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[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

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Original Scientific Paper

NEW RESULTS ON SEMICLOSED LINEAR RELATIONS

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Abstract. This paper has triple main objectives. The first objective is an analysis of some auxiliary results on closedness and boundedness of linear relations. The second objective is to provide some new characterization results on semiclosed linear relations. Here it is shown that the class of semiclosed linear relations is invariant under finite and countable sums, products, and limits. We have obtained fundamental new results as well as a Kato Rellich Theorem for semiclosed linear relations and essentially interesting generalizations. The last objective deals with semiclosed linear relation with closed range, where we have particularly established new characterizations of closable linear relation.
Keywords: Semiclosed linear relation, Closable, Countable sums and products, Limits, Kato Rellich Theorem, Closed range.

1. Introduction

Let H be a complex Hilbert space with its scalar product and associated hilbertian norm denoted by $\langle .;. \rangle$ and $\|.\|$, respectively. A linear relation or multivalued linear operator T is a linear mapping with linear domain $\mathcal{D}(T) \subseteq H$, that assigns to each $x \in \mathcal{D}(T)$ a nonempty set $Tx = \{y : (x, y) \in G(T)\} \subset H$. If Tx never contains more then one element, then T is (single-valued) linear operator

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on H. Note that $G(T)$ is the graph of T and it is a subset of $H \times H$ defined by $G(T) = \{(x, y) \in H \times H : x \in \mathcal{D}(T), y \in Tx\}$. The range $\mathcal{R}(T)$ of T is defined as the union of all $Tx, x \in \mathcal{D}(T)$. The null space $\mathcal{N}(T)$ and the multivalued part $T(0)$ of the linear relation T are respectively defined by

$$\mathcal{N}(T) = \{x \in H : (x, 0) \in G(T)\} \text{ and } T(0) = \{y \in H : (0, y) \in G(T)\}.$$

If $\mathcal{N}(T) = \{0\}$ (resp. $\mathcal{R}(T) = H$), we say that T is injective (resp. surjective). If T is injective and surjective, we say that it is a bijection. Let $LR(H)$ denotes the space of all linear relations on H.

Proposition 1.1. *[3],[11] Let $T \in LR(H)$. Then:*

$$\begin{aligned} \mathcal{N}(T) \times \{0\} &= G(T) \cap (H \times \{0\}); \\ \{0\} \times T(0) &= G(T) \cap (\{0\} \times H); \\ H \times \mathcal{R}(T) &= G(T) + (H \times \{0\}); \\ \mathcal{D}(T) \times H &= G(T) + (\{0\} \times H). \end{aligned}$$

For every $T \in LR(H)$, there exists a relation $T^{-1} \in LR(H)$ called the formal inverse of T defined by $G(T^{-1}) = \{(y, x) : (x, y) \in G(T)\}$. Obviously,

$$\mathcal{D}(T^{-1}) = \mathcal{R}(T),\ \mathcal{R}(T^{-1}) = \mathcal{D}(T),\ \mathcal{N}(T^{-1}) = T(0) \text{ and } T^{-1}(0) = \mathcal{N}(T).$$

The adjoint T^* of T is defined by

$$G(T^*) = \{(y, x) : \langle v, y\rangle = \langle u, x\rangle \text{ for some } (u, v) \in G(T)\}.$$

If S and T are two relations in $LR(H)$, then the sum $S + T$ and the product ST are also relations in $LR(H)$ and they are respectively defined by:

$$\begin{aligned} G(S+T) &= \{(x, u+v) : (x, u) \in G(S) \text{ and } (x, v) \in G(T)\} \\ G(ST) &= \{(x, y) : (x, v) \in G(T) \text{ and } (v, y) \in G(S) \text{ for some } v \in H\}. \end{aligned}$$

The identity relation defined on a nonempty subset M of H will be denoted by I_M.

For all $T \in LR(H)$, let Q_T denote the natural quotient map from H onto $H/\overline{T(0)}$ where $\overline{T(0)}$ is the closure of $T(0)$. Note that the quotient map Q_T is used to extend the definition of the operator norm to the linear relations class. Clearly $T_s = Q_T T$ is a linear operator with $\mathcal{D}(T_s) = \mathcal{D}(T)$. T_s is called a linear operator part (or a single valued part) of T. For $x \in \mathcal{D}(T)$, $\|Tx\| = \|T_s x\|$ and the norm of T is defined by $\|T\| = \|T_s\|$. A relation T is said to be continuous if $\|T\| < \infty$. If T is continuous with $\mathcal{D}(T) = H$, then we say that T is bounded. Given two relations $S, T \in LR(H)$, we say that T is an extension of S if

$$T_{|\mathcal{D}(S)} = S.$$

Clearly, if T is an extension of S, then $G(S) \subset G(T)$. However, the converse is not true in general only if $T(0) = S(0)$.

One main reason why linear relations are more convenient than operators is that one can define the inverse, the closure, the conjugates and the completion for a linear relation without any additional condition on the relation. See for example [3] and [1] for interesting works on linear relations.

We investigate in this paper the notion of semiclosed linear relations on Hilbert and Banach spaces, also called paracomplete linear relations by Alvarez and Wilcox in [2]. Paracomplete subspaces in Banach spaces were studied in the papers [4], [5], [10] and others. The notion of a semiclosed, or almost closed or quotient, operator introduced in [6], [7], [8] and [12] can be naturally generalized to linear relations. The class of semiclosed linear relations is closed under addition, product, inversion, restriction, and limits. We give some interesting new characterizations of these relations and we obtain certain interesting generalizations of results on the closedness, boundedness, product and some of semiclosed linear relations. Finally we establish a certain number of results concerning the closedness of $\mathcal{R}(T)$ where T is a semiclosed linear relation by using Neubauer's Lemma. The structure of this work is as follows. Throughout Section 2, we give some auxiliary results on linear relations, sometimes purely algebraic and topological, which are required in the sequel. In section 3, we define and obtain several properties of semiclosed linear relations via the concept of selection or single valued part of a linear relation in Hilbert spaces. A linear relation with semiclosed multivalued part is semiclosed if and only if it has a semiclosed selection. We considered the case where a semiclosed linear relation is closed, closable or bounded. Restriction, inverse, adjoint, finite sum, product and iteration of semiclosed linear relations are also studied as well as a Kato Rellich Theorem for semiclosed linear relations. Finally, in Section 4, we investigate semiclosed linear relations with closed range which gives in particular a new characterization of closable linear relations.

2. Some auxiliary results on linear relations

We commence with a recollection of some preliminary properties required in the sequel.

A relation $T \in LR(H)$ is said to be closed if its graph is closed in $H \times H$. The closure of T is the relation $\overline{T} \in LR(H)$ defined by $G(\overline{T}) = \overline{G(T)}$. Hence, T is closed if $T = \overline{T}$.

Lemma 2.1. *[3] Let $T \in LR(H)$. Then, T is closed if and only if T_s is closed linear operator and $T(0)$ is a closed subspace of H.*

Let H_T denote the vector space $\mathcal{D}(T)$ endowed with the graph inner product $\langle .,. \rangle_T$ of T defined by

$$\langle x, y\rangle_T = \langle x, y\rangle_H + \langle Tx, Ty\rangle_H \quad \text{for } x, y \in \mathcal{D}(T).$$

Clearly, $H_T = H_{T_s}$, also H_T is norm isomorphic to $G(T)$ when T is a linear operator. Thus, we have:

Proposition 2.1. *Let T be a densely defined linear relation on H with $T(0)$ is closed, then T is closed if and only if H_T is complete.*

Proof. One only has to see that $H_T = H_{T_s}$ which is norm isomorphic to the closed graph $G(T_s)$ in $H \times H/T(0)$. □

Proposition 2.2. *If T is a closed relation. Then T is assimilable to a continuous relation from H_T into H.*

Indeed, let $i : H_T \hookrightarrow H$ be a linear operator defined by:

$$\mathcal{D}(i) = H_T \text{ and } i(x) = x \text{ for all } x \in H_T.$$

(i is an injection mapping from H_T onto H). Now we need to show that the relation Ti is of a finite norm:

$$\|Ti\| = \sup_{x \in H_T} \frac{\|(Ti)x\|}{\|x\|_T} = \sup_{x \in \mathcal{D}(T)} \frac{\|Tx\|}{\|x\| + \|Tx\|} = \begin{cases} \dfrac{\|T\|}{1 + \|T\|} & \text{if } \|T\| < +\infty \\ 1 & \text{if } \|T\| = +\infty \end{cases}$$

Corollary 2.1. *If T is continuous such that $\mathcal{D}(T)$ and $T(0)$ are closed, then T is closed.*

A linear relation T is said to be closable if $\overline{T}$ is an extension of T.

Lemma 2.2. *[3] Let $T \in LR(H)$. The following properties are equivalent:*

1. *T is closable;*
2. *$T(0) = \overline{T}(0)$;*
3. *T_s is closable and $T(0)$ is closed.*

Proposition 2.3. *If T is closable linear relation, then $\mathcal{D}(\overline{T}) = \overline{\mathcal{D}(T)}$ and T is continuous on $\mathcal{D}(\overline{T})$.*

Proof.

$$\overline{\mathcal{D}(T)} = \overline{\mathcal{D}(T_s)} = \mathcal{D}(\overline{T_s}) = \mathcal{D}((\overline{T})_s) = \mathcal{D}(\overline{T}).$$

Hence $\mathcal{D}(\overline{T})$ is closed and using the closed graph theorem for linear relations ([3] Theorem III.4.2) we obtain that $\overline{T}$ is continuous. □

3. Main results on semiclosed linear relations

3.1. Characterization of semiclosed linear relation

A linear subspace M of a Hilbert space H is called semiclosed if there exists a norm $\|.\|_M$ such that $(M, \|.\|_M)$ is complete and continuously embedded in H, i.e, $\|x\| \leq \lambda \|x\|_M$ for any $x \in M$.

In the two following theorems, we collect some well known characterizations and properties of semiclosed linear subspaces in a Hilbert space H.

Theorem 3.1. *[9] Let M be a linear subspace of H. The following statements are equivalent:*

1. *M is semiclosed subspace of H.*
2. *M is the range of a bounded operator on H.*
3. *M is the range of a closed operator on H.*
4. *M is the domain of a closed operator on H.*

Theorem 3.2. *[11] Let M, N be two linear subspaces of H. Then:*

1. *M and N are semiclosed subspaces of H if and only if $M \times N$ is a semiclosed subspace of $H \times H$.*
2. *If M and N are semiclosed subspaces of H, then $M + N$ and $M \cap N$ are also semiclosed subspaces of H.*
3. ***Neubauer's Lemma:** If M, N are semiclosed subspaces and both of $M + N$ and $M \cap N$ are closed, then M and N are closed in H.*

A semiclosed linear relation can also be characterized by means of semiclosed subspaces.

Definition 3.1. A linear relation $T \in LR(H)$ is said to be semiclosed on H if its graph $G(T)$ is semiclosed in $H \times H$.

Let $SC(H)$ denote the set of all semiclosed linear relations on H.

Corollary 3.1. *Let $T \in SC(H)$. Then, $\mathcal{D}(T), N(T), \mathcal{R}(T)$ and $T(0)$ are semiclosed sets in H.*

Proof. The proof follows immediately from the proposition 1.1 and the theorem 3.2. $\square$

A linear operator A is called a selection (or single valued part) of T if

$$T = A + T - T \text{ and } \mathcal{D}(A) = \mathcal{D}(T).$$

In particular, a linear operator is a selection of itself. The singlevalued part T_s of a linear relation T is a natural selection of T, nevertheless, T admits other selections.

Proposition 3.1. *[3] Let A be a selection of T. Then*

1. *$\mathcal{R}(T) = \mathcal{R}(A) + T(0)$. However, this sum may not always be direct.*
2. *$G(A) \cap (\{0\} \times T(0)) = \{0\} \times \{0\}$.*

3. $G(T) = G(A) + (\{0\} \times T(0))$.

One of the basic results of this paper is the following:

Theorem 3.3. *Let T be a linear relation with $T(0)$ semiclosed in H. Then, T is semiclosed linear relation if and only if T has a semiclosed selection.*

Proof. Let T be a semiclosed linear relation, then $T(0)$ is semiclosed in H. Let P be the linear projection defined on $\mathcal{R}(T)$ such that $\mathcal{N}(P) = T(0)$. Then we have in one hand,

$$PT(0) = \{0\}, \text{ i.e } PT \text{ is a linear operator satisfying } \mathcal{R}(PT) \cap T(0) = \{0\}.$$

In the other hand, we have for all $y \in Tx$:

$$Tx = y + T(0) = Py + (I - P)y + T(0) = PTx + T(0).$$

Hence, $T = PT + T - T$ and $G(T) = G(PT) + (\{0\} \times T(0))$. Thus $T = PT \oplus T(0)$, therefore PT is a semiclosed selection of T.

Conversely, let A be a semiclosed selection of T. Then $T = A + T - T$, where $T - T$ is a linear relation defined by:

$$G(T - T) = \{0\} \times T(0).$$

Since $G(T) = G(A) + (\{0\} \times T(0))$ we obtain, $G(T)$ is semiclosed in $H \times H$, hence T is semiclosed linear relation. $\square$

The Proposition 1.8 of [2] is now an immediate consequence of the Theorem 3.3, where the authors supposed that $T(0)$ is closed. Indeed, it is shown in [2] that if $T \in LR(H)$ with $T(0)$ closed, then T is semiclosed if and only if $T_s = Q_T T$ is semiclosed. The theorem 3.3 generalizes this situation where $T(0)$ is considered only semiclosed.
So, since $H_{T_s} = H_T$, combining the definition 2 in [12] and Proposition 1.8 of [2], we deduce the following characterization result which is in fact, a natural generalization of Theorem 4.2 of [13].

Proposition 3.2. *Let $T \in LR(H)$ with $T(0)$ closed. Then T is semiclosed if and only if there exists a inner product $(.,.)$ such that $H_T = (\mathcal{D}(T), (.,.))$ is complete, $H_T \hookrightarrow H$ and T is continuous from H_T to H. H_T is called the auxiliary Hilbert space of T.*

Similarly, if T is a linear relation on a Banach space E with closed multivalued part, then we say that T is semiclosed on E if and only if there exists a norm $\|.\|_T$ on $\mathcal{D}(T)$ such that $E_T = (\mathcal{D}(T), \|.\|_T)$ is a Banach space continuously embedded in E and T is continuous from E_T to E.

Some essential characterizations on semiclosedness of linear relations are given below.

Proposition 3.3. *Let $T \in SC(H)$ such that both of $\mathcal{D}(T)$ and $T(0)$ are closed, then T is bounded.*

Proof. We have from the theorem 3.3, that T_s is semiclosed linear operator with $\mathcal{D}(T_s) = \mathcal{D}(T)$. Thus, there exists an inner product $(.,.)$ on $\mathcal{D}(T)$ such that the Hilbert space $H_{T_s} = H_T = (\mathcal{D}(T), (.,.))$ is continuously embedded in H and T_s is bounded from H_{T_s} to H. Since $\mathcal{D}(T) = \mathcal{D}(T_s)$ is closed, we obtain $\mathcal{D}(T) = H$ and T_s is bounded on H. Hence, T is bounded linear relation with $T(0)$ closed. Consequently, T is bounded closed linear relation. □

Obviously, every closed linear relation is semiclosed. Nevertheless, there exists semiclosed linear relations which are not closed. Indeed, the fact that T is semiclosed linear relation prove that $T(0)$ is a semiclosed subset in H, however $T(0)$ is not necessarily closed. Consequently, T is not necessarily closed.

The following proposition gives an important case of semiclosed linear relations which are not closed on H, especially when $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are semiclosed subspaces but non closed.

