋮	⋮	⋮
$e_1^{(n-2)/2}$	5n-20,5n+20,0	20
$e^{(n-2)/2}$	5n-15,5n+15,0	10
$e_1^{n/2}$	5n-10,5n+10,0	20
$e^{n/2}$	5n-5,5n+5,0	10
$e_1^{(n+2)/2}$	5n,5n,0	10

Table 2. The values of $n(u)$, $n(v)$ and equidistant vertices, where $n \geq 8$.

The exceptional cases are given in Table 3. Also, their Mostar indices are given in Table 4.

Type of edge	C_{40}			C_{60}			C_{80}		
e_1^1	12	12	16	12	12	36	12	12	56
e^1	9	20	11	9	39	12	9	59	12
e_1^2	15	22	3	18	40	2	18	60	2
e^2	15	25	0	15	45	0	15	65	0
e_1^3	18	18	4	23	34	3	24	54	2
e^3	-	-	-	25	35	0	25	55	0
e_1^4	-	-	-	29	29	2	32	47	1
e^4	-	-	-	-	-	-	35	45	0
e_1^5	-	-	-	-	-	-	39	39	2

Table 3. Exception of $n(u)$, $n(\mathrm{v})$, equidistant vertices.

n	4	6	8
$\mathrm{Mo}(C_{10n})$	350	1360	3140

Table 4. Special cases of Mostar index of fullerene C_{10n}..

Now consider the fullerene graph C_{10n}, where n is odd, as depicted in Figure 2.2. There are five symmetry elements of order two in $Aut(C_{10n})$ denoted by α_i, $1 \leq i \leq 5$. One can easily check that

$fix(\alpha_1) = \{1^1, 10^2, 10^3, \cdots, 10^n, 1^{n+1}, 5^2, 5^3, \cdots, 5^n\}$,

$fix(\alpha_2) = \{2^1, 2^2, 2^3, \cdots, 2^n, 2^{n+1}, 7^2, 7^3, \cdots, 7^n\}$,

$fix(\alpha_3) = \{3^1, 4^2, 4^3, \cdots, 4^n, 3^{n+1}, 9^2, 9^3, \cdots, 9^n\}$,

$fix(\alpha_4) = \{4^1, 6^2, 6^3, \cdots, 6^n, 4^{n+1}, 1^2, 1^3, \cdots, 1^n\}$,

$fix(\alpha_5) = \{5^1, 8^2, 8^3, \cdots, 8^n, 5^{n+1}, 3^2, 3^3, \cdots, 3^n, 2^{n+1}\}$.

This means that $fix(\alpha_i) = 2n$, $(1 \leqslant i \leqslant 5)$. Similar to the last case, the presentations of other elements of $Aut(C_{10n})$ are as follows:

$\beta_1 = (1^1, 1^{n+1})(2^1, 2^{n+1})(3^1, 3^{n+1})(4^1, 4^{n+1})(5^1, 5^{n+1})(1^2, 1^n)(2^2, 2^n)(3^2, 3^n)(4^2, 4^n)(5^2, 5^n)(6^2, 6^n)(7^2, 7^n)(8^2, 8^n)(9^2, 9^n)(10^2, 10^n)(1^3, 1^{n-1})(2^3, 2^{n-1})(3^3, 3^{n-1})(4^3, 4^{n-1})(5^3, 5^{n-1})(6^3, 6^{n-1})(7^3, 7^{n-1})(8^3, 8^{n-1})(9^3, 9^{n-1})(10^3, 10^{n-1}) \cdots (1^{(n+1)/2}, 1^{(n+3)/2})(2^{(n+1)/2}, 2^{(n+3)/2})(3^{(n+1)/2}, 3^{(n+3)/2})(4^{(n+1)/2}, 4^{(n+3)/2})(5^{(n+1)/2}, 5^{(n+3)/2})(6^{(n+1)/2}, 6^{(n+3)/2})(7^{(n+1)/2}, 7^{(n+3)/2})(8^{(n+1)/2}, 8^{(n+3)/2})(9^{(n+1)/2}, 9^{(n+3)/2})(10^{(n+1)/2}, 10^{(n+3)/2})$,

$\beta_2 = (1^1, 3^{n+1})(2^1, 2^{n+1})(3^1, 1^{n+1})(4^1, 5^{n+1})(5^1, 4^{n+1})(1^2, 3^n)(2^2, 2^n)(3^2, 1^n)(4^2, 10^n)(5^2, 9^n)(6^2, 8^n)(7^2, 7^n)(8^2, 6^n)(9^2, 5^n)(10^2, 4^n)(1^3, 3^{n-1})(2^3, 2^{n-1})(3^3, 1^{n-1})(4^3, 10^{n-1})(5^3, 9^{n-1})(6^3, 8^{n-1})(7^3, 7^{n-1})(8^3, 6^{n-1})(9^3, 5^{n-1})(10^3, 4^{n-1} \cdots (1^{(n+1)/2}, 3^{(n+3)/2})(2^{(n+1)/2}, 2^{(n+3)/2})(3^{(n+1)/2}, 1^{(n+3)/2})(4^{(n+1)/2}, 10^{(n+3)/2})$

$(5^{(n+1)/2},9^{(n+3)/2})(6^{(n+1)/2},8^{(n+3)/2})(7^{(n+1)/2},7^{(n+3)/2})(8^{(n+1)/2},6^{(n+3)/2})$ $(9^{(n+1)/2},5^{(n+3)/2})(10^{(n+1)/2},4^{(n+3)/2})$,

$\beta_{3}=(1^1,4^{n+1})(2^1,3^{n+1})(3^1,2^{n+1})(4^1,1^{n+1})(5^1,5^{n+1}(1^2,5^n)(2^2,4^n)(3^2,3^n)(4^2,2^n)(5^2,1^n)(6^2,10^n)(7^2,9^n)(8^2,8^n)(9^2,7^n)(10^2,6^n)(1^3,5^{n-1})(2^3,4^{n-1})(3^3,3^{n-1})(4^3,2^{n-1})(5^3,1^{n-1})(6^3,10^{n-1})(7^3,9^{n-1})(8^3,8^{n-1})(9^3,7^{n-1})(10^3,6^{n-1})\cdots(1^{(n+1)/2},5^{(n+3)/2})(2^{(n+1)/2},4^{(n+3)/2})(3^{(n+1)/2},3^{(n+3)/2})(4^{(n+1)/2},2^{(n+3)/2})(5^{(n+1)/2},1^{(n+3)/2})(6^{(n+1)/2},10^{(n+3)/2})(7^{(n+1)/2},9^{(n+3)/2})(8^{(n+1)/2},8^{(n+3)/2})(9^{(n+1)/2},7^{(n+3)/2})(10^{(n+1)/2},6^{(n+3)/2})$,

$\beta_{4}=(1^1,5^{n+1})(2^1,4^{n+1})(3^1,3^{n+1})(4^1,2^{n+1})(5^1,1^{n+1})(1^2,7^n)(2^2,6^n)(3^2,5^n)(4^2,4^n)(5^2,3^n)(6^2,2^n)(7^2,1^n)(8^2,10^n)(9^2,9^n)(10^2,8^n)(1^3,7^{n-1})(2^3,6^{n-1})(3^3,5^{n-1})(4^3,4^{n-1})(5^3,3^{n-1})(6^3,2^{n-1})(7^3,1^{n-1})(8^3,10^{n-1})(9^3,9^{n-1})(10^3,8^{n-1})\cdots(1^{(n+1)/2},7^{(n+3)/2})(2^{(n+1)/2},6^{(n+3)/2})(3^{(n+1)/2},5^{(n+3)/2})(4^{(n+1)/2},4^{(n+3)/2})(5^{(n+1)/2},3^{(n+3)/2})(6^{(n+1)/2},2^{(n+3)/2})(7^{(n+1)/2},1^{(n+3)/2})(8^{(n+1)/2},10^{(n+3)/2})(9^{(n+1)/2},9^{(n+3)/2})(10^{(n+1)/2},8^{(n+3)/2})$,

$\beta_{5}=(1^1,1^{n+1})(2^1,5^{n+1})(3^1,4^{n+1})(4^1,3^{n+1})(5^1,2^{n+1})(1^2,9^n)(2^2,8^n)(3^2,7^n)(4^2,6^n)(5^2,5^n)(6^2,4^n)(7^2,3^n(8^2,2^n)(9^2,1^n)(10^2,10^n)(1^3,9^{n-1})(2^3,8^{n-1})(3^3,7^{n-1})(4^3,6^{n-1})(5^3,5^{n-1})(6^3,4^{n-1})(7^3,3^{n-1})(8^3,2^{n-1})(9^3,1^{n-1})(10^3,10^{n-1})\cdots(1^{(n+1)/2},9^{(n+3)/2})(2^{(n+1)/2},8^{(n+3)/2})(3^{(n+1)/2},7^{(n+3)/2})(4^{(n+1)/2},6^{(n+3)/2})(5^{(n+1)/2},5^{(n+3)/2})(6^{(n+1)/2},4^{(n+3)/2})(7^{(n+1)/2},3^{(n+3)/2})(8^{(n+1)/2},2^{(n+3)/2})(9^{(n+1)/2},1^{(n+3)/2})(10^{(n+1)/2},10^{(n+3)/2})$,

$\beta_{6}=(1^1,2^{n+1})(2^1,1^{n+1})(3^1,5^{n+1})(4^1,4^{n+1})(5^1,3^{n+1})(1^2,1^n)(2^2,10^n)(3^2,9^n)(4^2,8^n)(5^2,7^n)(6^2,6^n)(7^2,5^n)(8^2,4^n)(9^2,3^n)(10^2,2^n)(1^3,1^{n-1})(2^3,10^{n-1})(3^3,9^{n-1})(4^3,8^{n-1})(5^3,7^{n-1})(6^3,6^{n-1})(7^3,5^{n-1})(8^3,4^{n-1})(9^3,3^{n-1})(10^3,2^{n-1})\cdots(1^{(n+1)/2},1^{(n+3)/2})(2^{(n+1)/2},10^{(n+3)/2})(3^{(n+1)/2},9^{(n+3)/2})(4^{(n+1)/2},8^{(n+3)/2})(5^{(n+1)/2},7^{(n+3)/2})(6^{(n+1)/2},6^{(n+3)/2})(7^{(n+1)/2},5^{(n+3)/2})(8^{(n+1)/2},4^{(n+3)/2})(9^{(n+1)/2},3^{(n+3)/2})(10^{(n+1)/2},2^{(n+3)/2})$,

$\gamma_{1}=(1^1,2^1,3^1,4^1,5^1)(1^2,3^2,5^2,7^2,9^2)(2^2,4^2,6^2,8^2,10^2)(1^3,3^3,5^3,7^3,9^3)(2^3,4^3,6^3,8^3,10^3)\cdots(1^{n-1},3^{n-1},5^{n-1},7^{n-1},9^{n-1})(2^{n-1},4^{n-1},6^{n-1},8^{n-1},10^{n-1})(1^n,3^n,5^n,7^n,9^n)(2^n,4^n,6^n,8^n,10^n)(1^{n+1},2^{n+1},3^{n+1},4^{n+1},5^{n+1})$,

$\gamma_{2}=(1^1,3^1,5^1,2^1,4^1)(1^2,5^2,9^2,3^2,7^2)(2^2,6^2,10^2,4^2,8^2)(1^3,5^3,9^3,3^3,7^3)(2^3,6^3,10^3,4^3,8^3)\cdots(1^{n-1},5^{n-1},9^{n-1},3^{n-1},7^{n-1})(2^{n-1},6^{n-1},10^{n-1},4^{n-1},8^{n-1})(1^n,5^n,9^n,3^n,7^n)(2^n,6^n,10^n,4^n,8^n)(1^{n+1},3^{n+1},5^{n+1},2^{n+1},4^{n+1})$,

$\gamma_{3}=(1^1,5^1,4^1,3^1,2^1)(1^2,9^2,7^2,5^2,3^2)(2^2,10^2,8^2,6^2,4^2)(1^3,9^3,7^3,5^3,3^3)(2^3,10^3,8^3,6^3,4^3)\cdots(1^{n-1},9^{n-1},7^{n-1},5^{n-1},3^{n-1})(2^{n-1},10^{n-1},8^{n-1},6^{n-1},4^{n-1})(1^n,9^n,7^n,5^n,3^n)(2^n,10^n,8^n,6^n,4^n)(1^{n+1},5^{n+1},4^{n+1},3^{n+1},2^{n+1})$,

$\gamma_{4}=(1^1,4^1,2^1,5^1,3^1)(1^2,7^2,3^2,9^2,5^2)(2^2,8^2,4^2,10^2,6^2)(1^3,7^3,3^3,9^3,5^3)(2^3,8^3,4^3,10^3,6^3)\cdots(1^{n-1},7^{n-1},3^{n-1},9^{n-1},5^{n-1})(2^{n-1},8^{n-1},4^{n-1},10^{n-1},6^{n-1})(1^n,7^n,3^n,9^n,5^n)(2^n,8^n,4^n,10^n,6^n)(1^{n+1},4^{n+1},2^{n+1},5^{n+1},3^{n+1})$,

$\sigma_{1}=(1^1,2^{n+1},3^1,4^{n+1},5^1,1^{n+1},2^1,3^{n+1},4^1,5^{n+1})(1^2,3^n,5^2,7^n,9^2,1^n,3^2,5^n,7^2,9^n)(2^2,4^n,6^2,8^n,10^2,2^n,4^2,6^n,8^2,10^n)(1^3,3^{n-1},5^3,7^{n-1},9^3,1^{n-1},3^3,5^{n-1},7^3,9^{n-1})(2^3,4^{n-1},6^3,8^{n-1},10^3,2^{n-1},4^3,6^{n-1},8^3,10^{n-1})\cdots(1^{(n+1)/2},3^{(n+3)/2},5^{(n+1)/2},7^{(n+3)/2},9^{(n+1)/2},1^{(n+3)/2},3^{(n+1)/2},5^{(n+3)/2},7^{(n+1)/2},9^{(n+3)/2})(2^{(n+1)/2},4^{(n+3)/2},6^{(n+1)/2},8^{(n+3)/2},10^{(n+1)/2},2^{(n+3)/2},4^{(n+1)/2},6^{(n+3)/2},8^{(n+1)/2},10^{(n+3)/2})$,

$\sigma_2=(1^1,3^{n+1},5^1,2^{n+1},4^1,1^{n+1},3^1,5^{n+1},2^1,4^{n+1})(1^2,5^n,9^2,3^n,7^2,1^n,5^2,9^n,3^2,7^n)(2^2,6^n,10^2,4^n,8^2,2^n,6^2,10^n,4^2,8^n)(1^3,5^{n-1},9^3,3^{n-1},7^3,1^{n-1},5^3,9^{n-1},3^3,7^{n-1})(2^3,6^{n-1},10^3,4^{n-1},8^3,2^{n-1},6^3,10^{n-1},4^3,8^{n-1})\cdots(1^{(n+1)/2},5^{(n+3)/2},9^{(n+1)/2},3^{(n+3)/2},7^{(n+1)/2},1^{(n+3)/2},5^{(n+1)/2},9^{(n+3)/2},3^{(n+1)/2},7^{(n+3)/2})(2^{(n+1)/2},6^{(n+3)/2},10^{(n+1)/2},4^{(n+3)/2},8^{(n+1)/2},2^{(n+3)/2},6^{(n+1)/2},10^{(n+3)/2},4^{(n+1)/2},8^{(n+3)/2})$,

$\sigma_3=(1^1,4^{n+1},2^1,5^{n+1},3^1,1^{n+1},4^1,2^{n+1},5^1,3^{n+1})(1^2,7^n,3^2,9^n,5^2,1^n,7^2,3^n,9^2,5^n)(2^2,8^n,4^2,10^n,6^2,2^n,8^2,4^n,10^2,6^n)(1^3,7^{n-1},3^3,9^{n-1},5^3,1^{n-1},7^3,3^{n-1},9^3,5^{n-1})(2^3,8^{n-1},4^3,10^{n-1},6^3,2^{n-1},8^3,4^{n-1},10^3,6^{n-1})\cdots(1^{(n+1)/2},7^{(n+3)/2},3^{(n+1)/2},9^{(n+3)/2},5^{(n+1)/2},1^{(n+3)/2},7^{(n+1)/2},3^{(n+3)/2},9^{(n+1)/2},5^{(n+3)/2})(2^{(n+1)/2},8^{(n+3)/2},4^{(n+1)/2},10^{(n+3)/2},6^{(n+1)/2},2^{(n+3)/2},8^{(n+1)/2},4^{(n+3)/2},10^{(n+1)/2},6^{(n+3)/2})$,

$\sigma_4=(1^1,5^{n+1},4^1,3^{n+1},2^1,1^{n+1},5^1,4^{n+1},3^1,2^{n+1})(1^2,9^n,7^2,5^n,3^2,1^n,9^2,7^n,5^2,3^n)(2^2,10^n,8^2,6^n,4^2,2^n,10^2,8^n,6^2,4^n)(1^3,9^{n-1},7^3,5^{n-1},3^3,1^{n-1},9^3,7^{n-1},5^3,3^{n-1})(2^3,10^{n-1},8^3,6^{n-1},4^3,2^{n-1},10^3,8^{n-1},6^3,4^{n-1})\cdots(1^{(n+1)/2},9^{(n+3)/2},7^{(n+1)/2},5^{(n+3)/2},3^{(n+1)/2},1^{(n+3)/2},9^{(n+1)/2},7^{(n+3)/2},5^{(n+1)/2},3^{(n+3)/2})(2^{(n+1)/2},10^{(n+3)/2},8^{(n+1)/2},6^{(n+3)/2},4^{(n+1)/2},2^{(n+3)/2},10^{(n+1)/2},8^{(n+3)/2},6^{(n+1)/2},4^{(n+3)/2})$.

So, the fullerene C_{10n} (n is odd) has $\frac{n-1}{2}\times 2+1=n$ orbits which each of them has 10 vertices, see Table 5.

Vertex	Members of orbit
1^1	$1^1,2^1,3^1,4^1,5^1,1^{n+1},2^{n+1},3^{n+1},4^{n+1},5^{n+1}$
1^2	$1^2,3^2,5^2,7^2,9^2,1^n,3^n,5^n,7^n,9^n$
2^2	$2^2,4^2,6^2,8^2,10^2,2^n,4^n,6^n,8^n,10^n$
$\vdots$	$\vdots$
$1^{(n+1)/2}$	$1^{(n+1)/2},3^{(n+1)/2},5^{(n+1)/2},7^{(n+1)/2},9^{(n+1)/2},$ $1^{(n+3)/2},3^{(n+3)/2},5^{(n+3)/2},7^{(n+3)/2},9^{(n+3)/2}$
$2^{(n+1)/2}$	$2^{(n+1)/2},4^{(n+1)/2},6^{(n+1)/2},8^{(n+1)/2},10^{(n+1)/2},$ $2^{(n+3)/2},4^{(n+3)/2},6^{(n+3)/2},8^{(n+3)/2},10^{(n+3)/2}$

Table 5. Members of orbits of C_{10n}, n is odd.

The values of $n(u)$, $n(v)$ and equidistant vertices in Table 6 can be obtained by a similar argument. Hence, we have the following theorem.

Theorem 2.5. *Consider the fullerene graph C_{10n} (n is odd), then there are $n+1$ orbits under the action of automorphism group on the set of edges. They are $O(e_1^1)$, $O(e^1)$, $O(e_1^2)$, $O(e^2)$, $\cdots$, $O(e^{(n-1)/2})$, $O(e_1^{(n+1)/2})$ and $O(e^{(n+1)/2})$.*

Theorem 2.6. *Consider the fullerene graph C_{10n} where n is odd and $n\geq 9$. Then*

$$Mo(C_{10n})=75n^2-100n+4005.$$

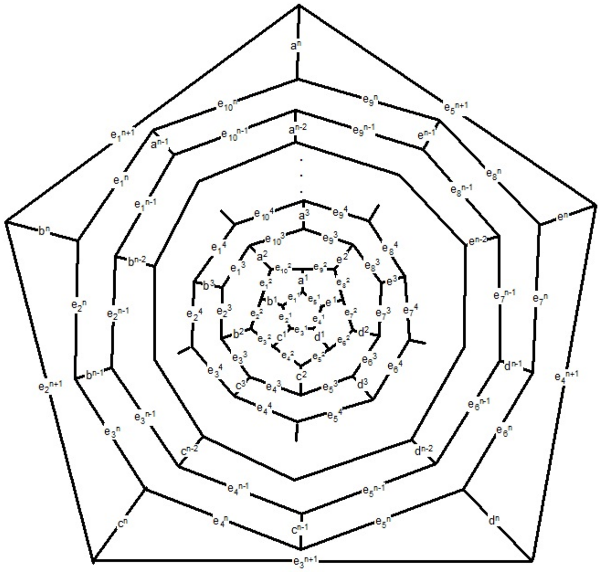

FIG. 2.4: C_{10n}, n is odd.

Type of edge	$n(u)$, $n(v)$, *equidistant*	Number
e_1^1	12,12,10n-24	10
e^1	9,10n-21,12	10
e_1^2	18,10n–20,2	20
e^2	15,10n-15,0	10
e_1^3	24,20n-26,2	20
e^3	25,10n–25,0	10
e_1^4	32,10n-33,1	20
e^4	35,10n-35,0	10
e_1^5	40,10n-41,1	20
e^5	45,10n-45,0	20
e_1^6	50,10n-50,0	20
e^6	55,10n-55,0	10
$\vdots$	$\vdots$	$\vdots$
$e_1^{(n-1)/2}$	5n-15,5n+15,0	20
$e^{(n-1)/2}$	5n-10,5n+10,0	10
$e_1^{(n+1)/2}$	5n-5,5n+5,0	20
$e^{(n+1)/2}$	5n,5n,0	5

Table 6. The values of $n(e)$, $n(v)$ and equidistant vertices, where $n \geq 7$.

The exceptional cases are given in Table 7. Also, their Mostar indices are given in Table 8.

Type of edge	C_{30}			C_{50}			C_{70}		
e_1^1	10	10	10	12	12	26	12	12	46
e^1	9	13	8	9	29	12	9	49	12
e_1^2	12	14	4	17	30	3	18	50	2
e^2	15	15	0	15	35	0	15	55	0
e_1^3	-	-	-	21	26	3	24	44	2
e^3	-	-	-	25	25	0	25	45	0
e_1^4	-	-	-	-	-	-	31	37	2
e^4	-	-	-	-	-	-	35	35	0

Table 7. Exception of $n(u)$, $n(\mathrm{v})$, equidistant vertices.

$\boldsymbol{n}$	**3**	**5**	**7**
$\mathrm{Mo}(C_{10n})$	80	760	2160

Table 8. Special cases of Mostar index of fullerene C_{10n}.

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TANGENT BUNDLE ENDOWED WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION ON AN ALMOST HERMITIAN MANIFOLD

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Abstract. In this paper, we have studied the tangent bundle endowed with quarter-symmetric non-metric connection obtained by vertical and complete lifts of a quarter-symmetric non-metric connection on the base manifold and, also, proposed the study of the tangent bundle of an almost Hermitian manifold and an almost Kaehler manifold. Finally, we obtained some theorems for Nijenhuis tensor on the tangent bundle of an almost Hermitian manifold and an almost Kaehler manifold.
Keywords: Almost Hermitian manifold, almost Kaehler manifold, vertical lift, complete lift, Nijenhuis tensor.

1. Introduction

The idea of quarter-symmetric linear connections in differentiable manifold was introduced by Golab [4] in 1975. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\widetilde{T}$ is of the for

$$\widetilde{T}(X,Y) = u(Y)\phi X - u(X)\phi Y$$

where u is 1-form and ϕ is a tensor of type (1, 1). Agashe and Chafle [1] studied a semi-symmetric non-metric connection on a Riemannian manifold in 1992. In 2007, the author [7] defined and studied a quarter-symmetric semi-metric connection on Sasakian manifold. In 2008, Chaturvedi and Pandey [2] studied Kaehler manifold equipped with a semi-symmetric non-metric connection.

The method of lift has an important role in modern differential geometry. With the lift function, it is possible to generalize to differentiable structures on any manifold to its extensions. The complete, vertical and horizontal lifts of tensor fields and connections on any manifold M to the tangent manifold TM were obtained by Yano and Ishihara [13] in 1973. In 1969, Tani [11] developed the theory of

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hypersurfaces prolonged to tangent bundle with respect to complete lift of metric tensor of Riemannian manifold. In 2005, Das and the author [3] obtained almost product structure by means of the complete, vertical and horizontal lifts of almost r-contact structures on the tangent bundle. The author [8] studied the lifts of hypersurfaces with quarter-symmetric semi-metric connection to the tangent bundles and obtained an important result (Theorem3 in [8]). We have used similar method in section 3 of this paper. Among some other authors who studied the differential geometry of the tangent bundle are the following [6, 9, 12, 14].

The paper is structured as follows. In Section 2, we will recall an almost Hermitian manifold, Quarter symmetric non-metric connection, Tangent bundle, Induced metric and connection. We will consider, in Section 3, the tangent bundle endowed with quarter-symmetric non-metric connection obtained by vertical and complete lifts of a quarter-symmetric non-metric connection on the base manifold and propose to study the tangent bundle of an almost Hermitian manifold and an almost Kaehler manifold. In Section 4, we will obtain some theorems for Nijenhuis tensor on tangent bundle of an almost Hermitian manifold and an almost Kaehler manifold. In the last Section, we will construct an example of a four dimensional almost Hermitian manifold on tangent bundle.

2. Preliminaries

2.1. An almost Hermitian manifold

A tensor field $\hat{F}$ of type (1,1) on an even dimensional differentiable manifold $M, n = 2m$ such that

$$\hat{F}^2\hat{X} + \hat{X} = 0. \tag{2.1}$$

If non-singular Hermitian metric of type (0,2) satisfies

$$\hat{g}(\hat{F}\hat{X}, \hat{F}\hat{Y}) = \hat{g}(\hat{X}, \hat{Y}) \tag{2.2}$$

for arbitrary vector fields $\hat{X}, \hat{Y}$, then $(M, \hat{F}, \hat{g})$ is called an almost Hermitian manifold with an almost Hermitian structure $(\hat{F}, \hat{g})$. Let $\hat{\nabla}$ be the Riemannian connection on M, then M is said to be a Kaehler manifold [2] if

$$(\hat{\nabla}_{\hat{X}}\hat{F})\hat{Y} = 0. \tag{2.3}$$

2.2. Quarter-symmetric non-metric connection

A linear connection $\overline{\nabla}$ on M is defined as

$$\overline{\nabla}_{\hat{X}}\hat{Y} = \hat{\nabla}_{\hat{X}}\hat{Y} + \hat{\omega}(\hat{Y})\hat{F}\hat{X}, \tag{2.4}$$

for arbitrary vector fields $\hat{X}, \hat{Y}$, 1-form $\hat{\omega}$ and $\hat{\nabla}$ denotes Riemannian connection on M.

The torsion tensor $\overline{T}$ is given by

$$\overline{T}(\hat{X},\hat{Y}) = \overline{\nabla}_{\hat{X}}\hat{Y} + \overline{\nabla}_{\hat{Y}}\hat{X} - [\hat{X},\hat{Y}].$$

The connection $\overline{\nabla}$ is symmetric if its torsion tensor vanishes, otherwise, it is non-symmetric. If there is a metric $\hat{g}$ in M such that $\overline{\nabla}\hat{g} = 0$, then the connection $\overline{\nabla}$ is a metric connection; otherwise it is non-metric [5].

A linear connection $\overline{\nabla}$ is said to be a quarter-symmetric linear connection if its torsion connection $\overline{T}$ is of the form

$$\overline{T}(\hat{X},\hat{Y}) = \hat{\omega}(\hat{Y})\hat{F}\hat{X} - \hat{\omega}(\hat{X})\hat{F}\hat{Y} \tag{2.5}$$

and

$$(\overline{\nabla}_{\hat{X}}\hat{g})(\hat{X},\hat{Y}) = -\hat{\omega}(\hat{Y})\hat{g}(\hat{F}\hat{X},\hat{Z}) - \hat{\omega}(\hat{Z})\hat{g}(\hat{F}\hat{X},\hat{Y}). \tag{2.6}$$

The connection $\overline{\nabla}$ satisfying (2.4), (2.5) and(2.6) is called quarter-symmetric non-metric connection [7].

2.3. Tangent Bundle

Let $T_p(M)$ be the tangent space of differentiable manifold M at a point of M, then the set $T(M) = \cup_{p\in M} T_p(M)$ is called the tangent bundle over the manifold M. For any point $\widetilde{p}$ of $T(M)$, the correspondence $\widetilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$. Thus $\pi(\widetilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \in M$ and M the base space [3, 13].

Vertical lifts: If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to M$, so that $f^V = fo\pi$. Thus, if a point $\widetilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then $f^V(\widetilde{p}) = f^V(x,y) = fo\pi(\widetilde{p}) = f(p) = f(x)$ thus the value of $f^V(\widetilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f.

Complete lifts: If f is a function in M, we write f^C for the function in $T(M)$ defined by $f^C = i(df)$ and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression $f^C = y^i\partial_i f = \partial f$ with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i\partial_i f$.

Suppose that $\hat{X} \in \mathrm{Im}_0^1(M)$. We define a vector field $\hat{X}^C$ in $T(M)$ by $\hat{X}^C f^C = (\hat{X}f)^C$, f being an arbitrary function in M and call $\hat{X}^C$ the complete lift of call $\hat{X}$ in $T(M)$.

Suppose that $\hat{\omega} \in \mathrm{Im}_0^1(M)$. Then a 1-form $\hat{\omega}^C$ in $T(M)$ defined by $\hat{\omega}^C(\hat{X}^C) = (\hat{\omega}(\hat{X}))^C$, $\hat{X}$ being an arbitrary vector field in M. We call $\hat{\omega}^C$ the complete lift of $\hat{\omega}$. Moreover, these lifts have the following properties [8]:

$$[\hat{X}^C, \hat{Y}^C] = [\hat{X}, \hat{Y}]^C \ ; \ \hat{F}^C(\hat{X}^C) = (\hat{F}(\hat{X}))^C$$
$$\hat{\omega}^V(\hat{X}^C) = (\hat{\omega}(\hat{X}))^V \ ; \ \hat{\omega}^C(\hat{X}^C) = (\hat{\omega}(\hat{X}))^C$$
$$\hat{g}^C(\hat{X}^V, \hat{Y}^C) = \hat{g}^C(\hat{X}^C, \hat{Y}^V) = (\hat{g}(\hat{X}, \hat{Y}))^V \ ; \ \hat{g}^C(\hat{X}^V, \hat{Y}^C) = (\hat{g}(\hat{X}, \hat{Y}))^C$$
$$\hat{T}^C(\hat{X}^C, \hat{Y}^C) = (\hat{T}(\hat{X}, \hat{Y}))^C$$

where $\hat{X}^V, \hat{\omega}^V, \hat{g}^V, \hat{T}^V$ and $\hat{X}^C, \hat{\omega}^C, \hat{g}^C, \hat{T}^C$ are vertical and complete lifts of $\hat{X}, \hat{\omega}, \hat{g}, \hat{T}$ respectively.

2.4. Induced metric and connection on $T(S)$

Let S be a manifold of $(n-1)$-dimension immersed in M by the immersion $\tau : S \to M$. Let us denote the differentiable mapping $d\tau$ of the immersion τ of B is a mapping from TS into TM, which is called tangent map of τ where $T(S)$ and $T(M)$ are the tangent bundles of S into M respectively. The tangent map of B is denoted by $\widetilde{B} : T(T(S)) \to T(T(M))$.

Let $\hat{g}$ be the Riemannian metric in M. The complete lift $\hat{g}^C$ of $\hat{g}$ in $T(M)$. If we denote by $\widetilde{g}$ the induced metric on $T(S)$ from $\hat{g}^C$ then we have $\widetilde{g}(X^C, Y^C) = \hat{g}^C(\tilde{B}X^C, \widetilde{B}Y^C)$ for $X, Y \in \mathrm{Im}_0^1(S)$. The complete lift $\hat{\nabla}^C$ of $\hat{\nabla}$ to $T(M)$ have the property $\hat{\nabla}^C_{\hat{X}^C}\hat{Y}^C = (\hat{\nabla}_{\hat{X}}\hat{Y})^C$; $\hat{\nabla}^C_{\hat{X}^C}\hat{Y}^V = (\hat{\nabla}_{\hat{X}}\hat{Y})^V$ for $X, Y \in \mathrm{Im}_0^1(M)$, $\hat{\nabla}$ being Riemannian connection of $T(M)$ with respect to $\hat{g}$, $\hat{\nabla}^C$ is Riemannian connection of $T(M)$ with respect to $\hat{g}^C$[14]. Similarly, the complete lift $\hat{\nabla}^C$ of the induce connection ∇ on (S, g) is also the Riemannian connection in $(T(M), \widetilde{g})$.

Yano and Ishihara [13] proved the following theorem:

THEOREM 2.1. *If $\hat{T}$ is torsion tensor of $\hat{\nabla}^C$ in $(M, \hat{g})$, then $\hat{T}^C$is torsion tensor of $\hat{\nabla}^C$ in $(T(M), \hat{g})$.*

In [11], using (3.10) we have

$$\hat{\omega}^V(\widetilde{B}X^C) = \hat{\omega}^V(BX)^{\overline{C}} = \hbar(\hat{\omega}^V(\hat{X}^C)) = \hbar(\hat{\omega}(\hat{X}))^V = (\hat{\omega}(BX))^{\overline{V}}$$
$$\hat{\omega}^C(\widetilde{B}X^C) = \hat{\omega}^C(BX)^{\overline{C}} = \hbar(\hat{\omega}^C(\hat{X}^C)) = \hbar(\hat{\omega}(\hat{X}))^C = (\hat{\omega}(BX))^{\overline{C}}$$

for arbitrary vector fields X, Y in S. Here, we denote the operation of restriction to $\pi_{\underline{M}}^{-1}(\tau(S))$ by $\hbar$ and the vertical and complete lift operations on $\pi_M^{-1}(\tau(S))$ by $\overline{V}$ and $\overline{C}$ respectively.

3. Tangent bundle endowed with quarter-symmetric non-metric connection on an almost Hermitian manifold

Let M be an almost Hermitian manifold and $T(M)$ be its tangent bundle. The complete lift of the $\hat{F}$ and $\hat{g}$ are $\hat{F}^C$ and $\hat{g}^C$ satisfying

(3.1) $$(\hat{F}^C)^2\hat{X} + \hat{X} = 0.$$

$$\hat{g}^C((\hat{F}\hat{X})^C, (\hat{F}\hat{Y})^C) = \hat{g}^C(\widetilde{B}X^C, \widetilde{B}Y^C) \tag{3.2}$$

An almost Hermitian manifold M is called
a Kaehler manifold if

$$(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) = 0 \tag{3.3}$$

a Nearly Kaehler manifold if

$$(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) = (\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)(\widetilde{B}Z^C, \widetilde{B}X^C), \tag{3.4}$$

an almost Kaehler manifold if

$$\begin{aligned}(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) + (\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)(\widetilde{B}Z^C, \widetilde{B}X^C)&\\ +(\hat{\nabla}^C_{\widetilde{B}Z^C}\widetilde{B}F^C)(\widetilde{B}X^C, \widetilde{B}Y^C) &= 0,\end{aligned} \tag{3.5}$$

a Quasi-Kaehler manifold if

$$(\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C, \widetilde{B}Z^C) + (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) = 0 \tag{3.6}$$

for arbitrary vector fields X, Y, Z and where $\hat{\nabla}^C$ is the Riemannian connection of $T(M)$.