Proposition 3.4. *Let $T \in SC(H)$, then $T^{-1}T$ and TT^{-1} are also semiclosed relations on H.*

Proof. The result follows immediately from the facts, $TT^{-1} = I_{\mathcal{R}(T)} + T(0)$ and $T^{-1}T = I_{\mathcal{D}(T)} + T^{-1}(0)$. □

It may be very important to note that there exists some closable linear relations which are not semiclosed and there exists some semiclosed linear relations which are not closable. Hence, one can confirm that there is no relation in terms of inclusion between the set of semiclosed linear relations and the set of closable linear relations. To clarify this situation, let us consider the two following original examples:

Example 3.1. The space $E = C([a,b])$ of continuous complex valued functions on $[a,b]$, equipped with the norm $\|x\|_\infty = \sup_{t\in[a,b]} |x(t)|$, $x \in E$, is a Banach space. Consider:

$$Tx = \int x(t)dt, \ x(t) \in E$$

with the polynomials $\mathcal{P}$ as its domain. T is a linear relation on E,

$$T(0) = \{y \in E : (0,y) \in G(T)\} = \mathbb{C}$$

where $G(T) = \{(x,y) \in E \times E : x \in D(T) = \mathcal{P}, \ y \in Tx\}$ is the graph of T. In particular, $T(0)$ is closed in E since on the complex constant polynomials the norm $\|.\|_\infty$ and the absolute value are equivalent. Furthermore,

$$T = T_s + T(0)$$

where the operator linear part T_s of T is given by:

$$T_s x(t) = \int_a^t x(t)dt$$

with domain $D(T_s) = D(T) = \mathcal{P}$, $T_s x$ is the primitive function of x which vanishes at the point $t = a$.
T is a closable linear relation on E since $T(0)$ is closed in E and T_s is closable on E. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(T)$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ are uniformly convergent to 0 and y respectively, then necessarily $y = 0$.

In fact, T is closable but not semiclosed linear relation on E, since T_s is a non-semiclosed linear operator on E.
Assume that T_s is semiclosed, then there exists a Banach space E_s such that the graph

$$G(T_s) = \{(x, T_s x) : x \in \mathcal{P},\ T_s x \in \mathcal{P}_0\},\ \mathcal{P}_0 = \{y \in \mathcal{P} : y(a) = 0\}$$

of T_s is closed in $E_s \times E$. Thus, $G(T_s)$ is a complete metric space. However, $G(T_s)$ is also the union of countably many finite-dimensional subspaces and is thus of first category. By Baire's theorem, complete metric spaces are of second category, which is a contradiction. Thus, the operator T_s with domain $\mathcal{P}$ is not semiclosed.

Example 3.2. Consider over the space $C([0,1])$ of all continuous functions on $[0,1]$ equipped with its usual norm, the linear operators T and S defined by: $T = \frac{d}{dx}$ with domain $\mathcal{D}(T) = C^1([0,1])$ and $Sf(x) = f(0)g(x)$ domain $\mathcal{D}(S) = C([0,1])$ where $g \neq 0$ is arbitrarily fixed in $C([0,1])$. Since T is closed and S is bounded, the product ST defined by $STf = \frac{df}{dx}(0)g(x)$ with domain $\mathcal{D}(ST) = \mathcal{D}(T)$ is a semiclosed linear operator on $C([0,1])$. Indeed, it is shown in [12] that the sum and the product of two semiclosed linear operators is also semiclosed. Now let $f_n(x) = -\frac{e^{-n}}{n}$. Then, for all $n \in \mathbb{N}^*$, $f_n \in \mathcal{D}(ST)$, for all $x \in [0,1], |f_n(x)|^2 = \frac{e^{-2n}}{n^2} \to 0$ and $|f_n(x)|^2 \leq 1$ with $1 \in L^1([0,1])$. Using the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_0^1 |f_n(x)|^2 dx = \int_0^1 \lim_{n\to+\infty} |f_n(x)|^2 dx = 0.$$

Hence, $(f_n)_n$ converge to 0 in $C([0,1])$. In other hand we have $STf_n = \frac{df_n}{dx}(0)g = g \neq 0$. Or, $(0, g)$ can not be in the graph of any linear operator, so ST is not closable.

3.2. Restriction, inverse and adjoint of semiclosed linear relations

Theorem 3.4. *Let $T \in SC(H)$. Then for all semiclosed subspace M of $\mathcal{D}(T)$, the restriction $T_{|M}$ of T to M is a semiclosed linear relation on H.*

Proof. Let $T \in SC(H)$, then $T(0)$ is semiclosed set in H and there exists a semiclosed selection A of T such that $T = A + T - T$.
Then we have: $T_{|M} = TI_M$, $T_{|M}(0) = T(0)$ and for all $x \in M$, $T_{|M}x = A_{|M}x + T(0)$ where $A_{|M}$ is the restriction of A to M. Hence, $G(T_{|M}) = G(A_{|M}) + (\{0\} \times T(0))$ is semiclosed subspace of $H \times H$ because $A_{|M}$ is semiclosed linear operator on H. This complete the proof. □

Proposition 3.5. *$T \in SC(H) \Leftrightarrow T^{-1} \in SC(H)$.*

Proof. Assume that $T \in SC(H)$ and let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be a linear operator defined on $H \times H$. Then J is a semiclosed operator on $H \times H$. Clearly, $J(G(T)) = G(T^{-1})$. Since T is supposed semiclosed, we obtain $J_{|G(T)}$ is semiclosed operator. Hence, $\mathcal{R}(J_{|G(T)}) = G(T^{-1})$ is a semiclosed subspace of $H \times H$. □

Corollary 3.2. *Let $T \in SC(H)$. The range and inverse range of any semiclosed subspace of H by T is semiclosed in H.*

Proposition 3.6. *Let $T \in SC(H)$, then $T^* \in SC(H)$.*

Proof. It follows immediately from the fact that $G(T^*) = [\mathcal{J}(G(T))]^{\perp}$ where $\mathcal{J} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. □

3.3. Finite sum, product and iteration of semiclosed linear relations

Theorem 3.5. *Let S, T be two semiclosed linear relations and $\alpha \in \mathbb{C}^*$. Then: $S+T$, ST and αT are semiclosed linear relations on H.*

Proof. Let A and B be two semiclosed selections of S and T respectively. Since $(S+T)(0) = S(0)+T(0)$ is semiclosed subset of H, it will be sufficient to show that $S+T$ has a semiclosed selection in order to prove that $S+T$ is semiclosed linear relation. Recall that the domain $\mathcal{D}_+$ of $S+T$ is $\mathcal{D}_+ = \mathcal{D}(T) \cap \mathcal{D}(S)$ and let $S_{|\mathcal{D}_+}$ and $T_{|\mathcal{D}_+}$ be respectively the restrictions of S and T to $\mathcal{D}_+$. Then we have, from the above proposition, for all $x \in \mathcal{D}_+$:

$$\begin{aligned} (S+T)x = S_{|\mathcal{D}_+}x + T_{|\mathcal{D}_+}x &= A_{|\mathcal{D}_+}x + S(0) + B_{|\mathcal{D}_+}x + T(0) \\ &= (A_{|\mathcal{D}_+} + B_{|\mathcal{D}_+})x + (S+T)(0). \end{aligned}$$

This implies that:

$$S+T = [A_{|\mathcal{D}_+} + B_{|\mathcal{D}_+}] + [(S+T) - (S+T)].$$

Thus, $A_{|\mathcal{D}_+} + B_{|\mathcal{D}_+}$ is a semiclosed selection of $S+T$. Hence, $S+T$ is semiclosed linear relation.

Let us denote by $\mathcal{D}_\times$ the domain of ST. Then, $\mathcal{D}_\times = T^{-1}(\mathcal{D}(S))$ and for all $x \in \mathcal{D}_\times$ we have:

$$STx = S(Tx) = ABx + ST(0).$$

Hence, AB is a semiclosed selection of ST because both of A and B are semiclosed operators. On the other hand, we have $ST(0) = S(T(0))$ is a semiclosed subset of H. Therefore, $ST \in SC(H)$.

□

This theorem provides the affirmative answer to the question formulated in [2] about the semiclosedness of product of two semiclosed linear relations and generalizes largely the Propositions 1.10 and 1.11 of [2].

Corollary 3.3.

1. *If S, T are closed relations, then $T+S$ and TS are semiclosed linear relations.*
2. *If T is a semiclosed relation such that $\mathcal{R}(T) \subset \mathcal{D}(T)$ and $n \in \mathbb{N}^*$, then T^n is also semiclosed relation.*
3. *The set of semiclosed linear relations is the smallest class closed under sum and product.*

3.4. Kato Rellich Theorem for semiclosed linear relations

In this paragraph, we give a new result about semiclosed linear relations which is a consequence of the Kato-Rellich theorem about relatively bounded (respectively relatively compact) linear operators. Before stating the theorem we shall make some definitions.

Definition 3.2. [3] Let $S, T \in LR(H)$. Then, S is said to be $T-$bounded if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exists a constant $c \geq 0$ such that

$$\|S(x)\| \leq c\,(\|x\| + \|T(x)\|) \text{ for all } x \in \mathcal{D}(T).$$

If S is $T-$bounded, then the inf of all numbers $b \geq 0$ for which a constant $a \geq 0$ exists such that

$$\|S(x)\| \leq a\|x\| + b\|T(x)\|,\ x \in \mathcal{D}(T),$$

is called the $T-$bound of S.

Theorem 3.6. *Let $S, T \in LR(H)$ such that $S(0) \subset T(0)$. If $T(0)$ is closed and S is $T-$bounded with $T-$bound less than 1, then*

$$S + T \in SC(H) \Leftrightarrow T \in SC(H).$$

Proof. We just have to note that $S(0) \subset T(0)$ implies that $(S+T)(0) = T(0)$ and then the theorem follows immediately from the Theorem 7 of [12] and the Theorem 3.6 of [13]. □

3.5. Limit and infinite sum of semiclosed linear relations

Let T_ε and S_n be two indexed collections of semiclosed linear relations on a Hilbert space H, with $\varepsilon > 0$ and $n \in \mathbb{N}$. Suppose that T_ε and S_n have the same multivalued part $\mathcal{T}(0)$ which is assumed to be closed and independent of ε and n and let H_ε and G_n be respectively the auxiliary Hilbert spaces of T_ε and S_n. Assume that there exists two Hilbert spaces K_1 and K_2 continuously embedded in E_ε and E_n for all $\varepsilon > 0,\ n \in \mathbb{N}$, respectively such that for all $x \in K_1,\ \sup_{\varepsilon>0} \|T_\varepsilon x\| < +\infty$ and for all $x \in K_2,\ \sup_N \|\sum_{n=0}^{N} S_n x\| < +\infty$ for every $N \in \mathbb{N}$. Then the following result holds.

Theorem 3.7. *If all of the above assumptions are satisfied, then:*

1. *the linear relation T defined by $Tx = \lim_{\varepsilon\to 0} T_\varepsilon x$ with the domain* $\mathcal{D}(T) = \left\{ x \in \left(\bigcap_{\varepsilon>0} \mathcal{D}(T_\varepsilon) \right) \cap K_1 : \lim_{\varepsilon\to 0} T_\varepsilon x \text{ exists in } H \right\}$ *is semiclosed on H,*

2. *the linear relation S defined by $Sx = \sum_{n=0}^{+\infty} S_n x$ with the domain* $\mathcal{D}(S) = \left\{ x \in \left(\bigcap_{n\in\mathbb{N}} \mathcal{D}(S_n) \right) \cap K_2 : \sum_{n=0}^{\infty} S_n x \text{ exists in } H \right\}$ *is semiclosed on H.*

Proof. 1. First note that $T(0) = \mathcal{T}(0)$ is closed and let us define on $\mathcal{D}(T)$ the following inner product:

$$\begin{aligned} (x,y) &= \langle x,y\rangle_{K_1} + \lim_{\varepsilon\to 0} \langle T_\varepsilon x, T_\varepsilon y\rangle_H \\ &= \langle x,y\rangle_{K_1} + \langle Tx, Ty\rangle_H \end{aligned}$$

and let $H_T = (\mathcal{D}(T), (.,.))$. Since K_1, H and H_ε are Hilbert spaces and T_ε is semiclosed for all $\varepsilon > 0$, then H_T is complete. In fact, let $(x_n)_n$ be a Cauchy sequence in H_T, then $(x_n)_n$ converges to x in K_1, H and H_ε, hence $x \in \left(\bigcap_{\varepsilon>0} \mathcal{D}(T_\varepsilon) \right) \cap K_1$ and from the semiclosedness of T_ε we obtain: $T_\varepsilon x_n$ converges to $T_\varepsilon x$ for all $\varepsilon > 0$. Since $(x_n)_n$ is a Cauchy sequence, there exists $\lambda > 0$ such that:

$$\|x_n\|_{H_T} = (x_n, x_n)^{1/2} < \lambda$$

and

$$\|x\|_{H_T}^2 = \lim_{n\to+\infty} \|x_n\|_{K_1}^2 + \lim_{\varepsilon\to 0} \lim_{n\to+\infty} \|T_\varepsilon x_n\|_H^2 < 2\lambda^2.$$

Hence, $x \in H_T$.

Let $\alpha > 0$. Then, by the assumption $\sup_{\varepsilon>0} \|T_\varepsilon x\|_H < +\infty$ on K_1 and the uniform boundedness principle, there exists $j \in \mathbb{N}$ such that for all $n, m \geq j$ and $\varepsilon > 0$,

$$\|x_n - x_m\|_{H_T} \leq \frac{\alpha}{2} \text{ and } \|T_\varepsilon x_n - T_\varepsilon x_m\|_{H_T} \leq \lambda \frac{\alpha}{2}$$

Moreover, we have

$$\begin{aligned}\|x_n - x\|_{H_T} &= \left[\lim_{m\to+\infty}\|x_n - x_m\|_{K_1}^2 + \lim_{\varepsilon\to 0}\lim_{m\to+\infty}\|T_\varepsilon x_n - T\varepsilon x_m\|_H^2\right]^{1/2} \\ &\leq \frac{\alpha}{2}(1+\lambda^2)^{1/2}.\end{aligned}$$

Consequently, H_T is a Hilbert space, continuously embedded in H and T is continuous from H_T onto H. Thus, we have from Corollary 3.2, T is semiclosed.

2. Let $Sx = \sum_{n=0}^{+\infty} S_n x$ with domain
$$\mathcal{D}(S) = \left\{ x \in \left(\bigcap_{n\in\mathbb{N}} \mathcal{D}(S_n)\right) \cap K_2 : \sum_{n=0}^{\infty} S_n x \text{ exists in } H \right\}.$$
Define $S_N = \sum_{n=0}^{N} S_n$ with domain $\mathcal{D}(S_N) = \left(\bigcap_{n=0}^{N} \mathcal{D}(S_n)\right) \cap K_2$. Then, S_N is semiclosed linear relation with closed multivalued part and auxiliary Hilbert space $H_{S_N} = (\mathcal{D}(S_N), (.,.)_{S_N})$ where
$$(x,y)_{S_N} = \langle x,y\rangle_{K_2} + \langle S_N x, S_N y\rangle \text{ for all } x,y \in \mathcal{D}(S_N).$$
Obviously, $Sx = \sum_{n=0}^{+\infty} S_n x = \lim_{N\to+\infty} S_N x$ and
$$D(S) = \left\{ x \in \left(\bigcap_{N\in\mathbb{N}} \mathcal{D}(S_N)\right) \cap K_2 : \lim_{N\to+\infty} S_N x \text{ exists in } H \right\}.$$
Hence, we have from the first assertion and the fact that $S(0) = \mathcal{T}(0)$, S is semiclosed linear relation on H.

□

4. Semiclosed linear relation with closed range

There are many important applications of the closedness of the range in the spectral study of differential operators and also in the context of perturbation theory, we have investigated in this section semiclosed linear relations with closed range.

Theorem 4.1. *Let $T \in SC(H)$. Then $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T) \oplus N$ is closed for some semiclosed subspace N in H.*

Proof. If $\mathcal{R}(T)$ is closed in H, it is then sufficient to choose $N = \{0\}$ to have the stated result.

Conversely, suppose that there exists an semiclosed subspace N of H such that $\mathcal{R}(T) \oplus N$ is closed in H. Since $T \in SC(H)$, then by virtue of the Corollary 3.1, $\mathcal{R}(T)$ is always a semiclosed subspace of H. Therefore, by the assertion 3 of the Theorem 3.2 $\mathcal{R}(T)$ is closed in H. □

In fact, semiclosed linear relations with closed null space and closed range in H are closed linear relations on H.

Theorem 4.2. *Let $T \in SC(H)$ such that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are closed in H, then T is a closed linear relation.*

Proof. Like $T \in SC(H)$, then the graph $G(T)$ of T is semiclosed in $H \times H$. Moreover, we have:

$$(H \times \{0\}) + G(T) = H \times \{0\} + \{0\} \times \mathcal{R}(T),$$
$$(H \times \{0\}) \cap G(T) = \mathcal{N}(T) \times \{0\}\,.$$

These two subspaces are closed in $H \times H$. Using the assertion 3 of Theorem 3.2, we deduce that $G(T)$ is closed in $H \times H$ and consequently T is a closed linear relation on H. □

Theorem 4.3. *Let $T \in SC(H)$ such that $\mathcal{R}(T)$ is closed in H. Then:*
$\overline{G(T)} = G(T) + \left(\overline{\mathcal{N}(T)} \times \{0\}\right)$.

Proof. $G(T)$ is semiclosed, $H \times \{0\}$ and $H \times \{0\} + G(T) = H \times \{0\} + \{0\} \times R(T)$ are closed subspaces of $H \times H$. Let's put

$$\begin{aligned} H_0 &= G(T) + \overline{G(T) \cap (H \times \{0\})} = G(T) + \overline{N(T) \times \{0\}} \\ &= G(T) + \overline{N(T)} \times \{0\}\,. \end{aligned}$$

H_0 is semiclosed in $H \times H$ and

$$H_0 + H \times \{0\} \subseteq H \times \{0\} + G(T) = H \times \{0\} + \{0\} \times R(T) \subseteq H_0 + H \times \{0\}\,.$$

Thus, $H_0 + H \times \{0\}$ is closed and by virtue of Neubauer's lemma we find that H_0 is in fact a closed subspace of $H \times H$. On the other hand,

$$G(T) \subseteq H_0 \subseteq \overline{G(T)},$$

so what $H_0 = \overline{G(T)}$. □

In the following, we will exploit the above result to give a new characterization of closable linear relations. Recall that a linear relation T is said to be closable if and only if $T(0)$ is closed and T_s is closable. Hence, if T is supposed semiclosed on H with $T(0)$ closed, then T_s is also semiclosed in H, in addition, if we assume that $\mathcal{R}(T_s)$ is closed we obtain from the above theorem:

$$\overline{G(T_s)} = G(T_s) + \left(\overline{\mathcal{N}(T_s)} \times \{0\}\right).$$

Theorem 4.4. *Let $T \in SC(H)$ such that $T(0)$ and $\mathcal{R}(T_s)$ are closed in H. Then, T is closable if and only if $\overline{\mathcal{N}(T)} \cap \mathcal{D}(T) = \mathcal{N}(T)$.*

Proof. Firstly, note that if $T(0)$ is closed, then $\mathcal{N}(T) = \mathcal{N}(T_s)$. Let T be closable (i.e T_s is closable) and $x \in \overline{\mathcal{N}(T)} \cap \mathcal{D}(T) = \overline{\mathcal{N}(T_s)} \cap \mathcal{D}(T_s)$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{N}(T_s)$ that converges to x in H. So, $(x - x_n) \to 0$ and $T_s(x - x_n) = T_s x \to T_s x$, from where $T_s x = 0$ and $x \in \mathcal{N}(T_s) = \mathcal{N}(T)$.

Conversely, let $(0, y) \in \overline{G(T_s)} = G(T_s) + \left(\overline{\mathcal{N}(T_s)} \times \{0\}\right)$. Then there is $x \in \mathcal{D}(T_s)$ and $t \in \overline{\mathcal{N}(T_s)}$ such that $x + t = 0$ and $T_s x = y$. Therefore, $x = -t \in \overline{\mathcal{N}(T_s)} \cap \mathcal{D}(T_s) = \mathcal{N}(T_s)$ and $y = T_s x = 0$. Which means that $\overline{G(T_s)}$ is the graph of a linear operator, ie T_s is closable on H. Hence T is closable linear relation □

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ITERATIVE COMPUTATION FOR SOLVING CONVEX OPTIMIZATION PROBLEMS OVER THE SET OF COMMON FIXED POINTS OF QUASI-NONEXPANSIVE AND DEMICONTRACTIVE MAPPINGS

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Abstract. In this paper, a new iterative method for solving convex minimization problems over the set of common fixed points of quasi-nonexpansive and demicontractive mappings is constructed. Convergence theorems are also proved in Hilbert spaces without any compactness assumption. As an application, we shall utilize our results to solve quadratic optimization problems involving bounded linear operator. Our theorems are significant improvements on several important recent results.
Keywords:Fixed point algorithm, Convex minimization problem, Quasi-nonexpansive mapping, Demicontractive mappings.

1. Introduction

Let H be a real Hilbert space, K be a nonempty subset of H. A map $T : K \to K$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \forall x, y \in K, \tag{1.1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in Fix(T)$. The

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mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

A mapping $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \forall x, y \in K.$$

A map T is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in K, \;\; p \in Fix(T).$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive $\subset$ k-strictly pseudo-contractive $\subset$ k-demi-contractive.

The function T in the following example is k-demi-contractive mapping but is not a k-strictly pseudo-contractive mapping.

Example 1.1. Let $H = \mathbb{R}$ and $K = [-1, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{2}{3}x\sin(\frac{1}{x}), & x \neq 0 \\ \\ 0 & x = 0. \end{cases} \tag{1.2}$$

Clearly $Fix(T) = \{0\}$. For $x \in K$, we have

$$\begin{aligned} |Tx - 0|^2 &= |\frac{2}{3}x\sin(\frac{1}{x})|^2 \\ &\leq |\frac{2}{3}x|^2 \\ &\leq |x|^2 \\ &\leq |x - 0|^2 + k|x - Tx|^2 \;\; \forall k \in [0, 1). \end{aligned}$$

Thus T is k demi-contractive for $k \in [0, 1)$. To see that T is not k strictly pseudo-contractive, choose $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty|^2 > |x - y|^2 + k|x - y - (Tx - Ty)|^2.$$

Hence, T is not k strictly pseudo-contractive mapping for $k \in [0, 1)$.

The function T in the following example is k-demi-contractive mapping but is not not quasi-nonexpansive.

Example 1.2. Let f be a real function defined by $f(x) = -x^2 - x$; it can be seen that $f : [-2, 1] \to [-2, 1]$. This function is demicontractive on $[-2, 1]$ and continuous. It is not quasi-nonexpansive and is not pseudocontractive on $[-2, 1]$ (check for instance the condition of pseudocontractivity for $x = -1.5$ and $y = -0.6$).

For several years, the study of fixed point theory for nonlinear mappings has attracted, and continues to attract the interest of several well known mathematicians (see, [9, 10, 13, 4]).

Interest in the study of fixed point theory for nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential Inclusions, Optimization theory.

Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K assigns to each $x \in H$ the unique $P_K x$ with the property

$$\|x - P_K x\| \leq \|y - x\|$$

for all $y \in K$. It is well known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2 \tag{1.3}$$

for all $y \in H$ and

$$\langle P_K z - y,\ z - P_K z\rangle \geq 0 \tag{1.4}$$

for all $z \in K$ and $y \in H$.

An operator $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay,\ x - y\rangle_H \geq 0, \quad \forall\ x, y \in K,$$

A is called k-*strongly monotone* if there exists $k \in (0,1)$ such that for each $x, y \in H$ such that

$$\langle Ax - Ay, x - y\rangle_H \geq k\|x - y\|^2.$$

An operator $A : H \to H$ is said to be *strongly positive bounded* if there exists a constant $c > 0$ such that

$$\langle Ax, x\rangle_H \geq c\|x\|^2, \quad \forall\ x \in H.$$

Remark 1.1. From the definion of A, we note that strongly positive bounded linear operator A is a $\|A\|$-Lipchitzian and c-strongly monotone operator.

Definition 1.1. Let H be a real Hilbert space. A function $g : H \to \mathbb{R}$ is said to be α-strongly convex if there exists $\alpha > 0$ such that for every $x, y \in H$ with $x \neq y$ and $\beta \in (0,1)$, the following inequality holds:

$$g(\beta x + (1-\beta)y) \leq \beta g(x) + (1-\beta)g(y) - \alpha\|x - y\|^2. \tag{1.5}$$

Lemma 1.1. *Let H be a real Hilbert space and $g : H \to \mathbb{R}$ a real-valued differentiable convex function. Assume that g is strongly convex. Then the differential map $\nabla g : H \to H$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle \nabla g(x) - \nabla g(y), x - y\rangle \geq k\|x-y\|^2 \ \forall\, x, y \in H. \tag{1.6}$$

Consider the following constrained optimization problem: Let H be a real Hilbert space. Given a convex objective function $g : H \to \mathbb{R}$ and $T : H \to H$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$, the problem can be expressed as

$$\begin{aligned} &Minimize\ g(x) \\ &subject\ \ to\ \ x \in Fix(T). \end{aligned} \tag{1.7}$$

Optimization problem for a convex objective function over the fixed points set of a nonexpansive mapping have been and will continue to be one of the central problems in nonlinear analysis and is one of the central issues in modern communication networks. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems. Recently, many iterative algorithms for solving these problems have been proposed, see [6, 2, 5, 11, 8] and the references therein.

Very recently, H. Iiduka [7] motivated by the fact that convex optimization problem for a strictly convex objective function over the fixed point set of a nonexpansive mapping includes a network bandwidth allocation problem, which is one of the central issues in modern communication networks, he proposed a fixed point optimization algorithm for solving Problem (1.7).

Algorithm 1.1. **Step 0.** *Choose $x_0 \in H$ arbitrarily, set $\lambda_0 \subset (0,1)$ $\alpha_0, \subset (0,1]$, and $d_0 = -\nabla g(x_0)$ arbitrarily and let $n := 0$.* **Step 1.** *Given $x_n \in H$ and $d_n \in H$, choose $\lambda_n \subset (0,1)$, $\alpha_n, \subset (0,1]$ and compute $x_{n+1} \in K$ as*

$$\begin{cases} y_n = T(x_n + \lambda_n d_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)y_n. \end{cases} \tag{1.8}$$

Step 2. *Choose $\beta_{n+1} \in (0,1]$ and compute the direction, $d_{n+1} \in H$, by*

$$d_{n+1} = -\nabla g(x_n) + \beta_{n+1} d_n.$$

Update $n := n+1$ and go to Step 1.

Under suitable conditions, he proved that $\{x_n\}_{n\in\mathbb{N}}$ in Algorithm 1.1 weakly converges to a unique solution to Problem (1.7).

Motivated by above results and the fact that the class of demicontractive mappings properly includes that of quasi-nonexpansive, strictly pseudocontractive mappings, we consider the following convex minimization problem : Let K be a nonempty, closed and convex subset a real Hilbert space H. Given a convex objective function $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function. Suppose the differential map $\nabla g : H \to H$ is L-Lipschitz. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. In the present paper, our main purpose is to solve the minimization problem:

$$\text{(1.9)} \qquad \textit{find } x^* \in \Gamma \ \textit{ such that } g(x^*) = \min_{x\in\Gamma} g(x).$$

We denote the set of solutions of Problem (1.9) by Ω.

2. Preliminaries

We start with the following demiclosedness principle for nonexpansive mappings.

Lemma 2.1. [1] *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at origin.*

Lemma 2.2. [3] *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda\|x - y\|^2, \ \ \lambda \in (0, 1).$$

Lemma 2.3. [12] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *(b)* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *[14] Let K be a nonempty, closed convex subset of a real Hilbert space H. Let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1, \ \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|, \ \forall\, x, y \in K.$$

Lemma 2.5. [9] *Assume K is a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a self-mapping of K. If T is a k-demicontractive mapping, then the fixed point set $Fix(T)$ is closed and convex.*

Lemma 2.6. *Let K be a nonempty, closed convex subset of a normed linear space E. Let $g : K \to \mathbb{R}$ a real valued differentiable convex function. Then x^* is a minimizer of g over K if and only if x^* solves the following variational inequality $\langle \nabla g(x^*), y - x^*\rangle \geq 0$ for all $y \in K$.*

Remark 2.1. By Lemma 2.6, $x^* \in \Omega$ if and only if x^* solves the following variational inequality problem :

$$\langle \nabla g(x^*), x^* - p\rangle \leq 0,\ \forall p \in \Gamma. \tag{2.1}$$

We denote the set of solutions of variational inequality problem (2.1) by $VI(\nabla g, \Gamma)$.

3. Main Results

In this section, we present our explicit iterative method for solving (1.9).

Lemma 3.1. *Let H be a real Hilbert space. Let K be a nonempty, closed convex subset of H and $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function. Suppose the differential map $\nabla g : K \to H$ is L-Lipschitz. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Then, $VI(\nabla g, \Gamma)$ is nonempty.*

Proof. Set η and τ two real numbers such that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Let t_0 be a fixed real number such that $t_0 \in \Big(0, min\{1,\ \frac{1}{\tau}\}\Big)$. We observe that $P_\Gamma(I - t_0\eta\nabla g)$ is a contraction. Indeed, for all $x, y \in K$, by Lemma 2.4, we have

$$\begin{aligned} \|P_\Gamma(I - t_0\eta\nabla g)x - P_\Gamma(I - t_0\eta\nabla g)y\| &\leq \|(I - t_0\eta\nabla g)x - (I - t_0\eta\nabla g)y\| \\ &\leq (1 - t_0\tau)\|x - y\|. \end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $P_\Gamma(I - t_0\eta\nabla g)$ has a unique fixed point, say $x_1 \in H$. That is, $x_1 = P_\Gamma(I - t_0\eta\nabla g)x_1$. Thus, in view of inequality (1.3), it is equivalent to the following variational inequality problem

$$\langle \nabla g(x_1), x_1 - p\rangle \leq 0,\ \ \forall p \in \Gamma.$$

By using Remark 2.1, we have $x_1 \in \Omega$. This completes this proof. □

We show the main result of this paper, that is, the strong convergence analysis for Algorithm 3.1.

Algorithm 3.1. **Step 0.** *Take $\{\alpha_n\} \subset (0, 1)$, $\{\theta_n\}, \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\eta > 0$ arbitrarily choose $x_0 \in K$; and let $n := 0$.*
Step 1. *Given $x_n \in K$, compute $x_{n+1} \in K$ as*

$$\begin{cases} z_n = \theta_n x_n + (1 - \theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1 - \beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n\nabla g)y_n, \end{cases} \tag{3.1}$$

Update $n := n + 1$ and go to Step 1.

Now we perform the convergence analysis for Algorithm 3.1.