If we define

$${}'F^C(\widetilde{B}X^C, \widetilde{B}Y^C) = \hat{g}^C(\widetilde{B}(FX)^C, \widetilde{B}(FY)^C) \tag{3.7}$$

taking the complete lift on the both sides of the equation (2.4) we get,

$$(\overline{\nabla}_{BX}BY)^{\overline{C}} = (\hat{\nabla}_{\widetilde{B}X}\widetilde{B}Y)^{\overline{C}} + (\hat{\omega}(BX)(BFY))^{\overline{C}}$$

$$\overline{\nabla}_{\widetilde{B}X^C}\widetilde{B}Y^C = \hat{\nabla}_{\widetilde{B}X^C}\widetilde{B}Y^C + \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}(FX)^V) + \hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}(FX)^C) \tag{3.8}$$

Now,

$$\begin{aligned}\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C - \overline{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}X^C - [X^C, Y^C] &= \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C\\ &+ \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}(FX)^V) + \hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}(FX)^C)\\ &- \hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}X^C - \hat{\omega}^C(\widetilde{B}X^C)(\widetilde{B}(FY)^V)\\ &- \hat{\omega}^V(\widetilde{B}X^V)(\widetilde{B}(FY)^C) - [X^C, Y^C].\end{aligned}$$

Using the theorem 2.1, we get

$$\begin{aligned}\overline{T}^C(\widetilde{B}X^C, \widetilde{B}Y^C) &= \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}(FX)^V) + \hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}(FX)^C)\\ &- \hat{\omega}^C(\widetilde{B}X^C)(\widetilde{B}(FY)^V) - \hat{\omega}^V(\widetilde{B}X^V)(\widetilde{B}(FY)^C),\end{aligned} \tag{3.9}$$

and

$$(\overline{\nabla}^C_{\widetilde{B}X^C})(\widetilde{B}Y^C, \widetilde{B}Z^C) = (\hat{\omega}(\widetilde{B}Y))^C(\hat{g}(\widetilde{B}(FX), \widetilde{B}Z))^V + (\hat{\omega}(\widetilde{B}Y))^V(\hat{g}(\widetilde{B}(FX), \widetilde{B}Z))^C + (\hat{\omega}(\widetilde{B}Z))^C(\hat{g}(\widetilde{B}(Y), \widetilde{B}(FX)))^V + (\hat{\omega}(\widetilde{B}Z))^V(\hat{g}(\widetilde{B}(Y), \widetilde{B}(FX)))^C. \tag{3.10}$$

The equation (3.8) can be written as

$$\overline{\nabla}_{\widetilde{B}X^C}\widetilde{B}Y^C = \hat{\nabla}_{\widetilde{B}X^C}\widetilde{B}Y^C + H(X,Y), \tag{3.11}$$

where

$$H(X,Y) = \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}(FX)^V) + \hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}(FX)^C). \tag{3.12}$$

If we define

$$'H(X,Y,Z) = \hat{g}^C(H(X,Y), \widetilde{B}Z^C) \tag{3.13}$$

then in the view of (3.12),(3.13) becomes

$$'H(X,Y,Z) = (\hat{\omega}(\widetilde{B}Y))^C(\hat{g}(\widetilde{B}(FX), \widetilde{B}Z))^V + (\hat{\omega}(\widetilde{B}Y))^V(\hat{g}(\widetilde{B}(FX), \widetilde{B}Z))^C. \tag{3.14}$$

Theorem 3.1. If an almost Hermitian manifold M admits a quarter symmetric non-metric connection $\overline{\nabla}^C$ with respect to the Riemannian connection $\hat{\nabla}^C$ in $(T(M), \hat{g}^C)$, then the necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is that metric connection $(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C)$ is hybrid in both slots, i.e. $(\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C = (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C)$.

Proof. Covariant derivative of $(FY)^C$ with respect to the connection $\overline{\nabla}^C$ gives

$$(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) + F^C(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C) = \overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}(FY)^C$$
.

In consequence of (3.1) and (3.8), the last expression becomes

$$(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) = (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) + \hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}X^V + \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}X^C + \hat{\omega}^C(\widetilde{B}(FY)^C)(\widetilde{B}(FX)^V) + \hat{\omega}^V(\widetilde{B}(FY)^V)(\widetilde{B}(FX)^C). \tag{3.15}$$

Replacing X by FX and Y by FY in (3.15) and using (3.1), we get

$$(\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) = (\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) \tag{3.16}$$

$$
\begin{aligned}
&+ \hat{\omega}^C(\widetilde{B}(FY)^C)\widetilde{B}(FX)^V \\
&+ \hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}X^V \\
&+ \hat{\omega}^V(\widetilde{B}(FY)^V)\widetilde{B}(FX)^C \\
&+ \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}X^C.
\end{aligned}
$$

Subtracting (3.15) from (3.16), we have

$$
\begin{aligned}
(\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) &- (\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) \\
&= (\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) \\
&- (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C).
\end{aligned}
\tag{3.17}
$$

A necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is

$$
(\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) = (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C).
\tag{3.18}
$$

In view of (3.17) and (3.18), we obtain the statement of the theorem.

Theorem 3.2. An almost Hermitian manifold M admits a quarter-symmetric non-metric connection $\overline{\nabla}^C$ with respect to the Riemannian connection $\hat{\nabla}^C$ in $(T(M), \hat{g}^C)$ is an almost Kaehler manifold if and only if $'F^C$ is closed with respect to the connection $\overline{\nabla}^C$.

Proof. We have

$$
\begin{aligned}
X^C('F^C(\widetilde{B}Y^C, \widetilde{B}Z^C)) &= \overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C(\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad +'F^C(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad +'F^C(\widetilde{B}Y^C, \overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Z^C) \\
&= \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C(\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad +'F^C(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad +'F^C(\widetilde{B}Y^C, \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Z^C),
\end{aligned}
$$

then

$$
\begin{aligned}
(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) &= \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C(\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad -'F^C(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C - \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C, \widetilde{B}Z^C) \\
&\quad -'F^C(\widetilde{B}Y^C, \overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Y^C - \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}Z^C).
\end{aligned}
$$

In consequence of (3.1), (3.2),(3.8), the last expression becomes

$$
(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C)(\widetilde{B}Y^C, \widetilde{B}Z^C) = \hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C(\widetilde{B}Y^C, \widetilde{B}Z^C)
\tag{3.19}
$$

$$
\begin{aligned}
&+\hat{\omega}^C(\widetilde{B}Y^C)\hat{g}^C(\widetilde{B}(X)^V,\widetilde{B}(Z)^C)\\
&+\hat{\omega}^V(\widetilde{B}Y^V)\hat{g}^C(\widetilde{B}(X)^C,\widetilde{B}(Z)^C)\\
&-\hat{\omega}^C(\widetilde{B}Z^C)\hat{g}^C(\widetilde{B}(FY)^C,\widetilde{B}(FX)^V)\\
&-\hat{\omega}^V(\widetilde{B}Z^V)\hat{g}^C(\widetilde{B}(FY)^C,\widetilde{B}(FX)^C)
\end{aligned}
$$

Taking the cycle sum of (3.19) in X, Y, Z, we get

$$
\begin{aligned}
(3.20)\qquad (\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C)(\widetilde{B}Y^C,\widetilde{B}Z^C) &+(\overline{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}'F^C)(\widetilde{B}Z^C,\widetilde{B}X^C)\\
&+(\overline{\nabla}^C_{\widetilde{B}Z^C}\widetilde{B}'F^C)(\widetilde{B}X^C,\widetilde{B}Y^C)\\
&=\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}'F^C(\widetilde{B}Y^C,\widetilde{B}Z^C)\\
&+\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}'F^C(\widetilde{B}Z^C,\widetilde{B}X^C)\\
&+\hat{\nabla}^C_{\widetilde{B}Z^C}\widetilde{B}'F^C(\widetilde{B}X^C,\widetilde{B}Y^C).
\end{aligned}
$$

In consequence of (3.5) and (3.20), we see that $'F^C$ is closed with respect to the connection $\overline{\nabla}^C$. Converse part is obvious from (3.20).

4. Theorems on Nijenhuis Tensor with M admits a quarter symmetric non-metric connection $\overline{\nabla}^C$

Theorem 4.1. An almost Hermitian manifold M admits a quarter-symmetric non-metric connection $\overline{\nabla}^C$ with respect to the Riemannian connection $\hat{\nabla}^C$ in $(T(M),\hat{g}^C)$ then the Nijenhuis Tensor of $\hat{\nabla}^C$ and $\overline{\nabla}^C$ coincide.

Proof. From (3.15), we have

$$
\begin{aligned}
(4.1)\quad (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) &= (\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C)-\hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}X^V\\
&- \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}X^C-\hat{\omega}^C(\widetilde{B}(FY)^C)(\widetilde{B}(FX)^V)\\
&- \hat{\omega}^V(\widetilde{B}(FY)^V)(\widetilde{B}(FX)^C).
\end{aligned}
$$

Replacing X by FX in (4.1) and using (3.1), we find

$$
\begin{aligned}
(4.2)\,(\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}Y^C) &= (\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}Y^C)\\
&- \hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}(FX)^V\\
&- \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}(FX)^C+\hat{\omega}^C(\widetilde{B}(FY)^C)(\widetilde{B}X^V)\\
&+ \hat{\omega}^V(\widetilde{B}(FY)^V)(\widetilde{B}X^C).
\end{aligned}
$$

Interchanging X and Y in (4.2), we get

$$
\begin{aligned}
(4.3)\,(\hat{\nabla}^C_{\widetilde{B}(FY)^C}\widetilde{B}F^C)(\widetilde{B}X^C) &= (\overline{\nabla}^C_{\widetilde{B}(FY)^C}\widetilde{B}F^C)(\widetilde{B}X^C)\\
&- \hat{\omega}^C(\widetilde{B}X^C)\widetilde{B}(FY)^V
\end{aligned}
$$

$$
\begin{aligned}
&- \hat{\omega}^V(\widetilde{B}X^V)\widetilde{B}(FY)^C + \hat{\omega}^C(\widetilde{B}(FX)^C)(\widetilde{B}Y^V)\\
&+ \hat{\omega}^V(\widetilde{B}(FX)^V)(\widetilde{B}Y^C).
\end{aligned}
$$

Operating F^C on whole equation of (4.1) and using (3.1), we get

$$
\begin{aligned}
(4.4)\ F^C(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) &= F^C(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C)\\
&- \hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}(FX)^V - \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}(FX)^C\\
&+ \hat{\omega}^C(\widetilde{B}(FY)^C)(\widetilde{B}X^V)\\
&+ \hat{\omega}^V(\widetilde{B}(FY)^V)(\widetilde{B}X^C).
\end{aligned}
$$

Interchanging X and Y in (4.2), we obtain

$$
\begin{aligned}
(4.5)\ F^C(\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)(\widetilde{B}X^C) &= F^C(\overline{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)(\widetilde{B}(FX)^C)\\
&- \hat{\omega}^C(\widetilde{B}X^C)\widetilde{B}(FY)^V - \hat{\omega}^V(\widetilde{B}X^V)\widetilde{B}(FY)^C\\
&+ \hat{\omega}^C(\widetilde{B}(FX)^C)(\widetilde{B}Y^V)\\
&+ \hat{\omega}^V(\widetilde{B}(FX)^V)(\widetilde{B}Y^C).
\end{aligned}
$$

The Nijenhuis Tensor in an almost Hermitian manifold is given by

$$
\begin{aligned}
(4.6)\quad \overline{N}(X^C, Y^C) &= (\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)\widetilde{B}Y^C - (\overline{\nabla}^C_{\widetilde{B}(FY)^C}\widetilde{B}F^C)\widetilde{B}X^C\\
&- \widetilde{B}F^C(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)\widetilde{B}Y^C + \widetilde{B}F^C(\overline{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)\widetilde{B}X^C.
\end{aligned}
$$

In view of (4.2), (4.3), (4.4), (4.5) and (4.6)

$$
\begin{aligned}
(4.7)\quad \overline{N}(X^C, Y^C) &= (\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)\widetilde{B}Y^C - (\hat{\nabla}^C_{\widetilde{B}(FY)^C}\widetilde{B}F^C)\widetilde{B}X^C\\
&- \widetilde{B}F^C(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)\widetilde{B}Y^C + \widetilde{B}F^C(\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)\widetilde{B}X^C
\end{aligned}
$$

is Nijenhuis tensor of connection $\hat{\nabla}^C$.

$$
\overline{N}(X^C, Y^C) = \hat{N}(X^C, Y^C),
$$

where

$$
\begin{aligned}
\hat{N}(X^C, Y^C) &= (\hat{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)\widetilde{B}Y^C - (\hat{\nabla}^C_{\widetilde{B}(FY)^C}\widetilde{B}F^C)\widetilde{B}X^C\\
&- \widetilde{B}F^C(\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)\widetilde{B}Y^C + \widetilde{B}F^C(\hat{\nabla}^C_{\widetilde{B}Y^C}\widetilde{B}F^C)\widetilde{B}X^C.
\end{aligned}
$$

Corollary 4.2. An almost Hermitian manifold M admits a quarter-symmetric non-metric connection $\overline{\nabla}^C$ with respect to the Riemannian connection $\hat{\nabla}^C$ in $(TM, \hat{g}^C$ to be a Hermitian manifold if the Nijenhuis tensor of connection $\overline{\nabla}^C$ vanishes, i.e. $\hat{N}(X^C, Y^C)$.

Corollary 4.3. On a Kaehler manifold, Nijenhuis tensor with respect to the quarter-symmetric non-metric connection $\overline{\nabla}^C$ vanishes, i.e. $\hat{N}(X^C, Y^C)$.

Theorem 4.4. On a Kaehler manifold, Nijenhuis tensor with respect to the quarter-symmetric non-metric connection $\overline{\nabla}^C$ satisfies the following relations:

$$(a)(\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C = (\hat{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) \text{ i.e. } (\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C)$$

is hybrid in both slots.

$$(b)(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) = 0 \Leftrightarrow \hat{\omega}^C(\widetilde{B}Y^C) = 0, \hat{\omega}^C(\widetilde{B}Y^C) = 0. \tag{4.8}$$

Proof. In view of (3.3), (3.15) becomes

$$\begin{aligned}(\overline{\nabla}^C_{\widetilde{B}X^C}\widetilde{B}F^C)(\widetilde{B}Y^C) &= \hat{\omega}^C(\widetilde{B}Y^C)\widetilde{B}X^V + \hat{\omega}^V(\widetilde{B}Y^V)\widetilde{B}X^C \\ &+ \hat{\omega}^C(\widetilde{B}(FY)^C)(\widetilde{B}(FX)^V) \\ . &+ \hat{\omega}^V(\widetilde{B}(FY)^V)(\widetilde{B}(FX)^C)\end{aligned} \tag{4.9}$$

Substituting FX in place of X and FY in the place of Y in (3.15) and using (3.15), we can find

$$\begin{aligned}(\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) &= \hat{\omega}^C(\widetilde{B}(FY)^C)\widetilde{B}(FX)^V \\ &+ \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}X^V) \\ &+ \hat{\omega}^V(\widetilde{B}(FY)^V)\widetilde{B}(FX)^C \\ &+ \hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}X^C)\end{aligned} \tag{4.10}$$

in consequence of (4.9) and (4.10), we can find (4.8).

Again, if $(\overline{\nabla}^C_{\widetilde{B}(FX)^C}\widetilde{B}F^C)(\widetilde{B}(FY)^C) = 0$, then (4.9) gives

$$\begin{aligned}\hat{\omega}^C(\widetilde{B}(FY)^C)\widetilde{B}(FX)^V \quad &+\hat{\omega}^V(\widetilde{B}(FY)^V)\widetilde{B}(FX)^C + \hat{\omega}^C(\widetilde{B}Y^C)(\widetilde{B}X^V) \\ &+\hat{\omega}^V(\widetilde{B}Y^V)(\widetilde{B}X^C) = 0.\end{aligned}$$

If X and FX are linearly independent, hence

$$\hat{\omega}^C(\widetilde{B}Y^C) = 0, \hat{\omega}^C(\widetilde{B}Y^C) = 0,$$

which proves the first part of the statement. The converse part is obvious.

5. Example

Let V^4 be a real vector space with a chosen basis $\{e_1, e_2, e_3, e_4\}$. Let ℓ denote the Lie algebra. Let G be a real connected four-dimensional Lie group and constructed with left-invariant an almost Hermitian structure $\hat{F}^C$ by

$$\hat{F}^C e_1 = -e_3, \hat{F}^C e_2 = -e_4, \hat{F}^C e_3 = e_1, \hat{F}^C e_4 = -e_2, \tag{5.1}$$

Define a left invariant non-singular Hermitian metric g in G by

$$\begin{aligned} \hat{g}^C(e_1, e_1) &= \hat{g}^C(e_2, e_2) = \hat{g}^C(e_3, e_3) = \hat{g}^C(e_4, e_4) = 1, \\ &\hat{g}^C(e_i, e_j) = 0, i \neq j, i, j = 1, 2, 3, 4. \end{aligned} \tag{5.2}$$

Thus, in consequence of (3.1), (3.2), (5.1), (5.2),
$\{G, \phi, \xi_p, \eta^p, g\}$ is an almost Hermitian manifold on a tangent bundle.

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CERTAIN GEOMETRIC PROPERTIES OF A NORMALIZED HYPER-BESSEL FUNCTION

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Abstract. In the present paper, by making use of previous results on analytic functions, certain geometric properties including starlikeness of order α and convexity of order α of a normalized hyper-Bessel function have been investigated. In addition, some conditions of the mentioned function which belongs to the Hardy space have been given. Moreover, specific examples of the results which refer to special values of the parameters have also been presented.
Keywords: analytic function, hyper-Bessel function, starlikeness of order α, convexity of order α, Hardy space.

1. Introduction

Special functions have a great importance in applied sciences. Due to their remarkable properties, a great number of mathematicians have investigated the geometric properties of special functions like Bessel, Struve and Lommel functions of the first kind. Especially, these geometric properties include univalence, starlikeness and convexity of the above mentioned functions. Actually, the first results on the univalence of Bessel function can be found in the papers [9] and [13] written by Brown and, Kreyszig and Todd in 1960, respectively. In recent years, the geometric properties of some special functions have become very interesting to mathematicians. More precisely, the authors in [3, 4, 5, 6] began to study the geometric properties of the above mentioned special functions and their generalizations. Also, the authors in [7, 14] studied Hardy space of generalized Bessel and hypergeometric functions, respectively. Especially, Bessel function has some extensions like q-Bessel and hyper-Bessel functions. Some geometric properties of these extensions may be found in [1, 2, 3] and the references therein.

Now, we would like to remind you about the definitions of the Bessel and hyper-Bessel functions, respectively. Bessel function is defined by the following infinite

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series (see [16]):

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu}, z \in \mathbb{C} \tag{1.1}$$

where $\Gamma(z)$ denotes the familiar gamma function. Similarly, the hyper-Bessel function is defined by (see [10]):

$$J_{\gamma_d}(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{d+1}\right)^{n(d+1)+\sum_{i=1}^{d}\gamma_i}}{n!\prod_{i=1}^{d}\Gamma(\gamma_i+n+1)}. \tag{1.2}$$

It is clear that, the hyper-Bessel function is reduced to the classical Bessel function given by (1.1) for $d=1$ and $\gamma_1=\nu$. Due to this relationship, the earlier studies on the classical Bessel function can be extended to the hyper-Bessel function. Motivated by some earlier works, our main aim in this study is to obtain new conditions on the starlikeness and convexity of the hyper-Bessel functions. Also, we will deal with the Hardy space of the hyper-Bessel function.

2. Starlikeness and convexity of hyper-Bessel function

In the beginning of this section, we are going to mention some basic notions in geometric function theory. Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$,

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{2.1}$$

which satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}_r$. Also, for $0 \leq \alpha < 1$, by $\mathcal{S}^\star(\alpha)$ and $\mathcal{C}(\alpha)$ we will denote the subclasses of $\mathcal{A}$ consisting of functions which are starlike and convex of order α, respectively. The analytic characterizations of these subclasses are

$$\mathcal{S}^\star(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for } z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \text{ for } z \in \mathbb{D} \right\},$$

respectively.

The following results on analytic functions given by Silverman in [15] will be used in order to prove our main results.

Lemma 2.1. *Let f is of the form (2.1). If*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le 1-\alpha, \tag{2.2}$$

then $f\in\mathcal{S}^{\star}(\alpha)$.

Lemma 2.2. *Let f is of the form (2.1). If*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le 1-\alpha, \tag{2.3}$$

then $f\in\mathcal{C}(\alpha)$.

Before revealing our main results concerning starlikeness and convexity of the hyper-Bessel function, we will apply the following natural normalization for the function $z\mapsto J_{\gamma_d}(z)$ given by (1.2):

$$\begin{aligned} F_{\gamma_d}(z) &= z\frac{\prod_{i=1}^{d}\Gamma(\gamma_i+1)}{\left(\frac{\sqrt[d+1]{z}}{d+1}\right)^{\sum_{i=1}^{d}\gamma_i}}J_{\gamma_d}(\sqrt[d+1]{z}) \\ &= z+\sum_{n\ge 2}\frac{(-1)^n}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}z^n, \end{aligned} \tag{2.4}$$

where $\rho=(d+1)^{d+1}$ and $(\beta)_n$ is the Pochhammer symbol which is defined by $(\beta)_0=1$ and $(\beta)_n=\beta(\beta+1)\ldots(\beta+n-1)$ for $n\ge 1$. As a consequence, the function $z\mapsto F_{\gamma_d}\in\mathcal{A}$.

Our first main result regarding starlikeness of order α of the function $F_{\gamma_d}(z)$ is the following:

Theorem 2.1. *Let* $\alpha\in[0,1)$, *d=1,2,...,* $\rho=(d+1)^{d+1}$ and $\mu=\prod_{i=1}^{d}(\gamma_i+1)$. *If*

$$\alpha\le\frac{4\rho^2\mu^2-12\rho\mu+3}{4\rho^2\mu^2-8\rho\mu+3},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ *is starlike of order* α *in* $\mathbb{D}$.

Proof. It is known from Lemma 2.1 that, if the function $f\in\mathcal{A}$ satisfies the inequality (2.2), then the function f is starlike of order α. Because of this, in order to prove starlikeness of order α of the function $F_{\gamma_d}(z)$, it is enough to show that

$$\sum_{n\ge 2}(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right|\le 1-\alpha$$

for $\alpha\in[0,1)$. Now using the well-known inequalities

$$2^{n-2}\le(n-1)! \tag{2.5}$$

and

(2.6) $$(\beta)_n \leq (\beta)^n$$

for $n \geq 2$, we can write that

$$\begin{aligned}
\sum_{n\geq 2}(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right| &= \sum_{n\geq 2}\frac{(n-\alpha)}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}} \\
&\leq 2\sum_{n\geq 2}\frac{n-\alpha}{(2\rho\mu)^{n-1}} \\
&= 2\sum_{n\geq 2}n\left(\frac{1}{2\rho\mu}\right)^{n-1} - 2\alpha\sum_{n\geq 2}\left(\frac{1}{2\rho\mu}\right)^{n-1}.
\end{aligned}$$

Considering the known series sums

$$\sum_{n\geq 2}t^{n-1} = \frac{t}{1-t} \text{ and } \sum_{n\geq 2}nt^{n-1} = \frac{t(2-t)}{(1-t)^2}$$

for $|t| < 1$ in the last step, we have

$$\sum_{n\geq 2}(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right| \leq \frac{4\rho\mu(2-\alpha)+2(\alpha-1)}{(2\rho\mu-1)^2} \leq 1-\alpha$$

for $\left|\frac{1}{2\rho\mu}\right| < 1$ and

$$\alpha \leq \frac{4\rho^2\mu^2 - 12\rho\mu + 3}{4\rho^2\mu^2 - 8\rho\mu + 3}.$$

So, the proof is completed. □

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 2.1, we have the following:

Corollary 2.1. *Let $\alpha \in [0,1)$. If*

$$\alpha \leq \frac{64\nu^2 + 80\nu + 19}{64\nu^2 + 96\nu + 35},$$

then the function $z \mapsto F_\nu(z) = 2^\nu z^{1-\frac{\nu}{2}}\Gamma(\nu+1)J_\nu(\sqrt{z})$ is starlike of order α.

It is well-known from [8, p.13–14] that the basic trigonometric functions can be represented by the classical Bessel function J_ν for the appropriate values of the parameter ν. Clearly, for $\nu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$, respectively, we have the following:

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z, J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z \text{ and } J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right).$$

Now, taking $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$ in Corollary 2.1 we have the following examples on the elemantary trigonometric functions.

Example 2.1. The following assertions are true for $z \in \mathbb{D}$:

a. If $\alpha \leq \alpha_1 \cong 0.75$, then the function $F_{\frac{1}{2}}(z) = \sqrt{z}\sin\sqrt{z}$ is in $\mathcal{S}^{\star}(\alpha)$.

b. If $\alpha \leq \alpha_2 \cong 0.88$, then the function $F_{\frac{3}{2}}(z) = 3\left(\frac{\sin\sqrt{z}}{\sqrt{z}} - \cos\sqrt{z}\right)$ is in $\mathcal{S}^{\star}(\alpha)$.

Our second main result concerning convexity of order α of the function $F_{\gamma_d}(z)$ is the following:

Theorem 2.2. *Let $\alpha \in [0,1)$, d=1,2,..., $\rho = (d+1)^{d+1}$ and$\mu = \prod_{i=1}^{d}(\gamma_i+1)$. If*

$$\alpha \leq \frac{(2\rho\mu-1)^3 - 8\rho^2\mu^2 + 12\rho\mu - 2}{(2\rho\mu-1)^3 - 16\rho^2\mu^2 + 12\rho\mu - 2},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ is convex of order α in $\mathbb{D}$.

Proof. By using the Lemma 2.2, it is possible to show that the function $F_{\gamma_d}(z)$ is convex of order α. For this reason, we need to show that the function $F_{\gamma_d}(z)$ satisfies the inequality (2.3). That is, it is sufficient to prove that

$$\sum_{n\geq 2} n(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right| \leq 1-\alpha.$$

By making use of the known inequalities (2.5) and (2.6), we can write that

$$\begin{aligned}\sum_{n\geq 2} n(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right| &= \sum_{n\geq 2}\frac{n(n-\alpha)}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}} \\ &\leq 2\sum_{n\geq 2}\frac{n(n-\alpha)}{(2\rho\mu)^{n-1}} \\ &= 2\sum_{n\geq 2} n^2\left(\frac{1}{2\rho\mu}\right)^{n-1} - 2\alpha\sum_{n\geq 2} n\left(\frac{1}{2\rho\mu}\right)^{n-1}.\end{aligned}$$

If we consider the known series sums

$$\sum_{n\geq 2} nt^{n-1} = \frac{t(2-t)}{(1-t)^2} \text{ and } \sum_{n\geq 2} n^2 t^{n-1} = \frac{t(t^2-3t+4)}{(1-t)^3}$$

for $|t| < 1$ in the above last inequality, we get

$$\sum_{n\geq 2} n(n-\alpha)\left|\frac{(-1)^{n-1}}{(n-1)!\rho^{n-1}\prod_{i=1}^{d}(\gamma_i+1)_{n-1}}\right| \leq \frac{8\rho^2\mu^2(1-2\alpha) + 2(6\rho\mu-1)(\alpha-1)}{(2\rho\mu-1)^3}$$

for $\left|\frac{1}{2\rho\mu}\right| < 1$. Since

$$\frac{8\rho^2\mu^2(1-2\alpha) + 2(6\rho\mu-1)(\alpha-1)}{(2\rho\mu-1)^3} \leq 1-\alpha$$

under asumption, the proof is thus completed. □

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 2.2, we have the following:

Corollary 2.2. *Let* $\alpha \in [0,1)$. *If*

$$\alpha \leq \frac{512\nu^3 + 1216\nu^2 + 968\nu + 261}{512\nu^3 + 1088\nu^2 + 712\nu + 133},$$

then the function $z \mapsto F_\nu(z) = 2^\nu z^{1-\frac{\nu}{2}} \Gamma(\nu+1) J_\nu(\sqrt{z})$ *is convex of order* α.

3. Hardy space of hyper-Bessel function

Let $\mathcal{H}$ denote the set of all analytic (holomorphic, regular) functions on the open unit disk $\mathbb{D}$. For any $\eta \in (0,\infty]$, any function $f \in \mathcal{H}$ and any $r \in [0,1)$ set

$$M_\eta(r,f) = \left\{ \begin{array}{ll} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\eta d\theta \right)^{\frac{1}{\eta}}, & \text{if } 0 < \eta < \infty \\ \max_{|z| \leq r} |f(z)|, & \text{if } \eta = \infty. \end{array} \right\}$$

By definition, the function $f \in \mathcal{H}$ is said to belong to the Hardy space $\mathcal{H}^\eta$, where $0 < \eta \leq \infty$, if the set $\{M_\eta(r,f) : r \in [0,1)\}$ is bounded. Note that for $1 \leq \eta \leq \infty, \mathcal{H}^\eta$ is a Banach space with the norm defined by $||f||_\eta = \lim_{r\to 1^-} M_\eta(r,f)$. Furthermore, $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathcal{H}$. We note that for $0 < p \leq q \leq \infty$, it can be shown that $\mathcal{H}^q$ is a subset of $\mathcal{H}^p$ (see[11]).

The following lemma due to Eenigenburg and Keogh [12] will be used in order derive our last main result.

Lemma 3.1. *Let* $\alpha \in [0,1)$. *If the function* $f \in \mathcal{C}(\alpha)$ *is not of the form*

$$\left\{ \begin{array}{ll} f(z) = k + lz\left(1 - ze^{i\theta}\right)^{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ f(z) = k + l\log\left(1 - ze^{i\theta}\right), & \alpha = \frac{1}{2}. \end{array} \right\}$$

for some $k, l \in \mathbb{C}$ *and* $\theta \in \mathbb{R}$, *then the following statements hold:*

- **a.** *There exist* $\delta = \delta(f) > 0$ *such that* $f' \in \mathcal{H}^{\delta + \frac{1}{2(1-\alpha)}}$.
- **b.** *If* $\alpha \in \left[0, \frac{1}{2}\right)$, *then there exist* $\tau = \tau(f) > 0$ *such that* $f \in \mathcal{H}^{\tau + \frac{1}{1-2\alpha}}$.
- **c.** *If* $\alpha \geq \frac{1}{2}$, *then* $f \in \mathcal{H}^\infty$.

Our main result concerning the Hardy space of the normalized hyper-Bessel function $F_{\gamma_d}(z)$ is as follows.

Theorem 3.1. *Let* $\alpha \in [0,1)$, *d=1,2,...*, $\rho = (d+1)^{d+1}$ and $\mu = \prod_{i=1}^d (\gamma_i + 1)$. *If*

$$\alpha \leq \frac{(2\rho\mu - 1)^3 - 8\rho^2\mu^2 + 12\rho\mu - 2}{(2\rho\mu - 1)^3 - 16\rho^2\mu^2 + 12\rho\mu - 2},$$

then the normalized hyper-Bessel function $F_{\gamma_d}(z)$ *has the following properties:*

i. $F_{\gamma_d}(z) \in \mathcal{H}^{\frac{1}{1-2\alpha}}$ *for* $\alpha \in \left[0, \frac{1}{2}\right)$,

ii. $F_{\gamma_d}(z) \in \mathcal{H}^{\infty}$ *for* $\alpha \in \left(\frac{1}{2}, 1\right)$.

Proof. Gauss hypergeometric function is defined by

$$ {}_2F_1(a,b,c;z) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{3.1}$$

By using the definetion of the Gauss hypergeometric function given by (3.1), it is easily shown that

$$\begin{aligned} k + \frac{lz}{(1-ze^{i\theta})^{1-2\alpha}} &= k + lz {}_2F_1(1,(1-2\alpha),1;ze^{i\theta}) \\ &= k + l\sum_{n\geq 0} \frac{(1-2\alpha)_n}{n!} e^{i\theta n} z^{n+1} \end{aligned}$$

for $k, l \in \mathbb{C}, \alpha \neq \frac{1}{2}$ and $\theta \in \mathbb{R}$. On the other hand, it can be easily shown that

$$\begin{aligned} k + l\log(1-ze^{i\theta}) &= k - lz {}_2F_1(1,1,2;ze^{i\theta}) \\ &= k - l\sum_{n\geq 0} \frac{1}{n+1} e^{i\theta n} z^{n+1}. \end{aligned}$$

As a result, the function $F_{\gamma_d}(z)$ given by (2.4) is not of the forms $k + \frac{lz}{(1-ze^{i\theta})^{1-2\alpha}}$ for $\alpha \neq \frac{1}{2}$ and $k + l\log(1-ze^{i\theta})$ for $\alpha = \frac{1}{2}$, respectively. Also, it is known from Theorem 2.2 that, the function $F_{\gamma_d}(z)$ is convex of order α. Hence, by Lemma 3.1 the proof is completed. □

By setting $d = 1$ and $\gamma_1 = \nu$ in the Theorem 3.1, we have the following:

Corollary 3.1. *Let* $\alpha \in [0,1)$. *If*

$$\alpha \leq \frac{512\nu^3 + 1216\nu^2 + 968\nu + 261}{512\nu^3 + 1088\nu^2 + 712\nu + 133},$$

then the function $z \mapsto F_\nu$ *has the following properties:*

i. $F_\nu(z) \in \mathcal{H}^{\frac{1}{1-2\alpha}}$ *for* $\alpha \in \left[0, \frac{1}{2}\right)$,

ii. $F_\nu(z) \in \mathcal{H}^{\infty}$ *for* $\alpha \in \left(\frac{1}{2}, 1\right)$.

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A NEW CHARACTERIZATION OF CURVES IN MINKOWSKI 4-SPACE $\mathbb{E}_1^4$

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Abstract. In this study, we attend to the curves whose position vectors are written as a linear combination of their Serret-Frenet vectors in Minkowski 4-space $\mathbb{E}_1^4$. We characterize such curves with regard to their curvatures. Further, we get certain consequences of T-constant and N-constant types of curves in $\mathbb{E}_1^4$.
Keywords: Constant ratio curves, T-constant curves, N-constant curves, Minkowski space.

1. Introduction

The term rectifying curves is presented by B.Y. Chen in [7]. Afterwards, Chen and Dillen gave the connection between these curves and centrodes that have a place in mechanics and kinematics as well as in differential geometry [10]. The rectifying curves in the Minkowski 3-space $\mathbb{E}_1^3$ were investigated in [12, 16, 17]. For a regular curve $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ given with the arclength parameter, the hyperplanes spanned by $\{T, N_1, N_3\}$ and $\{T, N_2, N_3\}$ are known as the first osculating hyperplane and the second osculating hyperplane, respectively. If x lies on its first (second) osculating hyperplane, then $x(s)$ is called as an osculating curve of first (second) kind. In [1], the authors considered the rectifying curves in Minkowski 4-space $\mathbb{E}_1^4$. They characterized the rectifying curves with the equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \upsilon(s)N_3(s)$$

for given differentiable functions $\lambda(s), \mu(s)$ and $\upsilon(s)$. Actually, these curves are osculating curves of a second kind. The rectifying curves in $\mathbb{E}_1^4$ are studied by the authors in [18, 19].

The notion of constant ratio curves in Minkowski spaces is given by B. Y. Chen in [9]. In the same paper, the author gave the necessary and sufficient conditions, $x^T = 0$ or the ratio $\left\|x^T\right\| : \|x\|$ is constant, for curves to become constant ratio.

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Moreover, in [8], the same author introduces *T-constant* and *N-constant* types of curves. If the norm of the tangential component (normal component) is constant, the curve is called as *T-constant* (*N-constant*). Also, if this norm is equal to zero, then the curve is a *T-constant* (*N-constant*) curve of first kind, otherwise second kind [15]. Recently, the authors have studied the mentioned curves in some spaces in [2, 3, 4, 5, 6, 15, 20, 21, 22, 28, 29, 30, 31].

In this study, we deal with spacelike curves with spacelike principal normal in $\mathbb{E}_1^4$ with respect to the their Frenet frame $\{T, N_1, N_2, N_3\}$. Since $\{T, N_1, N_2, N_3\}$ is an orthonormal basis in $\mathbb{E}_1^4$, we write the position vector of the curve as

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{1.1}$$

for some differentiable functions $m_i(s)$, $i = 0, 1, 2, 3$. We classify osculating curves of the first and the second kind with regard to their curvature functions $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$. We give W-curves in $\mathbb{E}_1^4$. Furthermore, we get certain consequences of these types curves to become ccr-curves. We consider T-constant and N-constant curves in $\mathbb{E}_1^4$.

2. Basic Consepts

Minkowski 4-space is 4-dimensional pseudo-Euclidean space defined by the Lorentzian inner product

$$\langle v, w\rangle_{\mathbb{L}} = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4,$$

where v_i, w_i, i=1,2,3,4 are the components of the vectors v and w. Any arbitrary vector v is called timelike, lightlike or spacelike if the Lorentzian inner product $\langle v, v\rangle_{\mathbb{L}}$ is negative definite, zero or positive definite, respectively. Then, the length of the vector $v \in \mathbb{E}_1^4$ is calculated by

$$\|v\| = \sqrt{|\langle v, v\rangle_{\mathbb{L}}|}.$$

The sets

$$\mathbb{S}_1^3(r^2) = \left\{v \in \mathbb{E}_1^4 : \langle v, v\rangle_{\mathbb{L}} = r^2\right\}$$

and

$$\mathbb{H}_0^3(-r^2) = \left\{v \in \mathbb{E}_1^4 : \langle v, v\rangle_{\mathbb{L}} = -r^2\right\}$$

are called pseudo-Riemannian and pseudo-Hyperbolic spaces in $\mathbb{E}_1^4$ for positive number r, respectively [11].

A curve $x = x(s) : I \to \mathbb{E}_1^4$ is timelike (lightlike (null), spacelike) if all tangent vectors $x'(s)$ are timelike (lightlike (null), spacelike). If $\|x'(s)\| = 1$, x is a unit speed curve [25].

The light cone $\mathcal{LC}$ of $\mathbb{E}_1^4$ is defined as

$$\mathcal{LC} = \left\{v \in \mathbb{E}_1^4,\ \ \langle v, v\rangle_{\mathbb{L}} = 0\right\}.$$

Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and $\{T, N_1, N_2, N_3\}$ be the Frenet frame of x in $\mathbb{E}_1^4$. Then, the Frenet formulas are

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \varepsilon\kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s) - \varepsilon\kappa_3(s)N_3(s), \\ N_3'(s) &= -\varepsilon\kappa_3(s)N_2(s), \end{aligned} \tag{2.1}$$

where $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ are the first, the second, and the third curvatures of the curve x and

$$\varepsilon = \langle N_2(s), N_2(s)\rangle_L = -\langle N_3(s), N_3(s)\rangle_L = \pm 1$$

[26].

Screw lines or helices, called as *W-curves* by F. Klein and S. Lie [23], are the curves with constant curvatures, and they are mentioned in [13, 14]. Moreover, a regular curve is a ccr-curve, constant curvature ratios, if its curvature's ratios are constants [24, 27].

3. Characterization of Spacelike Curves in $\mathbb{E}_1^4$

Now, we shall consider curves given with the equality (1.1) in $\mathbb{E}_1^4$. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal, and $\kappa_1(s) \neq 0$, $\kappa_2(s)$ and $\kappa_3(s)$ be the curvatures of x. Differentiating (1.1) according to s and using (2.1), we get

$$\begin{aligned} x'(s) = \; & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & +(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & +(m_2'(s) + \varepsilon\kappa_2(s)m_1(s) - \varepsilon\kappa_3(s)m_3(s))N_2(s) \\ & +(m_3'(s) - \varepsilon\kappa_3(s)m_2(s))N_3, \end{aligned}$$

which follows

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \varepsilon\kappa_2 m_1 - \varepsilon\kappa_3 m_3 &= 0, \\ m_3' - \varepsilon\kappa_3 m_2 &= 0. \end{aligned} \tag{3.1}$$

The following theorem determines the W-curves in $\mathbb{E}_1^4$.

Theorem 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike* *principal normal. If x is a W-curve in $\mathbb{E}_1^4$, then*

$$\begin{aligned}
m_0(s) &= -\frac{2\kappa_1}{\sqrt{-2\lambda+2\mu}}\left\{c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} - c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s}\right\} \\
&\quad -\frac{2\kappa_1}{\sqrt{2\lambda+2\mu}}\left\{c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} - c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}\right\}, \\
m_1(s) &= \frac{-1}{\kappa_1} + c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} + c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s} \\
&\quad + c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} + c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}, \\
m_2(s) &= \frac{1}{\kappa_2}\left\{\begin{array}{c}
-c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s}\left(\frac{-\lambda+\mu+2\kappa_1^2}{\sqrt{-2\lambda+2\mu}}\right) \\
+c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s}\left(\frac{-\lambda+\mu+2\kappa_1^2}{\sqrt{-2\lambda+2\mu}}\right) \\
-c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s}\left(\frac{\lambda+\mu+2\kappa_1^2}{\sqrt{2\lambda+2\mu}}\right) \\
+c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}\left(\frac{\lambda+\mu+2\kappa_1^2}{\sqrt{2\lambda+2\mu}}\right)
\end{array}\right., \\
m_3(s) &= \varepsilon\kappa_3\int m_2(s)ds.
\end{aligned}$$

Here, c_i $(1 \leq i \leq 4)$ are integral constants and

$$\begin{aligned}
\lambda &= \sqrt{\kappa_1^4 + 2\kappa_1^2\kappa_3^2 + 2\varepsilon\kappa_1^2\kappa_2^2 + \kappa_3^4 - 2\varepsilon\kappa_2^2\kappa_3^2 + \kappa_2^4}, \\
\mu &= -\kappa_1^2 + \kappa_3^2 - \varepsilon\kappa_2^2
\end{aligned}$$

are real constants.

Proof. Assume $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a unit speed spacelike curve with spacelike principal normal. From (3.1), we get the differential equation

$$m_1^{(iv)} + (\kappa_1^2 + \varepsilon\kappa_2^2 - \kappa_3^2)m_1'' - \kappa_1^2\kappa_3^2 m_1 - \kappa_1\kappa_3^2 = 0,$$

which has a solution

$$\begin{aligned}
m_1(s) &= \frac{-1}{\kappa_1} + c_1 e^{\frac{-1}{2}\sqrt{-2\lambda+2\mu}s} + c_2 e^{\frac{1}{2}\sqrt{-2\lambda+2\mu}s} \\
&\quad + c_3 e^{\frac{-1}{2}\sqrt{2\lambda+2\mu}s} + c_4 e^{\frac{1}{2}\sqrt{2\lambda+2\mu}s}.
\end{aligned}$$

Thus, the theorem is proved. □

3.1. Osculating Curve of First Kind in $\mathbb{E}_1^4$

Definition 3.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal. If x lies in the hyperplane spanned by $\{T, N_1, N_3\}$, then x is called an osculating curve of first kind in $\mathbb{E}_1^4$.

In [19], authors consider the osculating curves of first kind in $\mathbb{E}_1^4$. It means that the differentiable function $m_2(s)$ vanishes identically. Thus, from (3.1), the system

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 &= 0, \\ \kappa_2 m_1 - \kappa_3 m_3 &= 0, \\ m_3' &= 0 \end{aligned}$$

is obtained. Therefore,

$$\begin{aligned} m_0 &= \frac{-cH_2'}{\kappa_1}, \\ m_1 &= cH_2, \\ m_3 &= c, \end{aligned}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$, $c \in \mathbb{R}$. Thus, one can write x as in the following

$$x(s) = c\left\{\frac{-H_2'}{\kappa_1}(s)T(s) + H_2(s)N_1(s) + N_3(s)\right\}.$$

In [19], authors give the Lemma 3.1.

Lemma 3.1. *[19] Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal. The necessary and sufficient condition for x to correspond an osculating curve of first kind is*

$$\left(\frac{cH_2'}{\kappa_1}\right)' + c\kappa_1 H_2 + 1 = 0, \tag{3.2}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$, $c \in \mathbb{R}$.

Corollary 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and corresponds to an osculating curve of the first kind in $\mathbb{E}_1^4$. If x is a ccr-curve, then*

$$H_2 = -\frac{1}{c\kappa_1},$$

where $c = m_3$ is a real constant.

We give a classification assuming only one of the curvature functions is non-constant as follows:

Assume $\kappa_1(s) =$constant> 0, $\kappa_2(s) =$constant$\neq 0$ and $\kappa_3(s)$ is a non-constant function. From (3.2), we obtain the differential equation

$$c\kappa_3''(s) + c\kappa_1^2\kappa_3(s) + \kappa_1\kappa_2 = 0,$$

which has a solution

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1\cos(\kappa_1 s) + c_2\sin(\kappa_1 s).$$

Similarly, assume that $\kappa_1(s) = constant > 0$, $\kappa_3(s) = constant \neq 0$ and $\kappa_2(s)$ is a non-constant function. Then, (3.2) implies the differential equation

$$\frac{c\kappa_3}{\kappa_1}\left(\frac{1}{\kappa_2(s)}\right)'' + \frac{c\kappa_1\kappa_3}{\kappa_2(s)} + 1 = 0, \tag{3.3}$$

with solution

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{-c_1\kappa_3\cos(\kappa_1 s) + c_2\kappa_3\sin(\kappa_1 s) - 1}.$$

Summing up these calculations, we give the Theorem 3.2.

Theorem 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. Then x corresponds to an osculating curve of first kind if*

i) $\kappa_1(s) =$constant> 0, $\kappa_2(s) =$constant$\neq 0$ and

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1\cos(\kappa_1 s) + c_2\sin(\kappa_1 s),$$

ii) $\kappa_1(s) =$constant> 0, $\kappa_3(s) =$constant$\neq 0$ and

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{-c_1\kappa_3\cos(\kappa_1 s) + c_2\kappa_3\sin(\kappa_1 s) - 1},$$

where c, c_1 and $c_2 \in \mathbb{R}$.

3.2. Osculating Curve of the Second Kind in $\mathbb{E}_1^4$

Definition 3.2. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. If x lies in the hyperplane spanned by $\{T, N_2, N_3\}$, then x is an osculating curve of the second kind in $\mathbb{E}_1^4$.

In [1], the authors consider the spacelike osculating curve of the second kind in $\mathbb{E}_1^4$. Actually, they call them as rectifying curves in $\mathbb{E}_1^4$. In this case, the differentiable function $m_1(s)$ vanishes identically. Thus from (3.1), the equalities

$$\begin{aligned} m_0' &= 1, \\ \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' - \varepsilon\kappa_3 m_3 &= 0, \\ m_3' - \varepsilon\kappa_3 m_2 &= 0 \end{aligned} \tag{3.4}$$

hold. Therefore

$$\begin{aligned} m_0 &= s + b, \\ m_2 &= (s+b)H_1, \\ m_3 &= \frac{(s+b)H_1' + H_1}{\varepsilon\kappa_3}, \end{aligned} \tag{3.5}$$

where $b \in \mathbb{R}$ and $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ is the first harmonic curvature of x. Hence, x is

$$x(s) = (s+b)T(s) + (s+b)H_1 N_2(s) + \frac{(s+b)H_1' + H_1}{\varepsilon \kappa_3} N_3(s).$$

By the use of (3.4) and (3.5), we give the following results.

Theorem 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with a spacelike principal normal in $\mathbb{E}_1^4$. Then the necessary and sufficient condition for x to correspond an osculating curve of second kind is*

$$\left(\frac{(s+b)H_1' + H_1}{\varepsilon \kappa_3}\right)' - \varepsilon \kappa_3 (s+b) H_1 = 0 \tag{3.6}$$

for $H_1(s) = \frac{\kappa_1}{\kappa_2}(s), b \in \mathbb{R}$.

Corollary 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal and corresponds to an osculating curve of the second kind. If x is a ccr-curve, then*

$$\kappa_3(s) = \pm \frac{1}{\sqrt{c + s^2 + 2bs}}, \tag{3.7}$$

where $b, c \in \mathbb{R}$.

Proof. Let x be an osculating curve of second kind. If x is a ccr-curve, then the functions $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ are constants. Thus, by the use of (3.6), one can get

$$\kappa_3'(s) + (s+b)\kappa_3^3(s) = 0,$$

which has a solution (3.7). □

As a consequence of the differential equation (3.6), one can get the following solutions as in the previous section.

Corollary 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. Then, x is corresponds to an osculating curve of the second kind if*

i) $\kappa_1(s) =$ *constant* > 0, $\kappa_2(s) =$ *constant* $\neq 0$, *and* $\kappa_3(s) = \pm \frac{1}{\sqrt{c+s^2+2bs}}$,

ii) $\kappa_2(s) =$ *constant* $\neq 0$, $\kappa_3(s) =$ *constant* $\neq 0$, *and*

$$\kappa_1(s) = \frac{1}{s+b}\left(c_1 \sinh(\kappa_3 s) + c_2 \cosh(\kappa_3 s)\right),$$

iii) $\kappa_1(s) =$ *constant* > 0, $\kappa_3(s) =$ *constant* $\neq 0$, *and*

$$\kappa_2(s) = \frac{\kappa_3(s+b)}{c_1 \sinh(\kappa_3 s) - c_2 \cosh(\kappa_3 s)},$$

where $c_1, c_2, b \in \mathbb{R}$.

4. T-Constant Curves in $\mathbb{E}_1^4$

Definition 4.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_t^n$. If the norm of the tangential component of x, i.e.$\left\|x^T\right\|$, is constant, then x is a *T-constant curve* [8]. Moreover, if this norm is equal to zero, i.e. $\left\|x^T\right\| = 0$, then the curve is a *T-constant* curve of the first kind, otherwise the second kind [15].

In view of (3.1), we give the results that determine T-constant curves in $\mathbb{E}_1^4$.

Theorem 4.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. The necessary and sufficient condition for x to become a T-constant curve of the first kind is*

$$\varepsilon H_2 R' + \left(\frac{\left(-\frac{R'}{\kappa_2}\right)'}{\varepsilon \kappa_3} - \frac{R}{H_2} \right)' = 0,$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x is a T-constant curve of the first kind. From (3.1), we get

$$m_1 = -\frac{1}{\kappa_1}, m_2 = \frac{m_1'}{\kappa_2}, m_3 = \frac{m_2' + \varepsilon \kappa_2 m_1}{\varepsilon \kappa_3}.$$

Further, substituting these values into $m_3' - \varepsilon \kappa_3 m_2 = 0$, we yield the expected result. □

Theorem 4.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed spacelike curve with a spacelike principal normal in $\mathbb{E}_1^4$. The necessary and sufficient condition for x to become a T-constant curve of the second kind is*

$$\left(\frac{\left(\frac{-R'}{\kappa_2} + H_1 m_0\right)'}{\varepsilon \kappa_3} - \frac{R}{H_2} \right)' - \varepsilon H_2 \left(-R' + \kappa_1 m_0\right) = 0,$$

where $m_0 \in \mathbb{R}, H_1(s) = \frac{\kappa_1}{\kappa_2}(s), H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x is a T-constant curve of second kind. From (3.1), we get

$$m_1 = -\frac{1}{\kappa_1}, m_2 = \frac{m_1' + \kappa_1 m_0}{\kappa_2}, m_3 = \frac{m_2' + \varepsilon \kappa_2 m_1}{\varepsilon \kappa_3}.$$

Further, substituting these values into $m_3' - \varepsilon \kappa_3 m_2 = 0$, we get the result. □

Corollary 4.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed T-constant spacelike curve of second kind with spacelike principal normal in $\mathbb{E}_1^4$. If x is a W-curve of $\mathbb{E}_1^4$, then x has the parametrization of*

$$x(s) = \lambda T - RN_1 + H_1 \lambda N_2 - \frac{R}{H_2} N_3,$$

where $R = \frac{1}{\kappa_1}, H_1 = \frac{\kappa_1}{\kappa_2}, H_2 = \frac{\kappa_3}{\kappa_2}, \lambda \in \mathbb{R}$ and c is an integral constant.

Theorem 4.3 gives a simple characterization of T-constant curves of second kind of $\mathbb{E}_1^4$.

Theorem 4.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed T-constant spacelike curve of second kind with spacelike principal normal in $\mathbb{E}_1^4$. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{2\lambda s + c}, \tag{4.1}$$

for some real constants $\lambda = m_0$ and c.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s) \rangle$ and using (1.1), we get $\rho\rho' = m_0$. If x is a T-constant curve of second kind, then by definition, the differentiable function $m_0(s)$ of x is constant. It is easy to show that this differential equation has a non-trivial solution (4.1). $\square$

5. N-Constant Curves in $\mathbb{E}_1^4$

Definition 5.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_t^n$. If the norm of the normal component of x, i.e.$\left\|x^N\right\|$, is constant, then x is a *N-constant curve* [8]. Moreover, if this norm is equal to zero, i.e. $\left\|x^N\right\| = 0$, then the curve is a *N-constant* curve of the first kind, otherwise second kind [15].

Hence, for a N-constant curve x in $\mathbb{E}_1^4$

$$\left\|x^N(s)\right\|^2 = m_1^2(s) + \varepsilon m_2^2(s) - \varepsilon m_3^2(s)$$

becomes a constant function. Therefore, by differentiation

$$m_1 m_1' + \varepsilon m_2 m_2' - \varepsilon m_3 m_3' = 0. \tag{5.1}$$

The following proposition gives a characterization of N-constant curves of the first kind.

Proposition 5.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed N-constant spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$. If x is a N-constant curve of the first kind, then,*

i) x is congruent to a spacelike line which passes through the origin,

ii) x is a planar curve,

iii) x is an osculating curve of second kind,

iv) x lies in the hyperplane which is spanned by $\{T, N_1, N_2\}$.

Conversely, if $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a unit speed spacelike curve with spacelike principal normal in $\mathbb{E}_1^4$ with $\kappa_1 > 0$, and one of (i), (ii), (iii), (iv) holds, then x is a N-constant curve of the first kind.

Proof. Assume x is a N-constant curve of the first kind in $\mathbb{E}_1^4$. There are two possibilities; either $m_1 = m_2 = m_3 = 0$ or $m_1^2 + \varepsilon m_2^2 = \varepsilon m_3^2$. In the first case, $x(I)$ is congruent to a spacelike line which passes through the origin. Let $m_1^2 + \varepsilon m_2^2 = \varepsilon m_3^2$, then by the use of the equations (3.1), we get $\kappa_2 m_1 m_3 = 0$. If $\kappa_2 = 0$, x is a planar curve. If $m_1 = 0$, x is an osculating curve of second kind. Let $m_3 = 0$, then there are two possibilities; either $\kappa_3 = 0$ or $m_2 = 0$. If $m_2 = 0$, x is a planar curve. If $\kappa_3 = 0$, x lies in the hyperplane which is spanned by $\{T, N_1, N_2\}$. □

Further, for the N-constant curves of the second kind, we obtain the following result.

Theorem 5.1. *Let $x(s) \in \mathbb{E}_1^4$ be a spacelike curve with a spacelike principal normal given with the arclength function s and fully lies in $\mathbb{E}_1^4$. If x is a N-constant curve of the second kind, then x has a parametrization of*

$$x(s) = (s + c)T(s) + H_1(s + c)N_2(s) + \frac{H_1'(s + c) + H_1}{\varepsilon\kappa_3} N_3(s),$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s), c \in \mathbb{R}$.

Proof. Assume x is a N-constant curve of the second kind in $\mathbb{E}_1^4$. From the equalities (3.1) and (5.1), we get $m_1 = 0$, $m_0(s) = s + c$, $m_2(s) = \frac{\kappa_1}{\kappa_2}(s)m_0$ and $m_3(s) = \frac{m_2'(s)}{\varepsilon\kappa_3(s)}$ for some constant $c \in \mathbb{R}$. This completes the proof of the theorem. □

Remark 5.1. Every N-constant curve of the second kind is an osculating curve of second kind in $\mathbb{E}_1^4$.

Theorem 5.2 gives a simple characterization of N-constant curve of the second kind in $\mathbb{E}_1^4$.

Theorem 5.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a N-constant curve of second kind. Then, the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{s^2 + 2sc + 2b}, \tag{5.2}$$

for real constants b and c.

Example 5.1. Let us consider the W-curve $x(s) = \left(\sinh s, \cosh s, \sqrt{2}\sin s, -\sqrt{2}\cos s\right)$ in $\mathbb{E}_1^4$. The Frenet frame vectors and the curvatures of x are given as

$$\begin{aligned} T(s) &= \left(\cosh s, \sinh s, \sqrt{2}\cos s, \sqrt{2}\sin s\right), \\ N_1(s) &= \frac{1}{\sqrt{3}}\left(\sinh s, \cosh s, -\sqrt{2}\sin s, \sqrt{2}\cos s\right), \\ N_2(s) &= \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, \cos s, \sin s\right), \\ N_3(s) &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, \sin s, -\cos s\right) \end{aligned}$$

and

$$\kappa_1 = \sqrt{3}, \quad \kappa_2 = -\frac{2\sqrt{6}}{3}, \quad \kappa_3 = \frac{\sqrt{3}}{3},$$

respectively. We find the curvature functions as $m_0 = m_2 = 0$, $m_1 = -\frac{\sqrt{3}}{3}$ and $m_3 = \frac{2\sqrt{6}}{3}$, which shows that the curve x is a T-constant curve of the first kind and N-constant curve of the second kind.

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SOLVING THE FUZZY INITIAL VALUE PROBLEM WITH NEGATIVE COEFFICIENT BY USING FUZZY LAPLACE TRANSFORM

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Abstract. In this paper, the fuzzy initial value problem with negative coefficient is solved by using fuzzy Laplace transform and generalized differentiability. The solutions are found and the comparison results are given.
Keywords: fuzzy initial value problem, generalized differentiability, fuzzy Laplace transform.

1. Introduction

Fuzzy differential equations have been studied by many researchers. Fuzzy differential equations can be solved by several types. Hukuhara differentiability [10, 17, 23], generalized differentiability [7, 8, 9, 12], extension principle [10, 11], the concept of differential inclusion [16] and the fuzzy problem to be a set of crips problem [14]. Another types are numeric methods [1, 2, 3, 4, 15] and the fuzzy Laplace transform [5, 21, 22, 24].

This paper is about the solutions of the fuzzy initial value problem with negative coefficient by fuzzy Laplace transform. The aim of this study is to investigate solutions of problem using the properties fuzzy Laplace transform and generalized differentiability.

The paper is organized as follows: Section 2 delas with preliminaries, Section 3 focuses on findings and the main results, and Section 4 refers to conclusions.

2. Preliminaries

Definition 2.1. [20] A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ satisfying the following properties:

u is normal: $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

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u is convex fuzzy set: $u(\lambda x+(1-\lambda)y)\geqslant \min\{u(x),u(y)\}$ for all $x,y\in\mathbb{R},\lambda\in[0,1]$,

u is upper semi-continuous on $\mathbb{R}$,

$cl\{x\in\mathbb{R}\mid u(x)>0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2.2. [18] Let $u\in\mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0<\alpha\leq 1$, is $[u]^\alpha=\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$. If $\alpha=0$, $[u]^0=cl\{x\in\mathbb{R}\mid u(x)>0\}$. The notation, $[u]^\alpha=[\underline{u}_\alpha,\overline{u}_\alpha]$ denotes explicitly the α-level set of u, where $\underline{u}_\alpha$ and $\overline{u}_\alpha$ denote the left-hand endpoint and the right-hand endpoint of $[u]^\alpha$, respectively.

The following remark shows when $[\underline{u}_\alpha,\overline{u}_\alpha]$ is a valid α-level set.

Remark 2.1. [13, 18] The sufficient and necessary conditions for $[\underline{u}_\alpha,\overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

$\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

$\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,

$\underline{u}_\alpha\leq\overline{u}_\alpha$, $0\leq\alpha\leq 1$.

Definition 2.3. [20] If A is a symmetric triangular fuzzy number with support $[\underline{a},\overline{a}]$, the $\alpha-$level set of A is $[A]^\alpha=\left[\underline{A}_\alpha,\overline{A}_\alpha\right]=\left[\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$, $(\underline{A}_1=\overline{A}_1,\ \underline{A}_1-\underline{A}_\alpha=\overline{A}_\alpha-\overline{A}_1)$.

Definition 2.4. [15, 18, 23] Let $u,v\in\mathbb{R}_F$. If there exists $w\in\mathbb{R}_F$ such that $u=v+w$, then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by $w=u\ominus v$.

Definition 2.5. [6, 15, 18] Let $f:[a,b]\to\mathbb{R}_F$ and $t_0\in[a,b]$. We say that f is Hukuhara differentiable at t_0, if there exists an element $f^{'}(t_0)\in\mathbb{R}_F$ such that for all $h>0$ sufficiently small, $\exists f(t_0+h)\ominus f(t_0)$, $f(t_0)\ominus f(t_0-h)$ and the limits hold

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f^{'}(t_0).$$

Definition 2.6. [18] Let $f:[a,b]\to\mathbb{R}_F$ and $t_0\in[a,b]$. We say that f is (1)-differentiable at t_0, if there exists an element $f^{'}(t_0)\in\mathbb{R}_F$ such that for all $h>0$ sufficiently small near to 0, exist $f(t_0+h)\ominus f(t_0)$, $f(t_0)\ominus f(t_0-h)$ and the limits

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f^{'}(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0) \ominus f(t_0+h)$, $f(t_0-h) \ominus f(t_0)$ and the limits

$$\lim_{h\to 0} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h\to 0} \frac{f(t_0-h) \ominus f(t_0)}{-h} = f'(t_0).$$

Theorem 2.1. *[19] Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$.*

(i) If f is (1)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)\right]$,

(ii) If f is (2)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)\right]$.

Theorem 2.2. *[19] Let $f' : [a,b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0,1]$, f is (1)-differentiable or (2)-differentiable.*

(i) If f and f' are (1)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$,

(ii) If f is (1)-differentiable and f' is (2)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iii) If f is (2)-differentiable and f' is (1)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iv) If f and f' are (2)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$.

Definition 2.7. [22, 24] The fuzzy Laplace transform of fuzzy-valued function f is defined as follows:

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)\,dt,$$

$$F(s) = L(f(t)) = \left[\lim_{\tau\to\infty} \int_0^\tau e^{-st} \underline{f}(t)\,dt, \lim_{\tau\to\infty} \int_0^\tau e^{-st} \overline{f}(t)\,dt\right].$$

$$F(s,\alpha) = L(f(t,\alpha)) = \left[L\left(\underline{f}(t,\alpha)\right), L\left(\overline{f}(t,\alpha)\right)\right]$$

where,

$$L\left(\underline{f}(t,\alpha)\right) = \int_0^\infty e^{-st}\underline{f}(t,\alpha)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t,\alpha)\,dt,$$

$$L\left(\overline{f}(t,\alpha)\right) = \int_0^\infty e^{-st}\overline{f}(t,\alpha)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t,\alpha)\,dt.$$

Theorem 2.3. *[5, 22, 24] Suppose that f is continuous fuzzy-valued function on* $[0,\infty)$ *and exponential order* α *and that* $f^{'}$ *is piecewise continuous fuzzy-valued function on* $[0,\infty)$, *then*

$$L\left(f^{'}(t)\right) = sL(f(t)) \ominus f(0),$$

if f is (1) differentiable,

$$L\left(f^{'}(t)\right) = (-f(0)) \ominus (-sL(f(t))),$$

if f is (2) differentiable.

Theorem 2.4. *[22, 24] Suppose that f and* $f^{'}$ *are continuous fuzzy-valued functions on* $[0,\infty)$ *and of exponential order* α *and that* $f^{''}$ *is piecewise continuous fuzzy-valued function on* $[0,\infty)$, *then*

$$L\left(f^{''}(t)\right) = s^2L(f(t)) \ominus sf(0) \ominus f^{'}(0),$$

if f and $f^{'}$ *are (1) differentiable,*

$$L\left(f^{''}(t)\right) = -f^{'}(0) \ominus \left(-s^2\right)L(f(t)) - sf(0),$$

if f is (1) differentiable and $f^{'}$ *is (2) differentiable,*

$$L\left(f^{''}(t)\right) = -sf(0) \ominus \left(-s^2\right)L(f(t)) \ominus f^{'}(0),$$

if f is (2) differentiable and $f^{'}$ *is (1) differentiable,*

$$L\left(f^{''}(t)\right) = s^2L(f(t)) \ominus sf(0) - f^{'}(0),$$

if f and $f^{'}$ *are (2) differentiable.*

Theorem 2.5. *[5, 22] Let* $f(x)$, $g(x)$ *be continuous fuzzy-valued functions suppose that* c_1 *and* c_2 *are constant, then*

$$L(c_1f(x) + c_2g(x)) = (c_1L(f(x))) + (c_2L(g(x))).$$

Theorem 2.6. *[5] Let* $f(x)$ *be continuous fuzzy-valued function on* $[0,\infty)$ *and* $\lambda \geq 0$, *then*

$$L(\lambda f(x)) = \lambda(L(f(x))).$$

3. Findings and Main Results

In this section, we consider solutions of the fuzzy initial value problem

$$y''(t) = -\lambda y(t),\ t > 0, \tag{3.1}$$

$$y(0) = [A]^{\alpha}, \quad y^{'}(0) = [B]^{\alpha}, \tag{3.2}$$

by Laplace transform, where $\lambda > 0$, A and B are symmetric triangular fuzzy numbers with supports $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, respectively,

$$[A]^{\alpha} = \left[\underline{A}_{\alpha}, \overline{A}_{\alpha}\right] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right],$$

$$[B]^{\alpha} = \left[\underline{B}_{\alpha}, \overline{B}_{\alpha}\right] = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right],$$

(i,j) solution means that y is (i) differentiable, $y^{'}$ is (j) differentiable.

Case 1) If y and $y^{'}$ are (1) differentiable, since

$$s^2 L\left(y\left(t,\alpha\right)\right) \ominus sy\left(0,\alpha\right) \ominus y^{'}\left(0,\alpha\right) = -\lambda L\left(y\left(t,\alpha\right)\right),$$

we have the equations

$$s^2 L\left(\underline{y}\left(t,\alpha\right)\right) - s\underline{y}\left(0,\alpha\right) - \underline{y}^{'}\left(0,\alpha\right) = -\lambda \overline{y}\left(t,\alpha\right),$$

$$s^2 L\left(\overline{y}\left(t,\alpha\right)\right) - s\overline{y}\left(0,\alpha\right) - \overline{y}^{'}\left(0,\alpha\right) = -\lambda \underline{y}\left(t,\alpha\right).$$

Using the initial values and taking the necessary operations,

$$L\left(\underline{y}\left(t,\alpha\right)\right) = \frac{s^2}{s^4-\lambda^2}\underline{B}_{\alpha} + \frac{s^3}{s^4-\lambda^2}\underline{A}_{\alpha} - \frac{\lambda s}{s^4-\lambda^2}\overline{A}_{\alpha} - \frac{\lambda}{s^4-\lambda^2}\overline{B}_{\alpha},$$

$$L\left(\overline{y}\left(t,\alpha\right)\right) = \frac{s^2}{s^4-\lambda^2}\overline{B}_{\alpha} + \frac{s^3}{s^4-\lambda^2}\overline{A}_{\alpha} - \frac{\lambda s}{s^4-\lambda^2}\underline{A}_{\alpha} - \frac{\lambda}{s^4-\lambda^2}\underline{B}_{\alpha}.$$

From here,

$$\begin{aligned} \underline{y}\left(t,\alpha\right) &= L^{-1}\left(\frac{s^2}{s^4-\lambda^2}\right)\underline{B}_{\alpha} + L^{-1}\left(\frac{s^3}{s^4-\lambda^2}\right)\underline{A}_{\alpha} \\ &\quad - L^{-1}\left(\frac{\lambda s}{s^4-\lambda^2}\right)\overline{A}_{\alpha} - L^{-1}\left(\frac{\lambda}{s^4-\lambda^2}\right)\overline{B}_{\alpha}, \end{aligned}$$

$$\begin{aligned}\overline{y}(t,\alpha) &= L^{-1}\left(\frac{s^2}{s^4-\lambda^2}\right)\overline{B}_\alpha + L^{-1}\left(\frac{s^3}{s^4-\lambda^2}\right)\overline{A}_\alpha \\ &\quad -L^{-1}\left(\frac{\lambda s}{s^4-\lambda^2}\right)\underline{A}_\alpha - L^{-1}\left(\frac{\lambda}{s^4-\lambda^2}\right)\underline{B}_\alpha.\end{aligned}$$

Thus, the lower solution and the upper solution are obtained as

$$\begin{aligned}\underline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha-\overline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha-\overline{A}_\alpha\right) \\ &\quad +\frac{e^{-\sqrt{\lambda}t}}{4}\left(\frac{\overline{B}_\alpha-\underline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha-\overline{A}_\alpha\right) \\ &\quad +\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha+\overline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha+\overline{A}_\alpha\right),\end{aligned}$$

$$\begin{aligned}\overline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\overline{B}_\alpha-\underline{B}_\alpha}{\sqrt{\lambda}}+\overline{A}_\alpha-\underline{A}_\alpha\right) \\ &\quad +\frac{e^{-\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha-\overline{B}_\alpha}{\sqrt{\lambda}}+\overline{A}_\alpha-\underline{A}_\alpha\right) \\ &\quad +\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha+\overline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha+\overline{A}_\alpha\right).\end{aligned}$$

Case 2) If y is (1) differentiable and $y^{'}$ is (2) differentiable, since

$$-y^{'}(0,\alpha)\ominus\left(-s^2\right)L\left(y(t,\alpha)\right)-sy(0,\alpha)=-\lambda L\left(y(t,\alpha)\right),$$

we have the equations

$$-\overline{y}^{'}(0,\alpha)-\left(-s^2L\left(\overline{y}(t,\alpha)\right)\right)-s\overline{y}(0,\alpha)=-\lambda L\left(\overline{y}(t,\alpha)\right),$$

$$-\underline{y}^{'}(0,\alpha)-\left(-s^2L\left(\underline{y}(t,\alpha)\right)\right)-s\underline{y}(0,\alpha)=-\lambda L\left(\underline{y}(t,\alpha)\right).$$

Using the initial values, we get

$$L\left(\underline{y}(t,\alpha)\right)=\frac{1}{s^2+\lambda}\underline{B}_\alpha+\frac{s}{s^2+\lambda}\underline{A}_\alpha.$$

$$L\left(\overline{y}(t,\alpha)\right)=\frac{1}{s^2+\lambda}\overline{B}_\alpha+\frac{s}{s^2+\lambda}\overline{A}_\alpha.$$

Taking inverse Laplace transforms of the equations, we have

$$\underline{y}(t,\alpha)=L^{-1}\left(\frac{1}{s^2+\lambda}\right)\underline{B}_\alpha+L^{-1}\left(\frac{s}{s^2+\lambda}\right)\underline{A}_\alpha,$$

$$\overline{y}(t,\alpha) = L^{-1}\left(\frac{1}{s^2+\lambda}\right)\overline{B}_\alpha + L^{-1}\left(\frac{s}{s^2+\lambda}\right)\overline{A}_\alpha.$$

From this, the lower and the upper solutions are obtained as

$$\underline{y}(t,\alpha) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\underline{B}_\alpha + \cos\left(\sqrt{\lambda}t\right)\underline{A}_\alpha,$$

$$\overline{y}(t,\alpha) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\overline{B}_\alpha + \cos\left(\sqrt{\lambda}t\right)\overline{A}_\alpha.$$

Case 3) If y is (2) differentiable and $y^{'}$ is (1) differentiable, since

$$-sy(0,\alpha) \ominus \left(-s^2\right)L(y(t,\alpha)) \ominus y^{'}(0,\alpha) = -\lambda L(y(t,\alpha))$$

we have the equations

$$-s\overline{y}(0,\alpha) - \left(-s^2 L(\overline{y}(t,\alpha))\right) - \overline{y}^{'}(0,\alpha) = -\lambda L(\overline{y}(t,\alpha))$$

$$-s\underline{y}(0,\alpha) - \left(-s^2 L\left(\underline{y}(t,\alpha)\right)\right) - \underline{y}^{'}(0,\alpha) = -\lambda L\left(\underline{y}(t,\alpha)\right).$$

Using the initial values and taking inverse Laplace transforms of the equations, we have

$$\underline{y}(t,\alpha) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\overline{B}_\alpha + \cos\left(\sqrt{\lambda}t\right)\underline{A}_\alpha.$$

$$\overline{y}(t,\alpha) = \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\underline{B}_\alpha + \cos\left(\sqrt{\lambda}t\right)\overline{A}_\alpha.$$