Theorem 3.1. *Assume that* $I - T_1$ *and* $I - T_2$ *are demiclosed at origin. Suppose that* $\{\alpha_n\}$, $\{\theta_n\}$ *and* $\{\beta_n\}$ *are the sequences such that:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\theta_n \in]\lambda, 1[, \quad \lim_{n\to\infty} \inf(1-\theta_n)(\theta_n - \lambda) > 0$ *and* $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0.$

Assume that $0 < \eta < \frac{2k}{L^2}$, *then, the sequence* $\{x_n\}$ *defined by Algorithm 3.1 converges strongly to a unique solution of Problem (1.9).*

Proof. Firstly, we prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. From Lemma 3.1, we have $VI(\nabla g, \Gamma)$ is nonempty. In what follows, we denote x^* to be the unique solution of $VI(\nabla g, \Gamma)$. Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \frac{1}{\tau}\}\Big)$ where $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Let $p \in \Gamma$. By using (3.1) and Lemma 2.2, we have

$$\begin{aligned}\|z_n - p\|^2 &= \Big\|\theta_n(x_n - p) + (1-\theta_n)(T_1x_n - p)\Big\|^2 \\ &= \theta_n\|x_n - p\|^2 + (1-\theta_n)\|T_1x_n - p\|^2 \\ &\quad -\theta_n(1-\theta_n)\|T_1x_n - x_n\|^2.\end{aligned}$$

Using the fact that T_1 is λ-demi-contractive, we obtain

$$\begin{aligned}\|z_n - p\|^2 &\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)\Big(\|x_n - p\|^2 + \lambda\|Tx_n - x_n\|^2\Big) \\ &\quad -\theta_n(1-\theta_n)\|T_1x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 - (1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2.\end{aligned}$$

Since $\theta_n \in]\lambda, 1[$, we have,

$$\|z_n - p\Big\| \leq \|x_n - p\Big\|. \tag{3.2}$$

Hence, we obtain

$$\begin{aligned}\|y_n - p\| &= \|\beta_n z_n + (1-\beta_n)T_2z_n - p\| \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|T_2z_n - p\| \\ &\leq \|z_n - p\|.\end{aligned}$$

By using inequality (3.2), we have

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

From (3.1), Lemma 2.4 and inequality (3.3), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|P_K(I - \alpha_n\eta\nabla g)y_n - p\| \\ &\leq (1-\tau\alpha_n)\|x_n - p\| + \alpha_n\|\eta\nabla g(p)\| \\ &\leq \max\{\|x_n - p\|, \frac{\|\eta\nabla g(p)\|}{\tau}\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\eta\nabla g(p)\|}{\tau}\},\ \ n \ge 1.$$

Hence $\{x_n\}$ is bounded also are $\{y_n\}$ and $\{\nabla g(x_n)\}$.
Consequently, by Lemma 2.4 and inequality (3.2), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|(I - \eta\alpha_n\nabla g)(y_n - p) - \alpha_n\eta\nabla g(p))\|^2\\ &\le \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\tau\alpha_n)^2\|y_n - p\|^2\\ &\quad +2\alpha_n(1-\tau\alpha_n)\|\eta\nabla g(p)\|\|y_n - p\|\\ &\le \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\tau\alpha_n)^2\|x_n - p\|^2\\ &\quad -(1-\tau\alpha_n)^2(1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2\\ &\quad +2\alpha_n(1-\tau\alpha_n)\|\eta\nabla g(p)\|\|x_n - p\|.\end{aligned}$$

Thus,

$$\begin{aligned}(1-\tau\alpha_n)^2(1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\eta\nabla g(p)\|^2\\ &\quad +2\alpha_n(1-\tau\alpha_n)\|\eta\nabla g(p)\|\|x_n - p\|.\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $C > 0$ such that

$$(1-\tau\alpha_n)^2(1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n C. \tag{3.4}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\Big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big] = 0. \tag{3.5}$$

It then implies from (3.4) that

$$\lim_{n\to\infty}(1-\theta_n)(\theta_n - \lambda)\|T_1x_n - x_n\|^2 = 0. \tag{3.6}$$

Since $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n - \lambda) > 0$, we have

$$\lim_{n\to\infty}\Big\|x_n - T_1x_n\Big\| = 0. \tag{3.7}$$

Observing that,

$$\begin{aligned}\|z_n - x_n\| &= \|\theta_n x_n + (1-\theta_n)T_1x_n - x_n\|\\ &= \|\theta_n x_n + (1-\theta_n)T_1x_n - \theta_n x_n - (1-\theta_n)x_n\|\\ &= (1-\theta_n)\|T_1x_n - x_n\|\\ &\le \|T_1x_n - x_n\|.\end{aligned}$$

Therefore, from (3.7) we get that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.8}$$

Next, we prove that $\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle = \lim_{j\to+\infty}\langle \nabla g(x^*), x^* - x_{n_j})\rangle.$$

From (3.7) and $I - T_1$ is demiclosed, we obtain $a \in Fix(T_1)$. From Lemma 2.2, the fact that T_2 is nonexpansive and (3.3), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)T_2 z_n - p\|^2\\
&= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|T_2 z_n - p\|^2 - (1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2\\
&\le \|x_n - p\|^2 - (1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|(I - \alpha_n\eta\nabla g)y_n - p\|^2\\
&\le \|(I - \alpha_n\eta\nabla g)(y_n - p) - \alpha_n\eta\nabla g(p)\|^2\\
&\le \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\alpha_n\tau)^2\|y_n - p\|^2\\
&\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|y_n - p\|\\
&\le \alpha_n^2\|\eta\nabla g(p)\|^2 + (1-\alpha_n\tau)^2\|x_n - p\|^2\\
&\quad -(1-\alpha_n\tau)^2(1-\beta_n)\beta_n\|T_2 z_n - z_n\|^2\\
&\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|x_n - p\|.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(1-\alpha_n\tau)^2\beta_n(1-\beta_n)\|T_2 z_n - z_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\eta\nabla g(p)\|^2\\
&\quad +2\alpha_n(1-\alpha_n\tau)\|\eta\nabla g(p)\|\|x_n - p\|.
\end{aligned} \tag{3.9}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$(1-\alpha_n\tau)^2\beta_n(1-\beta_n)\|T_2 z_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n B. \tag{3.10}$$

It then implies from (3.10) and (3.5), that

$$\lim_{n\to\infty} \beta_n(1-\beta_n)\|T_2 z_n - z_n\| = 0. \tag{3.11}$$

Since $\lim_{n\to\infty}\inf \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} \|z_n - T_2 z_n\| = 0. \tag{3.12}$$

Since $z_{n_j} \rightharpoonup a$, it follows from (3.12) and Lemma 2.1, we have $a \in Fix(T_2)$. Therefore, $a \in \Gamma$. On the other hand, by using x^* solves $VI(\nabla g, \Gamma)$, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle \nabla g(x^*), x^* - x_n\rangle &= \lim_{j\to+\infty}\langle \nabla g(x^*), x^* - x_{n_j}\rangle \\ &= \langle \nabla g(x^*), x^* - a\rangle \le 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$.

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \langle (I - \eta\alpha_n\nabla g)y_n - x^*, x_{n+1} - x^*\rangle \\ &\le \langle (I - \eta\alpha_n\nabla g)y_n - x^* + \eta\alpha_n\nabla g(x^*) - \eta\alpha_n\nabla g(x^*), x_{n+1} - x^*\rangle \\ &\le \|(I - \alpha_n\eta\nabla g)(y_n - x^*)\|\|x_{n+1} - x^*\| + \alpha_n\langle \eta\nabla g(x^*), x^* - x_{n+1}\rangle \\ &\le (1 - \alpha_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\eta\langle \nabla g(x^*), x^* - x_{n+1}\rangle.\end{aligned}$$

From Lemma 2.3, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ B_k \le B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ *as* $n \to \infty$ and $B_{\tau(n)} \le B_{\tau(n)+1}$ for $n \ge n_0$. From (3.4), we have

$$(1 - \theta_{\tau(n)})(\theta_{\tau(n)} - \lambda)\left\|x_{\tau(n)} - T_1x_{\tau(n)}\right\|^2 \le \alpha_{\tau(n)}C.$$

Since $\theta_{\tau(n)} \in]\lambda, 1[$, we have

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - T_1x_{\tau(n)}\right\|^2 = 0. \tag{3.13}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in H and $\limsup_{\tau(n)\to+\infty}\langle \nabla gx^*, x^* - x_{\tau(n)})\rangle \le 0$. We have for all $n \ge n_0$,

$$0 \le \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \le \alpha_{\tau(n)}[-\tau\|x_{\tau(n)} - x^*\|^2 + 2\eta\langle \nabla gx^*, x^* - x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \le \frac{2\eta}{\tau}\langle \nabla gx^*, x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty}B_{\tau(n)} = \lim_{n\to\infty}B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \ge n_0$, we have $B_{\tau(n)} \le B_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \le j \le n$. As consequence, we have for all $n \ge n_0$,

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty}B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.1 for solving convex optimization problems over the set of common fixed point of two nonexpansive mappings without demiclosedness assumption.

Theorem 3.2. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $g : K \to \mathbb{R}$ be a differentiable, k-strongly convex real-valued function and suppose the differential map $\nabla g : K \to H$ is L-Lipschitz. Let $T_1 : K \to K$ and $T_2 : K \to K$ two nonexpansive mappings such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $0 < \eta < \frac{2k}{L^2}$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$(3.14) \qquad \begin{cases} z_n = \theta_n x_n + (1-\theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1-\beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n \nabla g)y_n, \end{cases}$$

Suppose that $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ are the sequences such that:
(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\theta_n \in]0,1[$, *and* $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$. *Then, the sequence $\{x_n\}$ defined by (3.14) converges strongly to a minimizer of g over $Fix(T_1) \cap Fix(T_2)$.*

Proof. Since every nonexpansive mapping is quasi-nonexpansive and 0-demicontractive. The proof follows Lemma 2.1 and Theorem 3.1. □

We apply Theorem 3.1 for solving the following quadratic optimization problem:

$$(3.15) \qquad \text{find } x^* \in \Gamma \text{ such that } g(x^*) = \min_{x\in\Gamma} g(x), \text{ where } g(x) = \frac{1}{2}\langle Ax, x\rangle.$$

Theorem 3.3. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be strongly bounded linear operator with coefficient $k > 0$. Let $T_1 : K \to K$ be a λ-demicontractive mapping and $T_2 : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Assume that $0 < \eta < \frac{2k}{\|A\|^2}$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$(3.16) \qquad \begin{cases} z_n = \theta_n x_n + (1-\theta_n)T_1 x_n, \\ y_n = \beta_n z_n + (1-\beta_n)T_2 z_n, \\ x_{n+1} = P_K(I - \eta\alpha_n A)y_n, \end{cases}$$

Assume that $I - T_1$ and $I - T_2$ are demiclosed at origin. Suppose that $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\beta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\theta_n \in]\lambda, 1[,$ *and* $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$. *Then, the sequence* $\{x_n\}$ *defined by (3.16) converges strongly to a solution of problem (3.15).*

Proof. The proof follows Theorem 3.1 and Remark 1.1 with $\nabla gx = Ax$. □

Now, we give some remarks on our results as follows:

(1) Our results improve many recent results using fixed point optimization algorithm to approximate minimizers of convex functions over the set of common fixed points of nonlinear mappings.

(2) Our results are applicable for solving variational inequality problems involving strongly monotone and Lipschitzian operator and fixed point problems involving quasi-nonexpansive and demicontractive mappings.

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REGULAR FRACTIONAL DIRAC TYPE SYSTEMS

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Abstract. In this paper, we study one dimensional fractional Dirac type systems which include the right-sided Caputo and the left-sided Riemann-Liouvile fractional derivatives of the same order $\alpha, \alpha \in (0,1)$. We investigate the properties of the eigenvalues and the eigenfunctions of this system.
Keywords: Fractional Dirac system, Riemann–Liouville and Caputo derivatives

1. Introduction

It is well known that classical calculus is based on integer order differentiation and integration. Fractional calculus generalizes integrals and derivatives to non-integer orders. The subject has a long history. Since 1695, many mathematicians, among them Liouville, Riemann, Leibniz, Grunwald, Letnikov Riesz and Caputo, have studied this subject. Fractional calculus has important applications to many real-world phenomena studies in engineering, chemistry, mechanics, physics, finance, etc. There is an extensive literature on this subject, see for example [9, 10, 16, 17, 19, 20, 22, 23, 24] and references therein.

Recently, the study of boundary value problems for fractional Sturm-Liouville equations recently has attracted a great deal of attention from many researchers. In

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[4], the authors investigated some basic spectral properties of the fractional Sturm-Liouville problem with Generalized Dirichlet conditions. They proved that this problem has an infinite sequence of real eigenvalues and the corresponding eigenfunctions form a complete orthonormal system in the Hilbert space $L_2[a,b]$. In [11], the authors studied the properties of the eigenfunctions and the eigenvalues of the regular Generalized Fractional Sturm-Liouville Problem. In [6], the authors studied the fractional Sturm–Liouville problem associated with the Weber fractional derivative of order α. In [15], the authors proved existence of strong solutions for the space–time fractional diffusion equations. Using the method of separating variables, they solved several types of fractional diffusion equations. Klimek et al. [13] studied to the regular fractional Sturm–Liouville eigenvalue problem. By applying the methods of fractional variational analysis, they proved the existence of a countable set of orthogonal solutions and corresponding eigenvalues. Klimek and Argawal [12] defined some fractional Sturm–Liouville operators and introduced two classes of fractional Sturm–Liouville problems namely regular and singular fractional Sturm–Liouville problems. They investigated the eigenvalue and eigenfunction properties of this classes. Baş [2] gave the theory of spectral properties for eigenvalues and eigenfunctions of Bessel type of fractional singular Sturm-Liouville problem. Baş and Metin [3] studied a fractional singular Sturm-Liouville operator having Coulomb potential of type. Klimek and Blasik [14] studied a regular fractional Sturm-Liouville problem with left and right Liouville-Caputo derivatives of order in the range $(1/2, 1)$. They proved that it has an infinite countable set of positive eigenvalues and its continuous eigenvectors form a basis in the space of square-integrable functions. Rivero et al. [21] studied some of the basic properties of the fractional version of the Sturm-Liouville problem. Zayernouri and Karniadakis [27] studied new classes of the regular and singular fractional Sturm–Liouville Problems and obtained some explicit forms of the eigenfunctions.

While the theory of fractional Sturm-Liouville equations is well developed, the literature involving fractional Dirac system is scarce. In [7], Ferreira and Vieira derived fundamental solutions for the fractional Dirac operator which factorizes the fractional Laplace operator. In [8], the authors obtained eigenfunctions and fundamental solutions for the three parameter fractional Laplace operator defined via fractional Liouville-Caputo derivatives. They also obtained a family of fundamental solutions of the corresponding fractional Dirac operator. In [5], the author proved Lieb–Thirring type bounds for fractional Schrödinger operators and Dirac operators with complex-valued potentials. In [1], the authors studied a regular q-fractional Dirac type system. In the present paper, we consider the fractional Dirac type system defined by

$$\begin{pmatrix} 0 & {}^{C}D_{b-}^{\alpha} \\ D_{a+}^{\alpha} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} \omega_1 y_1 \\ \omega_2 y_2 \end{pmatrix}$$

where p, r, ω_1 and ω_2 are real-valued continuous functions defined on $[a,b]$ and $\omega_i(x) > 0,\ \forall x \in [a,b],\ (i = 1,2),\ \lambda$ is a complex spectral parameter. If we take $\alpha \to 1$ in this system, then we get the one dimensional Dirac type system. This system is one of the basic models of one-dimensional quantum mechanics. For example, a relativistic electron in the electrostatic field $\Omega(x)$ is described by the system

$$\left(\begin{array}{cc} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{array}\right) f(x) + \left(\begin{array}{cc} \Omega(x) - \frac{mc}{h} & kx^{-1} \\ kx^{-1} & \Omega(x) + \frac{mc}{h} \end{array}\right) f(x) = \frac{\lambda}{hc} f(x) \tag{1.1}$$

where $c > 0$ is the velocity of light, $k \in \mathbb{Z} \backslash \{0\}$, $\Omega(x)$ is a spherically symmetric potential, $m > 0$ is the mass of the particle ([26]). Basic properties of the one dimensional Dirac systems have been considered in [18], [26], [25] and the references therein.

2. Preliminaries

In this section, we provide some basic definitions and properties of the fractional calculus theory. These concepts and properties can be found in [20],[16],[22],[10], and references therein.

Definition 2.1. (see [20]) Let $0 < \alpha \leq 1$ and $f \in L_1(a,b)$. The right-sided and left-sided Riemann-Liouville integrals of order α are given by the formulas, respectively

$$\left(I_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(s)(s-x)^{\alpha-1} ds, \quad x < b, \tag{2.1}$$

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(s)(x-s)^{\alpha-1} ds, \quad x > a, \tag{2.2}$$

where Γ denotes the gamma function.