Case 4) If y is (2) differentiable and $y^{'}$ is (2) differentiable, since

$$s^2 L(y(t,\alpha)) \ominus sy(0,\alpha) - y^{'}(0,\alpha) = -\lambda L(y(t,\alpha))$$

we have the equations

$$s^2 L\left(\underline{y}(t,\alpha)\right) - s\underline{y}(0,\alpha) - \underline{y}^{'}(0,\alpha) = -\lambda L(\overline{y}(t,\alpha))$$

$$s^2 L(\overline{y}(t,\alpha)) - s\overline{y}(0,\alpha) - \overline{y}^{'}(0,\alpha) = -\lambda L\left(\underline{y}(t,\alpha)\right)$$

Using the initial values and taking the necessary operations,

$$L\left(\underline{y}(t,\alpha)\right) = \frac{s^3}{s^4-\lambda^2}\underline{A}_\alpha + \frac{s^2}{s^4-\lambda^2}\overline{B}_\alpha - \frac{\lambda s}{s^4-\lambda^2}\overline{A}_\alpha - \frac{\lambda}{s^4-\lambda^2}\underline{B}_\alpha,$$

$$L\left(\overline{y}\left(t,\alpha\right)\right)=\frac{s^3}{s^4-\lambda^2}\overline{A}_\alpha+\frac{s^2}{s^4-\lambda^2}\underline{B}_\alpha-\frac{\lambda s}{s^4-\lambda^2}\underline{A}_\alpha-\frac{\lambda}{s^4-\lambda^2}\overline{B}_\alpha.$$

Taking inverse Laplace transforms of the equations, the lower and the upper solutions are obtained as

$$\begin{aligned}\underline{y}\left(t,\alpha\right) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\overline{B}_\alpha-\underline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha-\overline{A}_\alpha\right)\\ &\quad+\frac{e^{-\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha-\overline{B}_\alpha}{\sqrt{\lambda}}+\underline{A}_\alpha-\overline{A}_\alpha\right)\\ &\quad+\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha+\overline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha+\overline{A}_\alpha\right),\end{aligned}$$

$$\begin{aligned}\overline{y}\left(t,\alpha\right) &= \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha-\overline{B}_\alpha}{\sqrt{\lambda}}+\overline{A}_\alpha-\underline{A}_\alpha\right)\\ &\quad+\frac{e^{-\sqrt{\lambda}t}}{4}\left(\frac{\overline{B}_\alpha-\underline{B}_\alpha}{\sqrt{\lambda}}+\overline{A}_\alpha-\underline{A}_\alpha\right)\\ &\quad+\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha+\overline{B}_\alpha\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha+\overline{A}_\alpha\right).\end{aligned}$$

Theorem 3.1. *(1,1) solution of the initial value problem (3.1)-(3.2) is a valid $\alpha-$level set for $t>0$ satisfying the inequality*

$$e^{2\sqrt{\lambda}t}\geq\left(\frac{\left(\overline{b}-\underline{b}\right)-\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}{\left(\overline{b}-\underline{b}\right)+\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}\right).$$

Proof. If

$$\frac{\partial\underline{y}\left(t,\alpha\right)}{\partial\alpha}\geq0,\quad\frac{\partial\overline{y}\left(t,\alpha\right)}{\partial\alpha}\leq0,\quad\underline{y}\left(t,\alpha\right)\leq\overline{y}\left(t,\alpha\right)$$

(1,1) solution of the initial value problem (3.1)-(3.2) is valid $\alpha-$level set. Thus, it must be

$$e^{\sqrt{\lambda}t}\left(\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}}+\overline{a}-\underline{a}\right)-e^{-\sqrt{\lambda}t}\left(\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}}-\overline{a}-\underline{a}\right)\geq0.$$

Since

$$\left(\overline{b}-\underline{b}\right)+\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)\geq0,$$

we have

$$e^{\sqrt{\lambda}t}\geq e^{-\sqrt{\lambda}t}\left(\frac{\left(\overline{b}-\underline{b}\right)-\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}{\left(\overline{b}-\underline{b}\right)+\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}\right).$$

Consequently, (1,1) solution of the initial value problem (3.1)-(3.2) is a valid $\alpha-$level set for $t > 0$ satisfying the inequality

$$e^{2\sqrt{\lambda}t} \geq \left(\frac{(\overline{b} - \underline{b}) - \sqrt{\lambda}\,(\overline{a} - \underline{a})}{(\overline{b} - \underline{b}) + \sqrt{\lambda}\,(\overline{a} - \underline{a})} \right).$$

□

Theorem 3.2. *(1,2) solution of the initial value problem (3.1)-(3.2) is valid $\alpha-$level set, for $t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$ satisfying the inequality*

$$t \geq \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(-\sqrt{\lambda} \left(\frac{\overline{a} - \underline{a}}{\overline{b} - \underline{b}} \right) \right).$$

Proof. If

$$\frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}t\right) (\overline{b} - \underline{b}) + \cos\left(\sqrt{\lambda}t\right) (\overline{a} - \underline{a}) \geq 0,$$

(1,2) solution of the initial value problem (3.1)-(3.2) is valid $\alpha-$level set. For $\sqrt{\lambda}t \in \left(0, \frac{\pi}{2}\right) \Rightarrow t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$, we have

$$\tan\left(\sqrt{\lambda}t\right) \geq -\sqrt{\lambda} \left(\frac{\overline{a} - \underline{a}}{\overline{b} - \underline{b}} \right) \Rightarrow t \geq \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(-\sqrt{\lambda} \left(\frac{\overline{a} - \underline{a}}{\overline{b} - \underline{b}} \right) \right).$$

This completes the proof. □

Theorem 3.3. *(2,1) solution of the initial value problem (3.1)-(3.2) is a valid $\alpha-$level set for $t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right)$ satisfying the inequality*

$$t \leq \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\sqrt{\lambda} \left(\frac{\overline{a} - \underline{a}}{\overline{b} - \underline{b}} \right) \right).$$

Proof. The proof is similar. □

Theorem 3.4. *(2,2) solution of the initial value problem (3.1)-(3.2) is a valid $\alpha-$level set for $t > 0$ satisfying the inequality*

$$e^{-2\sqrt{\lambda}t} \geq \left(\frac{(\overline{b} - \underline{b}) - \sqrt{\lambda}\,(\overline{a} - \underline{a})}{(\overline{b} - \underline{b}) + \sqrt{\lambda}\,(\overline{a} - \underline{a})} \right).$$

Proof. The proof is similar. □

Theorem 3.5. *All of the solutions are symmetric triangular fuzzy numbers for any $t > 0$.*

Proof. For (1,1) solution, since

$$\underline{y}(t,1)=\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{a}+\underline{a}\right)=\overline{y}(t,1),$$

and

$$\begin{aligned}\underline{y}(t,1)-\underline{y}(t,\alpha) &= \frac{e^{\sqrt{\lambda}t}}{4}(1-\alpha)\left(\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}}+(\overline{a}-\underline{a})\right)\\ &\quad \frac{e^{-\sqrt{\lambda}t}}{4}(\alpha-1)\left(\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}}-(\overline{a}-\underline{a})\right)\\ &= \overline{y}(t,\alpha)-\overline{y}(t,1),\end{aligned}$$

(1,1) solution of the initial value problem (3.1)-(3.2) is a symmetric triangular fuzzy number for any $t>0$. For (1,2) solution, since

$$\underline{y}(t,1)=\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\left(\frac{\overline{b}+\underline{b}}{2}\right)+\cos\left(\sqrt{\lambda}t\right)\left(\frac{\overline{a}+\underline{a}}{2}\right)=\overline{y}(t,1),$$

and

$$\begin{aligned}\underline{y}(t,1)-\underline{y}(t,\alpha) &= (1-\alpha)\left(\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}t\right)\left(\frac{\overline{b}-\underline{b}}{2}\right)+\cos\left(\sqrt{\lambda}t\right)\left(\frac{\overline{a}-\underline{a}}{2}\right)\right)\\ &= \overline{y}(t,\alpha)-\overline{y}(t,1),\end{aligned}$$

(1,2) solution of the initial value problem (3.1)-(3.2) is a symmetric triangular fuzzy number for any $t>0$. For (1,2) and (2,2) solutions, the proof is similar. □

Example 3.1. Consider the solutions of the fuzzy initial value problem

$$y''(t)=-y(t),\quad y(0)=[1]^{\alpha},\quad y^{'}(0)=[2]^{\alpha} \tag{3.3}$$

by fuzzy Laplace transform, where $[1]^{\alpha}=[\alpha,2-\alpha]$, $[2]^{\alpha}=[1+\alpha,3-\alpha]$ with supports $[0,2]$ and $[1,3]$, respectively.
For (1,1) solution, the lower and the upper solutions are

$$\underline{y}(t,\alpha)=e^{t}(\alpha-1)+2\sin t+\cos t,$$

$$\overline{y}(t,\alpha)=e^{t}(1-\alpha)+2\sin t+\cos t.$$

For (1,2) solution, the lower and the upper solutions are

$$\underline{y}(t,\alpha)=(1+\alpha)\sin(t)+\alpha\cos(t),$$

$$\overline{y}(t,\alpha)=(3-\alpha)\sin(t)+(2-\alpha)\cos(t).$$

For (2,1) solution, the lower and the upper solutions are

$$\underline{y}(t,\alpha)=(3-\alpha)\sin(t)+\alpha\cos(t),$$

$$\overline{y}(t,\alpha) = (1+\alpha)\sin(t) + (2-\alpha)\cos(t).$$

For (2,2) solution, the lower and the upper solutions are

$$\underline{y}(t,\alpha) = e^{-t}(\alpha - 1) + 2\sin t + \cos t,$$

$$\overline{y}(t,\alpha) = e^{-t}(1-\alpha) + 2\sin t + \cos t.$$

(1,1) solution is a valid $\alpha-$level set since $e^t > 0$. (1,2) solution is a valid $\alpha-$level set since $t - \tan^{-1}(-1) > 0$ for $t \in \left(0, \frac{\pi}{2}\right)$ according to figure 3.1. (2,1) solution is a valid $\alpha-$level set since $t - \tan^{-1}(1) \leq 0$ for $t \in (0, 0.785398]$ on $\left(0, \frac{\pi}{2}\right)$ according to figure 3.2. (2,2) solution is a valid $\alpha-$level set since $e^{-t} > 0$. All of the solutions are symmetric triangular fuzzy numbers. Also, we can see graphics of solutions for $\alpha = 0.2$ in figure 3.3-figure 3.6.

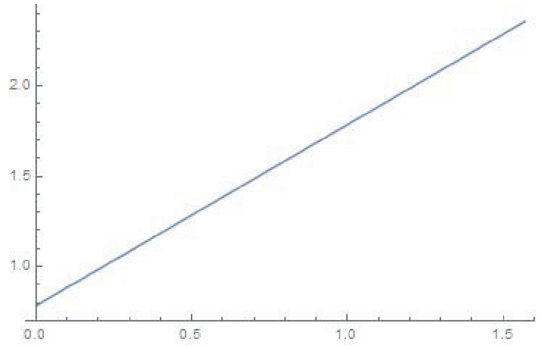

Figure 3.1. Graphic of the function $t - \tan^{-1}(-1)$

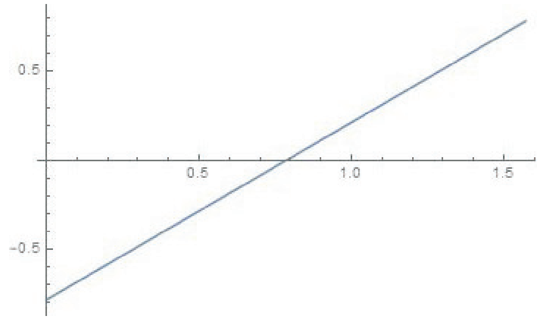

Figure 3.2. Graphic of the function $t - \tan^{-1}(1)$

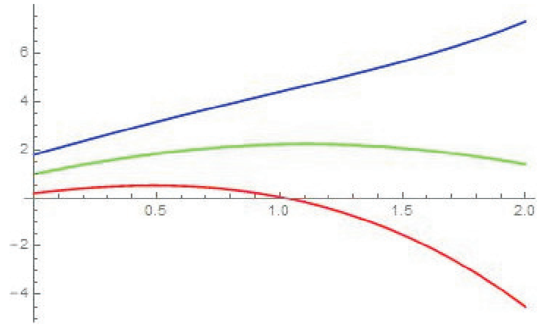

Figure 3.3. Graphic of (1,1) solution for $\alpha = 0.2$

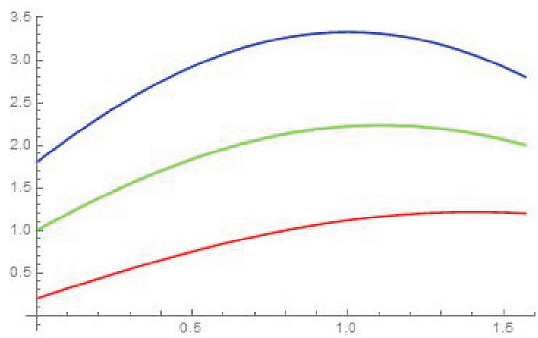

Figure 3.4. Graphic of (1,2) solution for $\alpha = 0.2$

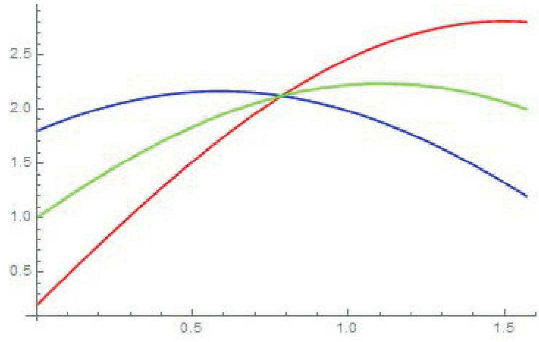

Figure 3.5. Graphic of (2,1) solution for $\alpha = 0.2$

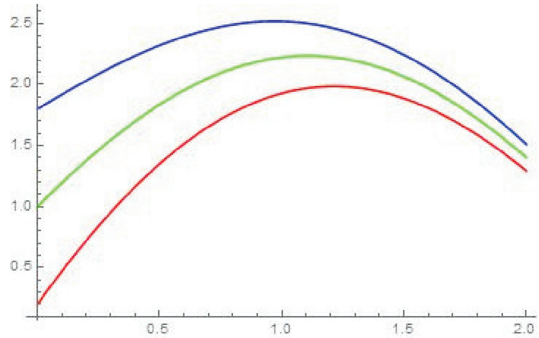

Figure 3.6. Graphic of (2,2) solution for $\alpha = 0.2$

Red $\rightarrow \underline{y}_{\alpha}(t)$, Blue $\rightarrow \overline{y}_{\alpha}(t)$, Green $\rightarrow \underline{y}_{1}(t) = \overline{y}_{1}(t)$

4. Conclusions

In this paper, fuzzy initial value problem with negative coefficient is studied by fuzzy Laplace transform and generalized differentiability. Solutions are found and comparison results are given. It has been proved that the solutions are valid fuzzy functions, which has been shown on an example. It has also been found that when (1,1), (2,1) and (2,2) solutions are valid $\alpha-$ level sets, (1,2) solution is a valid $\alpha-$ level set for $t \in (0, 0.785398]$. However, we can see that (1,1) solution behaves differently from the crips solution in figure 3.3. It means that (1,1) solution becomes fuzzier as time goes by.

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GENERALIZED BESSEL AND FRAME MEASURES

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Abstract. Considering a finite Borel measure μ on $\mathbb{R}^d$, a pair of conjugate exponents p, q, and a compatible semi-inner product on $L^p(\mu)$, we have introduced (p, q)-Bessel and (p, q)-frame measures as a generalization of the concepts of Bessel and frame measures. In addition, we have defined the notions of q-Bessel sequence and q-frame in the semi-inner product space $L^p(\mu)$. Every finite Borel measure ν is a (p, q)-Bessel measure for a finite measure μ. We have constructed a large number of examples of finite measures μ which admit infinite (p, q)-Bessel measures ν. We have showed that if ν is a (p, q)-Bessel/frame measure for μ, then ν is σ-finite and it is not unique. In fact, by using the convolutions of probability measures, one can obtain other (p, q)-Bessel/frame measures for μ. We have presented a general way of constructing a (p, q)-Bessel/frame measure for a given measure.
Keywords: Fourier frame, Plancherel theorem, spectral measure, frame measure, Bessel measure, semi-inner product.

1. Introduction

According to [5], a Borel measure ν on $\mathbb{R}^d$ is called a dual measure for a given measure μ on $\mathbb{R}^d$ if for every $f \in L^2(\mu)$,

$$\int_{\mathbb{R}^d} |\widehat{fd\mu}(t)|^2 d\nu(t) \simeq \int_{\mathbb{R}^d} |f(x)|^2 d\mu(x), \tag{1.1}$$

where for a function $f \in L^1(\mu)$ the Fourier transform is given by

$$\widehat{fd\mu}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i t\cdot x} d\mu(x) \qquad (t \in \mathbb{R}^d).$$

Precisely, the equivalence in equation (1.1) means that there are positive constants A and B independent of the function $f(x)$ such that

$$A \int_{\mathbb{R}^d} |f(x)|^2 d\mu(x) \le \int_{\mathbb{R}^d} |\widehat{fd\mu}(t)|^2 d\nu(t) \le B \int_{\mathbb{R}^d} |f(x)|^2 d\mu(x).$$

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Therefore, when $A = B = 1$, by Plancherel's theorem for Lebesgue measure λ on $\mathbb{R}^d$, λ is a dual measure to itself. Dual measures are in fact a generalization of the concept of Fourier frames and they are also called frame measures. According to [5], if μ is not an F-spectral measure (see Definition 2.3), then there cannot be any general statement about the existence of frame measures ν. Nevertheless, the authors showed that if one frame measure exists, then by using convolutions of measures, many frame measures can be obtained, especially a frame measure which is absolutely continuous with respect to Lebesgue measure. Moreover, they presented a general way of constructing Bessel/frame measures for a given measure.

In this paper, we generalize the notion of Bessel/frame measure from Hilbert spaces $L^2(\mu)$, $L^2(\nu)$ to Banach spaces $L^p(\mu)$, $L^q(\nu)$ (p, q are conjugate exponents) via a compatible semi-inner product defined on $L^p(\mu)$. Compatible semi-inner products are natural substitutes for inner products on Hilbert spaces. We introduce (p, q)-Bessel and (p, q)-frame measures, and we define notions of q-Bessel sequence and q-frame in the semi-inner product space $L^p(\mu)$. Then we investigate the existence and some general properties of them.

The rest of this paper is organized as follows: In Section 2, basic definitions and preliminaries are given. In Section 3, we investigate the existence of (p, q)-Bessel/frame measures. We show that every finite Borel measure ν is a (p, q)-Bessel measure for a finite measure μ. In addition, we construct a large number of examples of measures which admit infinite discrete (p, q)-Bessel measures, by F-spectral measures and applying the Riesz-Thorin interpolation theorem. In general, for every spectral measure (B-spectral measure, or F-spectral measure respectively) μ, there exists a discrete measure $\nu = \sum_{\lambda \in \Lambda_\mu} \delta_\lambda$ which is a Plancherel measure (Bessel measure or frame measure respectively) for μ. Then the Riesz-Thorin interpolation theorem yields that ν is also a (p, q)-Bessel measure for μ, where $1 \le p \le 2$ and q is the conjugate exponent to p. Moreover, this shows that if μ is a spectral measure (B-spectral measure, or F-spectral measure), then the set $\{e_\lambda\}_{\lambda \in \Lambda_\mu}$ forms a q-Bessel sequence for $L^p(\mu)$. It is known [13, 19] that if a measure μ is an F-spectral measure, then it must be of pure type, i.e., μ is either discrete, absolutely continuous or singular continuous. Therefore, we consider such measures in constructing the examples. The interested reader can refer to [3, 6, 7, 9, 13, 16, 18, 19, 20, 21, 23, 24] to see examples and properties of spectral measures (B-spectral measures, or F-spectral measures) and related concepts. Besides discrete (p, q)-Bessel measures $\nu = \sum_{\lambda \in \Lambda_\mu} \delta_\lambda$ associated to spectral measures (B-spectral measures, or F-spectral measures) μ, we prove that there exists an infinite absolutely continuous (p, q)-Bessel measure ν for some finite measures μ (see Proposition 3.12 and Example 4.1). We show that if ν is a (p_1, q_1)-Bessel/frame measure and (p_2, q_2)-Bessel/frame measure for μ, where $1 \le p_1, p_2 < \infty$ and q_1, q_2 are the conjugate exponents to p_1, p_2, respectively, then ν is a (p, q)-Bessel measure for μ too, where $p_1 < p < p_2$ and q is the conjugate exponent to p. Consequently, if ν is a Bessel/frame measure for μ, then it is a (p, q)-Bessel measure for μ too. In Proposition 3.10 we prove that there exists a measure μ which admits tight (p, q)-frame measures and (p, q)-Plancherel measures. Section 4 is devoted to investigating properties of (p, q)-Bessel/frame measures based on the results by Dutkay, Han, and Weber from [5].

2. Preliminaries

Definition 2.1. Let $t \in \mathbb{R}^d$. Denoted by e_t the exponential function

$$e_t(x) = e^{2\pi i t\cdot x} \qquad (x \in \mathbb{R}^d).$$

Definition 2.2. Let H be a Hilbert space. A sequence $\{f_i\}_{i\in I}$ of elements in H is called a *Bessel sequence* for H if there exists a positive constant B such that for all $f \in H$,

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

Here B is called the *Bessel bound* for the Bessel sequence $\{f_i\}_{i\in I}$.

The sequence $\{f_i\}_{i\in I}$ is called a *frame* for H, if there exist constants $A, B > 0$ such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

In this case, A and B are called *frame bounds.*

Frames are a natural generalization of orthonormal bases. It is easily seen from the lower bound that a frame is complete in H, so every f can be expressed using (infinite) linear combination of the elements f_i in the frame [2].

Definition 2.3. Let μ be a compactly supported probability measure on $\mathbb{R}^d$ and Λ be a countable set in $\mathbb{R}^d$, the set $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ is called a *Fourier frame* for $L^2(\mu)$ if for all $f \in L^2(\mu)$,

$$A\|f\|^2_{L^2(\mu)} \leq \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle_{L^2(\mu)}|^2 \leq B\|f\|^2_{L^2(\mu)}.$$

When $E(\Lambda)$ is an orthonormal basis (Bessel sequence, or frame) for $L^2(\mu)$, we say that μ is a *spectral measure (B-spectral measure, or F-spectral measure* respectively) and Λ is called a *spectrum (B-spectrum, or F-spectrum* respectively) for μ.

We give the following definition from [5], assuming that the given measure μ is a finite Borel measure on $\mathbb{R}^d$.

Definition 2.4. [[5]] Let μ be a finite Borel measure on $\mathbb{R}^d$. A Borel measure ν is called a *Bessel measure* for μ, if there exists a positive constant B such that for every $f \in L^2(\mu)$,

$$\|\widehat{fd\mu}\|^2_{L^2(\nu)} \leq B\|f\|^2_{L^2(\mu)}.$$

Here B is called a *(Bessel) bound* for ν.

The measure ν is called a *frame measure* for μ if there exist positive constants A, B such that for every $f \in L^2(\mu)$,

$$A\|f\|^2_{L^2(\mu)} \leq \|\widehat{fd\mu}\|^2_{L^2(\nu)} \leq B\|f\|^2_{L^2(\mu)}.$$

In this case, A and B are called *(frame) bounds* for ν. The measure ν is called a tight frame measure if $A = B$ and Plancherel measure if $A = B = 1$ (see also [8]).

The set of all Bessel measures for μ with fixed bound B is denoted by $\mathcal{B}_B(\mu)$ and the set of all frame measures for μ with fixed bounds A, B is denoted by $\mathcal{F}_{A,B}(\mu)$.

Remark 2.1. A compactly supported probability measure μ is an F-spectral measure if and only if there exists a countable set Λ in $\mathbb{R}^d$ such that $\nu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a frame measure for μ.

Definition 2.5. A finite set of contraction mappings $\{\tau_i\}_{i=1}^n$ on a complete metric space is called an *iterated function system (IFS)*. Hutchinson [15] proved that, for the metric space $\mathbb{R}^d$, there exists a unique compact subset X of $\mathbb{R}^d$, which satisfies $X = \bigcup_{i=1}^n \tau_i(X)$. Moreover, if the IFS is associated with a set of probability weights $\{\rho_i\}_{i=1}^n$ (i.e., $0 < \rho_i < 1$, $\sum_{i=1}^n \rho_i = 1$), then there exists a unique Borel probability measure μ supported on X such that $\mu = \sum_{i=1}^n \rho_i(\mu \circ \tau_i^{-1})$. The corresponding X and μ are called the *attractor* and the *invariant measure* of the IFS, respectively. It is well known that the invariant measure is either absolutely continuous or singular continuous with respect to Lebesgue measure. In an affine IFS each τ_i is affine and represented by a matrix. If R is a $d \times d$ expanding integer matrix (i.e., all eigenvalues λ satisfy $|\lambda| > 1$), and $\mathcal{A} \subset \mathbb{Z}^d$, with $\#\mathcal{A} =: N \geq 2$, then the following set (associated with a set of probability weights) is an affine iterated function system.

$$\tau_a(x) = R^{-1}(x + a) \quad (x \in \mathbb{R}^d, a \in \mathcal{A}).$$

Since R is expanding, the maps τ_a are contractions (in an appropriate metric equivalent to the Euclidean one). In some cases, the invariant measure $\mu_{\mathcal{A}}$ is a *fractal measure* (see [3]). For example singular continuous invariant measures supported on Cantor type sets are fractal measures (see [15, 14]).

Definition 2.6. [[22]](*Semi-inner product spaces*)
Let X be a vector space over the filed F of complex (real) numbers. If a function $[\cdot,\cdot] : X \times X \to F$ satisfies the following properties:

1. $[x + y, z] = [x, z] + [y, z]$, for $x, y, z \in X$;
2. $[\lambda x, y] = \lambda[x, y]$, for $\lambda \in F$ and $x, y \in X$;
3. $[x, x] > 0$, for $x \neq 0$;
4. $|[x, y]|^2 \leq [x, x][y, y]$,

then $[\cdot,\cdot]$ is called a *semi-inner product* and the pair$(X,[\cdot,\cdot])$ is called a *semi-inner product space.* It is easy to observe that $\|x\| = [x,x]^{\frac{1}{2}}$ is a norm on X. So every semi-inner product space is a normed linear space. On the other hand, one can generate a semi-inner product in a normed linear space, in infinitely many different ways.

As a matter of fact, semi-inner products provide the possibility of carrying over Hilbert space type arguments to Banach spaces.

Every Banach space has a semi-inner product that is compatible. For example consider the Banach function space $L^p(X,\mu),\ p \geq 1$, a compatible semi-inner product on this space is defined by (see [12])

$$[f,g]_{L^p(\mu)} := \frac{1}{\|g\|_{L^p(\mu)}^{p-2}} \int_X f(x)|g(x)|^{p-1}\overline{\mathrm{sgn}(g(x))}d\mu(x),$$

for every $f,g \in L^p(X,\mu)$ with $\|g\|_{L^p(\mu)} \neq 0$, and $[f,g]_{L^p(\mu)} = 0$ for $\|g\|_{L^p(\mu)} = 0$.

To construct frames in a Hilbert space H the sequence space l^2 is required. Similarly, to construct frames in a Banach space X one needs a Banach space of scaler valued sequences X_d (in fact a BK-space X_d, see [1] and the references therein). According to Zhang and Zhang [26] frames in Banach spaces can be defined via a compatible semi-inner product in the following way:

Definition 2.7. Let X be a Banach space with a compatible semi-inner product $[\cdot,\cdot]$ and norm $\|\cdot\|_X$. Let X_d be an associated BK-space with norm $\|\cdot\|_{X_d}$. A sequence of elements $\{f_i\}_{i\in I} \subseteq X$ is called an *X_d-frame* for X if $\{[f,f_i]\}_{i\in I} \in X_d$ for all $f \in X$, and there exist constants $A,B > 0$ such that for every $f \in X$,

$$A\|f\|_X \leq \|\{[f,f_i]\}_{i\in I}\|_{X_d} \leq B\|f\|_X.$$

See also [25].

Based on Definition 2.7, we present the next definition. We consider the function space $L^p(\mu)$ and the sequence space $l^q(I)$ (where $p > 1$ and q is the conjugate exponent to p) as the Banach space and the BK- space, respectively.

Definition 2.8. Suppose that $1 < p,q < \infty$ and $1/p + 1/q = 1$. Let μ be a finite Borel measure on $\mathbb{R}^d$ and let $[\cdot,\cdot]$ be the compatible semi-inner product on $L^p(\mu)$ as defined above. We say that a sequence $\{f_i\}_{i\in I}$ is a *q-Bessel sequence* for $L^p(\mu)$ if there exists a constant $B > 0$ such that for every $f \in L^p(\mu)$,

$$\sum_{i\in I} |[f,f_i]_{L^p(\mu)}|^q \leq B\|f\|_{L^p(\mu)}^q.$$

We call B a *(q-Bessel) bound.*

We say the sequence $\{f_i\}_{i\in I}$ is a *q-frame* for $L^p(\mu)$ if there exist constants $A, B > 0$ such that for every $f \in L^p(\mu)$,

$$A\|f\|^q_{L^p(\mu)} \leq \sum_{i\in I} |[f, f_i]_{L^p(\mu)}|^q \leq B\|f\|^q_{L^p(\mu)}.$$

We call A, B *(q-frame) bounds*. We call the sequence $\{f_i\}_{i\in I}$ a *tight q-frame* if $A = B$ and *Parseval q-frame* if $A = B = 1$.

We extend the notions of Bessel and frame measures as follows.

Definition 2.9. Suppose that $1 \leq p < \infty, 1 < q \leq \infty$ and $1/p+1/q = 1$. Let μ be a finite Borel measure on $\mathbb{R}^d$, and let $[\cdot,\cdot]$ be the compatible semi-inner product on $L^p(\mu)$ as defined above. We say that a Borel measure ν is a (p, q)-*Bessel measure* for μ, if there exists a constant $B > 0$ such that for every $f \in L^p(\mu)$,

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) \leq B\|f\|^q_{L^p(\mu)} \quad (p \neq 1, q \neq \infty)$$

and

$$\|\widehat{fd\mu}\|_\infty \leq B\|f\|_{L^1(\mu)} \quad (p = 1, q = \infty).$$

We call B a ((p, q)-*Bessel*) *bound* for ν.

We say the Borel measure ν is a (p, q)-*frame measure* for μ, if there exist constants $A, B > 0$ such that for every $f \in L^p(\mu)$,

$$A\|f\|^q_{L^p(\mu)} \leq \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) \leq B\|f\|^q_{L^p(\mu)} \quad (p \neq 1, q \neq \infty)$$

and

$$A\|f\|_{L^1(\mu)} \leq \|\widehat{fd\mu}\|_\infty \leq B\|f\|_{L^1(\mu)} \quad (p = 1, q = \infty).$$

We call A, B ((p, q)-*frame*) *bounds* for ν. If $A = B$, we call the measure ν a *tight* (p, q)-*frame measure* and if $A = B = 1$, we call it a (p, q)-*Plancherel measure.*

We denote the set of all (p, q)-Bessel measures for μ with fixed bound B by $\mathcal{B}_B(\mu)_{p,q}$ and the set of all (p, q)-frame measures for μ with fixed bounds A, B by $\mathcal{F}_{A,B}(\mu)_{p,q}$.

Remark 2.2. Since $[f, e_t]_{L^p(\mu)} = \int_{R^d} f(x)e^{-2\pi it\cdot x}d\mu(x) = \widehat{fd\mu}(t)$ for any $f \in L^p(\mu)$ and $t \in \mathbb{R}^d$, we can also write $\widehat{fd\mu}(t)$ instead of $[f, e_t]_{L^p(\mu)}$. If there exists a (p, q)-Bessel/frame measure ν for μ, then the function $T_\nu : L^p(\mu) \to L^q(\nu)$ defined by $T_\nu f = \widehat{fd\mu}$ is linear and bounded. For $p = 1$, $q = \infty$, every σ-finite measure ν on $\mathbb{R}^d$ is a $(1,\infty)$-Bessel measure for μ, since we always have $\|\widehat{fd\mu}\|_\infty \leq \|f\|_{L^1(\mu)}$. More precisely, $\nu \in \mathcal{B}_1(\mu)_{(1,\infty)}$.

Theorem 2.1. [10] (Riesz-Thorin interpolation theorem) *Let $1 \leq p_0$, p_1, q_0, $q_1 \leq \infty$, where $p_0 \neq p_1$ and $q_0 \neq q_1$, and let T be a linear operator. Suppose that for some measure spaces (Y, ν), (X, μ), $T : L^{p_0}(X, \mu) \to L^{q_0}(Y, \nu)$ is bounded with norm C_0, and $T : L^{p_1}(X, \mu) \to L^{q_1}(Y, \nu)$ is bounded with norm C_1. Then for all $\theta \in (0, 1)$ and p, q defined by $1/p = (1 - \theta)/p_0 + \theta/p_1$; $1/q = (1 - \theta)/q_0 + \theta/q_1$, there exists a constant C such that $C \leq C_0^{(1-\theta)} C_1^\theta$ and $T : L^p(X, \mu) \to L^q(Y, \nu)$ is bounded with norm C.*

3. Existence and Examples

In this section, we will investigate the existence of (p,q)-Bessel and (p,q)-frame measures and also the existence of q-Bessel sequences and q-frames. In addition, we will construct the examples of measures which admit (p,q)-Bessel measures.

Proposition 3.1. *Suppose that $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let μ be a finite Borel measure. Then every finite Borel measure ν is a (p,q)-Bessel measure for μ.*

Proof. Take $f \in L^p(\mu)$ and $t \in \mathbb{R}^d$. Then by applying Holder's inequality

$$|[f, e_t]_{L^p(\mu)}| \leq \int_{\mathbb{R}^d} |f(x)e^{-2\pi i t\cdot x}| d\mu(x) \leq (\mu(\mathbb{R}^d))^{\frac{1}{q}} \|f\|_{L^p(\mu)}.$$

Thus,

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) \leq \mu(\mathbb{R}^d)\nu(\mathbb{R}^d)\|f\|^q_{L^p(\mu)}.$$

Therefore $\nu \in \mathcal{B}_{\mu(\mathbb{R}^d)\nu(\mathbb{R}^d)}(\mu)_{(p,q)}$. For $p = 1$, $q = \infty$, as we mentioned in Remark 2.2 $\nu \in \mathcal{B}_1(\mu)_{(1,\infty)}$. □

Proposition 3.2. *Suppose that $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let $\Lambda \subset \mathbb{R}^d$, $\#\Lambda < \infty$ and let μ be a finite Borel measure. Then the finite sequence $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-Bessel sequence for $L^p(\mu)$.*

Proof. Consider the finite discrete measure $\nu = \sum_{\lambda\in\Lambda} \delta_\lambda$. Since

$$\sum_{\lambda\in\Lambda} |[f, e_\lambda]_{L^p(\mu)}|^q = \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t),$$

then the assertion follows from Proposition 3.1. □

Remark 3.1. Proposition 3.1 shows that the Bessel bound may change for different measures ν. So if we consider Borel probability measures ν, then we have a fixed Bessel bound $\mu(\mathbb{R}^d)$ for all ν. Moreover, this Bessel bound does not depend on p, q, i.e., for every probability measure ν we have $\nu \in \mathcal{B}_{\mu(\mathbb{R}^d)}(\mu)_{(p,q)}$, where $1 < p < \infty$ and q is the conjugate exponent to p. In addition, we obtain from Proposition 3.1 that for all conjugate exponents $p, q > 1$ the set $\mathcal{B}_{\mu(\mathbb{R}^d)}(\mu)_{p,q}$ is infinite, since there are infinitely many probability measures ν (such as every measure $\nu = \frac{1}{\lambda(S)}\chi_S d\lambda$ where $S \subset \mathbb{R}^d$ with the finite Lebesgue measure $\lambda(S)$, every finite discrete measure $\nu = \frac{1}{n}\sum_{a=1}^n \delta_a$ where δ_a denotes the Dirac measure at the point a, every invariant measure obtained from an iterated function system, and others).