Definition 2.2. (see [20]) Let $0 < \alpha \leq 1$ and $f \in L_1(a,b)$. The right-sided and respectively left-sided Riemann-Liouville derivatives of order α are defined, respectively, as follows

$$\left(D_{b^-}^{\alpha} f\right)(x) = -D\left(I_{b^-}^{1-\alpha} f\right)(x), \ x < b, \tag{2.3}$$

$$\left(D_{a^+}^{\alpha} f\right)(x) = D\left(I_{a^+}^{1-\alpha} f\right)(x), \ x > a. \tag{2.4}$$

Analogous formulas yield the right-sided and left-sided Liouville-Caputo derivatives of order α, respectively:

$$\left({}^{C}D_{b^-}^{\alpha} f\right)(x) = \left(I_{b^-}^{1-\alpha}(-D) f\right)(x), \ x < b, \tag{2.5}$$

$$\left({}^{C}D_{a^+}^{\alpha} f\right)(x) = \left(I_{a^+}^{1-\alpha} D f\right)(x), \ x > a. \tag{2.6}$$

Property 1: Let $f, g \in C[a,b]$. Then, the fractional differential operators defined in (2.3)-(2.5) satisfy the following identities:

$$(i) \int_a^b f(x) D_{b^-}^{\alpha} g(x) dx = \int_a^b g(x)^{C} D_{a^+}^{\alpha} f(x) dx - f(x) I_{b^-}^{1-\alpha} g(x) \Big|_a^b, \tag{2.7}$$

$$(ii)\int_a^b f(x)\,D_{a+}^{\alpha}g(x)\,dx=\int_a^b g(x)^{C}D_{b-}^{\alpha}f(x)\,dx+f(x)\,I_{a+}^{1-\alpha}g(x)\,|_a^b\,. \tag{2.8}$$

Property 2 (see [11]): Assume that $\alpha\in(0,1)$, $\beta>\alpha$, and $f\in C[a,b]$. Then the relations

$$\begin{aligned} D_{a+}^{\alpha}I_{a+}^{\alpha}f(x) &= f(x),\\ {}^{C}D_{a+}^{\alpha}I_{a+}^{\alpha}f(x) &= f(x), \end{aligned} \tag{2.9}$$

$$D_{a+}^{\alpha}I_{a+}^{\beta}f(x)=I_{a+}^{\beta-\alpha}f(x), \tag{2.10}$$

$$D_{b+}^{\alpha}I_{b-}^{\beta}f(x)=I_{b-}^{\beta-\alpha}f(x), \tag{2.11}$$

$$\begin{aligned} D_{b-}^{\alpha}I_{b-}^{\alpha}f(x) &= f(x),\\ {}^{C}D_{b-}^{\alpha}I_{b-}^{\alpha}f(x) &= f(x), \end{aligned} \tag{2.12}$$

hold for any $x\in[a,b]$. Furthermore, the integral operators defined in (2.1)-(2.2) satisfy the following semi-group properties:

$$I_{a+}^{\alpha}I_{a+}^{\beta}=I_{a+}^{\alpha+\beta}; \tag{2.13}$$

$$I_{b-}^{\alpha}I_{b-}^{\beta}=I_{b-}^{\alpha+\beta}. \tag{2.14}$$

Now, we introduce convenient Hilbert space $L_{\omega}^{2}((a,b);E)$ $(E:=\mathbb{C}^{2})$ of vector-valued functions using the inner product

$$\begin{aligned} (f,g) &:= \int_a^b f_1(x)\overline{g_1(x)}\omega_1(x)\,dx\\ &\quad+\int_a^b f_2(x)\overline{g_2(x)}\omega_2(x)\,dx, \end{aligned}$$

where

$$f(x)=\begin{pmatrix} f_1(x)\\ f_2(x)\end{pmatrix},\ g(x)=\begin{pmatrix} g_1(x)\\ g_2(x)\end{pmatrix},$$

f_i,g_i and ω_i are real-valued continuous functions defined on $[a,b]$ and $\omega_i(x)>0,\ \forall x\in[a,b],\ (i=1,2)$.

3. Main Results

In the present section, our goal is to study the fractional Dirac type system which includes the right-sided Liouville-Caputo and the left-sided Riemann-Liouvile fractional derivatives of same order α. Throughout this section, we assume $\alpha\in(0,1)$.

Let

$$\begin{aligned} \Upsilon y &= \begin{pmatrix} 0 & {}^{C}D_{b-}^{\alpha}\\ D_{a+}^{\alpha} & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}+\begin{pmatrix} p(x) & 0\\ 0 & r(x)\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}\\ &= \begin{pmatrix} {}^{C}D_{b-}^{\alpha}y_2+p(x)\,y_1\\ D_{a+}^{\alpha}y_1+r(x)\,y_2\end{pmatrix}, \end{aligned}$$

where $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. With this notation, we consider the fractional Dirac type system:

$$\Upsilon y_\lambda = \lambda \omega y_\lambda, \ a \le x \le b < \infty, \tag{3.1}$$

where $y_\lambda = \begin{pmatrix} y_{\lambda 1} \\ y_{\lambda 2} \end{pmatrix}$, p, r are real-valued continuous functions defined on $[a, b]$, $\omega(x) = \begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix}$, ω_i are real-valued continuous functions defined on $[a, b]$ and $\omega_i(x) > 0, \ \forall x \in [a, b], \ (i = 1, 2), \ \lambda$ is a complex spectral parameter and boundary conditions

$$a_{11} I_{a^+}^{1-\alpha} y_{\lambda 1}(a) + a_{12} y_{\lambda 2}(a) = 0, \tag{3.2}$$

$$a_{21} I_{a^+}^{1-\alpha} y_{\lambda 1}(b) + a_{22} y_{\lambda 2}(b) = 0, \tag{3.3}$$

with $a_{11}^2 + a_{12}^2 \neq 0$ and $a_{21}^2 + a_{22}^2 \neq 0$.

Theorem 3.1. *The operator $T := \omega^{-1}\Upsilon$ generated by fractional Dirac type system (FD) defined by (3.1)-(3.3) is formally self-adjoint on $L^2_\omega((a, b); E)$.*

Proof. Let $y(.), z(.) \in L^2((a, b); E)$. Then, we have

$$\begin{aligned}(Ty, z) - (y, Tz) &= \int_a^b \left(D_{a^+}^\alpha y_1 + r(x) y_2\right) \overline{z_2} dx \\ &+ \int_a^b \left({}^C D_{b^-}^\alpha y_2 + p(x) y_1\right) \overline{z_1} dx \\ &- \int_a^b y_2 \overline{\left(D_{a^+}^\alpha z_1 + r(x) z_2\right)} dx \\ &- \int_a^b y_1 \overline{\left({}^C D_{b^-}^\alpha z_2 + p(x) z_1\right)} dx\end{aligned}$$

$$= \int_a^b \left(D_{a^+}^\alpha y_1\right) \overline{z_2} dx + \int_a^b \left({}^C D_{b^-}^\alpha y_2\right) \overline{z_1} dx$$

$$- \int_a^b y_2 \overline{\left(D_{a^+}^\alpha z_1\right)} dx - \int_a^b y_1 \overline{\left({}^C D_{b^-}^\alpha z_2\right)} dx$$

Since

$$\int_0^a \left({}^C D_{b^-}^\alpha y_2\right) \overline{z_1} \omega_1 dx = \int_a^b y_2 \overline{\left(D_{a^+}^\alpha z_1\right)} \omega_1 dx \\ - \left[y_2(b) \overline{I_{a^+}^{1-\alpha} z_1(b)} - y_2(a) \overline{I_{a^+}^{1-\alpha} z_1(a)}\right]$$

and

$$\int_a^b y_1 \overline{\left({}^C D_{b^-}^\alpha z_2\right)} dx = \int_a^b \left(D_{a^+}^\alpha y_1\right) \overline{z_2} dx$$

$$- \left[\overline{z_2(b)} I_{a^+}^{1-\alpha} y_1(b) - \overline{z_2(a)} z_2(a) I_{a^+}^{1-\alpha} y_1(a)\right]$$

we get

$$(Ty,z)-(y,Tz)=[y,z]_b-[y,z]_a, \tag{3.4}$$

where $[y,z]_x := \overline{z_2(x)}I_{a^+}^{1-\alpha}y_1(x) - y_2(x)\overline{I_{a^+}^{1-\alpha}z_1(x)}$. We proceed to show that the equality $(Ty,z)=(y,Tz)$ for any $y(.),z(.)\in L^2((a,b);E)$. From the boundary conditions (3.2) and (3.3), we get $[y,z]_b=0$ and $[y,z]_a=0$. Consequently,

$$(Ty,z)=(y,Tz). \tag{3.5}$$

This completes the proof. □

Lemma 3.1. *All eigenvalues of the FD system defined by (3.1)-(3.3) are real.*

Proof. Let μ be an eigenvalue with an eigenfunction $z(x)$. From the equality (3.5), we get

$$(Tz,z)=(z,Tz)=(z,\mu z)=\overline{\mu}(z,z). \tag{3.6}$$

On the other hand,

$$(Tz,z)=(\mu z,z)=\mu(z,z). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\mu(z,z)=\overline{\mu}(z,z),\ \ (\mu-\overline{\mu})(z,z)=0.$$

Since $z(x)\neq 0$, we get $\mu=\overline{\mu}$.

Lemma 3.2. *If μ_1 and μ_2 are two different eigenvalues of the FD system defined by (3.1)-(3.3), then the corresponding eigenfunctions θ and η are orthogonal in the space $L_\omega^2((a,b);E)$.*

□

Proof. Let μ_1 and μ_2 be two different real eigenvalues with corresponding eigenfunctions θ and η, respectively. From (3.5), we obtain

$$(T\theta,\eta)=(\theta,T\eta),(\mu_1\theta,\eta)=(\theta,\mu_2\eta),(\mu_1-\mu_2)(\theta,\eta)=0.$$

Since $\mu_1\neq\mu_2$, we obtain that $\theta(x)$ and $\eta(x)$ are orthogonal in $L_\omega^2((a,b);E)$. □

Now let $y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}$, $z(x)=\begin{pmatrix} z_1(x)\\ z_2(x)\end{pmatrix}\in L^2((a,b);E)$. Then, we define the Wronskian of $y(x)$ and $z(x)$ by

$$W(y,z)(x)=I_{a^+}^{1-\alpha}y_1(x)z_2(x)-I_{a^+}^{1-\alpha}z_1(x)y_2(x).$$

Theorem 3.2. *The Wronskian of any solution of Eq. (3.1) is independent of x.*

Proof. Let $y(x)$ and $z(x)$ be two solutions of Eq. (3.1). By Green's formula (3.4), we have

$$(Ty, z) - (y, Tz) = [y, z]_b - [y, z]_a \,.$$

Since $Ty = \lambda y$ and $Tz = \lambda z$, we have

$$\begin{aligned} (\lambda y, z) - (y, \lambda z) &= [y, z]_b - [y, z]_a \,, \\ \left(\lambda - \overline{\lambda}\right)(y, z) &= [y, z]_b - [y, z]_a \,. \end{aligned}$$

Since $\lambda \in \mathbb{R}$, we have $[y, z]_b = [y, z]_a = W(y, \overline{z})(a)$, i.e., the Wronskian is independent of x. □

Corollary 3.1. *If $y(x)$ and $z(x)$ are both solutions of Equation (3.1), then either $W(y, z)(x) = 0$ or $W(y, z)(x) \neq 0$ for all $x \in [a, b]$.*

Theorem 3.3. *Any two solutions of the equation (3.1) are linearly dependent if and only if their Wronskian is zero.*

Proof. Let $y(x)$ and $z(x)$ be two linearly dependent solutions of Equation (3.1). Then, there exists a constant $c > 0$ such that $y(x) = c\, z(x)$. Hence

$$W(y, z) = \begin{vmatrix} I_{a^+}^{1-\alpha} y_1(x) & y_2(x) \\ I_{a^+}^{1-\alpha} z_1(x) & z_2(x) \end{vmatrix} = \begin{vmatrix} cI_{a^+}^{1-\alpha} z_1(x) & cz_2(x) \\ I_{a^+}^{1-\alpha} z_1(x) & z_2(x) \end{vmatrix} = 0.$$

Conversely, the Wronskian $W(y, z) = 0$ and therefore, $y(x) = cz(x)$, i.e., $y(x)$ and $z(x)$ are linearly dependent. □

Before proceeding further, we need the following auxiliary functions.

We introduce the function $\phi(x) := \begin{pmatrix} (I_{a^+}^{\alpha} 1)(x) \\ (I_{b^-}^{\alpha} 1)(x) \end{pmatrix}$. Further, the general solution of the equation $\Upsilon \psi = 0$, i.e.,

$$\begin{pmatrix} 0 & {}^{C}D_{b^-}^{\alpha} \\ D_{a^+}^{\alpha} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

is given by

$$\psi = \begin{pmatrix} \xi_1 \Phi(\alpha, a, x) \\ \xi_2 \end{pmatrix},$$

where

$$\Phi(\alpha, a, x) = \frac{(x - a)^{\alpha - 1}}{\Gamma(\alpha)}. \tag{3.8}$$

Lemma 3.3. *Let*

$$\Delta := a_{11}a_{12} - a_{11}a_{21}$$

and

$$Y_\lambda(y) := \{V - \lambda\omega\} y_\lambda, \tag{3.9}$$

where $V(x) := \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix}$. *Assume* $\Delta \neq 0$. *Then on the space* $C[a,b]$, *the FD system defined by (3.1)-(3.3) is equivalent to the integral equation*

$$y_\lambda(x) = -MY_\lambda(y) + A(x)T + B(x)Z,$$

where the coefficients M, A, T, B *and* Z *are*

$$\begin{aligned} M &:= \begin{pmatrix} 0 & I_{a+}^{\alpha} \\ I_{b-}^{\alpha} & 0 \end{pmatrix}, \\ A(x) &:= \begin{pmatrix} \frac{a_{12}a_{22}}{\Delta}\Phi(\alpha,a,x) \\ -\frac{a_{21}a_{12}}{\Delta} \end{pmatrix}, \\ T &:= -I_{b-}^{\alpha} Y_{\lambda 1}(y)\,|_{x=a}, \\ B(x) &:= \begin{pmatrix} -\frac{a_{12}a_{21}}{\Delta}\Phi(\alpha,a,x) \\ \frac{a_{21}a_{11}}{\Delta} \end{pmatrix}, \\ Z &:= -I_{a+}^{1} Y_{\lambda 2}(y)\,|_{x=b}, \end{aligned}$$

and the function $\Phi(\alpha,a,x)$ *is defined in (3.8).*

Proof. Using fractional composition rules and (3.9), we can rewrite the equation (3.1) as follows:

$$\Upsilon\left[y_\lambda(x) + MY_\lambda(y)\right] = 0.$$

Thus, we get

$$y_\lambda(x) + MY_\lambda(y) = \begin{pmatrix} \xi_1\Phi(\alpha,a,x) \\ \xi_2 \end{pmatrix},$$

i.e.,

$$y_\lambda(x) = -MY_\lambda(y) + \begin{pmatrix} \xi_1\Phi(\alpha,a,x) \\ \xi_2 \end{pmatrix}. \tag{3.10}$$

Now, we shall connect the coefficients ξ_i $(i = 1,2)$ to the values a_{ij} $(i,j = 1,2)$ in the boundary conditions (3.2)-(3.3). From the equation (3.10), we obtain

$$Ky_\lambda(x) = -KMY_\lambda(y) + K\begin{pmatrix} \xi_1\Phi(\alpha,a,x) \\ \xi_2 \end{pmatrix},$$

where $K := \begin{pmatrix} I_{a^+}^{1-\alpha} & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

$$\begin{pmatrix} I_{a^+}^{1-\alpha} y_{\lambda 1} \\ y_{\lambda 2} \end{pmatrix} = -\begin{pmatrix} 0 & I_{a^+}^{1} \\ I_{b^-}^{\alpha} & 1 \end{pmatrix} Y_\lambda(y) + \begin{pmatrix} I_{a^+}^{1-\alpha}\left[\xi_1 \Phi(\alpha, a, x)\right] \\ \xi_2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} I_{a^+}^{1-\alpha} y_{\lambda 1} \\ y_{\lambda 2} \end{pmatrix} = \begin{pmatrix} -I_{a^+}^{1} Y_{\lambda 2}(y) \\ -I_{b^-}^{\alpha} Y_{\lambda 1}(y) \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

By virtue of (3.2) and (3.3), we conclude that

$$\begin{aligned} I_{a^+}^{1-\alpha} y_{\lambda 1}(a) &= \xi_1, \\ I_{a^+}^{1-\alpha} y_{\lambda 1}(b) &= -I_{a^+}^{1} Y_{\lambda 2}(y) \mid_{x=b} + \xi_1, \\ y_{\lambda 2}(a) &= -I_{b^-}^{\alpha} Y_{\lambda 1}(y) \mid_{x=a} + \xi_2, \\ y_{\lambda 2}(b) &= \xi_2. \end{aligned}$$

This leads to the system of equations

$$\begin{aligned} a_{11}\xi_1 + a_{12}\xi_2 &= a_{12} T \\ a_{21}\xi_1 + a_{22}\xi_2 &= a_{12} Z. \end{aligned}$$

Since $\Delta \neq 0$, the solution for coefficients $\xi_j, j = 1, 2$ is unique:

$$\begin{aligned} \xi_1 &= \tfrac{a_{11}(a_{22}T - a_{21}Z)}{\Delta}, \\ \xi_2 &= \tfrac{a_{21}(a_{11}Z - a_{12}T)}{\Delta}. \end{aligned}$$

We have finished the proof of the lemma. □

Now, we prove the existence and uniqueness of eigenfunction of the regular FD system defined by (3.1)-(3.3). In the next result, we use the following notation:

$$A := \|A(x)\|_C, \; B := \|B(x)\|_C, \; S_\phi := \|\phi(x)\|_C,$$

where $\|.\|_C$ denotes the supremum norm on the space $C([a, b], E)$.