Proposition 3.3. *Suppose that $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let ν be a finite Borel measure. Then ν is a (p,q)-Bessel measure for every finite Borel measure μ. In addition, $\nu \in \mathcal{B}_{\nu(\mathbb{R}^d)}(\mu)_{(p,q)}$ for all probability measures μ.*

Proof. See the proof of Proposition 3.1. □

Corollary 3.1. *Suppose that $1 < p, q < \infty$ and $1/p + 1/q = 1$. A finite Borel measure ν is a (p, q)-Bessel measure for a finite Borel measure μ, if and only if μ is a (p, q)-Bessel measure for ν. In particular, every finite Borel measure μ is a (p, q)-Bessel measure to itself.*

Proof. The statements are direct consequences of Propositions 3.1 and 3.3. □

Lemma 3.1. *Suppose that $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let μ be a finite Borel measure. Then the following assertions hold.*

(i) If there exists a countable set Λ in $\mathbb{R}^d$ such that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-frame for $L^p(\mu)$, then $\nu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a (p, q)-frame measure for μ.

(ii) If ν is purely atomic, i.e., $\nu = \sum_{\lambda\in\Lambda} d_\lambda \delta_\lambda$, and a (p, q)-frame measure for the probability measure μ, then $\{\sqrt[q]{d_\lambda}\, e_\lambda\}_{\lambda\in\Lambda}$ is a q-frame for $L^p(\mu)$.

Proof. (i) Let $\nu = \sum_{\lambda\in\Lambda} \delta_\lambda$. Then for all $f \in L^p(\mu)$,

$$\sum_{\lambda\in\Lambda} |[f, e_\lambda]_{L^p(\mu)}|^q = \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t).$$

(ii) Since for all $f \in L^p(\mu)$,

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) = \sum_{\lambda\in\Lambda} d_\lambda |[f, e_\lambda]_{L^p(\mu)}|^q = \sum_{\lambda\in\Lambda} |[f, \sqrt[q]{d_\lambda} e_\lambda]|^q.$$

□

Example 3.1. Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent to p. If $f \in L^p([0, 1]^d)$, then from the Hausdorff-Young inequality we have $\hat{f} \in l^q(\mathbb{Z}^d)$ and $\|\hat{f}\|_q \leq \|f\|_p$. Therefore, the measure $\nu = \sum_{t\in\mathbb{Z}^d} \delta_t$ is a (p, q)-Bessel measure for $\mu = \chi_{\{[0,1]^d\}} dx$. Besides, $\{e_t\}_{t\in\mathbb{Z}^d}$ is a q-Bessel sequence for $L^p(\mu)$, since $\sum_{t\in\mathbb{Z}^d} |[f, e_t]_{L^p(\mu)}|^q \leq \|f\|_p^q$, where $1 < p \leq 2$ and q is the conjugate exponent to p.

Proposition 3.4. *Suppose $1 < p \leq 2$ and q is the conjugate exponent to p. Let $\mu = \chi_{\{[0,1]^d\}} dx$ and let $0 < a \leq \phi(x) \leq b < \infty$ on $[0, 1]^d$. If $\phi_t(x) := \phi(x)$ for all $t \in \mathbb{Z}^d$, then $\{\phi_t e_t\}_{t\in\mathbb{Z}^d}$ is a q-Bessel sequence for $L^p(\mu)$.*

Proof. Take $f \in L^p(\mu)$. We have $\frac{1}{\|\phi\|_p^{p-2}} \phi^{p-1} f \in L^p(\mu)$, since

$$\int_{\mathbb{R}^d} |f(x)|^p \left| \frac{\phi^{p-1}(x)}{\|\phi\|_p^{p-2}} \right|^p d\mu(x) \leq \frac{b^{(p-1)p}}{a^{(p-2)p}} \int_{\mathbb{R}^d} |f(x)|^p d\mu(x) < \infty.$$

Hence by Example 3.1,

$$\begin{aligned}
\sum_{t\in\mathbb{Z}^d} \left|[f, \phi_t e_t]_{L^p(\mu)}\right|^q &= \sum_{t\in\mathbb{Z}^d} \left| \frac{1}{\|\phi\|_p^{p-2}} \int_{\mathbb{R}^d} f(x) \left|\phi(x) e_t(x)\right|^{p-1} e_{-t}(x) d\mu(x) \right|^q \\
&\leq \left| \int_{\mathbb{R}^d} |f(x)|^p \left| \frac{\phi^{p-1}(x)}{\|\phi\|_p^{p-2}} \right|^p d\mu(x) \right|^{q/p} \leq \frac{b^p}{a^{p-q}} \|f\|_p^q.
\end{aligned}$$

□

Corollary 3.2. *Suppose that $1 < p,q < \infty$ and $1/p + 1/q = 1$. Let μ be a probability measure. Let $0 < a \leq \phi(x) \leq b < \infty$ on $\operatorname{supp}\mu$ and $\phi_i(x) := \phi(x)$ for all $i \in I$. If $\{f_i\}_{i\in I}$ is a q-frame for $L^p(\mu)$, then $\{\phi_i f_i\}_{i\in I}$ is also a q-frame for $L^p(\mu)$ and for every $f \in L^p(\mu)$,*

$$\frac{a^p}{b^{p-q}} A \parallel f \parallel^q_{L^p(\mu)} \leq \|\{[f, \phi_i f_i]_{L^p(\mu)}\}_{i\in I}\|^q \leq \frac{b^p}{a^{p-q}} B \|f\|^q_{L^p(\mu)}.$$

Remark 3.2. Example 3.1 cannot be extended to the case $p > 2$, since there exist continuous functions f such that $\sum_{n\in\mathbb{Z}} |[f, e_n]_{L^p(\mu)}|^{2-\epsilon} = \infty$ for all $\epsilon > 0$. Therefore, $\nu = \sum_{n\in\mathbb{Z}} \delta_n$ is not a (p,q)-Bessel measure for $\mu = \chi_{[0,1]}dx$ where $p > 2$ and also $\{e_n\}_{n\in\mathbb{Z}}$ is not a q-Bessel sequence for $L^p(\mu)$. As an example take $f(x) = \sum_{n=2}^{\infty} \frac{e^{in\log n}}{n^{1/2}(\log n)^2} e^{inx}$ (see [17]).

Proposition 3.5. *Suppose that $1 < p,q < \infty$ and $1/p + 1/q = 1$. Let μ be a compactly supported Borel probability measure. Consider two subsets of $\mathbb{R}^d$, $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ and $\Omega = \{\omega_n : n \in \mathbb{N}\}$ with the property that there exists a positive constant C such that $|\lambda_n - \omega_n| \leq C$ for $n \in \mathbb{N}$.*

(i) If $\{e_{\lambda_n}\}_{n\in\mathbb{N}}$ is a q-Bessel sequence for $L^p(\mu)$, then $\{e_{\omega_n}\}_{n\in\mathbb{N}}$ is a q-Bessel sequence too.

(ii) If $\{e_{\lambda_n}\}_{n\in\mathbb{N}}$ is a q-frame for $L^p(\mu)$, then there exists a $\delta > 0$ such that if $C \leq \delta$ then $\{e_{\omega_n}\}_{n\in\mathbb{N}}$ is a q-frame too (see [3]).

Proof. We need only consider the case, when all $\omega_n = ((\omega_n)_1, \ldots, (\omega_n)_d)$ differ from $\lambda_n = ((\lambda_n)_1, \ldots, (\lambda_n)_d)$ just on the first component, then the assertion follows by induction on the number of components.
Let $\operatorname{supp}\mu \subseteq [-M, M]^d$ for some $M > 0$. Let $f \in L^p(\mu)$ and $x \in \mathbb{R}^d$. The function $\widehat{fd\mu}$ is analytic in each variable $t_1, \ldots, t_d$. Moreover, for every $t \in \mathbb{R}^d$

$$\frac{\partial^k \widehat{fd\mu}}{\partial t_1^k}(t) = \int f(x)(-2\pi i x_1)^k e^{-2\pi i t\cdot x} d\mu(x) = \left[(-2\pi i x_1)^k f, e_t\right]_{L^p(\mu)}.$$

Writing the Taylor expansion at $(\lambda_n)_1$ in the first variable and using Holder's inequality, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left|\widehat{fd\mu}(\omega_n) - \widehat{fd\mu}(\lambda_n)\right|^q &= \left|\sum_{k=1}^{\infty} \frac{\frac{\partial^k \widehat{fd\mu}}{\partial t_1^k}(\lambda_n)}{k!} ((\omega_n)_1 - (\lambda_n)_1)^k\right|^q \\
&\leq \sum_{k=1}^{\infty} \frac{\left|\frac{\partial^k \widehat{fd\mu}}{\partial t_1^k}(\lambda_n)\right|^q}{k!} \cdot \left(\sum_{k=1}^{\infty} \frac{|(\omega_n)_1 - (\lambda_n)_1|^{pk}}{k!}\right)^{q/p} \\
&\leq \sum_{k=1}^{\infty} \frac{\left|\frac{\partial^k \widehat{fd\mu}}{\partial t_1^k}(\lambda_n)\right|^q}{k!} \cdot \left(\sum_{k=1}^{\infty} \frac{C^{pk}}{k!}\right)^{q-1} \\
&= \sum_{k=1}^{\infty} \frac{\left|\frac{\partial^k \widehat{fd\mu}}{\partial t_1^k}(\lambda_n)\right|^q}{k!} \cdot \left(e^{C^p} - 1\right)^{q-1}.
\end{aligned}$$

Considering the q-Bessel sequence $\{e_{\lambda_n}\}_{n\in\mathbb{N}}$ with a bound B, we obtain

$$\begin{aligned}\sum_{n\in\mathbb{N}}\left|\frac{\partial^k\widehat{fd\mu}}{\partial t_1^k}(\lambda_n)\right|^q &= \sum_{n\in\mathbb{N}}\left|\left[(-2\pi i x_1)^k f, e_{\lambda_n}\right]_{L^p(\mu)}\right|^q \le B\|(-2\pi i x_1)^k f\|^q_{L^p(\mu)}\\ &\le B(2\pi M)^{qk}\|f\|^q_{L^p(\mu)}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\left|\widehat{fd\mu}(\omega_n)-\widehat{fd\mu}(\lambda_n)\right|^q &\le B\left(e^{C^p}-1\right)^{q-1}\|f\|^q_{L^p(\mu)}\sum_{k=1}^{\infty}\frac{(2\pi M)^{qk}}{k!}\\ &= B\left(e^{C^p}-1\right)^{q-1}\left(e^{(2\pi M)^q}-1\right)\|f\|^q_{L^p(\mu)}.\end{aligned}$$

Hence by Minkowski's inequality,

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}}|\widehat{fd\mu}(\omega_n)|^q\right)^{1/q} &\le \left(\sum_{n\in\mathbb{N}}|\widehat{fd\mu}(\lambda_n)|^q\right)^{1/q}+\left(\sum_{n\in\mathbb{N}}\left|\widehat{fd\mu}(\omega_n)-\widehat{fd\mu}(\lambda_n)\right|^q\right)^{1/q}\\ &\le \left(B^{1/q}+\left(B\left(e^{C^p}-1\right)^{q-1}\left(e^{(2\pi M)^q}-1\right)\right)^{1/q}\right)\|f\|_{L^p(\mu)},\end{aligned}$$

and this implies that $\{e_{\omega_n}\}_{n\in\mathbb{N}}$ is a q-Bessel sequence for $L^p(\mu)$.
To show that $\{e_{\omega_n}\}_{n\in\mathbb{N}}$ is also a q-frame for $L^p(\mu)$, let A be a lower bound for $\{e_{\lambda_n}\}_{n\in\mathbb{N}}$. Take $\delta>0$ small enough such that for $0<C\le\delta$,

$$A^{1/q}-\left(B\left(e^{C^p}-1\right)^{q-1}\left(e^{(2\pi M)^q}-1\right)\right)^{1/q}>0.$$

Then, by Minkowski's inequality,

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}}|\widehat{fd\mu}(\omega_n)|^q\right)^{1/q} &\ge \left(\sum_{n\in\mathbb{N}}|\widehat{fd\mu}(\lambda_n)|^q\right)^{1/q}-\left(\sum_{n\in\mathbb{N}}\left|\widehat{fd\mu}(\omega_n)-\widehat{fd\mu}(\lambda_n)\right|^q\right)^{1/q}\\ &\ge \left(A^{1/q}-\left(B\left(e^{C^p}-1\right)^{q-1}\left(e^{(2\pi M)^q}-1\right)\right)^{1/q}\right)\|f\|_{L^p(\mu)}.\end{aligned}$$

Thus the assertion follows. □

Proposition 3.6. *Suppose that $1\le p_0,p_1<\infty$ and q_0,q_1 are the conjugate exponents to p_0,p_1 respectively. If ν is a (p_0,q_0)-Bessel measure and a (p_1,q_1)-Bessel measure for μ, then ν is also a (p,q)-Bessel measure for μ, where $p_0<p<p_1$ and q is the conjugate exponent to p.*

Proof. If ν is a (p_0,q_0)-Bessel measure for μ with bound C and also a (p_1,q_1)-Bessel measure with bound D, we have

$$\forall f\in L^{p_0}(\mu)\quad \|\widehat{fd\mu}\|^{q_0}_{L^{q_0}(\nu)}\le C\|f\|^{q_0}_{L^{p_0}(\mu)},$$

and

$$\forall f \in L^{p_1}(\mu) \quad \|\widehat{fd\mu}\|^{q_1}_{L^{q_1}(\nu)} \le D\|f\|^{q_1}_{L^{p_1}(\mu)}.$$

Now if $1/p = (1-\theta)/p_0 + \theta/p_1;\ 1/q = (1-\theta)/q_0 + \theta/q_1$, where $0 < \theta < 1$ (i.e., $p_0 < p < p_1$ and $1/p + 1/q = 1$), then the Riesz-Thorin interpolation theorem yields

$$\forall f \in L^{p}(\mu) \qquad \|\widehat{fd\mu}\|^{q}_{L^{q}(\nu)} \le B^q\|f\|^{q}_{L^{p}(\mu)}.$$

where $B \le C^{\frac{1}{q_0}(1-\theta)} D^{\frac{1}{q_1}\theta}$ (Considering the fact that if $p_0 = 1$ and $q_0 = \infty$, then $C^{\frac{1}{q_0}}$ changes to C, and if $p_1 = 1$ and $q_1 = \infty$, then $D^{\frac{1}{q_1}}$ changes to D). Hence ν is a (p,q)-Bessel measure for μ, where $p_0 < p < p_1$ and q is the conjugate exponent to p. □

Corollary 3.3. *If ν is a Bessel/frame measure for μ, then ν is also a (p,q)-Bessel measure for μ, where $1 \le p \le 2$ and q is the conjugate exponent to p.*

Proof. Let $p_0 = 1, q_0 = \infty, p_1 = 2, q_1 = 2$ in the assumption of Proposition 3.6, then the conclusion follows. □

Proposition 3.7. *If $\nu \in \mathcal{F}_{A,B}(\mu)$, then for any constant $\alpha > 0$, ν is a frame measure for $\alpha\mu$. More precisely $\nu \in \mathcal{F}_{\alpha A,\alpha B}(\alpha\mu)$.*

Proof. Since $\nu \in \mathcal{F}_{A,B}(\mu)$ for all $f \in L^2(\mu)$,

$$A\|f\|^2_{L^2(\mu)} \le \|\widehat{fd\mu}\|^2_{L^2(\nu)} \le B\|f\|^2_{L^2(\mu)},$$

and we have

$$\|\widehat{\alpha fd\mu}\|^2_{L^2(\nu)} = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x)e_{-t}(x)d\alpha\mu(x)\right|^2 d\nu(t) = \|\widehat{fd\alpha\mu}\|^2_{L^2(\nu)}.$$

Since $\alpha f \in L^2(\mu)$,

$$A\|\alpha f\|^2_{L^2(\mu)} \le \|\widehat{\alpha fd\mu}\|^2_{L^2(\nu)} \le B\|\alpha f\|^2_{L^2(\mu)} \quad \text{for all } f \in L^2(\mu).$$

Therefore,

$$\alpha A\|f\|^2_{L^2(\alpha\mu)} \le \|\widehat{fd\alpha\mu}\|^2_{L^2(\nu)} \le \alpha B\|f\|^2_{L^2(\alpha\mu)} \quad \text{for all } f \in L^2(\alpha\mu).$$

Hence $\nu \in \mathcal{F}_{\alpha A,\alpha B}(\alpha\mu)$. □

Theorem 3.1. [23] *There exists positive constants c, C such that for every set $S \subset \mathbb{R}^d$ of finite measure, there is a discrete set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda)$ is a frame for $L^2(S)$ with frame bounds $c|S|$ and $C|S|$, where $|S|$ denotes the measure of S.*

Theorem 3.2. *Let S be a subset (not necessarily bounded) of $\mathbb{R}^d$ with finite Lebesgue measure $|S|$. Then the probability measure $\mu = \frac{1}{|S|}\chi_S dx$ has an infinite discrete (p,q)-Bessel measure ν, where $1 \le p \le 2$ and q is the conjugate exponent to p.*

Proof. By Theorem 3.1, there are positive constants c, C such that for every set $S \subset \mathbb{R}^d$ of finite Lebesgue measure $|S|$, there is a discrete set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda)$ is a frame for $L^2(S)$ with frame bounds $c|S|$ and $C|S|$. Then by considering the upper bound of the frame, we have

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leq C|S| \|f\|^2_{L^2(S)} \quad \text{for all } f \in L^2(S).$$

Let $\mu = \frac{1}{|S|}\chi_S dx$, and then by Proposition 3.7,

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leq C \|f\|^2_{L^2(\mu)} \quad \text{for all } f \in L^2(\mu).$$

In addition, $\|\{[f, e_\lambda]_{L^1(\mu)}\}_{\lambda \in \Lambda}\|_\infty \leq \|f\|_{L^1(\mu)}$, for every f in $L^1(\mu)$. Now if $1/p = 1 - \theta/2$; $1/q = \theta/2$, for $0 < \theta < 1$ (i.e., $1 < p < 2$ and q is the conjugate exponent to p), then the Riesz-Thorin interpolation theorem yields

$$\sum_{\lambda \in \Lambda} |[f, e_\lambda]_{L^p(\mu)}|^q \leq \mathcal{C}^q \|f\|^q_{L^p(\mu)} \quad \text{for all } f \in L^p(\mu),$$

where $\mathcal{C} \leq C^{\frac{1}{2}\theta}$. Therefore, $\nu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is a (p,q)-Bessel measure for $\mu = \frac{1}{|S|}\chi_S dx$, and we have $\nu \in \mathcal{B}_{\mathcal{C}^q}(\mu)_{(p,q)}$, where $1 < p < 2$ and q is the conjugate exponent to p. Moreover, $\nu \in \mathcal{B}_C(\mu)_{(2,2)}$ and $\nu \in \mathcal{B}_1(\mu)_{(1,\infty)}$. On the other hand for every $1 < p < 2$ and q (the conjugate exponent to p), $\{e_\lambda\}_{\lambda \in \Lambda}$ is a q-Bessel sequence for $L^p(\mu)$, with bound $\mathcal{C}^q$. $\square$

If $S \subset \mathbb{R}^d$ is a compact set with positive Lebesgue measure, then by Theorem 3.1, we always have the measure $\mu = \frac{1}{|S|}\chi_S dx$ is an F-spectral measure, but regardless of the fact whether it is a spectral measure, it is related to Fuglede's conjecture [11]. In the following example, we will consider a spectral measure of this type.

Example 3.2. Let $\mu = \chi_{\{[0,1]\cup[2,3]\}}dx$. The set of exponential functions $\{e_\lambda : \lambda \in \Lambda := \mathbb{Z} \cup \mathbb{Z} + \frac{1}{4}\}$ is an orthogonal basis for $L^2(\mu)$ (see [9]). We will consider the probability measure $\mu' = \frac{1}{2}\chi_{\{[0,1]\cup[2,3]\}}dx$. Then for every f in $L^2(\mu')$, we have $\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle_{L^2(\mu')}|^2 = \|f\|^2_{L^2(\mu')}$. In addition, for every $f \in L^1(\mu')$, we have $\|\{[f, e_\lambda]_{L^1(\mu')}\}_{\lambda \in \Lambda}\|_\infty \leq \|f\|_{L^1(\mu')}$. Now by applying the Riesz-Thorin interpolation theorem $\sum_{\lambda \in \Lambda} |[f, e_\lambda]_{L^2(\mu')}|^q \leq \|f\|^q_{L^p(\mu')}$, for all $f \in L^p(\mu')$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. Hence, $\nu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is a (p,q)-Bessel measure for μ', especially $\nu \in \mathcal{B}_1(\mu')_{(p,q)}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. Besides, $\{e_\lambda\}_{\lambda \in \Lambda}$ is a q-Bessel sequence for $L^p(\mu')$ with bound 1, where $1 < p \leq 2$ and q is the conjugate exponent to p.

Proposition 3.8. [20] *Let $\mu(x) = \phi(x)dx$ be a compactly supported absolutely continuous probability measure. Then μ is an F-spectral measure if and only if the density function $\phi(x)$ is bounded above and below almost everywhere on the support (see also [8]).*

Corollary 3.4. *If the density function of a compactly supported absolutely continuous probability measure μ is essentially bounded above and below on the support, then the following assertions hold.*

(i) There exists an infinite (p,q)-Bessel measure $\nu = \sum_{\lambda\in\Lambda_\mu} \delta_\lambda$ for μ, where $1 \le p \le 2$ and q is the conjugate exponent to p. Moreover, when μ is a spectral measure, we have $\nu \in \mathcal{B}_1(\mu)_{p,q}$, where $1 \le p \le 2$ and q is the conjugate exponent to p.

(ii) There exists a q-Bessel sequence $\{e_\lambda\}_{\lambda\in\Lambda_\mu}$ for $L^p(\mu)$, where $1 < p \le 2$ and q is the conjugate exponent to p. In addition, when μ is a spectral measure, $\{e_\lambda\}_{\lambda\in\Lambda_\mu}$ is a q-Bessel sequence for $L^p(\mu)$ with bound 1, where $1 < p \le 2$ and q is the conjugate exponent to p.

Proof. The conclusion follows from Proposition 3.8 and the Riesz-Thorin interpolation theorem (see the proof of Theorem 3.2 and also, see Example 3.2). □

By Proposition 3.3, if $1 < p,q < \infty$ and $1/p + 1/q = 1$, then a fixed finite Borel measure ν is a (p,q)-Bessel measure for every finite measure μ, especially $\nu \in \mathcal{B}_{\nu(\mathbb{R}^d)}(\mu)_{(p,q)}$ for all probability measures μ. In the following part, we will give an example of a discrete spectral measure μ such that it has a finite discrete (p,q)-Bessel measure ν with Bessel bound less than $\nu(\mathbb{R}^d)$, precisely $\nu \in \mathcal{B}_1(\mu)_{(p,q)}$, where $1 \le p \le 2$ and q is the conjugate exponent to p.

Example 3.3. Consider the atomic measure $\mu := \frac{1}{2}(\delta_0 + \delta_{\frac{1}{2}})$, the set $\{e_l : l \in L := \{0,1\}\}$ is an orthonormal basis for $L^2(\mu)$. Hence $\sum_{l\in L} |\langle f, e_l\rangle_{L^2(\mu)}|^2 = \|f\|^2_{L^2(\mu)}$ for all $f \in L^2(\mu)$. Moreover, $\|\{[f,e_l]_{L^1(\mu)}\}_{l\in L}\|_\infty \le \|f\|_{L^1(\mu)}$ for every f in $L^1(\mu)$. Now by applying the Riesz-Thorin interpolation theorem $\sum_{l\in L} |[f,e_l]_{L^p(\mu)}|^q \le \|f\|^q_{L^p(\mu)}$, for all $f \in L^p(\mu)$, where $1 \le p \le 2$ and q is the conjugate exponent to p. Therefore, $\{e_l\}_{l\in L}$ is a finite q-Bessel sequence for $L^p(\mu)$ with bound 1, and $\nu = \sum_{l\in L} \delta_l$ is a finite discrete (p,q)-Bessel measure for μ, especially $\nu \in \mathcal{B}_1(\mu)_{(p,q)}$, where $1 \le p \le 2$ and q is the conjugate exponent to p. When $p > 2$ and q is the conjugate exponent to p, based on Proposition 3.3 $\nu \in \mathcal{B}_2(\mu)_{(p,q)}$ and $\{e_l\}_{l\in L}$ is a finite q-Bessel sequence for $L^p(\mu)$ with bound 2.

Proposition 3.9. [13] *Let $\mu = \sum_{c\in C} p_c\delta_c$ be a discrete probability measure on $\mathbb{R}^d$. μ is an F-spectral measure with an F-spectrum Λ if and only if $\#C < \infty$ and $\#\Lambda < \infty$.*

Corollary 3.5. *Let $1 < p,q < \infty$ and $1/p + 1/q = 1$. If μ is any probability measure, then the following assertions hold.*

(i) A finite discrete measure $\nu = \sum_{\lambda\in\Lambda} \delta_\lambda$ is a (p,q)-Bessel measure for μ, precisely $\nu \in \mathcal{B}_{\nu(\mathbb{R}^d)}(\mu)_{(p,q)}$. If $\mu = \sum_{c\in C} p_c\delta_c$ and if μ is an F-spectral measure with the F-spectrum Λ, then for every $1 < p \le 2$ there exists a positive constant $\mathcal{C}$ such that we have $\nu \in \mathcal{B}_{\mathcal{C}}(\mu)_{(p,q)}$ (q is the conjugate exponent to p). In addition, If $\mu = \sum_{c\in C} p_c\delta_c$ is a spectral measure with the spectrum Λ, then we have $\nu \in \mathcal{B}_1(\mu)_{(p,q)}$, where $1 < p \le 2$ and q is the conjugate exponent to p.

(ii) A finite sequence $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-Bessel sequence for $L^p(\mu)$ with bound $\nu(\mathbb{R}^d)$ ($\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$). If $\mu = \sum_{c\in C} p_c\delta_c$ and if μ is an F-spectral measure with the F-spectrum Λ, then for every $1 < p \leq 2$ there exists a constant $\mathcal{C}$ such that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-Bessel sequence for $L^p(\mu)$ with bound $\mathcal{C}$ (q is the conjugate exponent to p). In addition, If $\mu = \sum_{c\in C} p_c\delta_c$ is a spectral measure with the spectrum Λ, then $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-Bessel sequence for $L^p(\mu)$ with bound 1, where $1 < p \leq 2$ and q is the conjugate exponent to p.

Proof. The conclusion follows from Propositions 3.3, 3.9, 3.2, and the Riesz-Thorin interpolation theorem. In fact, the corollary says that if a probability measure μ is also a discrete F-spectral measure, then beside the bound $\nu(\mathbb{R}^d)$, we can find other bounds by applying Riesz-Thorin interpolation theorem (where $1 < p \leq 2$ and q is the conjugate exponent to p). As we can see in Example 3.3, for all $p > 1$ and q (the conjugate exponent to p), we have $\nu \in \mathcal{B}_2(\mu)_{(p,q)}$ and since μ is a spectral measure with the spectrum L, we also have $\nu \in \mathcal{B}_1(\mu)_{(p,q)}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. $\square$

Theorem 3.3. [4] *Let R be a $d \times d$ expansive integer matrix, $0 \in \mathcal{A} \subset \mathbb{Z}^d$. Let $\mu_{\mathcal{A}}$ be an invariant measure associated to the iterated function system*

$$\tau_a(x) = R^{-1}(x + a) \quad (x \in \mathbb{R}^d, a \in \mathcal{A})$$

and the probabilities $(\rho_a)_{a\in\mathcal{A}}$. Then μ has an infinite B-spectrum of positive Beurling dimension (Beurling dimension is used as a method of investigating existence of Bessel spectra for singular measures).

Theorem 3.4. *Any fractal measure μ obtained from an affine iterated function system has an infinite discrete (p, q)-Bessel measure ν, where $1 \leq p \leq 2$ and q is the conjugate exponent to p.*

Proof. Suppose that R is a $d \times d$ expansive integer matrix, $0 \in \mathcal{A} \subset \mathbb{Z}^d$. If $\mu_{\mathcal{A}}$ is an invariant measure associated to the iterated function system

$$\tau_a(x) = R^{-1}(x + a) \quad (x \in \mathbb{R}^d, a \in \mathcal{A})$$

and the probabilities $(\rho_a)_{a\in\mathcal{A}}$, then according to Theorem 3.3 there exists an infinite subset Λ of $\mathbb{R}^d$ and a constant $B > 0$ such that

$$\sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle_{L^2(\mu_{\mathcal{A}})}|^2 \leq B\|f\|^2_{L^2(\mu_{\mathcal{A}})} \quad \text{for all } f \in L^2(\mu_{\mathcal{A}}).$$

We also have $\|\{[f, e_\lambda]_{L^1(\mu_{\mathcal{A}})}\}_{\lambda\in\Lambda}\|_\infty \leq \|f\|_{L^1(\mu_{\mathcal{A}})}$, for every $f \in L^1(\mu_{\mathcal{A}})$. Now if $1/p = 1 - \theta/2$; $1/q = \theta/2$, for $0 < \theta < 1$ (i.e., $1 < p < 2$ and q is the conjugate exponent to p), then the Riesz-Thorin interpolation theorem yields

$$\sum_{\lambda\in\Lambda} |[f, e_\lambda]_{L^p(\mu_{\mathcal{A}})}|^q \leq B'^q\|f\|^q_{L^p(\mu_{\mathcal{A}})} \quad \text{for all } f \in L^p(\mu_{\mathcal{A}}),$$

where $B' \leq B^{\frac{1}{2}\theta}$. Thus, $\nu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is a (p,q)-Bessel measure for $\mu_{\mathcal{A}}$, and $\nu \in \mathcal{B}_{B'^q}(\mu_{\mathcal{A}})_{(p,q)}$, where $1 < p < 2$ and q is the conjugate exponent to p. Moreover, we have $\nu \in \mathcal{B}_B(\mu_{\mathcal{A}})_{(2,2)}$ and $\nu \in \mathcal{B}_1(\mu_{\mathcal{A}})_{(1,\infty)}$. On the other hand, for every $1 < p < 2$ and q (the conjugate exponent to p), $\{e_\lambda\}_{\lambda\in\Lambda}$ is a q-Bessel sequence for $L^p(\mu_{\mathcal{A}})$ with bound B'^q. □

If a measure μ is an F-spectral measure, then it must be of pure type, i.e., μ is either discrete, singular continuous or absolutely continuous [19, 13]. The case when the measure μ is singular continuous, is not precisely known. The first known example of a singular continuous spectral measure supported on a non-integer dimension set (a fractal measure), was given by Jorgensen and Pedersen [16]. They showed that the measure μ_4 (the Cantor measures supported on Cantor set of 1/4 contraction), is spectral. A spectrum of μ_4 is $\Lambda = \left\{\sum_{m=0}^{k} 4^m d_m \ : d_m \in \{0,1\}, k \in \mathbb{N}\right\}$. They also showed that μ_{2k} (the Cantor measures with even contraction ratio) is spectral, but μ_{2k+1} (the Cantor measures with odd contraction ratio) is not.

Remark 3.3. Since Cantor type measures are fractal measures, by applying Theorem 3.4 one can obtain that every Cantor type measure μ admits a (p,q)-Bessel measure $\nu = \sum_{\lambda\in\Lambda_\mu} \delta_\lambda$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. Moreover, for every spectral Cantor type measure μ_{2k}, we have $\nu \in \mathcal{B}_1(\mu_{2k})_{p,q}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p.

In [21] the author presents a method for constructing many examples of continuous measures μ (including fractal ones) which have components of different dimensions, but nevertheless they are F-spectral measures. In the following part, we will provide some results by [21]. By applying the Riesz-Thorin interpolation theorem, one can obtain infinite discrete (p,q)-Bessel measures $\nu = \sum_{\lambda\in\Lambda_\mu} \delta_\lambda$ (where $1 \leq p \leq 2$ and q is the conjugate exponent to p), for such F-spectral measures μ.

Definition 3.1. [[21]] Let μ and μ' be positive and finite measures on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A *mixed type measure* ρ is a measure which is constructed on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ and defined by

$$\rho = \mu \times \delta_0 + \delta_0 \times \mu',$$

where δ_0 denotes the Dirac measure at the origin. Equivalently, the measure ρ may be defined by the requirement that

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x,y)d\rho(x,y) = \int_{\mathbb{R}^n} f(x,0)d\mu(x) + \int_{\mathbb{R}^m} f(0,y)d\mu'(y),$$

for every continuous, compactly supported function f on $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 3.5. [21] *Let μ and μ' be continuous F-spectral measures. Then the mixed type measure $\rho = \mu \times \delta_0 + \delta_0 \times \mu'$ is also an F-spectral measure.*

Theorem 3.6. [21] *If μ is the sum of the k-dimensional area measure on $[0,1]^k \times \{0\}^{d-k}$, and the j-dimensional area measure on $\{0\}^{d-j} \times [0,1]^j$ where $1 \leq j,k \leq d-1$, then μ is an F-spectral measure.*

The following theorem provides many examples of single dimensional measures which are F-spectral measures:

Theorem 3.7. [21] *Let $\phi : \mathbb{R}^k \to \mathbb{R}^{d-k}$ be a smooth function $(1 \leq k \leq d-1)$. If μ is the k-dimensional area measure on a compact subset of the graph $\{(x, \phi(x)) : x \in \mathbb{R}^k\}$ of ϕ, then μ is an F-spectral measure.*

The next proposition shows that if $1 < p,q < \infty$ and $1/p + 1/q = 1$, then considering any countable subset (finite or infinite) Λ of $\mathbb{R}^d$, one can obtain tight (p,q)-frame measures and (p,q)-Plancherel measures ν_Λ for δ_0. In addition, there exists tight and Parseval q-frames for $L^p(\delta_0)$.

Proposition 3.10. *Suppose that $1 < p,q < \infty$ and $1/p + 1/q = 1$. Then there exists a measure μ which admits tight (p,q)-frame measures and (p,q)-Plancherel measures. Moreover, there exists tight and Parseval q-frames for $L^p(\mu)$.*

Proof. Let $\mu = \delta_0$. For a countable subset Λ of $\mathbb{R}^d$, Let $\nu_\Lambda = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda$ where $c_\lambda > 0$.

If $\sum_{\lambda \in \Lambda} c_\lambda = m \neq 1$, then for all $f \in L^p(\mu)$,

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) = \sum_{\lambda \in \Lambda} c_\lambda |f(0)|^q = m \|f\|^q_{L^p(\mu)}.$$

If $\sum_{\lambda \in \Lambda} c_\lambda = 1$, then for all $f \in L^p(\mu)$,

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) = \sum_{\lambda \in \Lambda} c_\lambda |f(0)|^q = \|f\|^q_{L^p(\mu)}.$$

On the other hand, for all $f \in L^p(\mu)$ we have

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) = \sum_{\lambda \in \Lambda} c_\lambda |[f, e_\lambda]_{L^p(\mu)}|^q = \sum_{\lambda \in \Lambda} |[f, \sqrt[q]{c_\lambda} e_\lambda]_{L^p(\mu)}|^q.$$

Hence, If $\sum_{\lambda \in \Lambda} c_\lambda = m \neq 1$, then $\{\sqrt[q]{c_\lambda} e_\lambda\}_{\lambda \in \Lambda}$ is a tight q-frame for $L^p(\mu)$, and If $0 < c_\lambda < 1$, $\sum_{\lambda \in \Lambda} c_\lambda = 1$, then $\{\sqrt[q]{c_\lambda} e_\lambda\}_{\lambda \in \Lambda}$ is a Parseval q-frame for $L^p(\mu)$. $\square$

Proposition 3.11. *Let μ be a finite Borel measure and let B be a positive constant. Then there exists a (p,q)-Bessel measure ν for μ for all $1 < p,q < \infty$ and $1/p + 1/q = 1$, such that $\nu \in \mathcal{B}_B(\mu)_{p,q}$. In addition, for every $1 < p,q < \infty$ and $1/p + 1/q = 1$, there exists a q-Bessel sequence with bound B for $L^p(\mu)$.*

Proof. Let $\nu = \sum_{i\in I} c_i\delta_{\lambda_i}$ for some $\lambda_i \in \mathbb{R}^d$ such that $\sum_{i\in I} c_i \leq \frac{B}{\mu(\mathbb{R}^d)}$. Let $p > 1$ and $f \in L^p(\mu)$. If q is the conjugate exponent to p, then by applying Holder's inequality we have

$$\int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t) \leq \sum_{i\in I} c_i \parallel f \parallel^q_{L^p(\mu)} \mu(\mathbb{R}^d) \leq B \parallel f \parallel^q_{L^p(\mu)} . \tag{3.1}$$

Hence $\nu \in \mathcal{B}_B(\mu)_{p,q}$.