Theorem 3.4. *Let $\alpha \in (0, 1)$ and assume $\Delta \neq 0$. Then unique continuous function y_λ for the regular FD system defined by (3.1)-(3.3) corresponding to each eigenvalue obeying*

$$\|V - \lambda\omega\|_C \leq \frac{1}{S_\phi + A\|\phi(a)\|_C + B(b-a)} \tag{3.11}$$

exists and such eigenvalue is simple.

Proof. Let us define the mapping $L : C([a,b],E) \to C([a,b],E)$ by

$$Lf := -MY_{\lambda}(f) + A(x)T + B(x)Z,$$

Now, we show that the equation (3.1) can be interpreted as a fixed point condition on the space $C([a,b],E)$. Using the following estimate

$$\|Y_{\lambda}(g) - Y_{\lambda}(h)\|_C \leq \|g-h\|_C \|V-\lambda\omega\|_C ,$$

we conclude that

$$\begin{aligned} \|Lg - Lh\|_C \quad & \leq \|g-h\|_C \|V-\lambda\omega\|_C S_{\phi} + A\|g-h\|_C \|\phi(a)\|_C \\ & \quad + B(b-a)\|g-h\|_C \|V-\lambda\omega\|_C \\ & = \|V-\lambda\omega\|_C \|g-h\|_C (S_{\phi} + A\|\phi(a)\|_C + B(b-a)) \\ & = \Pi \|g-h\|_C , \end{aligned}$$

where $\Pi = \|V-\lambda\omega\|_C (S_{\phi} + A\|\phi(a)\|_C + B(b-a))$. By the condition (3.11), the mapping L is a contraction on the space $C([a,b],E)$ so it has a unique fixed point. Therefore, such eigenvalue is simple. □

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THE HOMEOMORPHIC PROPERTY OF THE STOCHASTIC FLOW GENERATED BYTHE ONE-DEFAULT MODEL IN ONE DIMENSIONAL CASE

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Abstract. In this paper, we will try to study the same result proved in [10]. So, on the same model and with some assumptions, we will study the property of homeomorphism of the stochastic flow generated by the natural model in a one-dimensional case and with some modifications, based on an important theory of Hiroshi Kunita. This is the main motivation of our research.
Keywords: Credit risk, Stochastic flow, Stochastic differential geometry, Diffeomorphism.

1. Introduction

The notion of flow in the deterministic case for ordinary differential equations has been studied by Blagovescenski and Freindlin [11]. Under stronger assumptions of regularity of coefficients, such solutions determine a stochastic flow of diffeomorphisms. This question was discussed under variety of assumptions by Baxendal [12], Bismut [6], Elworthy [3], Kunita [2], Malliavin [4] and others. See also Kunita [13] for extensive literature on the subject. While the method of Bismut and Kunita, is primarily an extension of the original one of Blagovescenski and Freindlin on using Kolmogorov extension theorem, the original method of Elworthy [3] is based

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on using theory of stochastic integration on some appropriate Hilbert manifold of diffeomorphisms. These method originated from a similar approach in the deterministic case (but still in the framework of Hilbert manifolds) by Ebin and Marsden [14], see also Ebin [15].

The notion of the stochastic flow associated with a stochastic differential equation has been studied by several authors, e.g. Elworthy [3], Malliavin [4], Ikeda-Watanabe [5], Bismut [6]. In this work, we are interested in the stochastic flow generated by the so-called $\natural$-model, it is one-default model which gives the conditional law of a random time with respect to a reference filtration. This models are widely applied in modeling financial risk and price valuation of financial products.

Precisely, it is proved in [1] that, for any continuous local martingale Y, for any Lipschitz function f on $\mathbb{R}$ null at the origin, there exists a probability measure $\mathbb{Q}$ and a random time $\tau > 0$ on an extension of $(\Omega, \mathbb{F}, \mathbb{P})$, such that the survival probability of τ, i.e., $\mathbb{Q}[\tau > t|\mathcal{F}_t]$ is equal to Z_t for $t \geq 0$. In the last reference, it has also been shown that there exists several solutions and that an increasing family of martingales, combined with a stochastic differential equation, constitutes a natural way to construct these solutions, which means that $X_t^u = \mathbb{Q}[\tau \leq |\mathcal{F}_t], 0 < u, t < \infty$ satisfies the following stochastic differential equation :

$$(\natural_u) : \begin{cases} dX_t = X_t \left(-\dfrac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), & t \in [u, \infty) \\ X_u = x \end{cases}$$

where the initial condition x can be any $\mathcal{F}_u$-mesurable random variable.

The main result of this paper is to prove the homeomorphism property of the stochastic flow generated by the stochastic flow associated with the $\natural$-equation based on Hiroshi Kunita theory, but we impose the following hypotheses:

The first hypothesis:

We keep the same naturel model, but we assume that all the processes indicated in the $\natural$-equation take real values. Thus, we impose that the coefficients of this equation are Lipschitz continuous.

The second hypothesis:

We always assume the hypothesis mentioned in [1], which denoted that the stochastic integral $\displaystyle\int_u^t \frac{e^{-\Lambda_s}}{1 - Z_s} dN_s$, $u \leq t < \infty$, exists and defines a local martingale.

Remark 1.1. With these assumptions, we recall that the solution of the $\natural$-equation is continuous according to the article [7].

Remark 1.2. It is reported here that the H. Kunita theory appearing in Section 2 was done for multidimensional processes. Therefore, to obtain our result in the one-default model, it suffices to apply the unidimensional version of the Itô's formula.

2. The stochastic flow of stochastic differential equation

This section is borrowed from [2].

Let $G_1(x), ..., G_r(x)$ be continuous mappings from $\mathbb{R}^d$ into itself and $M_t^1, ..., M_t^r$ be continuous semimartingales defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P}; \mathbb{F}_t)$. Here $\mathbb{F}_t, 0 \leq t < \infty$ is an increasing family of sub σ-fields of $\mathbb{F}$ such that $\wedge_{\varepsilon>0}\mathbb{F}_{t+\varepsilon} = \mathbb{F}_t$ holds for each t. Consider an Itô stochastic differential equation (SDE) on $\mathbb{R}^d$;

$$d\xi_t = \sum_{j=1}^{r} G_j(\xi_t) dM_t^j \tag{2.1}$$

A sample continuous $\mathbb{F}_t$-adapted stochastic process ξ_t with values in $\mathbb{R}^d$ is called a solution of (2.1), if it satisfies

$$\xi_t = \xi_0 + \sum_{j=1}^{d} \int_0^t G_j(\xi_s) dM_s^j \tag{2.2}$$

where the right hand side is the Ito integral.

Concerning coefficients of the equation, we will assume in this section that they are Lipschitz continuous, i.e., there is a positive constant L such that

$$|G_j^i(x) - G_j^i(y)| \leq L|x-y|, \qquad \forall x, y \in \mathbb{R}^d$$

holds for all indices i, j, where $G_j^i(x)$ is the i-th component of the vector function $G_j(x)$. Then for a given point x of $\mathbb{R}^d$, the equation has a unique solution such that $\xi_0 = 0$. We denote it as $\xi_t(x)$ or $\xi_t(x, \omega)$. It is continuous in (t, x) a.s. In fact, the following proposition is well known.

Proposition 2.1. *[8]. $\xi_t(x, \omega)$ is continuous in $[0, \infty) \times \mathbb{R}^d$ for almost all ω. Furthermore, for any $T > 0$ and $p \geq 2$, there is a positive constant $K_{p,T}^{(1)}$ such that*

$$\mathbb{E}|\xi_t(x) - \xi_s(y)|^p \leq K_{p,T}^{(1)} \left(|x-y|^p + |t-s|^{\frac{p}{2}}\right) \tag{2.3}$$

holds for all x, y of $\mathbb{R}^d$ and t, s of $[0, T]$.

We thus regard that for fixed t, $\xi_t(\cdot, \omega)$ is a continuous map from $\mathbb{R}^d$ into itself for almost all ω. The purpose of this section is to prove that map $\xi_t(\cdot, \omega)$ is one to one and onto, and that the inverse map $\xi_t^{-1}(\cdot, \omega)$ is also continuous.

Theorem 2.1. *[2]. Suppose that $G_1, ..., G_r$ of equation (2.1) are Lipschitz continuous. Then the solution map $\xi_t(\cdot, \omega)$ is a homeomorphism of $\mathbb{R}^d$ for all t, a.s. ω.*

Remark 2.1. In case of one dimensional SDE, Ogura and Yamada [9] has shown the same result under a weaker condition, using a strong comparison theorem of solutions. In fact, if coefficients are Lipschitz continuous on any finite interval (local Lipschitzan) and if they are of linear growth, i.e., $|G_j(x)| \leq C(1 + |x|)$ holds for all x with some positive C, then the solution $\xi_t(\cdot, \omega)$ is homeomorphism for any t a.s.

Remark 2.2. The (*local*) Lipschitz continuity of coefficients is crucial for the theorem. Ogura and Yamada [9] has given an example of one dimensional SDE with α-Hölder continuous coefficients ($\frac{1}{2} < \alpha < 1$), which has a unique strong solution but does not have the "one to one" property.

Remark 2.3. It is enough to prove the theorem in case that M_t^i, $i = 1, ..., r$ satisfies the properties below: Let $M_t^j = B_t^j + A_t^j$ be the decomposition of semimartingale such that B_t^j is a continuous local martingale and A_t^j is a continuous process of bounded variation. Let $< B^j >_t$ be the quadratic variation of B_t^j. Then it holds for each j and $\forall s < t$,

$$A_t^j - A_s^j \leq t - s \ , \ < B^j >_t - < B^j >_s \leq t - s, \qquad \forall s < t \tag{2.4}$$

In the following discussion, condition (2.4) is always assumed. We will first show the "one to one" property. Our approach is based on several elementary inequalities.

Lemma 2.1. *[2]. Let $T > 0$ and p be any real number. Then there is a positive constant $K_{p,T}^{(2)}$ such that $\forall x, y \in \mathbb{R}^d$ and $\forall t \in [0, T]$,*

$$\mathbb{E}|\xi_t(x) - \xi_s(y)|^p \leq K_{p,T}^{(2)}|x - y|^p, \quad \forall x, y \in \mathbb{R}^d, \quad \forall t \in [0, T] \tag{2.5}$$

The above lemma shows that if $x \neq y$ then $\xi_t(x) \neq \xi_t(y)$ holds for all t a.s. But it does not conclude that $\xi_t(\cdot, \omega)$ is "one to one", since the exceptional null set $N_{x,y} = \{\omega \,|\xi_t(x) = \xi_t(y)\, for\, some\ t\}$ depends on the pair (x, y). To overcome this point, we shall prove the following lemma.

Lemma 2.2. *[8]. Set*

$$\eta(x, y) = \frac{1}{|\xi_t(x) - \xi_t(y)|} \tag{2.6}$$

Then $\eta_t(x, y)$ is continuous in $[0, \infty) \times \{(x, y) \in \mathbb{R}^{2d}|x \neq y\}$.

The above lemma leads immediately to the "one to one" property of the map $\xi_t(\cdot, \omega)$ for all t a.s. We shall next consider the onto property. We first establish

Lemma 2.3. *[2]. Let $T > 0$ and p be any real number. Then there is a positive constant $K_{p,T}^{(3)}$ such that*

$$\mathbb{E}(1 + |\xi_t(x)|^2)^p \leq K_{p,T}^{(3)}(1 + |x|^2)^p, \quad \forall x \in \mathbb{R}^d, \ \forall t \in [0, T] \tag{2.7}$$

Remark 2.4. It holds $(1 + |x|^2) \leq (1 + |x|)^2 \leq 2(1 + |x|^2)$. Therefore, inequality (2.7) implies

$$\mathbb{E}(1 + |\xi_t(x)|)^{2p} \leq 2^{|p|}K_{p,T}^{(3)}(1 + |x|)^{2p} \tag{2.8}$$

Now taking negative p in the above lemma, we see that $|\xi_t(x)|$ tends to infinity in probability as x tends sequentially to infinity. We shall prove a stronger convergence. We claim

Lemma 2.4. *[2]. Let $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$. Set*

$$\eta_t(x) = \begin{cases} \dfrac{1}{1+|\xi_t(x)|} & if\ x \in \mathbb{R}^d \\ 0 & if\ x = \infty \end{cases}$$

Then $\eta_t(x,\omega)$ is a continuous map from $[0,\infty) \times \overline{\mathbb{R}^d}$ into $\mathbb{R}$ a.s.

Lemma 2.5. *[2]. Define a stochastic process $\overline{\xi}_t$ on $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ by*

$$\overline{\xi}_t(x) = \begin{cases} \xi_t(x) & if\ x \in \mathbb{R}^d \\ \infty & if\ x = \infty \end{cases}$$

Then $\overline{\xi}_t(x)$ is continuous in $[0,\infty) \times \overline{\mathbb{R}^d}$.

Now the map $\overline{\xi}_t$ is a homeomorphism of $\overline{\mathbb{R}^d}$, since it is one to one, onto and continuous. Since ∞ is the invariant point of the map $\overline{\xi}_t$, we see that ξ_t is a homeomorphism of $\mathbb{R}^d$. This completes the proof of Theorem 2.2.

3. Main result

In our model and with the assumptions set out in Section 1, we show the homeomorphic property of the solution of the $\natural$-equation by applying the lemmas introduced by H.Kunita presented in the previous section. We take $\varepsilon = p$ and $\beta = p - n$ with $p > 0$, we have for $u \leq s \leq t$:

$$X_t^u(x) = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right) dN_s + \int_u^t X_s f(X_s - (1-Z_s)) dY_s$$

We know that the quantity $f(X_s - (1-Z_s))$ is bounded because f is a Lipschitz function, but as we do not know a priori if the quantity $\left(-\dfrac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not, we introduce the stopping time $\tau_n = \inf\{t, 1-Z_t < \frac{1}{n}\}$. Therefore, we assume the process $\tilde{X}$ instead of X:
$d\tilde{X}_t = \tilde{X}_t(-\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} dN_t + f(\tilde{X}_t - (1-Z_t))dY_t)$, Such as $\tilde{X}_t = X_t, \quad \forall t \leq \tau_n$, $n \in \mathbb{N}$.

3.1. Proof of the one to one property

In this part, we will apply the lemma 2.1 to the one-default model. So if $x = y$ the inequality is clearly satisfied for any constant $\tilde{K}^2_{p,T}$. We shall assume $x = y$. Let $\tilde{\varepsilon}$ be an arbitrary positive number and:

$$\sigma_{\tilde{\varepsilon}} = \inf\{t > 0, \, |\tilde{X}_t^u(x) - \tilde{X}_t^u(y)| < \tilde{\varepsilon}\}$$

denote $A_t = \tilde{X}_t^u(x) - \tilde{X}_t^u(y)$, and we shall apply Itô's formula to the function $f(z) = |z|^p$. Then it holds for $t < \tilde{\varepsilon}$;

$$\tilde{X}_t^u(x) = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \tilde{X}_s f \left(\tilde{X}_s - (1 - Z_s) \right) dY_s$$

$$d\tilde{X}_t^u(x) = \tilde{X}_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} dN_t + f \left(\tilde{X}_t - (1 - Z_t) \right) dY_t \right)$$

$$\left| \tilde{X}_t^u(x) - \tilde{X}_t^u(y) \right|^p - |x - y|^p = \int_u^t \frac{\partial f}{\partial z} \left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y) \right) \times$$

$$\left(\tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(x) f \left(\tilde{X}_s(x) - (1 - Z_s) \right) dY_s - \right.$$

$$\left. \tilde{X}_s(y) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(y) f \left(\tilde{X}_s(y) - (1 - Z_s) \right) dY_s \right) +$$

$$\frac{1}{2} \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times$$

$$\left[\tilde{X}_s(x) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(x) f \left(\tilde{X}_s(x) - (1 - Z_s) \right) dY_s - \right.$$

$$\left. \tilde{X}_s(y) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \tilde{X}_s(y) f \left(\tilde{X}_s(y) - (1 - Z_s) \right) dY_s \right]^2$$

$$\left| \tilde{X}_t^u(x) - \tilde{X}_t^u(y) \right|^p - |x - y|^p = \int_u^t \frac{\partial f}{\partial z} \left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y) \right) \times$$

$$\left[\left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \right.$$

$$\left. \left(\tilde{X}_s(x) f \left(\tilde{X}_s(x) - (1 - Z_s) \right) - \tilde{X}_s(y) f \left(\tilde{X}_s(y) - (1 - Z_s) \right) \right) dY_s \right]$$

$$+ \frac{1}{2} \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times$$

$$\left[\left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \right.$$

$$\left. \left(\tilde{X}_s(x) f \left(\tilde{X}_s(x) - (1 - Z_s) \right) - \tilde{X}_s(y) f \left(\tilde{X}_s(y) - (1 - Z_s) \right) \right) dY_s \right]^2$$

$$\left|\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right|^p - |x-y|^p = \tilde{I}_t + \tilde{J}_t$$

we start with $\tilde{I}_t$:

$$\tilde{I}_t = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right) \times$$

$$\left[\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s +\right.$$

$$\left.\left(\tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) - \tilde{X}_s(y) f\left(\tilde{X}_s(y) - (1 - Z_s)\right)\right) dY_s\right]$$

Noting

$$\tilde{V}(\tilde{X}_s^x) = \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right)$$

$$\tilde{V}(\tilde{X}_s^y) = \tilde{X}_s(y) f\left(\tilde{X}_s(y) - (1 - Z_s)\right)$$

such that

$$\left|\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right| \le \tilde{L}\left|\tilde{X}_s^x - \tilde{X}_s^y\right|$$

and

$$\frac{\partial f}{\partial z} = p|z|^{p-1}$$

we put

$$\tilde{I}_t = \tilde{I}_t^1 + \tilde{I}_t^2$$

such that

$$\tilde{I}_t^1 = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right)\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s$$

$$\tilde{I}_t^2 = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right)\left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right) dY_s$$

For $\tilde{I}_t^1$, we have:

$$\begin{aligned}
\left|\frac{\partial f}{\partial z}\left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y)\right)\left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)\right| &\le |p||z|^{p-1}\left|\tilde{X}_s(x) - \tilde{X}_s(y)\right| \\
&\le |p|\left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p
\end{aligned}$$

Therefore

$$\tilde{I}_t^1 \le |p|\int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \times \int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s$$

Noting
$Q_t = \int_u^t -\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} dN_s$, it is a local martingale according to hypothesis 1. (so called the hypothesis $H_Y(C)$ [1]). So

$$\tilde{I}_t^1 \leq |p| \, Q_t \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds$$

For $\tilde{I}_t^2$, we have:

$$\begin{aligned} \left| \frac{\partial f}{\partial z} \left(\tilde{X}_t^u(x) - \tilde{X}_t^u(y) \right) \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y) \right) \right| &\leq |p||z|^{p-1} \tilde{L} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right| \\ &\leq |p| \tilde{L} \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p \end{aligned}$$

Therefore

$$\tilde{I}_t^2 \leq |p| \tilde{L} \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds$$

So, we have

$$\begin{aligned} \tilde{I}_t = \tilde{I}_t^1 + \tilde{I}_t^2 &\leq |p| \, Q_t \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds + |p| \, \tilde{L} \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds \\ &\leq |p| \int_u^t \left| \tilde{X}_s(x) - \tilde{X}_s(y) \right|^p ds \, (Q_t + \tilde{L}) \end{aligned}$$

Therefore, we have

$$\left| \mathbb{E} \, \tilde{I}_{t\wedge\sigma_{\tilde{\varepsilon}}} \right| \leq |p| \, (Q_{t\wedge\sigma_{\tilde{\varepsilon}}} + \tilde{L}) \int_u^t \mathbb{E} \left| \tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(x) - \tilde{X}_{s\wedge\sigma_{\tilde{\varepsilon}}}(y) \right|^p ds \tag{3.1}$$

Next,

$$\tilde{J}_t = \frac{1}{2} \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times$$

$$\left[\left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y) \right) dY_s \right]^2$$

$$\tilde{J}_t = \frac{1}{2} \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times$$

$$\left[\left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right)^2 dN_s dN_s + \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y) \right)^2 dY_s dY_s \right.$$

$$\left. +2 \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y) \right) dN_s dY_s \right]$$

Noting $\tilde{J}_t = \frac{1}{2} \left[\tilde{J}_t^1 + \tilde{J}_t^2 + \tilde{J}_t^3 \right]$ such that:

$$\tilde{J}_t^1 = \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y) \right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y) \right)^2 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right)^2 dN_s dN_s$$

$$\tilde{J}_t^2 = \int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right)^2 dY_s dY_s$$

$$\tilde{J}_t^3 = 2\int_u^t \frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right) \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right) dN_s dY_s$$

and note that

$$\frac{\partial^2 f}{\partial z^2} = p(p-1)|z|^{p-2}$$

For $\tilde{J}_t^1$ we have

$$\left|\frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)^2\right| \leq \left|p(p-1)|z|^{p-2} \times \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right)^2\right|$$

$$\leq |p||p-1| \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p$$

Therefore

$$\tilde{J}_t^1 \leq |p||p-1| \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds \int_u^t \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s$$

The hypothesis 1. is always assumed, so

$$\tilde{J}_t^1 \leq |p||p-1| Q_t^2 \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds$$

For $\tilde{J}_t^2$ we have

$$\left|\frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right)^2\right| \leq \left|p(p-1)|z|^{p-2} \tilde{L}^2 \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right)^2\right|$$

$$\leq |p||p-1| \, \tilde{L}^2 \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p$$

So

$$\tilde{J}_t^2 \leq |p||p-1| \, \tilde{L}^2 \int_u^t \left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|^p ds$$

For $\tilde{J}_t^3$ we have

$$\left|\frac{\partial^2 f}{\partial z^2} \left(\tilde{X}_s^u(x) - \tilde{X}_s^u(y)\right) \times \left(\tilde{X}_s(x) - \tilde{X}_s(y)\right) \left(\tilde{V}(\tilde{X}_s^x) - \tilde{V}(\tilde{X}_s^y)\right)\right|$$

$$\leq \left| p(p-1)z^{p-2}\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\tilde{L}\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)\right|$$

$$\leq \left| p(p-1)\tilde{L}\left(\tilde{X}_s(x)-\tilde{X}_s(y)\right)^p\right|$$

The hypothesis 1. is always assumed, so

$$\tilde{J}_t^3 \leq 2|p||p-1|\,\tilde{L}\,Q_t\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

Therefore

$$\tilde{J}_t = \frac{1}{2}\left[\tilde{J}_t^1+\tilde{J}_t^2+\tilde{J}_t^3\right]$$

$$\tilde{J}_t = \frac{1}{2}\left[|p||p-1|Q_t^2\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds + |p||p-1|\,\tilde{L}^2\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\right.$$

$$\left.+2|p||p-1|\,\tilde{L}\,Q_t\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds\right]$$

$$\tilde{J}_t \leq \frac{1}{2}|p||p-1|(Q_t+\tilde{L})^2\int_u^t \left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

Therefore

$$\left|\mathbb{E}\tilde{J}_{t\wedge\sigma_{\tilde{\varepsilon}}}\right| \leq \frac{1}{2}|p||p-1|(Q_t+\tilde{L})^2\int_u^t \mathbb{E}\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds \tag{3.2}$$

Summing up these two inequalities 3.1 and 3.2, we obtain

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p \leq |x-y|^p + \tilde{C}_p\int_u^t \mathbb{E}\left|\tilde{X}_s(x)-\tilde{X}_s(y)\right|^p ds$$

where $\tilde{C}_p$ is a positive constant.

By Gronwall's inquality, we have:

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(x)-\tilde{X}^u_{t\wedge\sigma_{\tilde{\varepsilon}}}(y)\right|^p \leq \tilde{K}^{(2)}_{p,u}\ |x-y|^p, \quad u\leq t\leq\infty$$

such that

$$\tilde{K}^{(2)}_{p,u}\ |x-y|^p = \exp\left(\tilde{C}_p u\right)$$

Letting ε tend to 0, we have:

$$\mathbb{E}\left|\tilde{X}^u_{t\wedge\sigma}(x)-\tilde{X}^u_{t\wedge\sigma}(y)\right|^p \leq \tilde{K}^{(2)}_{p,u}\ |x-y|^p$$

where σ is the first time such that $\tilde{X}_t^u(x) = \tilde{X}_t^u(y)$. However, it holds $\sigma = \infty$ a.s, since otherwise the left hand side would be infinity if $p < 0$. The proof is complete.

The above lemma shows that if $x \neq y$ then $\tilde{X}_t^u(x) \neq \tilde{X}_t^u(y)$ holds for all t a.s. But it does not conclude that $\tilde{X}_t(., \omega)$ is one to one, since the exceptional null set $\tilde{N}_{x,y} = \{\omega / \tilde{X}_t^u(x) = \tilde{X}_t^u(y) \,\text{for some t}\}$ depends on the pair (x, y). To overcome this point, we shall apply the lemma 2.2.

In this case we have:

$$\tilde{X}_t^u(x) = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s f\left(\tilde{X}_s - (1 - Z_s)\right) dY_s$$

$$\tilde{X}_{\acute{t}}^u(\acute{x}) = \acute{x} + \int_u^{\acute{t}} \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^{\acute{t}} \tilde{X}_s f\left(\tilde{X}_s - (1 - Z_s)\right) dY_s$$

$$\tilde{X}_t^u(y) = y + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s f\left(\tilde{X}_s - (1 - Z_s)\right) dY_s$$

$$\tilde{X}_{\acute{t}}^u(\acute{y}) = \acute{y} + \int_u^{\acute{t}} \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^{\acute{t}} \tilde{X}_s f\left(\tilde{X}_s - (1 - Z_s)\right) dY_s$$

Putting

$$\tilde{\eta}_t(x, y) = \frac{1}{|\tilde{X}_t^u(x) - \tilde{X}_t^u(y)|}$$

$$\tilde{\eta}_{t'}(x', y') = \frac{1}{|\tilde{X}_{t'}^u(x') - \tilde{X}_{t'}^u(y')|}$$

So

$$|\tilde{\eta}_t(x, y) - \tilde{\eta}_{t'}(x', y')|^p = \left| \frac{1}{|\tilde{X}_t^u(x) - \tilde{X}_t^u(y)|} - \frac{1}{|\tilde{X}_{t'}^u(x') - \tilde{X}_{t'}^u(y')|} \right|^p$$

$$\leq 2^p \left(\frac{1}{|\tilde{X}_t^u(x) - \tilde{X}_t^u(y)|}\right)^p \left(\frac{1}{|\tilde{X}_{t'}^u(x') - \tilde{X}_{t'}^u(y')|}\right)^p \left[|\tilde{X}_t^u(x) - \tilde{X}_{t'}^u(x')|^p + |\tilde{X}_t^u(y) - \tilde{X}_{t'}^u(y')|^p\right]$$

By Hölder inequality

$$\mathbb{E}\,|\tilde{\eta}_t(x, y) - \tilde{\eta}_{t'}(x', y')|^p \leq 2^p \left(\mathbb{E}(\tilde{\eta}_t(x, y)^{4p})\mathbb{E}(\tilde{\eta}_{t'}(x', y')^{4p})\right)^{\frac{1}{4}} \times$$

$$\left[\left(\mathbb{E}|\tilde{X}_t^u(x) - \tilde{X}_{t'}^u(x')|^{2p}\right)^{\frac{1}{2}} + \left(\mathbb{E}|\tilde{X}_t^u(y) - \tilde{X}_{t'}^u(y')|^{2p}\right)^{\frac{1}{2}}\right]$$

By lemme 2.1 and proposition 2.1, we have

$$\mathbb{E}\,|\tilde{\eta}_t(x, y) - \tilde{\eta}_{t'}(x', y')|^p \leq \tilde{C}_{p,T}|x-y|^{-p}|x'-y'|^{-p}\left(|x - x'|^p + |y - y'|^p + 2|t - t'|^{\frac{p}{2}}\right)$$

$$\leq \tilde{C}_{p,T}\tilde{\delta}^{-2p}\left(|x - x'|^p + |y - y'|^p + 2|t - t'|^{\frac{p}{2}}\right)$$

if $|x-y| \geq \tilde{\delta}$ and $|x'-y'| \geq \tilde{\delta}$, where $\tilde{C}_{p,T}$ is a positive constant. Then by Kolmogorov theorem 2.1, $\tilde{\eta}_t(x,y)$ is continuous in $[0,T] \times \{(x,y)/|x-y| \geq \tilde{\delta}\}$. Since T and $\tilde{\delta}$ are arbitrary positive numbers, we get the assertion. The proof is complete.

The above calculus leads immediately the one to one property of the map $\tilde{X}^u_t(.,\omega)$ for all t a.s. We shall next consider the onto property.

3.2. Proof of the onto property

In this part we will apply the lemmas 2.3, 2.4, and 2.5 to our model.