Since

$$\sum_{i\in I} |[f, \sqrt[q]{c_i} e_{\lambda_i}]_{L^p(\mu)}|^q = \sum_{i\in I} c_i |[f, e_{\lambda_i}]_{L^p(\mu)}|^q = \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu(t),$$

the second statement follows from (3.1). □

All infinite (p, q)-Bessel measures ν we observed were discrete. Now the question is whether we can find a finite measure μ which admits a continuous infinite (p, q)-Bessel measure ν. In the following we show that the answer is affirmative (see also Example 4.1).

Proposition 3.12. *If $\nu = \lambda$ (the Lebesgue measure on $\mathbb{R}^d$) and $\mu = \lambda|_{[0,1]^d}$, then λ is a (p, q)-Bessel measure for μ where $1 \leq p \leq 2$ and q is the conjugate exponent to p.*

Proof. According to Plancherel's theorem the following equation is satisfied:

$$\int_{\mathbb{R}^d} |\hat{f}(t)|^2 d\lambda(t) = \int_{\mathbb{R}^d} |f(x)|^2 d\lambda(x) \quad \text{for all } f \in L^2(\lambda).$$

If f is supported on $[0, 1]^d$, then

$$\int_{\mathbb{R}^d} |\widehat{fd\mu}|^2 d\lambda(t) = \int_{\mathbb{R}^d} |f(x)|^2 d\mu(x) \quad \text{for all } f \in L^2(\mu).$$

Moreover, we have $\|\widehat{fd\mu}\|_\infty \leq \|f\|_{L^1(\mu)}$ for all f in $L^1(\mu)$. Now by applying the Riesz-Thorin interpolation theorem

$$\int_{\mathbb{R}^d} |\widehat{fd\mu}|^q d\lambda(t) \leq \|f\|^q_{L^p(\mu)} \quad \text{for all } f \in L^p(\mu),$$

where $1 \leq p \leq 2$ and q is the conjugate exponent to p. Hence $\lambda \in \mathcal{B}_1(\mu)_{p,q}$.

(likewise, for every $\mu = \lambda|_S$, where S is a subset of $\mathbb{R}^d$ with finite Lebesgue measure we have $\lambda \in \mathcal{B}_1(\mu)_{p,q}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p) □

Corollary 3.6. The measure $\mu = \lambda|_{[0,1]^d}$ has infinite continuous and discrete (p, q)-Bessel measures, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. More precisely, if $\nu_1 = \sum_{t\in\mathbb{Z}^d} \delta_t$ and $\nu_2 = \lambda$, then $\nu_1, \nu_2 \in \mathcal{B}_1(\mu)_{p,q}$.

Proof. The conclusion follows from Example 3.1 and Proposition 3.12. □

Corollary 3.7. *Every $\mu = \lambda|_S$, where S is a subset of $\mathbb{R}^d$ with finite Lebesgue measure, has infinite continuous and discrete (p,q)-Bessel measures, where $1 \leq p \leq 2$ and q is the conjugate exponent to p.*

Proof. The approach is similar to Proposition 3.12 and Theorem 3.2. □

4. Properties and Structural Results

In this section our assertions are based on the results by Dutkay, Han, and Weber from [5]. We generalize the results and we give some of the proofs for completeness.

Proposition 4.1. Let μ be a Borel probability measure. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. If ν is a (p,q)-Bessel measure for μ, then there exists a constant C such that $\nu(K) \leq C diam(K)^d$ for any compact subset K of $\mathbb{R}^d$. Accordingly, ν is σ-finite.

Proof. It is easy to check that $\widehat{d\mu} : \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous and $\widehat{d\mu}(0) = \mu(\mathbb{R}^d) = 1$. So for every $\eta > 0$ there exists $\epsilon > 0$ such that for $x \in \mathsf{B}(0,\epsilon)$ we have $|\widehat{d\mu}(0)| - |\widehat{d\mu}(x)| \leq |\widehat{d\mu}(0) - \widehat{d\mu}(x)| \leq \eta$, and then $|\widehat{d\mu}(x)| \geq 1 - \eta$. If $\delta := (1-\eta)^q$, then $|\widehat{d\mu}(x)|^q \geq \delta$ for $x \in \mathsf{B}(0,\epsilon)$. Thus, for any $t \in \mathbb{R}^d$,

$$\begin{aligned} B = B\|e_t\|^q_{L^p(\mu)} &\geq \int_{\mathbb{R}^d} |[e_t, e_x]|^q d\nu(x) = \int_{\mathbb{R}^d} |[1, e_{x-t}]|^q d\nu(x) \\ &= \int_{\mathbb{R}^d} |\widehat{d\mu}(x-t)|^q d\nu(x) \geq \int_{\mathsf{B}(t,\epsilon)} |\widehat{d\mu}(x-t)|^q d\nu(x) \\ &\geq \nu(\mathsf{B}(t,\epsilon))\delta. \end{aligned}$$

Now Let $K \subseteq \mathbb{R}^d$ be compact and $r = diam(K)$. Then there exists a point $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$ such that $K \subset \prod_{i=1}^d [x_i - r, x_i + r]$. We may assume that $\epsilon < 2r$ and $2r/\epsilon \in \mathbb{N}$. Let $M = 2r/\epsilon$. We have $\prod_{i=1}^d [x_i - r, x_i + r] = \bigcup_{\alpha=1}^{M^d} C_\alpha$ where C_αs are d-dimensional cubes of side length ϵ. For any $\alpha \in \{1, \ldots, M^d\}$, let t_α be the center point of C_α. Then $C_\alpha \subset \mathsf{B}(t_\alpha, \epsilon)$. Now if $C := (2/\epsilon)^d B/\delta$, then

$$\nu(K) \leq \nu\left(\bigcup_{\alpha=1}^{M^d} \mathsf{B}(t_\alpha,\epsilon)\right) \leq \sum_{\alpha=1}^{M^d} \nu(\mathsf{B}(t_\alpha,\epsilon)) \leq \left(\frac{2r}{\epsilon}\right)^d \frac{B}{\delta} = r^d \left(\frac{2}{\epsilon}\right)^d \frac{B}{\delta} = Cr^d.$$

Hence the assertion follows. □

Theorem 4.1. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let $B > A > 0$. Then the set $\mathcal{F}_{A,B}(\mu)_{p,q}$ is empty for some finite compactly supported Borel measures μ.

Proof. Let $\mu = \chi_{[0,1]}dx + \delta_2$. Suppose $\nu \in \mathcal{F}_{A,B}(\mu)_{p,q}$. Let $f := \chi_{\{2\}}$. Then $\|f\|_{L^p(\mu)} = 1$ and $|[f, e_t]_{L^p(\mu)}| = 1$ for all $t \in \mathbb{R}$. In addition, the upper bound implies that $\nu(\mathbb{R}) \le B < \infty$. Then from the inner regularity of Borel measures we obtain that for any $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}$ and a positive constant R such that $\nu(\mathbb{R}) - \epsilon < K \le \nu(\mathsf{B}(0, R))$. Therefore, $\nu(\mathbb{R} \setminus \mathsf{B}(0, R)) < \epsilon$.

Choose some T large, arbitrary and let $g(x) := e^{-2\pi i T x}\chi_{[0,1]}$. Then

$$|[g, e_t]_{L^p(\mu)}|^q = \left|\int_{[0,1]} e^{-2\pi i(T+t)x}dx\right|^q = \left|\frac{sin(\pi(T+t))}{\pi(T+t)}\right|^q \qquad (t \in \mathbb{R}).$$

The substitution $z := -2\pi x$ gives the last equality. Consequently, for all $t \in \mathbb{R}$, $|[g, e_t]_{L^p(\mu)}|^q \le 1$ and if we take $T \ge 2R$, then for all $t \in (-R, R)$ we have

$$|[g, e_t]_{L^p(\mu)}|^q \le \frac{1}{\pi^q(T-R)^q}.$$

Hence from the lower bound we obtain

$$\begin{aligned} A = A\|g\|^q_{L^p(\mu)} &\le \int_{\mathbb{R}} |[g, e_t]_{L^p(\mu)}|^q d\nu(t) \\ &= \int_{\mathsf{B}(0,R)} |[g, e_t]_{L^p(\mu)}|^q d\nu(t) + \int_{\mathbb{R}\setminus\mathsf{B}(0,R)} |[g, e_t]_{L^p(\mu)}|^q d\nu(t) \\ &\le \frac{1}{\pi^q(T-R)^q} \cdot \nu(\mathbb{R}) + \epsilon. \end{aligned}$$

Now if $T \to \infty$ and $\epsilon \to 0$, then $A = 0$. This is a contradiction. $\square$

The next proposition shows that if there exists a (p, q)-Bessel/frame measure, then many others can be constructed.

Proposition 4.2. Let μ be a finite Borel measure and A, B be positive constants. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Then both sets $\mathcal{B}_B(\mu)_{p,q}$ and $\mathcal{F}_{A,B}(\mu)_{p,q}$ are convex and closed under convolution with Borel probability measures.

Proof. Let $\nu_1, \nu_2 \in \mathcal{B}_B(\mu)_{p,q}$ and $0 < \lambda < 1$. For all $f \in L^p(\mu)$,

$$\begin{aligned} \int_{\mathbb{R}^d} |\widehat{fd\mu}|^q d(\lambda\nu_1 + (1-\lambda)\nu_2) &= \lambda \int_{\mathbb{R}^d} |\widehat{fd\mu}|^q d\nu_1 + (1-\lambda)\int_{\mathbb{R}^d} |\widehat{fd\mu}|^q d\nu_2 \\ &\le B\|f\|^q_{L^p(\mu)}. \end{aligned}$$

Then $\lambda\nu_1 + (1-\lambda)\nu_2 \in \mathcal{B}_B(\mu)_{p,q}$. Similarly, if $\nu_1, \nu_2 \in \mathcal{F}_{A,B}(\mu)_{p,q}$, then we have $\lambda\nu_1 + (1-\lambda)\nu_2 \in \mathcal{F}_{A,B}(\mu)_{p,q}$.

Let $s \in \mathbb{R}^d$. Then for all $f \in L^p(\mu)$,

$$\|e_s f\|^p_{L^p(\mu)} = \int_{\mathbb{R}^d} |e_s(x)f(x)|^p d\mu(x) = \int_{\mathbb{R}^d} |f(x)|^p d\mu(x) = \|f\|^p_{L^p(\mu)}.$$

In addition, let $\nu \in \mathcal{B}_B(\mu)_{p,q}$ and let ρ be a Borel probability measure on $\mathbb{R}^d$. Then for any $t \in \mathbb{R}^d$ and $f \in L^p(\mu)$,

$$\begin{aligned} [e_{-s}f, e_t]_{L^p(\mu)} &= \int_{\mathbb{R}^d} e_{-s}(x)f(x)e^{-2\pi it\cdot x}d\mu(x) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i(s+t)\cdot x}d\mu(x) \\ &= [f, e_{s+t}]_{L^p(\mu)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu * \rho(t) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |[f, e_{t+s}]_{L^p(\mu)}|^q d\nu(t)\ d\rho(s) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |[e_{-s}f, e_t]_{L^p(\mu)}|^q d\nu(t)\ d\rho(s) \\ &\leq \int_{\mathbb{R}^d} B\|e_{-s}f\|^q_{L^p(\mu)} d\rho(s) = B\int_{\mathbb{R}^d} \|f\|^q_{L^p(\mu)} d\rho(s) \\ &= B\|f\|^q_{L^p(\mu)}. \end{aligned}$$

For $\nu \in \mathcal{F}_{A,B}(\mu)_{p,q}$ one can obtain the lower bound analogously. □

Corollary 4.1. *Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. If there exists a (p, q)-Bessel/frame measure for μ, then there exists one which is absolutely continuous with respect to the Lebesgue measure and whose Radon-Nikodym derivative is C^∞.*

Proof. Let ν be a (p, q)-Bessel/frame measure for μ. Convoluting ν with the Lebesgue measure on $[0, 1]$ we have

$$\begin{aligned} \nu * \chi_{[0,1]}d\lambda(E) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_E(x+y)d\nu(x)\chi_{[0,1]}(y)d\lambda(y) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_E(t)\chi_{[0,1]}(t-x)d\nu(x)d\lambda(t-x) \\ &= \int_{\mathbb{R}} \chi_E(t)\nu([t-1,t])d\lambda(t) = \int_E \nu([t-1,t])d\lambda(t), \end{aligned}$$

where E is any Borel subset of $\mathbb{R}$. Thus, we obtained a (p, q)-Bessel/frame measure for μ which is absolutely continuous with respect to the Lebesgue measure.

Now consider the following two propositions from [10].
(i) If $d\nu = fd\lambda$ and $d\mu = gd\lambda$, then $d(\nu * \mu) = (f * g)d\lambda$.
(ii) If $f \in L^1$ (or f is locally integrable on $\mathbb{R}^d$), $g \in C^k$, and $\partial^\alpha g$ is bounded for $|\alpha| \leq k$, then $f * g \in C^k$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.

Let $g \geq 0$ be a compactly supported C^∞-function with $\int g(t)d\lambda(t) = 1$. Let $d\nu_0 = \nu * \chi_{[0,1]}d\lambda$ and $d\mu_0 = gd\lambda$. Then we have $d(\nu_0 * \mu_0) = (\nu([\cdot - 1, \cdot]) * g)d\lambda$ and $\nu([\cdot - 1, \cdot]) * g \in C^\infty$. □

Definition 4.1. [[5]] A sequence of Borel probability measures $\{\lambda_n\}$ is called an *approximate identity* if

$$sup\{\| t \|: t \in supp\lambda_n\} \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 4.1. [5] *Let $\{\lambda_n\}$ be an approximate identity. If f is a continuous function on $\mathbb{R}^d$, then for any $x \in \mathbb{R}^d$, we have $\int f(x+t)\ d\lambda_n(t) \to f(x)$ as $n \to \infty$.*

By Proposition 4.2, if ν is a (p,q)-Bessel/frame measure for μ, then $\nu * \rho$ is also a (p,q)-Bessel/frame measure for μ with the same bound(s), where ρ is any Borel probability measure. An obvious question is under what conditions the converse is true. The next theorem gives an answer.

Theorem 4.2. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let $\{\lambda_n\}$ be an approximate identity. Suppose ν is a σ-finite Borel measure, and suppose all measures $\nu * \lambda_n$ are (p,q)-Bessel/frame measures for μ with uniform bounds, independent of n. Then ν is a (p,q)-Bessel/frame measure for μ.

Proof. Take $f \in L^p(\mu)$. Since $|[f, e_\cdot]_{L^p(\mu)}|^q$ (or $|\widehat{fd\mu}|^q$) is continuous on $\mathbb{R}^d$, by Lemma 4.1 and Fatou's lemma we have

$$\begin{aligned}\int_{\mathbb{R}^d} |[f, e_x]_{L^p(\mu)}|^q d\nu(x) &\le \liminf_n \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |[f, e_{x+t}]_{L^p(\mu)}|^q d\lambda_n(t)\ d\nu(x)\\ &= \liminf_n \int_{\mathbb{R}^d} |[f, e_y]_{L^p(\mu)}|^q d(\nu * \lambda_n)(y)\\ &\le B\|f\|^q_{L^p(\mu)}.\end{aligned}$$

Hence ν is a (p,q)-Bessel measure with the same bound B as $\nu * \lambda_n$.

Now showing that

$$\int_{\mathbb{R}^d} |[f, e_x]_{L^p(\mu)}|^q d(\nu * \lambda_n) \to \int_{\mathbb{R}^d} |[f, e_x]_{L^p(\mu)}|^q d\nu,$$

gives the lower bound (see [5]). □

We need the following two propositions from [5] to present a general way of constructing (p,q)-Bessel/frame measures for a given measure.

Proposition 4.3. [5] Let μ and μ' be Borel probability measures. For $f \in L^1(\mu)$, the measure $(fd\mu) * \mu'$ is absolutely continuous w.r.t. $\mu * \mu'$ and if the Radon-Nikodym derivative is denoted by Pf, then

$$Pf = \frac{(fd\mu) * \mu'}{d(\mu * \mu')}.$$

Proposition 4.4. [5] Let μ, μ' be two Borel probability measures and $1 \le p \le \infty$. if $f \in L^p(\mu)$, then the function Pf is in $L^p(\mu * \mu')$ and

$$\|Pf\|_{L^p(\mu*\mu')} \le \|f\|_{L^p(\mu)}.$$

Now we will show that if a convolution of two measures admits a (p,q)-Bessel/frame measure, then one can obtain a (p,q)-Bessel/frame measure for one of the measures in the convolution by using the Fourier transform of the other measure in the convolution.

Proposition 4.5. *Let μ, μ' be two Borel probability measures. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. If ν is a (p,q)-Bessel measure for $\mu * \mu'$, then $|\hat{\mu'}|^q d\nu$ is a (p,q)-Bessel measure for μ with the same bound.*

If in addition ν is a (p,q)-frame measure for $(\mu * \mu')$ with bounds A and B, and for all $f \in L^p(\mu)$, $c\|f\|^q_{L^p(\mu)} \leq \|Pf\|^q_{L^p(\mu*\mu')}$, then $|\hat{\mu}'|^q d\nu$ is a (p,q)-frame measure for μ with bounds cA and B.

Proof. If $\mu, \nu \in M(\mathbb{R}^d)$, then $\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}$ (see[10]). Take $f \in L^p(\mu)$. Then

$$\int_{\mathbb{R}^d} |\widehat{(f d\mu)}|^q \cdot |\hat{\mu'}|^q d\nu = \int_{\mathbb{R}^d} |(\widehat{f d\mu) * \mu'}|^q d\nu = \int_{\mathbb{R}^d} |P f \widehat{d(\mu * \mu')}|^q d\nu.$$

Thus, we have

$$\begin{aligned} cA\|f\|^q_{L^p(\mu)} &\leq A\|Pf\|^q_{L^p(\mu*\mu')} \leq \int_{\mathbb{R}^d} |Pf\widehat{d(\mu * \mu')}|^q d\nu \\ &\leq B\|Pf\|^q_{L^p(\mu*\mu')} \leq B\|f\|^q_{L^p(\mu)}. \end{aligned}$$

□

Now by Proposition 4.5, we will show that there exists a singular continuous measure which admits continuous and discrete (p,q)-Bessel measures.

Example 4.1. Let λ be the Lebesgue measure on $\mathbb{R}$ and $\mu = \lambda|_{[0,1]}$. If μ_4 is the invariant measure for the affine IFS with $R = 4$ and $\mathcal{A} = \{0, 2\}$, and if μ'_4 is the invariant measure for the affine IFS with $R = 4$ and $\mathcal{A}' = \{0, 1\}$, then convolution of measures μ_4 and μ'_4 is the Lebesgue measure on $[0, 1]$ (see Corollary 4.7 from [5]). By Corollary 3.6, $\nu_1 = \sum_{t\in\mathbb{Z}} \delta_t$ and $\nu_2 = \lambda$ are in $\mathcal{B}_1(\mu)_{p,q}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p. Hence by Proposition 4.5, $\nu'_1 = \sum_{t\in\mathbb{Z}} |\widehat{\mu'_4}(t)|^2 \delta_t$ and $\nu'_2 = |\widehat{\mu'_4}(x)|^2 d\lambda(x)$ are in $\mathcal{B}_1(\mu_4)_{p,q}$, where $1 \leq p \leq 2$ and q is the conjugate exponent to p.

In the next theorem, we have some stability results. In fact, this theorem is a generalization of Proposition 3.5.

Theorem 4.3. *Let μ be a compactly supported Borel probability measure. Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. If ν is a (p,q)-Bessel measure for μ, then for any $r > 0$ there exists a constant $D > 0$ such that*

$$\int_{\mathbb{R}^d} \sup_{|y|\leq r} |[f, e_{x+y}]_{L^p(\mu)}|^q d\nu(x) \leq D\|f\|^q_{L^p(\mu)}, \qquad \textit{for all } f \in L^p(\mu).$$

If ν is a (p,q)-frame measure for μ, then there exist constants $\delta > 0$ and $C > 0$ such that

$$C\|f\|^q_{L^p(\mu)} \leq \int_{\mathbb{R}^d} \inf_{|y|\leq\delta} |[f, e_{x+y}]_{L^p(\mu)}|^q d\nu(x), \qquad \textit{for all } f \in L^p(\mu).$$

Proof. The approach is completely similar to the proof of Theorem 2.10 from [5]. □

We show that by using this stability of (p,q)-frame measures, one can obtain atomic (p,q)-frame measures from a general (p,q)-frame measure.

Definition 4.2. Let $Q=[0,1)^d$ and $r>0$. If ν is a Borel measure on $\mathbb{R}^d$ and if $(x_k)_{k\in\mathbb{Z}^d}$ is a set of points such that for all $k\in\mathbb{Z}^d$ we have $x_k\in r(k+Q)$ and $\nu(r(k+Q))<\infty$, then a *discretization of the measure* ν is defined by

$$\nu' := \sum_{k\in\mathbb{Z}^d} \nu(r(k+Q))\delta_{x_k}.$$

Theorem 4.4. *Let $1<p,q<\infty$ and $1/p+1/q=1$. If a compactly supported Borel probability measure μ has a (p,q)-Bessel/frame measure ν, then it also has an atomic one. More precisely, if ν is a (p,q)-Bessel measure for μ and if ν' is a discretization of the measure ν, then ν' is a (p,q)-Bessel measure for μ.*

If ν is a (p,q)-frame measure for μ and $r>0$ is small enough, then ν' is a (p,q)-frame measure for μ.

Proof. Let $Q=[0,1)^d$. Let $(x_k)_{k\in\mathbb{Z}^d}$ be a set of points such that $x_k\in r(k+Q)$ for all $k\in\mathbb{Z}^d$. For every $x\in r(k+Q)$ define $\epsilon(x):=x_k-x$. Thus, $|\epsilon(x)|\le r\sqrt{d}=:r'$ and for any $f\in L^p(\mu)$,

$$\begin{aligned}\int_{\mathbb{R}^d} |[f,e_{x+\epsilon(x)}]_{L^p(\mu)}|^q d\nu(x) &= \sum_{k\in\mathbb{Z}^d}\int_{r(k+Q)} |[f,e_{x_k}]_{L^p(\mu)}|^q d\nu(x)\\ &= \sum_{k\in\mathbb{Z}^d} \nu(r(k+Q))|[f,e_{x_k}]_{L^p(\mu)}|^q.\end{aligned}$$

Since we have

$$\begin{aligned}\int_{\mathbb{R}^d} \inf_{|y|\le r'} |[f,e_{x+y}]_{L^p(\mu)}|^q d\nu(x) &\le \int_{\mathbb{R}^d} |[f,e_{x+\epsilon(x)}]_{L^p(\mu)}|^q d\nu(x)\\ &\le \int_{\mathbb{R}^d} \sup_{|y|\le r} |[f,e_{x+y}]_{L^p(\mu)}|^q d\nu(x),\end{aligned}$$

the upper and lower bounds follow from Theorem 4.3. □

By Lemma 3.1, if there exists a purely atomic (p,q)-frame measure ν for a probability measure μ, then there exists a q-frame for $L^p(\mu)$. Now we conclude that if there exists a (p,q)-frame measure ν (not necessarily purely atomic) for a compactly supported probability measure μ, then there exists a q-frame for $L^p(\mu)$.

Corollary 4.2. *Let μ be a compactly supported Borel probability measure. Let $1<p,q<\infty$ and $1/p+1/q=1$. If ν is a (p,q)-frame measure for μ with bounds A,B and $r>0$ is sufficiently small, then there exist positive constants C,D such that $\{c_k e_{x_k}: k\in\mathbb{Z}^d\}$ is a q-frame for $L^p(\mu)$ with bounds C,D, where $x_k\in r(k+Q)$ and $c_k=\sqrt[q]{\nu(r(k+Q))}$.*

Proof. Let $\nu \in \mathcal{F}_{A,B}(\mu)_{p,q}$. Then by Theorems 4.4 and 4.3, $\nu' = \sum_{k\in\mathbb{Z}^d} c_k^q \delta_{x_k}$ is a (p,q)-frame measure for μ. More precisely, $\nu' \in \mathcal{F}_{C,D}(\mu)_{p,q}$. Hence for all $f \in L^p(\mu)$,

$$\begin{aligned} C\|f\|^q_{L^p(\mu)} &\leq \int_{\mathbb{R}^d} |[f, e_t]_{L^p(\mu)}|^q d\nu'(t) = \sum_{k\in\mathbb{Z}^d} c_k^q |[f, e_{x_k}]_{L^p(\mu)}|^q \\ &= \sum_{k\in\mathbb{Z}^d} |[f, c_k e_{x_k}]_{L^p(\mu)}|^q \leq D\|f\|^q_{L^p(\mu)}. \end{aligned}$$

□

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THE ANALYTIC SOLUTION OF INITIAL BOUNDARY VALUE PROBLEM INCLUDING TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. The motivation of this study is to determine the analytic solution of initial boundary value problem including time fractional differential equation with Neumann boundary conditions in one dimension. By making use of seperation of variables, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem.
Keywords: Caputo fractional derivative, space-fractional diffusion equation, Mittag-Leffler function, initial-boundary-value problems, spectral method.

1. Introduction

As PDEs of fractional order play an important role in modelling numerous processes and systems in various scientific research areas such as applied mathematics, physics chemistry etc., the interest in this topic has become enourmous. Since the fractional derivative is non-local, the model with fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non linear processes. That is why this has attracted an increasing number of researchers. The derivatives in the sense of Caputo is one of the most common since modelling of physical processes with fractional differential equations including Caputo derivative is much better than other models. In literature, increasing number of studies can be found supporting this conclusion [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [16], [17]. Especially there are various studies on fractional diffusion equations: Exact analytical solutions of heat equations are obtained by using operational method [18]. The existence, uniqueness and regularity of solution of impulsive sub-diffusion equation are established by means of eigenfunction expansion [19]. The anomalous diffusion models with non-singular power-law kernel have been investigated and constructed [20]. Moreover, the Caputo derivative of constant is zero which is not hold by many fractional derivatives. The solutions of fractional PDEs and ODEs are determined in terms of Mittag-Leffler function.

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2. Preliminary Results

In this section, we recall fundamental definition and well known results about fractional derivative in Caputo sense.

Definition 2.1. The q^{th} order fractional derivative of $u(t)$ in Caputo sense is defined as

$$D^{q}u\left(t\right)=\frac{1}{\Gamma\left(n-q\right)}\int_{t_0}^{t}\left(t-s\right)^{n-q-1}u^{(n)}(s)ds, t\in\left[t_0,t_0+T\right] \tag{2.1}$$

where $u^{(n)}\left(t\right)=\frac{d^n u}{dt^n}$, $n-1<q<n$. Note that Caputo fractional derivative is equal to integer order derivative when the order of the derivative is integer.

Definition 2.2. The q^{th} order Caputo fractional derivative for $0<q<1$ is defined as

$$D^{q}u\left(t\right)=\frac{1}{\Gamma\left(1-q\right)}\int_{t_0}^{t}\left(t-s\right)^{-q}u'(s)ds, t\in\left[t_0,t_0+T\right] \tag{2.2}$$

The two-parameter Mittag–Leffler function which is taken into account in eigenvalue problem, is given by

$$E_{\alpha,\beta}\left(\lambda(t-t_0)^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{\left(\lambda(t-t_0)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\beta\right)}, \alpha,\beta>0 \tag{2.3}$$

including constant λ. Especially, for $t_0=0$, $\alpha=\beta=q$ we have

$$E_{\alpha,\beta}\left(\lambda t^{q}\right)=\sum_{k=0}^{\infty}\frac{\left(\lambda t^{q}\right)^{k}}{\Gamma\left(qk+q\right)}, q>0. \tag{2.4}$$

Mittag–Leffler function coincides with exponential function i.e., $E_{1,1}\left(\lambda t\right)=e^{\lambda t}$ for $q=1$. For details see [13, 21].

We determined the solution of following time fractional differential equation with Neumann boundary and initial conditions in this study:

$$D_t^{\alpha}u\left(x,t;\alpha\right)=u_{xx}\left(x,t;\alpha\right)-\gamma u\left(x,t;\alpha\right), \tag{2.5}$$

$$u_x\left(0,t\right)=u_x\left(l,t\right)=0, \tag{2.6}$$

$$u\left(x,0\right)=f(x) \tag{2.7}$$

where $0<\alpha<1,\ 0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T, \gamma\in\mathbb{R}$.

3. Main Results

By means of separation of variables method, the solution to the problem (2.5)-(2.7) is constructed in analytical form. Thus, a solution to the problem (2.5)-(2.7) has the following form:

$$u\left(x,t;\alpha\right) = X(x)\ T(t;\alpha) \tag{3.1}$$

where $0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T$.

Plugging (3.1) into (2.5) and arranging it, we have

$$\frac{D_t^\alpha\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)} + \gamma = \frac{X''\left(x\right)}{X\left(x\right)} = -\lambda^2 \tag{3.2}$$

The equation (3.2) produces a fractional differential equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (3.2). Hence, with boundary conditions (2.6), we have the following problem:

$$X''\left(x\right) + \lambda^2 X\left(x\right) = 0 \tag{3.3}$$

$$X'\left(0\right) = X'\left(l\right) = 0 \tag{3.4}$$

The solution of eigenvalue problem (3.3)-(3.4) is accomplished by making use of the exponantial function of the following form:

$$X\left(x\right) = e^{rx} \tag{3.5}$$

Hence, the characteristic equation is computed in the following form:

$$r^2 + \lambda^2 = 0 \tag{3.6}$$

Case 1. If $\lambda = 0$, the Eq.(3.6) has two coincident roots $r_1 = r_2$, leading to the general solution of the eigenvalue problem (3.3)-(3.4) having the following form:

$$X\left(x\right) = k_1 x + k_2 \tag{3.7}$$

$$X'\left(x\right) = k_1 \tag{3.8}$$

The first boundary condition yields

$$X'(0) = k_1 = 0 \Rightarrow k_1 = 0 \tag{3.9}$$

This result leads to

$$X(x) = k_2 \tag{3.10}$$

Similarly, the second boundary condition leads to

$$X'(l) = k_1 = 0 \Rightarrow k_1 = 0 \tag{3.11}$$

Hence, we obtain the solution as follows:

$$X_0(x) = k_2 \tag{3.12}$$

Case 2. If $\lambda < 0$, the Eq.(3.6) has distinct real roots $r_1,\ r_2$ leading to the general solution of the eigenvalue problem (3.3)-(3.4) and having the following form:

$$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{3.13}$$

$$X'(x) = r_1 c_1 e^{r_1 x} + r_2 c_2 e^{r_2 x} \tag{3.14}$$

The first boundary condition yields

$$X'(0) = r_1 c_1 + r_2 c_2 = 0 \Rightarrow c_1 = -\frac{r_2}{r_1} c_2 \tag{3.15}$$

This result leads to

$$X(x) = -\frac{r_2}{r_1} c_2 e^{r_1 x} + c_2 e^{r_2 x} \tag{3.16}$$

Similarly, the last boundary condition leads to

$$X(l) = -\frac{r_2}{r_1} c_2 e^{r_1 l} + c_2 e^{r_2 l} = 0 \Rightarrow c_2 = 0 \tag{3.17}$$

which implies that $c_1 = 0$. Therefore, $X(x) = 0$ which means that we don't have any solution for $\lambda < 0$.

Case 3. If $\lambda > 0$, the Eq.(3.6) has two complex conjugate roots lead to the general solution of the eigenvalue problem (3.3)-(3.4) and have the following form:

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \tag{3.18}$$

$$X'(x) = -c_1 \lambda \sin(\lambda x) + c_2 \lambda \cos(\lambda x) \tag{3.19}$$

The first boundary condition yields

$$X'(0) = 0 = c_2 \lambda \Rightarrow c_2 = 0 \tag{3.20}$$

This result leads to

$$X(x) = c_1 \cos(\lambda x) \tag{3.21}$$

Similarly, the last boundary condition leads to

$$X'(l) = -c_1 \lambda \sin(\lambda l) = 0 \tag{3.22}$$

which implies that

$$\sin(\lambda l) = 0 \tag{3.23}$$

Let $n\pi = \lambda_n l$. Hence, the eigenvalues can be determined as follows:

$$\lambda_n = \frac{n\pi}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \dots \tag{3.24}$$

The representation of the solution is obtained as follows:

$$X_n(x) = \cos\left(\frac{n\pi x}{l}\right), n = 1, 2, 3, \dots \tag{3.25}$$

The second equation in (3.2) for every eigenvalue λ_n is determined as follows:

$$\frac{D_t^\alpha (T(t;\alpha))}{T(t;\alpha)} = -\left(\lambda^2 + \gamma\right) \tag{3.26}$$

which yields the following solution

$$T_n(t;\alpha) = E_{\alpha,1}\left(-\left(\left(\frac{n\pi x}{l}\right)^2 + \gamma\right) t^\alpha\right) n = 1, 2, 3, \dots \tag{3.27}$$

The solution for every eigenvalue λ_n is constructed as follows:

$$u_n(x,t;\alpha) = X_n(x) T_n(t;\alpha) = E_{\alpha,1}\left(-\left(\left(\frac{n\pi x}{l}\right)^2 + \gamma\right) t^\alpha\right) \cos\left(\frac{n\pi x}{l}\right), n = 0, 1, 2, 3, \dots \tag{3.28}$$

Hence the general solution becomes

$$u(x,t;\alpha) = d_0 + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi x}{l}\right) E_{\alpha,1}\left(-\left(\left(\frac{n\pi x}{l}\right)^2 + \gamma\right) t^\alpha\right) \tag{3.29}$$

Note that boundary conditions and fractional differential equation are satisfied by this solution. The coefficients in (3.29) are obtained by making use of initial condition (2.7):

$$u\left(x,0\right)=f\left(x\right)=d_0+\sum_{n=1}^{\infty}d_n\cos\left(\frac{n\pi x}{l}\right) \tag{3.30}$$

$$\left\langle f\left(x\right),\ \cos\left(\frac{k\pi x}{l}\right)\right\rangle=\left\langle d_o,\ \cos\left(\frac{k\pi x}{l}\right)\right\rangle+\sum_{n=1}^{\infty}d_n\left\langle\cos\left(\frac{n\pi x}{l}\right),\cos\left(\frac{k\pi x}{l}\right)\right\rangle \tag{3.31}$$

We obtain the coefficients d_n for $n=0,1,2,3,\ldots$ as follows:

$$d_0=\frac{1}{l}\int_0^l f\left(x\right)dx \tag{3.32}$$

$$d_n=\frac{2}{l}\int_0^l\cos\left(\frac{n\pi x}{l}\right)f\left(x\right)dx \tag{3.33}$$

4. Illustrative Example

In this part, we first take the following partial differential equation with Neumann boundary and initial conditions:

$$u_t\left(x,t\right)=u_{xx}\left(x,t\right)-u(x,t),0\leqslant x\leqslant 1,0\leqslant t\leqslant T$$

$$u_x\left(0,t\right)=0,\ \ u_x\left(1,t\right)=0,0\leqslant t\leqslant T$$

$$u\left(x,0\right)=\cos(\pi x)\ \ 0\leqslant x\leqslant 1 \tag{4.1}$$

which has the solution in the following form:

$$u\left(x,t\right)=\cos(\pi x)\ e^{-\left(\pi^2+1\right)t}. \tag{4.2}$$

Secondly, we take the following time fractional differential equation with Neumann boundary and initial conditions:

$$D_t^{\alpha}u\left(x,t\right)=u_{xx}\left(x,t\right)-u(x,t),\ \ 0<\alpha<1,0\leqslant x\leqslant 1,\ \ 0\leqslant t\leqslant T \tag{4.3}$$

$$u_x\left(0,t\right)=u_x\left(1,t\right)=0,0\leqslant t\leqslant T \tag{4.4}$$

$$u\left(x,0\right)=cos(\pi x),0\leqslant x\leqslant 1 \tag{4.5}$$

The application of seperation of variables method yields the following equation:

$$\frac{D_t^{\alpha}\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)}+1=\frac{X''\left(x\right)}{X\left(x\right)}=-\lambda^2 \tag{4.6}$$

The equation (4.6) produces a fractional differential equation with respect to time and a differential equation with respect to space. The first fractional differential equation is obtained by taking the equation on the right hand side of Eq. (4.6). Hence, with boundary conditions (4.4), we have the following problem:

$$X''\left(x\right)+\lambda^2X\left(x\right)=0 \tag{4.7}$$

$$X'\left(0\right)=0,\ X'\left(1\right)=0 \tag{4.8}$$

Hence the eigenvalue problem (4.7)-(4.8) yields the following solution:

$$X_n\left(x\right)=\cos\left(n\pi x\right)\ ,n=1,2,3,\ldots \tag{4.9}$$

By using the similar calculations as in (3.27), $T_n\left(t;\alpha\right)$ for $n=1,2,3,\ldots$ is determined in the following form:

$$T_n\left(t;\alpha\right)=E_{\alpha,1}\left(-\left(\left(n\pi\right)^2+1\right)t^{\alpha}\right)\ n=1,2,3,\ldots \tag{4.10}$$

For each eigenvalue λ_n, we obtain the following solution:

$$u_n\left(x,t;\alpha\right)=X_n\left(x\right)T_n\left(t;\alpha\right)=E_{\alpha,1}\left(-\left(\left(n\pi\right)^2+1\right)t^{\alpha}\right)\cos\left(n\pi x\right)\ n=0,1,2,3,\ldots \tag{4.11}$$

Hence, the general solution is established as follows:

$$u\left(x,t;\alpha\right)=d_0+\sum_{n=1}^{\infty}d_n\cos\left(n\pi x\right)\ E_{\alpha,1}\left(-\left(\left(n\pi\right)^2+1\right)t^{\alpha}\right) \tag{4.12}$$

Note that the general solution (4.12) satisfies both boundary conditions (4.4) and the fractional equation (4.3). We determine the coefficients d_n in such a way that the general solution (4.12) satisfes the initial condition (4.5). Plugging $t=0$ in to the general solution (4.12) and making equal to the initial condition (4.5), we have

$$u\left(x,0\right)=d_0+\sum_{n=1}^{\infty}d_n\cos\left(n\pi x\right) \tag{4.13}$$

Via the inner product we obtain the coefficients d_n for $n=0,1,2,3,\ldots$ as follows:

$$d_0=\frac{1}{l}\int_0^1 f(x)dx=\int_0^1\cos(\pi x)\ dx=\frac{1}{\pi}\sin(\pi x)\Big|_{x=0}^{x=1}=0 \tag{4.14}$$

and

$$d_n = \frac{2}{l}\int_0^1 \cos\left(\frac{n\pi x}{l}\right) f(x)dx = 2\int_0^1 \cos\left(n\pi x\right)\cos\left(\pi x\right) dx \tag{4.15}$$

Thus $d_n = 0$ for $n \neq 1$.

For $n = 1$ we get

$$d_1 = 2\int_0^1 \cos^2\left(\pi x\right) dx = 2\left[\frac{x}{2} + \frac{1}{4\pi}\sin\left(2\pi x\right)\right]\Bigg|_{x=0}^{x=1} = 1 \tag{4.16}$$

Thus

$$u\left(x,t;\alpha\right) = \cos(\pi x)\ E_{\alpha,1}\left(-\left(\pi^2+1\right)t^{\alpha}\right) \tag{4.17}$$

It is important to note that plugging $\alpha = 1$ in to the solution (4.17) gives the solution (4.2) which confirm the accuracy of the method we apply.

5. Conclusion

In this research, the analytic solution of initial boundary value problem with Neumann boundary conditions in one dimension has been constructed. By using the separation of variables, the solution is formed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem.

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ON THE BASIC STRUCTURES OF DUAL SPACE

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Abstract. Topology studies the properties of spaces that are invariant under any continuous deformation. Topology is needed to examine the properties of the space. Fundamentally, the most basic structure required to do math in the space is topology. There exists little information on the expression of the basis and topology on dual space. The main point of the research is to explain how to define the basis and topology on dual space D^n. Then, we will study the geometric constructions corresponding to the open balls in D and D^2, respectively.
Keywords: dual space; dual numbers; topological structure.

1. Introduction

Topology, as a well-determined mathematical field, emerged at the beginning of the 20th century although some isolated conclusions are traced back to a few centuries ago. The term topology belongs to a special mathematical opinion central to the field of mathematics named topology. Topology tells us how components of a group concern spatially with one another. Fundamentally, in the modern version of geometry, the study of all different kinds of spaces can be regarded as topology. The thing that distinguishes different sorts of geometry from each other is in the kinds of transformations that are allowed before you really consider something changed. Topology investigates the properties of spaces that are invariant under any continuous deformation. This is almost the most basic form of geometry available. It is used in nearly all branches of mathematics in one form or another. Besides, topology is applied in biology, computer science, physics, robotics, geography and landscape ecology, fiber art, games, and puzzles. For more details, we refer the readers to ([1], [3]−[10], [18]−[20]).

Dual numbers were defined by W. K. Clifford (1845−1879) as a tool for his geometrical studies, and their first applications were given by Kotelnikov [15]. Eduard Study [21] used dual numbers and dual vectors in his research on line geometry and kinematics. He proved that there exists a one-to-one correspondence between the

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points of the dual unit sphere in D^3 and the directed lines of Euclidean 3-space. In 1994, by using dual numbers, Cheng [5] introduced the C^H programming language. These numbers play an important role in field theory as well [12]. The most interesting use of dual numbers in field theory can be shown in a series of articles by Wald *et al.* [22]. Furthermore, Gromov, in a series of articles, applied the dual numbers in several ways: in contractions and analytical continuations of classical groups [13], and then in quantum group formalism [14]. Dual numbers have their application in various fields such as computer modelling of rigid body, mechanism design, kinematics, modelling human body, dynamics, etc. ([11] and [16]). For example, in kinematics, using dual numbers, it is possible to explain the screw theory [17].

The basis and topology concepts of dual space have not been investigated in detail, although dual numbers and dual space are used in many articles about mathematics, kinematics, and physics. In order to study the mathematical structure of dual space, we need its topological structure. Furthermore, there is no order relation on the dual numbers system. In this case, how does the dual absolute value and norm provide triangular inequality? The answer to this question is given by this study.

The main aim of this article is to give the basis and topology concepts on dual space D^n. The order relation on dual numbers is defined to achieve this aim. By using this order relation, the concepts of dual inner product, norm, and metric are examined again in detail. This study will provide an insight into the structure of dual space.

2. Basic Concepts

Let the set of the pair (x, x^*) be

$$D = \mathbb{R} \times \mathbb{R} = \{\overline{x} = (x, x^*) \mid x, x^* \in \mathbb{R}\}.$$

Two inner operations and an equality on D are described as follows:

(i) $\oplus : D \times D \to D$ for $\overline{x} = (x, x^*)$ and $\overline{y} = (y, y^*)$ defined as

$$\overline{x} \oplus \overline{y} = (x + y, x^* + y^*)$$

is called the addition in D.

(ii) $\odot : D \times D \to D$ for $\overline{x} = (x, x^*)$ and $\overline{y} = (y, y^*)$ defined as

$$\overline{x} \odot \overline{y} = (xy, xy^* + x^*y)$$

is called the multiplication in D.

(iii) For $\overline{x} = (x, x^*)$ and $\overline{y} = (y, y^*)$, if $x = y$, $x^* = y^*$, $\overline{x}$ and $\overline{y}$ are equal, and it is indicated as $\overline{x} = \overline{y}$.

If the two operators and the equality on D with a set of real numbers $\mathbb{R}$ are defined as above, the set D is called the dual numbers system and the element

$\overline{x} = (x, x^*)$ is called a dual number. For $\overline{x} = (x, x^*) \in D$, the real number x is called the real part of $\overline{x}$, and the real number x^* is called the dual part of $\overline{x}$. The dual numbers $(1,0) = 1$ and $(0,1) = \varepsilon$ are called the unit element of multiplication operation in D, and the dual unit which satisfies the condition that

$$\varepsilon \neq 0,\ \varepsilon^2 = 0,\ \varepsilon 1 = 1\varepsilon = \varepsilon,$$

respectively. If we use the multiplication property and $\varepsilon = (0,1)$, we have the expression $\overline{x} = x + \varepsilon x^*$. The set of all dual numbers is written as follows:

$$D = \left\{ \overline{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}.$$

The set D forms a commutative ring according to the operations

$$(x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon (x^* + y^*)$$

and

$$(x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon (xy^* + x^* y).$$

For the dual numbers $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$, if $y \neq 0$, then the division $\dfrac{\overline{x}}{\overline{y}}$ is defined as follows:

$$\frac{\overline{x}}{\overline{y}} = \frac{x}{y} + \varepsilon \frac{x^* y - x y^*}{y^2}.$$

The absolute value of the dual number $\overline{x} = x + \varepsilon x^*$ can be given as

$$|\overline{x}|_D = |x| + \varepsilon \frac{x x^*}{|x|},\ (x \neq 0).$$

Clearly, $|\overline{x}|_D = 0$ if $\overline{x} = 0$ [24].

The set of

$$D^3 = \{ \widetilde{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \mid \overline{x}_i \in D,\ 1 \leq i \leq 3 \}$$

gives all triples of dual numbers. The elements of D^3 are called dual vectors and a dual vector can be represented by

$$\widetilde{x} = x + \varepsilon x^* = (x, x^*),$$

where x and x^* are the vectors of $\mathbb{R}^3$. Let us take $\widetilde{x} = x + \varepsilon x^*$, $\widetilde{y} = y + \varepsilon y^* \in D^3$, and $\overline{\lambda} = \lambda + \varepsilon \lambda^* \in D$. Then, the addition and multiplication operations on D^3 are as below:

$$\begin{aligned} \widetilde{x} + \widetilde{y} &= x + y + \varepsilon (x^* + y^*), \\ \overline{\lambda} \widetilde{x} &= \lambda x + \varepsilon (\lambda x^* + \lambda^* x). \end{aligned}$$

According to these operations, the set D^3 is a module over the ring D entitled by a $D-$module or dual space D^3 [11].

The set of dual vectors on D^n is represented by

$$D^n = \{ \widetilde{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n) \mid \overline{x}_i \in D,\ i = 1, ..., n \}.$$

These vectors can be given in the form $\widetilde{x} = x + \varepsilon x^* = (x, x^*)$, where x and x^* are the vectors of $\mathbb{R}^n$.

Let $\widetilde{x} = x + \varepsilon x^*$ and $\widetilde{y} = y + \varepsilon y^*$ be dual vectors of D^n, and let $\overline{\lambda} = \lambda + \varepsilon \lambda^*$ be a dual number. Then we define the following operations that make D^n a module called dual space D^n. These axioms are as follows:

$$\begin{aligned} \widetilde{x} + \widetilde{y} &= x + y + \varepsilon (x^* + y^*), \\ \overline{\lambda}\widetilde{x} &= \lambda x + \varepsilon (\lambda x^* + \lambda^* x). \end{aligned}$$

Formally, a vector space V over the field F together with a function

$$\langle , \rangle : V \times V \to F$$

is called an inner product space satisfying the following three axioms for $x, y, z \in V$ and $\lambda, \mu \in F$:

$i)$ Symmetric Property:

$$\langle x, y \rangle = \langle y, x \rangle .$$

$ii)$ Linearity:

$$\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$$

and

$$\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle .$$

$iii)$ Positive Definite Property:

$$\langle x, x \rangle \geq 0$$

and

$$\langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

A vector space V is normed vector space if there is a norm function that transforms V to non-negative real numbers, symbolized as $\|x\|$, for all vectors $x, y \in V$ and all scalars $\lambda \in F$, and satisfies the following conditions:

$i)$ $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

$ii)$ $\|\lambda x\| = |\lambda| . \|x\|$,

$iii)$ $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

In 1906, M. Frechet showed that given any non-empty set W, a distance function $d : W \times W \to \mathbb{R}$ may be described. The pair (W, d) is named a metric space, where W is a non-empty set and d is a real valued function on $W \times W$ called a metric that satisfies the following axioms for $x, y, z \in W$:

$d.1)$ $d(x, y) \geq 0$,

$d.2)$ $d(x, y) = 0$ if and only if $x = y$,

$d.3)$ $d(x, y) = d(y, x)$ (Symmetry property),

$d.4)$ $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

Let (W, d) be a metric space. In parallel to $\mathbb{R}^n$, the set

$$\mathbf{B}_d(a, r) = \{x \in W \mid d(a, x) < r\}$$

with $r > 0$ is named an open ball with centre a and radius r. Similarly, the set

$$\mathbf{S}_d(a, r) = \{x \in W \mid d(a, x) \leq r\}$$

with $r > 0$ is entitled as a closed ball with centre a and radius r. In the metric space (W, d), a subset $A \subseteq W$ is called an open set if and only if for all points $a \in A$, there is an $r > 0$, such that the open ball $\mathbf{B}_d(a, r)$ is a subset of A.

Lemma 2.1. *An open ball $\mathbf{B}_d(a, r)$ in a metric space (W, d) is open.*

Definition 2.1. If W is a set, a collection β of subsets of W is a basis for a topology on W, such that

(1) $\bigcup\limits_{\mathbf{B} \in \beta} \mathbf{B} = W$

(2) $\mathbf{B}_1 \cap \mathbf{B}_2 = \bigcup\limits_{\mathbf{B} \in \beta} \mathbf{B}$ for $\forall \mathbf{B}_1, \mathbf{B}_2 \in \beta$, where $\mathbf{B}_1 \cap \mathbf{B}_2 \neq \varnothing$.

Let $\beta = \{\mathbf{B}_d(a, r) \mid a \in W, r \in \mathbb{R}^+\}$. A collection β on W is a topological basis. Assume that the topology obtained from this basis β is symbolized as τ. This topology τ is defined as the metric topology reduced from the metric d on the set W ([2], [18] and [23]).

Assume that A and B are any two sets. An ordering for the Cartesian product $A \times B$ is determined as follows:
If not only (a_1, b_1) but also (a_2, b_2) are the elements of $A \times B$, we can write $(a_1, b_1) < (a_2, b_2)$ if and only if either

1) $a_1 <_A a_2$

or

2) if $a_1 = a_2$, $b_1 <_B b_2$,

where $<_A$and $<_B$ are order relations on any two sets A and B, respectively. Specifically, let two partially ordered sets A and B be given. The lexicographical order on the Cartesian product $A \times B$ is described as follows:

$$(a_1, b_1) \leq (a_2, b_2) \text{ if and only if } a_1 <_A a_2 \text{ or } (\text{if } a_1 = a_2,\ b_1 \leq_B b_2).$$

If $A = B = \mathbb{R}$ is taken into consideration,

$$\beta = \{(a_1 \times b_1, a_2 \times b_2) \mid a_1 < a_2 \text{ or } (\text{if } a_1 = a_2,\ b_1 < b_2)\}$$

is a basis of $\mathbb{R} \times \mathbb{R}$ with reference to order relation.

3. Inner Product, Norm and Metric on Dual Space

Theorem 3.1. *Let $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$ be dual numbers. In this case, the relation $<_D$ defined as*

$$\overline{x} <_D \overline{y} \Leftrightarrow x < y \text{ or if } x = y,\ x^* < y^*$$

is an order relation on D, where $<$ is the order relation on $\mathbb{R}$.

Proof. Let us take $\overline{x} = x + \varepsilon x^*$, $\overline{y} = y + \varepsilon y^*$, $\overline{z} = z + \varepsilon z^* \in D$ and

$$\overline{x} <_D \overline{y} \Leftrightarrow x < y \text{ or if } x = y,\ x^* < y^*. \tag{3.1}$$

In this case, by taking account of dual inequality (3.1), it is possible to write the below expressions:

i) if $\overline{x} \neq \overline{y}$, then either $\overline{x} <_D \overline{y}$ or $\overline{y} <_D \overline{x}$,

ii) if $\overline{x} <_D \overline{y}$, then $\overline{x} \neq \overline{y}$,

iii) if $\overline{x} <_D \overline{y}$ and $\overline{y} <_D \overline{z}$, then $\overline{x} <_D \overline{z}$.

Thus, the proof is completed. □

Furthermore, for $\overline{x} = x + \varepsilon x^*$, $\overline{y} = y + \varepsilon y^*$ and $\overline{z} = z + \varepsilon z^* \in D$, the relation $\leq_D$ defined as

$$\overline{x} \leq_D \overline{y} \Leftrightarrow x < y \text{ or if } x = y,\ x^* \leq y^*$$

provides the following expressions:

i) $\overline{x} \leq_D \overline{x}$,

ii) if $\overline{x} \leq_D \overline{y}$ and $\overline{y} \leq_D \overline{x}$, then $\overline{x} = \overline{y}$,

iii) if $\overline{x} \leq_D \overline{y}$ and $\overline{y} \leq_D \overline{z}$, then $\overline{x} \leq_D \overline{z}$,

where $\leq$ is the partial order relation on $\mathbb{R}$. This relation is called the partial order relation on D.

In this section, using the above defined order relations on D, we will reconsider the concepts of dual norm and metric on D^n obtained from the dual inner product.

A dual inner product on dual space D^n is a function

$$\begin{aligned} \langle , \rangle_D \quad &: \quad D^n \times D^n \to D, \\ \langle \widetilde{x}, \widetilde{y} \rangle_D \quad &= \quad \langle x, y \rangle + \varepsilon \left(\langle x, y^* \rangle + \langle x^*, y \rangle \right), \end{aligned} \tag{3.2}$$

where $\widetilde{x} = x + \varepsilon x^*$, $\widetilde{y} = y + \varepsilon y^* \in D^n$, and the notation $\langle , \rangle$ is an inner product on $\mathbb{R}^n$, such that for the dual vectors $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z} \in D^n$ and the dual numbers $\overline{\lambda} = \lambda + \varepsilon \lambda^*$, $\overline{\mu} = \mu + \varepsilon \mu^* \in D$,the following conditions exist:

i) $\langle \widetilde{x}, \widetilde{x} \rangle_D \geq_D 0$ and $\langle \widetilde{x}, \widetilde{x} \rangle_D = 0$ if $x = 0$.

ii) This inner product $\langle , \rangle_D$ satisfies symmetry property, i.e.,

$$\langle \widetilde{x}, \widetilde{y} \rangle_D = \langle \widetilde{y}, \widetilde{x} \rangle_D .$$

iii) This inner product $\langle , \rangle_D$ provides bilinear property, i.e.,

$$\left\langle \overline{\lambda}\widetilde{x} + \overline{\mu}\widetilde{y}, \widetilde{z} \right\rangle_D = \overline{\lambda} \left\langle \widetilde{x}, \widetilde{z} \right\rangle_D + \overline{\mu} \left\langle \widetilde{y}, \widetilde{z} \right\rangle_D ,$$

and

$$\left\langle \widetilde{x}, \overline{\lambda}\widetilde{y} + \overline{\mu}\widetilde{z} \right\rangle_D = \overline{\lambda} \left\langle \widetilde{x}, \widetilde{y} \right\rangle_D + \overline{\mu} \left\langle \widetilde{x}, \widetilde{z} \right\rangle_D .$$

A dual norm on dual space D^n is defined as follows:

$$\begin{aligned} \|.\|_D \quad &: \quad D^n \to D, \\ \|\widetilde{x}\|_D \quad &= \quad \begin{cases} 0 & , x = 0 \\ \|x\| + \varepsilon \frac{\langle x, x^* \rangle}{\|x\|} & , x \neq 0, \end{cases} \end{aligned}$$

where $\widetilde{x} = x + \varepsilon x^* \in D^n$ and $x,\ x^* \in \mathbb{R}^n$. The dual norm has the following three properties:

i) For $\widetilde{x} \in D^n$,

$$\|\widetilde{x}\|_D \geq_D 0.$$

ii) For $\widetilde{x} \in D^n$ and $\overline{\lambda} \in D$,

$$\left\| \overline{\lambda}\widetilde{x} \right\|_D = \left| \overline{\lambda} \right|_D \cdot \|\widetilde{x}\|_D .$$

iii) We can write the following dual inequalities:
If these conditions $x = 0,\ y \neq 0$ (or $x \neq 0,\ y = 0$) and $\langle x^*, y \rangle \geq 0$ (or $\langle x, y^* \rangle \geq 0$) are satisfied, there exists the below dual inequality

$$\|\widetilde{x} + \widetilde{y}\|_D \geq_D \|\widetilde{x}\|_D + \|\widetilde{y}\|_D .$$

In all other cases, it is possible to write the following expression

$$\|\widetilde{x} + \widetilde{y}\|_D \leq_D \|\widetilde{x}\|_D + \|\widetilde{y}\|_D .$$

Here, the third property is called the dual triangle inequality, and these properties are proved by using the definition of dual norm.

A dual distance on dual space D^n is a function

$$\begin{aligned} \overline{d} \quad &: \quad D^n \times D^n \to D, \\ \overline{d}\left(\widetilde{x}, \widetilde{y}\right) \quad &= \quad \|\widetilde{x} - \widetilde{y}\|_D = \begin{cases} 0 & , x = y \\ \|x - y\| + \varepsilon \frac{\langle x - y, x^* - y^* \rangle}{\|x - y\|} & , x \neq y \end{cases} \end{aligned}$$

that satisfies the following conditions for $\widetilde{x},\ \widetilde{y},\ \widetilde{z} \in D^n$:

i) $\overline{d}\left(\widetilde{x}, \widetilde{y}\right) \geq_D 0$, and $\overline{d}\left(\widetilde{x}, \widetilde{y}\right) = 0$ if $x = y$.

ii) If the function $\overline{d}$ is taken into consideration, the below symmetry property is satisfied:

$$\overline{d}\left(\widetilde{x}, \widetilde{y}\right) = \overline{d}\left(\widetilde{y}, \widetilde{x}\right).$$

iii) It is possible to say that the dual inequalities exist:
If these conditions $x-y=0,\ y-z\neq 0$ (or $x-y\neq 0,\ y-z=0$) and $\langle x^*-y^*, y-z\rangle \geq 0$ (or $\langle x-y, y^*-z^*\rangle \geq 0$) are satisfied, it is clear that

$$\overline{d}\left(\widetilde{x},\widetilde{z}\right) \geq_D \overline{d}\left(\widetilde{x},\widetilde{y}\right) + \overline{d}\left(\widetilde{y},\widetilde{z}\right).$$

In all other cases, the following dual inequality is written

$$\overline{d}\left(\widetilde{x},\widetilde{z}\right) \leq_D \overline{d}\left(\widetilde{x},\widetilde{y}\right) + \overline{d}\left(\widetilde{y},\widetilde{z}\right).$$

4. Basis and Topology on Dual Space

The aim in this section is to present how to introduce the concepts of basis and topology on dual space D^n. Then, the geometric modellings for the open balls of D and D^2 are shown in detail.

Let $\left(D^n, \overline{d}\right)$ be a dual metric space. Given a dual point $\widetilde{a} = a + \varepsilon a^* \in D^n$ and a dual constant $\overline{r} = r + \varepsilon r^*$, where $r > 0$, the sets

$$\begin{aligned}\widetilde{B}_{\overline{d}}(\widetilde{a},\overline{r}) &= \left\{\widetilde{x} = x+\varepsilon x^* \in D^n \mid \overline{d}\left(\widetilde{x},\widetilde{a}\right) <_D \overline{r},\ \overline{r} = r+\varepsilon r^*\right\} \\ &= \left\{\begin{array}{c}\widetilde{x} = x+\varepsilon x^* \in D^n \mid \|x-a\| < r \text{ or if } \|x-a\| = r \\ ,\ \frac{\langle x-a, x^*-a^*\rangle}{\|x-a\|} < r^*\end{array}\right\} \cup \{a+\varepsilon x^*\}\end{aligned}$$

and

$$\begin{aligned}\widetilde{S}_{\overline{d}}(\widetilde{a},\overline{r}) &= \left\{\widetilde{x} = x+\varepsilon x^* \in D^n \mid \overline{d}\left(\widetilde{x},\widetilde{a}\right) \leq_D \overline{r},\ \overline{r} = r+\varepsilon r^*\right\} \\ &= \left\{\begin{array}{c}\widetilde{x} = x+\varepsilon x^* \in D^n \mid \|x-a\| < r \text{ or if } \|x-a\| = r \\ ,\ \frac{\langle x-a, x^*-a^*\rangle}{\|x-a\|} \leq r^*\end{array}\right\} \cup \{a+\varepsilon x^*\}\end{aligned}$$

are called a dual open ball and dual closed ball with radius $\overline{r}$ and center $\widetilde{a}$, respectively.

Each dual distance function $\overline{d}$ on dual space D^n generates a topology $\widetilde{\tau}_{\overline{d}}$ on D^n, which has a basis on the family of dual open balls

$$\widetilde{\beta} = \left\{\widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right) \mid \widetilde{a} \in D^n \text{ and } \overline{r} = r+\varepsilon r^*\right\}.$$

Now, we will demonstrate that $\widetilde{\beta}$ is a basis on D^n:

i) Due to $\widetilde{a} \in \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right)$, it is possible to write $\{\widetilde{a}\} \subset \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right)$, and thus we obtain the following equality:

$$D^n = \bigcup_{\widetilde{a}\in D^n} \{\widetilde{a}\} \subset \bigcup_{\widetilde{B}\in\widetilde{\beta}} \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right) \subset D^n,$$

i.e.,

$$\bigcup_{\widetilde{B}\in\widetilde{\beta}} \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right) = D^n.$$

ii) For all open balls $\widetilde{B}_{1_{\overline{d}}} = \widetilde{B}_{1_{\overline{d}}}(\widetilde{a}_1, \overline{r}_1)$ and $\widetilde{B}_{2_{\overline{d}}} = \widetilde{B}_{2_{\overline{d}}}(\widetilde{a}_2, \overline{r}_2)$ except for $\widetilde{B}_{1_{\overline{d}}} \cap \widetilde{B}_{2_{\overline{d}}} = \phi$, we must show the existence of the following equality:

$$\widetilde{B}_{1_{\overline{d}}}(\widetilde{a}_1, \overline{r}_1) \cap \widetilde{B}_{2_{\overline{d}}}(\widetilde{a}_2, \overline{r}_2) = \bigcup_{\widetilde{B} \in \widetilde{\beta}} \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r}).$$

Assume that $\widetilde{y} \in \widetilde{B}_{1_{\overline{d}}} \cap \widetilde{B}_{2_{\overline{d}}}$. Since $\widetilde{y} \in D^n$ is the element of not only $\widetilde{B}_{1_{\overline{d}}}$ but also $\widetilde{B}_{2_{\overline{d}}}$, we can write $\overline{d}(\widetilde{y}, \widetilde{a}_1) <_D \overline{r}_1$ and $\overline{d}(\widetilde{y}, \widetilde{a}_2) <_D \overline{r}_2$ with respect to the definition of dual open ball, respectively. Thus, dual inequalities are obtained as follows:

$$\overline{d}(\widetilde{y}, \widetilde{a}_1) = \begin{cases} 0 & , y = a_1 \\ \|y - a_1\| + \varepsilon \frac{\langle y - a_1, y^* - a_1^* \rangle}{\|y - a_1\|} & , y \neq a_1 \end{cases} <_D \overline{r}_1 = r_1 + \varepsilon r_1^*$$

and

$$\overline{d}(\widetilde{y}, \widetilde{a}_2) = \begin{cases} 0 & , y = a_2 \\ \|y - a_2\| + \varepsilon \frac{\langle y - a_2, y^* - a_2^* \rangle}{\|y - a_2\|} & , y \neq a_2 \end{cases} <_D \overline{r}_2 = r_2 + \varepsilon r_2^*.$$

Situation 1. Consider that $y = a_1 = a_2$. This gives the below dual inequalities

$$\overline{d}(\widetilde{y}, \widetilde{a}_1) = 0 <_D r_1 + \varepsilon r_1^*$$

and

$$\overline{d}(\widetilde{y}, \widetilde{a}_2) = 0 <_D r_2 + \varepsilon r_2^*.$$

Taking $a_1 = a_2 = a$ and dualmin $\{\overline{r}_1, \overline{r}_2\} = \overline{r}$ in the above dual inequalities, we find $\overline{d}(\widetilde{y}, \widetilde{a}) <_D \overline{r}$, that is, $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r})$.

Situation 2. Let $y = a_1$ and $y \neq a_2$. We can write

$$\overline{d}(\widetilde{y}, \widetilde{a}_1) = 0 <_D \overline{r}_1$$

and

$$\overline{d}(\widetilde{y}, \widetilde{a}_2) <_D \overline{r}_2 \Leftrightarrow \|y - a_2\| < r_2 \text{ or if } \|y - a_2\| = r_2, \ \frac{\langle y - a_2, y^* - a_2^* \rangle}{\|y - a_2\|} < r_2^*.$$

When $y = a_1 = a$ is taken into consideration, we get $\overline{d}(\widetilde{y}, \widetilde{a}) <_D \overline{r}_1$, that is, $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r}_1)$. Now, let us think $y \neq a_2 = a$. In this case, we have

$$\begin{aligned} \overline{d}(\widetilde{y}, \widetilde{a}) &= \|y - a\| + \varepsilon \frac{\langle y - a, y^* - a^* \rangle}{\|y - a\|} \\ &= \|y - a_2\| + \varepsilon \frac{\langle y - a_2, y^* - a^* \rangle}{\|y - a_2\|}. \end{aligned}$$

By taking account of the inequality $\|y - a_2\| < r_2$, it is seen that $\overline{d}(\widetilde{y}, \widetilde{a}) <_D \overline{r}_2$, that is, $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r}_2)$. Assume that $\|y - a_2\| = r_2$. Thus, the following inequality is obtained

$$\frac{\langle y - a_2, y^* - a^* \rangle}{\|y - a_2\|} < r_2^* + \frac{\langle y - a_2, a_2^* - a^* \rangle}{\|y - a_2\|} = r_3^*. \tag{4.1}$$

If we use the inequality (4.1), the below expression can be obtained

$$\overline{d}\left(\widetilde{y},\widetilde{a}\right) <_D r_2+\varepsilon r_3^* = \overline{r}_3. \tag{4.2}$$

Therefore, the dual inequality (4.2) implies $\widetilde{y}\in \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}_3\right)$.

Situation 3. The proof for this situation is as in the situation 2.

Situation 4. Suppose that $y\neq a_1$ and $y\neq a_2$. Using the order relation on D, it is allowed to write

$$\left\|\widetilde{y}-\widetilde{a}_1\right\|_D <_D \ \overline{r}_1 \Leftrightarrow \left\|y-a_1\right\| < r_1 \text{ or if } \left\|y-a_1\right\| = r_1,\ \frac{\left\langle y-a_1, y^*-a_1^*\right\rangle}{\left\|y-a_1\right\|} < r_1^*$$

and

$$\left\|\widetilde{y}-\widetilde{a}_2\right\|_D <_D \ \overline{r}_2 \Leftrightarrow \left\|y-a_2\right\| < r_2 \text{ or if } \left\|y-a_2\right\| = r_2,\ \frac{\left\langle y-a_2, y^*-a_2^*\right\rangle}{\left\|y-a_2\right\|} < r_2^*.$$

In order to analyze this situation, four cases exist. These cases are as follows:

Situation 4.1. By considering $\left\|y-a_1\right\|<r_1$ and $\left\|y-a_2\right\|<r_2$, the following inequalities can be written

$$\left\|y-a\right\| \leq \left\|y-a_1\right\| + \left\|a_1-a\right\| < r_1 + \left\|a_1-a\right\| = r_{11} \tag{4.3}$$

and

$$\left\|y-a\right\| \leq \left\|y-a_2\right\| + \left\|a_2-a\right\| < r_2 + \left\|a_1-a\right\| = r_{22}. \tag{4.4}$$

From the inequalities (4.3) and (4.4), we deduce $\left\|y-a\right\|<r$, where $\min\left\{r_{11},r_{22}\right\}=r$. Thus, we can express the dual inequality

$$\left\|\widetilde{y}-\widetilde{a}\right\|_D = \left\|y-a\right\| + \varepsilon\frac{\left\langle y-a, y^*-a^*\right\rangle}{\left\|y-a\right\|} <_D r+\varepsilon r^* = \overline{r},$$

which implies that $\widetilde{y}\in \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right)$.

Situation 4.2. Let $\left\|y-a_1\right\|<r_1$ and if $\left\|y-a_2\right\|=r_2$, $\frac{\left\langle y-a_2,y^*-a_2^*\right\rangle}{\left\|y-a_2\right\|}<r_2^*$. In this case, we have

$$\left\|y-a\right\| \leq \left\|y-a_1\right\| + \left\|a_1-a\right\| < r_1 + \left\|a_1-a\right\| = r_{11}$$

and

$$\left\|y-a\right\| \leq \left\|y-a_2\right\| + \left\|a_2-a\right\| = r_2 + \left\|a_2-a\right\| = r_{22}.$$

If we consider the above inequalities, it is easy to see that $\left\|y-a\right\|<r$, where $\min\left\{r_{11},r_{22}\right\}=r$. This immediately implies $\overline{d}\left(\widetilde{y},\widetilde{a}\right)<_D \overline{r}$, that is, $\widetilde{y}\in \widetilde{B}_{\overline{d}}\left(\widetilde{a},\overline{r}\right)$.

Situation 4.3. The proof for this case is as in the situation 4.2.

Situation 4.4. Consider that

$$\text{if } \left\|y-a_1\right\| = r_1,\ \frac{\left\langle y-a_1, y^*-a_1^*\right\rangle}{\left\|y-a_1\right\|} < r_1^*$$

and

$$\text{if } \|y-a_2\| = r_2,\ \frac{\langle y-a_2, y^*-a_2^*\rangle}{\|y-a_2\|} < r_2^*.$$

Similarly, we can write the following inequalities:

$$\|y-a\| \leq \|y-a_1\| + \|a_1-a\| = r_1 + \|a_1-a\| = r_{11}$$

and

$$\|y-a\| \leq \|y-a_2\| + \|a_2-a\| = r_2 + \|a_2-a\| = r_{22}.$$

It is seen from the above inequalities that $\|y-a\| \leq r$, where $\min\{r_{11}, r_{22}\} = r$. Therefore, we have the expressions below:

$$\|y-a\| < r \text{ or } \|y-a\| = r\ .$$

By considering $\|y-a\| < r$, it is certain that $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r})$. Now, let us assume that $\|y-a\| = r$. The following inequalities can be written

$$\frac{\langle y-a, y^*-a^*\rangle}{\|y-a\|} < \frac{r_1 r_1^* + \langle y-a_1, a_1^*-a^*\rangle + \langle a_1-a, y^*-a^*\rangle}{\|y-a\|} = r_{11}^*$$

and

$$\frac{\langle y-a, y^*-a^*\rangle}{\|y-a\|} < \frac{r_2 r_2^* + \langle y-a_2, a_2^*-a_0^*\rangle + \langle a_2-a, y^*-a^*\rangle}{\|y-a\|} = r_{22}^*.$$

If we choose $r^* = \min\{r_{11}^*, r_{22}^*\}$, the below inequality can be written:

$$\frac{\langle y-a, y^*-a^*\rangle}{\|y-a\|} < r^*. \tag{4.5}$$

From the inequality (4.5) and $\|y-a\| = r$, we obtain $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r})$.

Consequently, considering four situations together, it is clear that

$$\widetilde{y} \in \underset{\widetilde{B}\in\widetilde{\beta}}{\cup} \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r})\,.$$

On the contrary, assume that $\widetilde{y} \in \underset{\widetilde{B}\in\widetilde{\beta}}{\cup} \widetilde{B}_{\overline{d}}(\widetilde{a}, \overline{r})$. Thus, there exist $\overline{r}_0 = r_0 + \varepsilon r_0^* \in D$ and $\widetilde{a}_0 = a_0 + \varepsilon a_0^* \in D^n$, such that it is possible to write $\widetilde{y} \in \widetilde{B}_{\overline{d}}(\widetilde{a}_0, \overline{r}_0)$, where $r_0 > 0$. In this case, we have the expression

$$\overline{d}(\widetilde{y}, \widetilde{a}_0) = \begin{cases} 0 & , y = a_0 \\ \|y-a_0\| + \varepsilon\frac{\langle y-a_0, y^*-a_0^*\rangle}{\|y-a_0\|} & , y \neq a_0 \end{cases} <_D r_0 + \varepsilon r_0^*. \tag{4.6}$$

Therefore, we need to study the following cases:

Case A: Let us consider that $y \neq a_0$. By using the order relation on D, we can write

$$\|y-a_0\| < r_0 \text{ or if } \|y-a_0\| = r_0,\ \frac{\langle y-a_0, y^*-a_0^*\rangle}{\|y-a_0\|} < r_0^*.$$

Case A.1. Assume that $\|y-a_0\| < r_0$. We have the following inequalities:

$$\|y-a_1\| \leq \|y-a_0\| + \|a_0-a_1\| < r_0 + \|a_0-a_1\| = r_1$$

and

$$\|y-a_2\| \leq \|y-a_0\| + \|a_0-a_2\| < r_0 + \|a_0-a_2\| = r_2.$$

If $y = a_1$ and $y = a_2$ are taken, then it is obvious that

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

If $y \neq a_1$ and $y \neq a_2$ are considered, due to

$$\|y-a_1\| < r_1$$

and

$$\|y-a_2\| < r_2,$$

it is clear that

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

We can express $\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}$ by means of similar calculations for the situations $y = a_2$, $y \neq a_1$ and $y = a_1$, $y \neq a_2$.

Case A.2. Suppose that if $\|y-a_0\| = r_0$, $\frac{\langle y-a_0, y^*-a_0^*\rangle}{\|y-a_0\|} < r_0^*$. Due to

$$\|y-a_1\| \leq \|y-a_0\| + \|a_0-a_1\| = r_0 + \|a_0-a_1\| = r_1$$

and

$$\|y-a_2\| \leq \|y-a_0\| + \|a_0-a_2\| = r_0 + \|a_0-a_2\| = r_2,$$

the following inequalities are obtained

$$\|y-a_1\| \leq r_1$$

and

$$\|y-a_2\| \leq r_2.$$

By taking account of the above inequalities, we have four cases:

Case A.2.1. Let us think that $\|y-a_1\| < r_1$ and $\|y-a_2\| < r_2$. Then we obtain

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case A.2.2. Let us consider that $\|y-a_1\| = r_1$ and $\|y-a_2\| < r_2$. From the second expression, we attain

$$\widetilde{y} \in \widetilde{B}_{2\overline{d}}. \tag{4.7}$$

On the other hand, the inequality

$$\frac{\langle y-a_1, y^*-a_1^*\rangle}{\|y-a_1\|} < \frac{r_0 r_0^* + \langle a_0-a_1, y^*-a_1^*\rangle + \langle y-a_0, a_0^*-a_1^*\rangle}{\|y-a_1\|} = r_1^* \tag{4.8}$$

can be written. Because of $\|y-a_1\| = r_1$ and the inequality (4.8), it is possible to write

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}}. \tag{4.9}$$

Thus, it is easy to see that

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case A.2.3. The proof for this case is as in the case A.2.2.

Case A.2.4. Assume that $\|y-a_1\| = r_1$ and $\|y-a_2\| = r_2$. In this case, the following inequalities are calculated

$$\frac{\langle y-a_1, y^*-a_1^*\rangle}{\|y-a_1\|} < \frac{r_0 r_0^* + \langle a_0-a_1, y^*-a_1^*\rangle + \langle y-a_0, a_0^*-a_1^*\rangle}{\|y-a_1\|} = r_1^*$$

and

$$\frac{\langle y-a_2, y^*-a_2^*\rangle}{\|y-a_2\|} < \frac{r_0 r_0^* + \langle a_0-a_2, y^*-a_2^*\rangle + \langle y-a_0, a_0^*-a_2^*\rangle}{\|y-a_2\|} = r_2^*.$$

If we take account of the above inequalities, these imply that

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case B: Let us assume that $y = a_0$. By revisiting the dual inequality (4.6), we have

$$\overline{d}\left(\widetilde{y}, \widetilde{a}_0\right) = 0 <_D r_0 + \varepsilon r_0^*.$$

Case B.1. If $a_1 = a_2 = a_0$ is considered, we obtain the following dual inequalities:

$$\overline{d}\left(\widetilde{y}, \widetilde{a}_1\right) = 0 <_D \overline{r}_1$$

and

$$\overline{d}\left(\widetilde{y}, \widetilde{a}_2\right) = 0 <_D \overline{r}_2.$$

Thus, we have

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case B.2. We can make similar computations for the situations $a_1 = a_0,\ a_2 \neq a_0$ and $a_2 = a_0$, $a_1 \neq a_0$.

Case B.3. Consider that $a_0 \neq a_1$ and $a_0 \neq a_2$. Because of $y = a_0$, the following equalities can be written:

$$\|y-a_1\| = \|a_0-a_1\|$$

and

$$\|y-a_2\| = \|a_0-a_2\|.$$

Case B.3.1. For $0 < \delta_1, \delta_2 < 1$, if we choose

$$r_1 = \|a_0-a_1\| + \delta_1$$

and

$$r_2 = \|a_0 - a_2\| + \delta_2,$$

it is seen that

$$\|y - a_1\| < r_1$$

and

$$\|y - a_2\| < r_2.$$

Thereby, the below expression is attained

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case B.3.2. Let us take $r_1 = \|a_0 - a_1\| + \delta_1$ and $r_2 = \|a_0 - a_2\|$. In this case, we have the following expressions:

$$\|y - a_1\| < r_1$$

and

$$\|y - a_2\| = r_2.$$

The first inequality implies that $\widetilde{y} \in \widetilde{B}_{1\overline{d}}$. On the other hand, we can write

$$\frac{\langle y - a_2, y^* - a_2^* \rangle}{\|y - a_2\|} = \|y^* - a_2^*\| \,.\cos\varphi,$$

where the angle φ is between the vectors $y - a_2$ and $y^* - a_2^*$. Since $-1 \leq \cos\varphi \leq 1$, we can find $r_2^* \in \mathbb{R}$, such that

$$\frac{\langle y - a_2, y^* - a_2^* \rangle}{\|y - a_2\|} < r_2^*. \tag{4.10}$$

From the inequality (4.10) and $\|y - a_2\| = r_2$, we deduce $\widetilde{y} \in \widetilde{B}_{2\overline{d}}$. Thus, considering the above statements, it is obvious that

$$\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}.$$

Case B.3.3. The proof for this case is as in the Case B.3.2.

Case B.3.4. Assume that $r_1 = \|a_0 - a_1\|$ and $r_2 = \|a_0 - a_2\|$. Therefore, there are the following inequalities:

$$\frac{\langle y - a_1, y^* - a_1^* \rangle}{\|y - a_1\|} < r_1^*$$

and

$$\frac{\langle y - a_2, y^* - a_2^* \rangle}{\|y - a_2\|} < r_2^*,$$

where r_1^*, $r_2^* \in \mathbb{R}$. Hence, $\widetilde{y} \in \widetilde{B}_{1\overline{d}} \cap \widetilde{B}_{2\overline{d}}$.

Consequently, since

$$\widetilde{B}_{1\overline{d}}(\widetilde{a}_1,\overline{r}_1)\cap\widetilde{B}_{2\overline{d}}(\widetilde{a}_2,\overline{r}_2)\subset\bigcup_{\widetilde{B}\in\widetilde{\beta}}\widetilde{B}_{\overline{d}}(\widetilde{a},\overline{r})$$

and

$$\bigcup_{\widetilde{B}\in\widetilde{\beta}}\widetilde{B}_{\overline{d}}(\widetilde{a},\overline{r})\subset\widetilde{B}_{1\overline{d}}(\widetilde{a}_1,\overline{r}_1)\cap\widetilde{B}_{2\overline{d}}(\widetilde{a}_2,\overline{r}_2),$$

we can write

$$\widetilde{B}_{1\overline{d}}(\widetilde{a}_1,\overline{r}_1)\cap\widetilde{B}_{2\overline{d}}(\widetilde{a}_2,\overline{r}_2)=\bigcup_{\widetilde{B}\in\widetilde{\beta}}\widetilde{B}_{\overline{d}}(\widetilde{a},\overline{r}).$$

In this case, from (i) and (ii), it is seen that $\widetilde{\beta}$ is a basis on D^n. Consider that the topology obtained from this basis $\widetilde{\beta}$ is indicated by $\widetilde{\tau}_{\overline{d}}$. The topology $\widetilde{\tau}_{\overline{d}}$ is called the dual metric topology reduced from the dual distance function $\overline{d}$ on the set D^n.

Now, the geometric modellings that correspond to the dual open balls on D and D^2 will be investigated. Firstly, on D, we study the geometric modelling for the dual open ball $\widetilde{B}(0,\overline{r})$ with center origin and dual radius $\overline{r}=r+\varepsilon r^*$, where $r>0$:

$$\begin{aligned}\widetilde{B}(0,\overline{r}) &= \left\{\widetilde{x}\in D \mid \overline{d}(\overline{x},0)<_D\overline{r}\right\}\\ &= \left\{\overline{x}=x+\varepsilon x^*\in D \mid |\overline{x}|_D<_D\overline{r}\right\}\\ &= \left\{\widetilde{x}=x+\varepsilon x^*\in D \mid |x|<r \text{ or if } |x|=r,\ \frac{x.x^*}{|x|}<r^*\right\}\cup\{0+\varepsilon x^*\}.\end{aligned}$$

The geometric modelling of the dual open ball $\widetilde{B}(0,\overline{r})$ is as follows:

i) Let us consider that $|x|<r$. In this case, we have $-r<x<r$.

ii) Let us take $|x|=r$. In this situation, we get

$$x=r\Rightarrow x^*<r^*$$

and

$$x=-r\Rightarrow x^*>-r^*.$$

Thus, if we take the situations (i) and (ii) together, the geometric modellings of the dual open balls are indicated by Fig.1. according to the situations of r^*.

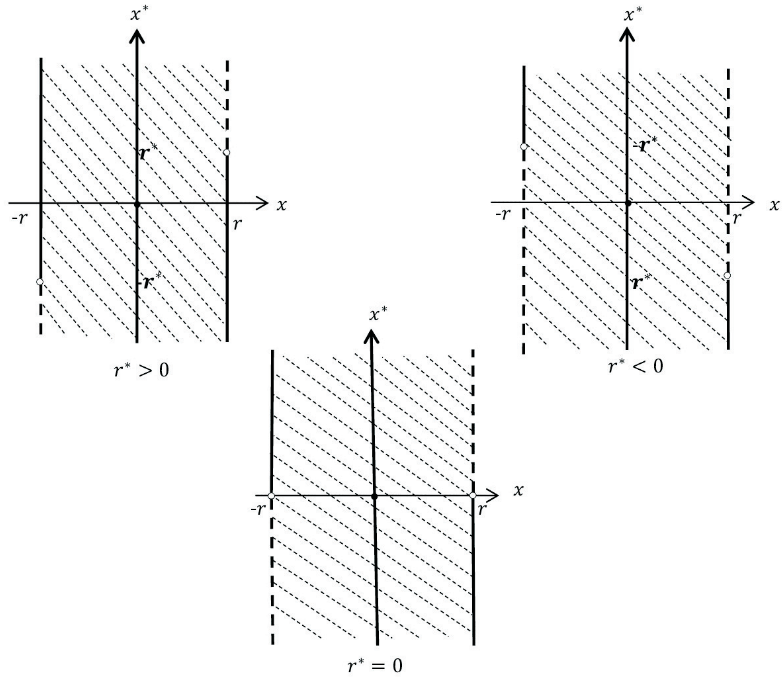

Fig. 1. *The geometric modellings in* $\mathbb{R}\times\mathbb{R}$*, for the dual open balls of D.*

Then, on D^2, we demonstrate the geometric structure corresponding to the dual open ball $\widetilde{B}(0,\overline{r})$ with center origin and dual radius $\overline{r} = r + \varepsilon r^*$, where $r > 0$:

$$\begin{aligned}\widetilde{B}(0,\overline{r}) &= \left\{\widetilde{x}\in D^2 \mid \overline{d}(\widetilde{x},0) <_D \overline{r}\right\}\\ &= \left\{\widetilde{x} = x+\varepsilon x^*\in D^2 \mid \|\widetilde{x}\|_D <_D \overline{r}\right\}\\ &= \left\{\widetilde{x} = x+\varepsilon x^*\in D^2 \mid \|x\| < r \text{ or if } \|x\| = r, \frac{\langle x, x^*\rangle}{\|x\|} < r^*\right\}\cup\{0+\varepsilon x^*\}.\end{aligned}$$

i) Let us take $\|x\| < r$. This inequality states the interior of the circle with radius r. x^* takes any value in $\mathbb{R}^2$.

ii) Assume that $\|x\| = r$. Thus, $\|x^*\|\cdot\cos\theta < r^*$ is obtained, where θ is the angle between the vectors x and x^*. To see the modellings of D^2 in $\mathbb{R}^3$, let us take $x^* = (x_1^*, 0)$. In this case, we can write

$$|x_1^*|\cdot\cos\theta < r^*. \tag{4.11}$$

Because $-1\leq\cos\theta\leq 1$, we have the following equality:

$$\cos\theta = \begin{cases} -\lambda^2 & , -1\leq\cos\theta<0\\ \lambda^2 & , 0<\cos\theta\leq 1\\ 0 & , \cos\theta = 0,\end{cases}$$

where $0 < \lambda \leq 1$. According to the situations of r^* and $\cos\theta$, we will investigate the geometric modellings of $\widetilde{B}(0, \overline{r})$:

Case 1. Let us consider $r^* > 0$. For $0 \neq \mu \in \mathbb{R}$, $r^* = \mu^2$ can be written. From the inequality (4.11), it is obvious that

$$|x_1^*| \cdot \cos\theta < \mu^2.$$

If $\cos\theta = \lambda^2$ is taken into consideration, the following inequality is written:

$$|x_1^*| < \frac{\mu^2}{\lambda^2}.$$

In this situation, the geometric modelling of $\widetilde{B}(0, \overline{r})$ is shown by Fig.2. If $\cos\theta = -\lambda^2$ is taken, it is possible to attain the below inequality:

$$|x_1^*| > -\frac{\mu^2}{\lambda^2}.$$

For $\forall x_1^* \in \mathbb{R}$, the above situation is provided. Geometric modelling of this situation is described by Fig.3. Also, if $\cos\theta = 0$ is taken into consideration, the following inequality is obtained

$$|x_1^*| \cdot 0 < \mu^2.$$

For $\forall x_1^* \in \mathbb{R}$, the above inequality can be written. The geometric modelling for this situation is as in Fig.3.

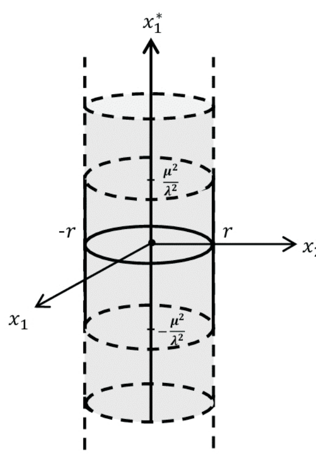

Fig. 2. $\left\{ \begin{array}{c} \textit{For the situation } (i) \textit{ and case } 1. \\ (\cos\theta = \lambda^2) \end{array} \right.$

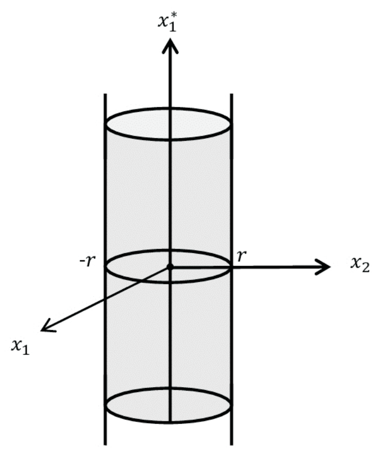

Fig. 3. $\left\{ \begin{array}{c} \textit{For the situation } (i) \textit{ and case } 1. \\ (\cos\theta = -\lambda^2 \textit{ or } \cos\theta = 0) \end{array} \right.$

Case 2. Suppose that $r^* < 0$. We can write $r^* = -\mu^2$, for $0 \neq \mu \in \mathbb{R}$. If we use the inequality (4.11), we have

$$|x_1^*| \cdot \cos\theta < -\mu^2.$$

This is only valid in the event of $\cos\theta = -\lambda^2$. In that case, the following inequality

$$|x_1^*| > \frac{\mu^2}{\lambda^2}$$

is obtained. Thus, the geometric modelling of this situation is represented by Fig.4.

Case 3. Finally, let us take $r^* = 0$. From the inequality (4.11), we have

$$|x_1^*| \cdot \cos\theta < 0.$$

Similarly, this is only possible in case of $\cos\theta = -\lambda^2$. So,

$$|x_1^*| > 0$$

is obtained and the geometric modelling of this situation is showed in Fig.5.

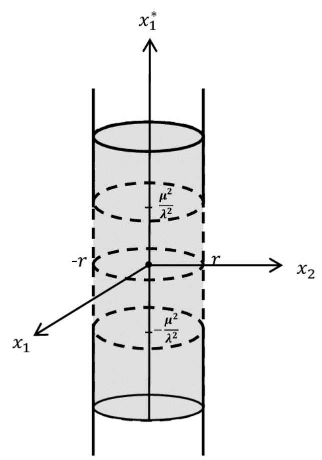

Fig.4. For the situation (i) and case 2.

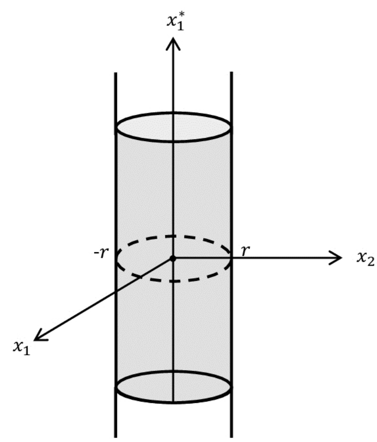

Fig.5. For the situation (i) and case 3.

Conclusion 4.1. *Throughout this paper, the order relations on D is introduced with reference to the lexicographical order relation on the Cartesian product. According to this order relation on D, the concepts of dual inner product, norm, and metric have been investigated. After that, by using the order relation, the notation of dual basis has been studied in detail and the geometric modelings for the open balls of D and D^2 have been given in the last section, respectively.*

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SOME RESULTS ON (ϵ)- KENMOTSU MANIFOLDS

Arpan Sardar

Abstract. We have studied curvature symmetries in (ϵ)-Kenmotsu manifolds. Next, we have proved the non-existence of a non-zero parallel 2-form in an (ϵ)-Kenmotsu manifold. Moreover, we have characterised ϕ-Ricci symmetric (ϵ)-Kenmotsu manifolds and finally, we have proved that under certain restriction on the scalar curvature $divR$=0 and $divC$=0 are equivalent, where 'div' denotes divergence.
Keywords: (ϵ)-Kenmotsu manifold, curvature symmetries, ϕ-Ricci symmetric manifold, Weyl curvature tensor.

1. Introduction

The basic difference between Riemannian and semi-Riemannian geometry is the existence of a null vector. In a Riemannian manifold (M, g), the signature of the metric tensor is positive definite, whereas the signature of a semi-Riemannian manifold is indefinite. With the help of indefinite metric Bejancu and Duggal [1] introduced (ϵ)-Sasakian manifolds. Then Xufeng and Xiaoli [16] proved that every (ϵ)-Sasakian manifold must be a real hyperface of some indefinite Kähler manifolds. Since Sasakian manifolds with indefinite metric have applications in Physics [4], we are interested to study various contact manifolds with indefinite metric. Geometry of Kenmotsu manifolds originated from Kenmotsu [10]. In [3] De and Sarkar introduced the notion of (ϵ)-Kenmotsu manifolds with indefinite metric. On the other hand, in [6] Eisenhart proved that if a Riemannian manifold admits a second order parallel syemmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. Later on, several authors investigated the Eisenhart problem on various spaces and obtained some fruitful results. Recently, Haseeb and De [7] have studied η-Ricci solitons in (ϵ)-Kenmotsu manifolds. (ϵ)-Kenmotsu manifolds have also been studied by several authors such as ([2],[8],[9],[13],[15]) and many others. So far, our knowledge about curvature symmetries have not been studied in semi-Riemannian manifolds. In this paper, we are going tol study curvature symmetries in (ϵ)-Kenmotsu manifolds. For curvature symmetries we refer

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the book of Duggal and Sharma [5].

In [7] Haseeb and De proved the following:

Theorem 1. Let M be an n-dimensional (ϵ)-Kenmotsu manifold. If the manifold has a symmetric parallel second order covariant tensor α, then α is a constant multiple of the metric tensor g.

Using the above theorem, we obtained the following statements.
Proposition 1.1. If a vector field X is an affine Killing in an (ϵ)-Kenmotsu manifold, then the vector field X is homothetic.
Proposition 1.2. An affine conformal vector field in an (ϵ)-Kenmotsu manifold is reduced to a conformal vector field.

Sharma[12] characterised a class of contact manifold admitting a vector field keeping the curvature tensor invariant.
In this paper, wel have considered the same problem in (ϵ)-Kenmotsu manifolds and proved the following:

Theorem 2. In an (ϵ)-Kenmotsu manifold a curvature collineation is Killing.

The nature of a parallel 2-form has been considered by several authors in contact manifolds. In the present paper we consider a parallel 2-form in the context of (ϵ)-Kenmotsu manifolds and prove the following:

Theorem 3. There is no non-zero parallel 2-form in an (ϵ)-Kenmotsu manifold.
As for example $d\eta$ is a 2-form in an (ϵ)-Kenmotsu manifold which is zero.
Next we prove:

Theorem 4. An (ϵ)-Kenmotsu manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold.

In a Riemannian or semi-Riemannian manifold of dimension n, $divR$ is obtained from the Bianchi identity and given by

$$(divR)(U,V)W = (\nabla_U S)(V,W) - (\nabla_V S)(U,W),$$

where R denotes the curvature tensor, S is the Ricci tensor, ∇ is the Riemannian connection and 'div' denotes the divergence.
Also it is known that

$$(divC)(U,V)W = \frac{n-2}{n-3}[\{(\nabla_U S)(V,W) - (\nabla_V S)(U,W)\} + \frac{1}{2(n-1)}\{dr(U)g(V,W) - dr(V)g(U,W)\}],$$

where C is the Weyl curvature tensor of type (1,3), r is the scalar curvature.

From the above definitions, it follows that $divR = 0$ implies $divC = 0$. However the converse, is not necessarily true. We address

Theorem 5. In an (ϵ)-Kenmotsu manifold $divR = 0$ and $divC = 0$ are equivalent provided the scalar curvature r is invariant under the characteristic vector field ξ.

2. (ϵ)-KENMOTSU MANIFOLDS

Duggal [4] introduced a larger class of contact metric manifolds.

Let M^{2n+1} be a (2n+1)-dimensional differentiable manifold of class C^{∞}. Then a quadruple (ϕ, ξ, η, g) defined on M^{2n+1} satisfying

$$\phi^2(U) = -U + \eta(U)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\xi, \xi) = \epsilon, \qquad \eta(U) = \epsilon\, g(U, \xi), \tag{2.2}$$

$$g(\phi U, \phi V) = g(U, V) - \epsilon\eta(U)\eta(V), \tag{2.3}$$

where ϕ is a tensor field of type (1,1) , η a tensor field of type (0,1), the Reeb vector field ξ and ϵ is 1 or -1 according as ξ is space like or time like vector field, is called an (ϵ)-almost contact metric manifold. If $d\eta(U, V) = g(U, \phi V)$, for every $U, V \in \chi(M)$, then we say that M is an (ϵ)-contact metric manifold. It can be easily seen that $\phi\xi = 0$, $\eta\phi = 0$.

Moreover, if the manifold satisfies

$$(\nabla_U \phi)V = -g(U, \phi V) - \epsilon\eta(V)\phi U, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g , then we shall call the manifold an (ϵ)-Kenmotsu manifold.

In an (ϵ)-Kenmotsu manifold the following relations hold([3],[7]) :

$$\nabla_U \xi = \epsilon(U - \eta(U)\xi), \tag{2.5}$$

$$(\nabla_U \eta)V = g(U, V) - \epsilon\eta(U)\eta(V), \tag{2.6}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.7}$$

$$(U, \xi) = -2n\eta(U). \tag{2.8}$$

Example. Let us consider $M^5 = \{(u_1, u_2, u_3, u_4, \mathrm{w}) : u_1, u_2, u_3, u_4, \mathrm{w}$ belongs to $\mathbb{R}$ and $\mathrm{w} \neq 0$ $\}$ and take the basis vector field $\{e_1, e_2, e_3, e_4, e_5\}$, where

$$e_1 = \mathrm{w}\frac{\partial}{\partial u_1},\ e_2 = \mathrm{w}\frac{\partial}{\partial u_2},\ e_3 = \mathrm{w}\frac{\partial}{\partial u_3},\ e_4 = \mathrm{w}\frac{\partial}{\partial u_4},\ e_5 = -\epsilon\ \mathrm{w}\frac{\partial}{\partial w} = \xi.$$

Let us define g as follows :

$$g(e_i, e_j) = 0, i \neq j, i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = 1, g(e_5, e_5) = \epsilon.$$

Then we obtain

$$[e_1,e_2] = [e_1,e_3] = [e_1,e_4] = [e_2,e_3] = [e_2,e_4] = [e_3,e_4] = 0,$$

$$[e_1,e_5] = \epsilon e_1,\ [e_2,e_5] = \epsilon e_2,\ [e_3,e_5] = \epsilon e_3, [e_4,e_5] = \epsilon e_4.$$

By Koszul's formula we have

$$\nabla_{e_1} e_1 = -e_5,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = \epsilon e_1,$$

$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_5,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = \epsilon e_2,$$

$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_5,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = \epsilon e_3,$$

$$\nabla_{e_4} e_1 = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_5,\ \nabla_{e_4} e_5 = \epsilon e_4,$$

$$\nabla_{e_5} e_1 = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

We can easily verify that $(M^5,\ \phi, \xi, \eta, g)$ satisfies all the properties of (ϵ)-Kenmotsu manifolds.

Definition 2.1. A vector field X is said to be an affine Killing vector field if it satisfies

$$\mathcal{L}_X \nabla = 0,$$

where $\mathcal{L}_X$ denotes the Lie differentiation along the vector field X.

Definition 2.2. A vector field X that leaves the Riemann curvature tensor invariant, that is,

$$(\mathcal{L}_X R)(U, V)W = 0$$

is called curvature collineation.

Definition 2.3. A conformal vector field X in a Riemannian or semi-Riemannian manifold (M, g) is defined by

$$\mathcal{L}_X g = 2\rho g, \tag{2.9}$$

for a smooth function ρ on M. If $\rho =$ constant, then the vector field X is called homothetic. If ρ vanishes identically, then X is Killing vector field.

Equation (2.9) yields

$$(\mathcal{L}_X \nabla)(U, V) = (U\rho)V + (V\rho)U - g(U, V)D\rho, \tag{2.10}$$

where $\nabla(U, V) = \nabla_U V$ for any vector field U, V on M and $D\rho$ is the gradient vector field of ρ.

Thus (2.9) implies (2.10), but not conversly.

The vector field X satisfying (2.10) is called conformal collineation and X is then called an affine conformal vector field.

Definition 2.4 An (ϵ)-Kenmotsu manifold is said to be ϕ-Ricci symmetric if

$$\phi^2((\nabla_U Q)W) = 0,$$

where Q is the Ricci operator defined by $g(QU, V) = S(U, V)$.

ϕ-Ricci symmetric manifold is weaker than Ricci symmetric ($\nabla S = 0$) manifold.

If U, W are orthogonal to the characteristic vector field ξ, then ϕ-Ricci symmetric manifold is called locally ϕ-Ricci symmetric. The notion of locally ϕ-symmetric for Sasakian manifolds was introduced by Takahashi[14].

3. PROOFS OF THE RESULTS

Proof of Proposition 1.1. If X is a affine Killing vector field, then

$$\mathcal{L}_X \nabla = 0,$$

which implies that

$$\mathcal{L}_X(\nabla g) = 0.$$

That is,

$$\nabla \mathcal{L}_X g = 0.$$

Thus $\mathcal{L}_X g$ is symmetric second order parallel tensor. Thus, from Theorem 1 we infer that

$$\mathcal{L}_X g = \lambda g,$$

where λ is constant. This implies X is homothetic.

Proof of Proposition 1.2. In [11] Sharma and Duggal prove that a vector field X on a manifold (M, g) is an affine conformal vector field if and only if

$$\mathcal{L}_X g = 2\rho g + K,$$

where K is a second order covariant constant ($\nabla K = 0$) symmetric tensor field. Hence from Theorem 1, we obtain $K = \lambda g$, λ is constant.
Therefore,

$$\mathcal{L}_X g = 2\rho g + \lambda g.$$

This implies

$$\mathcal{L}_X g = 2\sigma g,$$

where $2\sigma = 2\rho + \lambda$, a smooth function. This completes the proof.

Proof of Theorem 2. By definition of curvature collineation, we get

$$\mathcal{L}_X R)(U, V)W = 0, \tag{3.1}$$

which implies

$$(\mathcal{L}_X g)(R(Z, U)V, W) + (\mathcal{L}_X g)(R(Z, U)W, V) = 0. \tag{3.2}$$

Putting $Z = V = W = \xi$ in (3.2), we get

$$(\mathcal{L}_X g)(R(\xi, U)\xi, \xi) + (\mathcal{L}_X g)(R(\xi, U)\xi, \xi) = 0,$$

which implies

$$(\mathcal{L}_X g)(R(\xi, U)\xi, \xi) = 0.$$

Now, using (2.7) in the foregoing equation, we get

$$(\mathcal{L}_X g)(U, \xi) = \eta(U)(\mathcal{L}_X g)(\xi, \xi). \tag{3.3}$$

Again putting $Z = V = \xi$ in (3.2) it follows

$$(\mathcal{L}_X g)(R(\xi, U)\xi, W) + (\mathcal{L}_X g)(R(\xi, U)W, \xi) = 0.$$

Using (2.7) in the above equation we infer that

$$\begin{aligned}(\mathcal{L}_X g)(U,W) - \eta(U)(\mathcal{L}_X g)(\xi,W) + \eta(W)(\mathcal{L}_X g)(U,\xi)\\ -\epsilon(\mathcal{L}_X g)(\xi,\xi)g(U,W) = 0.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we get

$$(\mathcal{L}_X g)(U,W) = \epsilon(\mathcal{L}_X g)(\xi,\xi)g(U,W).$$

This implies

$$(\mathcal{L}_X g)(U,W) = \epsilon[\mathcal{L}_X g(\xi,\xi) - 2g(\xi,\mathcal{L}_X\xi)]g(U,W). \tag{3.5}$$

Since $(\mathcal{L}_X R)(U,V)W = 0$ implies $(\mathcal{L}_X S)(V,W) = 0$. Therefore,

$$(\mathcal{L}_X S)(\xi,\xi) = 0,$$

which implies

$$S(\xi,\mathcal{L}_X\xi) = 0.$$

That is,

$$g(Q\xi,\mathcal{L}_X\xi) = 0.$$

Now using (2.8) in the above equation, we obtain

$$g(\xi,\mathcal{L}_X\xi) = 0. \tag{3.6}$$

Using (3.6) in (3.5) we conclude that

$$(\mathcal{L}_X g)(U,W) = 0,$$

that is, X is Killing vector field. Therefore, the Theorem is proved.

Proof of Theorem 3. Let α be a parallel 2-form in an (ϵ)-Kenmotsu manifold. This means α is skew-symmetric and $\nabla\alpha = 0$.
Therefore

$$\alpha(U,V) = -\alpha(V,U). \tag{3.7}$$

Putting $U = V = \xi$ in (3.7) we get

$$\alpha(\xi,\xi) = 0. \tag{3.8}$$

Differentiating (3.8) along U, we get

$$\alpha(\nabla_U \xi, \xi) = 0.$$

Using (2.5) in the above gives

$$\epsilon\alpha(U, \xi) - \epsilon\eta(U)\alpha(\xi, \xi) = 0.$$

Finally, using (3.8), we obtain

$$\alpha(U, \xi) = 0. \tag{3.9}$$

Again, differentiating along V in the foregoing equation we get

$$\alpha(\nabla_V U, \xi) + \alpha(U, \nabla_V \xi) = 0. \tag{3.10}$$

Replacing U by $\nabla_V U$ in (3.9) we get

$$\alpha(\nabla_V U, \xi) = 0. \tag{3.11}$$

Using (3.11), (2.5) in (3.10) and after some calculation we obtain

$$\alpha(U, V) = 0,$$

that is, $\alpha = 0$. This completes the proof.

Proof of Theorem 4. Let M be an (2n+1)-dimensional ϕ-Ricci symmetric (ϵ)-Kenmotsu manifold. Then

$$\phi^2((\nabla_U Q)V) = 0,$$

for arbitary vector fields U, V, which implies

$$-(\nabla_U Q)V + \eta((\nabla_U Q)V)\xi = 0. \tag{3.12}$$

Putting $V = \xi$ in (3.12) and using (2.8), we get

$$2n\nabla_U \xi + Q(\nabla_U \xi) + \eta(-2n\nabla_U \xi - Q(\nabla_U \xi))\xi = 0. \tag{3.13}$$

Now using (2.5) in (3.13) and after some calculations, we obtain

$$S(U, V) = -2ng(U, V),$$

which implies that the manifold is an Einstein manifold.
Conversely, if the manifold is an Einstein manifold, then obviously it becomes ϕ-Ricci symmetric manifold. This completes the proof.

Proof of Theorem 5. Let us assume that $divC = 0$. Hence

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = \frac{1}{2(n-1)}[dr(U)g(V,W) - dr(V)g(U,W)]. \tag{3.14}$$

We know

$$S(U,\xi) = -2n\eta(U).$$

Then

$$(\nabla_U S)(V,\xi) = \nabla_U S(V,\xi) - S(\nabla_U V,\xi) - S(V,\nabla_U \xi).$$

Using (2.5) and (2.8) in the above equation, we get

$$(\nabla_U S)(V,\xi) - (\nabla_V S)(U,\xi) = -4nd\eta(U,V).$$

But in an (ϵ)-Kenmotsu manifold $d\eta = 0$,therefore, the above equation implies that

$$(\nabla_U S)(V,\xi) - (\nabla_V S)(U,\xi) = 0. \tag{3.15}$$

Substituting $W = \xi$ in (3.14) and using (3.15), we have

$$dr(U)\eta(V) - dr(V)\eta(U) = 0.$$

Replacing V by ξ in the above equation, it follows

$$dr(U) = dr(\xi)\eta(U). \tag{3.16}$$

Suppose the scalar curvature is invariant under the characteristic vector field ξ , that is,

$$\mathcal{L}_\xi r = 0,$$

which implies

$$\mathrm{dr}(\xi) = 0.$$

Hence (3.16) gives r = constant.
Therefore from (3.14) we get

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = 0,$$

which implies

$$(divR)(U,V)W = 0.$$

This completes the proof.

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