Let $T > 0$ and p any real number:

$$\tilde{X}^u_t(x) = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s f\left(\tilde{X}_s - (1 - Z_s)\right) dY_s$$

$$d\tilde{X}^u_t(x) = \tilde{X}_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} dN_t + f\left(\tilde{X}_t - (1 - Z_t)\right) dY_t\right)$$

We shall apply Itô's formula to the function $f(z) = (1 + |z|^2)^p$. It holds

$$f(\tilde{X}^u_t(x)) - f(x) = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \times$$

$$\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s\right] +$$

$$\frac{1}{2}\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}^u_s(x)\right) \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s\right]^2$$

$f(\tilde{X}^u_t(x)) - f(x) = \tilde{I}_t + \tilde{J}_t$ such that

$$\tilde{I}_t = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s\right]$$

$$\tilde{J}_t = \frac{1}{2}\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}^u_s(x)\right) \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s\right]^2$$

For $\tilde{I}_t$, we have

$$\tilde{I}_t = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \times \left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s\right]$$

$$\tilde{I}_t = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \tilde{X}_s(x) f\left(\tilde{X}_s(x) - (1 - Z_s)\right) dY_s$$

$= \tilde{I}^1_t + \tilde{I}^2_t$ such that

$$\tilde{I}^1_t = \int_u^t \frac{\partial f}{\partial z}\left(\tilde{X}^u_s(x)\right) \tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s$$

$$\tilde{I}_t^2 = \int_u^t \frac{\partial f}{\partial z} \left(\tilde{X}_s^u(x) \right) \tilde{X}_s(x) f \, \left(\tilde{X}_s(x) - (1 - Z_s) \right) dY_s$$

For $\tilde{I}_t^1$, note $\frac{\partial f}{\partial z} = 2\, p\, z\, (1 + |z|^2)^{p-1}$ and the hypothesis 1. is always assumed, so

$$\tilde{I}_t^1 = \int_u^t \frac{\partial f}{\partial z} \left(\tilde{X}_s^u(x) \right) \tilde{X}_s(x) \left(- \frac{e^{-\Lambda_s}}{1 - Z_{s \wedge \tau_n}} \right) dN_s$$

$$\begin{aligned} \left| \frac{\partial f}{\partial z} \left(\tilde{X}_s^u(x) \right) \tilde{X}_s(x) \right| &\leq 2\,|p|\,|z| \, \left(1 + |z|^2\right)^{p-1} \left| \tilde{X}_s(x) \right| \\ &\leq 2\,|p| \left(1 + \left| \tilde{X}_s(x) \right|^2 \right)^p \end{aligned}$$

Therefore

$$\tilde{I}_t^1 \leq 2\,|p|\, Q_t \int_u^t \left(1 + \left| \tilde{X}_s(x) \right|^2 \right)^p ds$$

For $\tilde{I}_t^2$, we have

$$\tilde{I}_t^2 = \int_u^t \frac{\partial f}{\partial z} \left(\tilde{X}_s^u(x) \right) \tilde{X}_s(x) f \, \left(\tilde{X}_s(x) - (1 - Z_s) \right) dY_s$$

Noting

$$\tilde{V}(\tilde{X}_s^x) = \tilde{X}_s(x) f \left(\tilde{X}_s(x) - (1 - Z_s) \right)$$

Let $\tilde{K}$ be a positive constant such that

$$V(\tilde{X}_s^x) \leq \tilde{K} \left(1 + |\tilde{X}_s(x)|^2 \right)^{\frac{1}{2}}$$

$$\begin{aligned} \left| \frac{\partial f}{\partial z} \left(\tilde{X}_s^u(x) \right) \tilde{X}_s(x) \times \tilde{V}(\tilde{X}_s^x) \right| &\leq 2\,|p|\,|z| \, \left(1 + |z|^2\right)^{p-1} \tilde{K} \left(1 + |\tilde{X}_s(x)|^2 \right)^{\frac{1}{2}} \\ &\leq 2\,|p|\, \tilde{K} \left(1 + |\tilde{X}_s(x)|^2 \right)^p \end{aligned}$$

So

$$\tilde{I}_t^2 \leq 2\,|p|\, \tilde{K} \int_u^t \left(1 + |\tilde{X}_s(x)|^2 \right)^p ds$$

Therefore

$$\begin{aligned} \tilde{I}_t &\leq 2\,|p|\, Q_t \int_u^t \left(1 + \left| \tilde{X}_s(x) \right|^2 \right)^p ds + 2\,|p|\, \tilde{K} \int_u^t \left(1 + |\tilde{X}_s(x)|^2 \right)^p ds \\ &\leq 2\,|p| \left(Q_t + \tilde{K} \right) \int_u^t \left(1 + |\tilde{X}_s(x)|^2 \right)^p ds \end{aligned}$$

We have

$$\left| \mathbb{E}\, \tilde{I}_t \right| \leq 2|p|\, (Q_t + \tilde{K}) \int_u^t \mathbb{E} \left(1 + |\tilde{X}_s(x)|^2 \right)^p ds \tag{3.3}$$

Next, for $\tilde{J}_t$ we have

$$\tilde{J}_t = \frac{1}{2}\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\left[\tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \tilde{X}_s(x)f\left(\tilde{X}_s(x)-(1-Z_s)\right)dY_s\right]^2$$

$$\tilde{J}_t = \frac{1}{2}\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\left[\tilde{X}_s(x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s + \tilde{V}(\tilde{X}_s^x)^2\,dY_s dY_s\right.$$
$$\left. +2\,\tilde{X}_s^u(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}(\tilde{X}_s^x)\,dN_s dY_s\right]$$

Noting $\tilde{J}_t = \frac{1}{2}\left[\tilde{J}_t^1 + \tilde{J}_t^2 + \tilde{J}_t^3\right]$, such that

$$\tilde{J}_t^1 = \int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\tilde{X}_s(x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s$$

$$\tilde{J}_t^2 = \int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\tilde{V}(\tilde{X}_s^x)^2\,dY_s dY_s$$

$$\tilde{J}_t^3 = 2\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\tilde{X}_s^u(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}(\tilde{X}_s^x)\,dN_s dY_s$$

and note that

$$\frac{\partial^2 f}{\partial z^2} = 2\,p\,\left(1+|z|^2\right)^{p-1} + 4\,p\,(p-1)\,z^2(1+|z|^2)^{p-2}$$

Then for $\tilde{J}_t^1$ we have

$$\tilde{J}_t^1 = \int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\tilde{X}_s(x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s$$

$$\begin{aligned}\left|\frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right)\times\tilde{X}_s(x)^2\right| &\leq \left|\left(2\,p\,\left(1+|z|^2\right)^{p-1} + 4\,p\,(p-1)\,z^2(1+|z|^2)^{p-2}\right)\tilde{X}_s(x)^2\right| \\ &\leq 2\,|p|\,(2(p-1)+1)\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p\end{aligned}$$

Therefore

$$\tilde{J}_t^1 \leq 2\,|p|\,(2(p-1)+1)\int_u^t\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds\int_u^t\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 dN_s dN_s$$

By hypothesis 1., we have

$$\tilde{J}_t^1 \leq 2\,|p|\,(2(p-1)+1)\;Q_t^2\int_u^t\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds$$

For $\tilde{J}_t^2$, we have

$$\tilde{J}_t^2 = \int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right) \times \tilde{V}(\tilde{X}_s^x)^2\, dY_s dY_s$$

$$\begin{aligned}\left|\frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right) \times \tilde{V}(\tilde{X}_s^x)^2\right| &\leq \left|\left(2\,p\,\left(1+|z|^2\right)^{p-1} + 4\,p\,(p-1)\,z^2(1+|z|^2)^{p-2}\right)\right. \\ &\quad \left.\times \tilde{K}^2\left(1+\left|\tilde{X}_s(x)\right|^2\right)\right| \\ &\leq 2\,|p|\,(2(p-1)+1)\,\tilde{K}^2\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p\end{aligned}$$

Therefore

$$\tilde{J}_t^2 \leq 2\,|p|\,(2(p-1)+1)\,\tilde{K}^2\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds$$

For $\tilde{J}_t^3$, we have

$$\tilde{J}_t^3 = 2\int_u^t \frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\tilde{V}(\tilde{X}_s^x)\, dN_s dY_s$$

$$\begin{aligned}\left|\frac{\partial^2 f}{\partial z^2}\left(\tilde{X}_s^u(x)\right) \times \tilde{X}_s(x)\tilde{V}(\tilde{X}_s^x)\right| &\leq \left|\left(2\,p\,\left(1+|z|^2\right)^{p-1} + 4\,p\,(p-1)\,z^2(1+|z|^2)^{p-2}\right)\right. \\ &\quad \left.\times \tilde{K}^2\left(1+\left|\tilde{X}_s(x)\right|^2\right)^{\frac{1}{2}}\tilde{X}_s(x)\right| \\ &\leq 2\,|p|\,(2(p-1)+1)\,\tilde{K}\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p\end{aligned}$$

The hypothesis 1. is always assumed, so

$$\tilde{J}_t^3 \leq 4\,|p|\,(2(p-1)+1)\,\tilde{K}\,Q_t\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds$$

Therefore

$$\tilde{J}_t = \frac{1}{2}\left[\tilde{J}_t^1 + \tilde{J}_t^2 + \tilde{J}_t^3\right]$$

$$\begin{aligned}\tilde{J}_t = \frac{1}{2}\Bigg[&2\,|p|\,(2(p-1)+1)\;Q_t^2\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds + \\ &2\,|p|\,(2(p-1)+1)\,\tilde{K}^2\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds \\ &+ 4\,|p|\,(2(p-1)+1)\,\tilde{K}\,Q_t\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds\Bigg]\end{aligned}$$

$$\tilde{J}_t \leq |p|\,(2(p-1)+1)\left(Q_t+\tilde{K}\right)^2\int_u^t \left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds$$

So

$$\left|\mathbb{E}\,\tilde{J}_t\right| \leq |p|\,(2(p-1)+1)\left(Q_t+\tilde{K}\right)^2 \int_u^t \mathbb{E}\left(1+\left|\tilde{X}_s(x)\right|^2\right)^p ds \tag{3.4}$$

Summing up these two inequalities 3.3 and 3.4, we obtain

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \left(1+|x|^2\right)^p + \,const \times \int_u^t \mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds$$

By Gronwall's inequality, we have

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \left(1+|x|^2\right)^p \times \exp\left(\tilde{C}_{p,u}\right)$$

such that

$$\tilde{C}_{p,u} = \,const \times \int_u^t \mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p ds$$

and

$$\tilde{K}^3_{p,u} = \exp\left(\tilde{C}_{p,u}\right)$$

So, we have the inequality of the lemma 2.3

$$\mathbb{E}\left(1+|\tilde{X}_s(x)|^2\right)^p \leq \tilde{K}^3_{p,u}\left(1+|x|^2\right)^p$$

Now, taking negative p in the above calculus, we see that $\left|\tilde{X}_t(x)\right|$ tends to infinity in probability as x tends sequencially to infinity. We shall prove a stronger convergence.

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one point compactification of $\mathbb{R}$. Set

$$\tilde{X}^u_t(x) = x + \int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s f\,\left(\tilde{X}_s - (1-Z_s)\right) dY_s$$

$$\tilde{\eta}_t(x) = \begin{cases} \dfrac{1}{1+\left|\tilde{X}_t(x)\right|} & \text{if } x \in \mathbb{R} \\ 0 & \text{if } x = \infty \end{cases}$$

Evidently $\tilde{\eta}_t(x)$ is continuous in $[0,\infty)\times\mathbb{R}$. Thus just to prove the continuity in the vicinity of infinity. Suppose $p > 2$. It holds

$$|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p \leq \tilde{\eta}_t(x)^p \tilde{\eta}_s(y)^p \left|\tilde{X}_t(x) - \tilde{X}_s(y)\right|^p$$

By Hölder inequality, proposition 2.1 and lemma 2.3, we have

$$\begin{aligned} \mathbb{E}\,|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p &\leq \left(\mathbb{E}\,\tilde{\eta}_t(x)^{4p}\right)^{\frac{1}{4}} \left(\mathbb{E}\,\tilde{\eta}_s(y)^{4p}\right)^{\frac{1}{4}} \left(\mathbb{E}|\tilde{X}_t(x) - \tilde{X}_s(y)|^{2p}\right)^{\frac{1}{2}} \\ &\leq \tilde{C}_{p,T}\,(1+|x|)^{-p}\,(1+|y|)^{-p}\left(|x-y|^p + |t-s|^{\frac{p}{2}}\right) \end{aligned}$$

if $t, s \in [0,T]$ and $x, y \in \mathbb{R}$, where $\tilde{C}_{p,T}$ is a positive constant. Set

$$\frac{1}{x} = x^{-1}$$

Since

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \leq \left|\frac{1}{x} - \frac{1}{y}\right|$$

We get the inequality

$$\mathbb{E}\,|\tilde{\eta}_t(x) - \tilde{\eta}_s(y)|^p \leq \tilde{C}_{p,T}\left(\left|\frac{1}{x} - \frac{1}{y}\right|^p + |t-s|^{\frac{p}{2}}\right)$$

Define

$$\bar{\eta}_t(x) = \begin{cases} \tilde{\eta}_t(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then the above inequality implies

$$\mathbb{E}|\bar{\eta}_t(x) - \bar{\eta}_s(y)|^p \leq \tilde{C}_{p,T}\left(|x-y|^p + |t-s|^{\frac{p}{2}}\right),\ x \neq 0,\ y \neq 0$$

In case $y = 0$, we have

$$\mathbb{E}|\bar{\eta}_t(x)|^p \leq \tilde{C}_{p,T}\,|x|^p$$

Therefore $\bar{\eta}_t(x)$ is continuous in $[0,\infty) \times \mathbb{R}$ by Kolmogorov's theorem. This proves that $\tilde{\eta}_t(x)$ is continuous in $[0,\infty) \times$ neighborhood of infinity.

So, define a stochastic process $\bar{X}_t$ on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by

$$\bar{X}_t(x) = \begin{cases} \tilde{X}_t(x) & \text{if } x \in \mathbb{R} \\ \infty & \text{if } x = \infty \end{cases}$$

Then $\bar{X}_t(x)$ is continuous sur $[0,\infty) \times \mathbb{R}$ by the previous lemma. Thus, for each $t > 0$, the map $\bar{X}_t(.,\omega)$ is homotopic to the identity map on $\bar{\mathbb{R}}$. Then $\bar{X}_t(.,\omega)$ is an onto map of $\bar{\mathbb{R}}$ by a well known homotopic theory. Now, the map $\bar{X}_t$ is a homeomorphism of $\bar{\mathbb{R}}$, since it is one to one, onto and continuous. Since ∞ is the invariant point of the map $\bar{X}_t$, we see that $\tilde{X}_t$ is a homeomorphism of $\mathbb{R}$. This completes the proof of theorem 2.1.

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SOME NEW IDENTITIES FOR THE SECOND COVARIANT DERIVATIVE OF THE CURVATURE TENSOR

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Abstract. In this paper, we have studied the second covariant derivative of Riemannian curvature tensor. Some new identities for the second covariant derivative have been given. Namely, identities obtained by cyclic sum with respect to three indices have been given. In the first case, two curvature tensor indices and one covariant derivative index participate in the cyclic sum, while in the second case one curvature tensor index and two covariant derivative indices participate in the cyclic sum.
Keywords: covariant derivative, curvature tensor, Riemannian manifold, second order identity

1. Introduction

The Riemannian curvature tensor R^i_{jmn} is very important in Riemannian manifold, especially when studying the theory of general relativity and quantum gravity (see [1, 8, 23]). Knowledge of the properties of curvature tensor is of great importance when studying the manifolds mentioned. Some other geometric object can be defined using curvature tensor, for example Ricci curvature tensor, scalar curvature, Weyl tensor, etc. In the articles [2, 3, 20], the curvature tensor was studied at various mappings and transformations (see also the monographs [4] and [7]).

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Initially, the idea was to use three indices in cyclic sum, and thus some of the properties of the Riemannian curvature tensor were proved (the first and the second Bianchi identities). The idea of a cyclic sum was continued in the paper [6], but in the summation four indices were used: two indices of curvature tensor and two indices of covariant derivative. In the present aricle we have given the new identities for cyclic summing of the second covariant derivatives with respect to three indices. We will see that one of these identities implies Lovelock differential identity.

2. Preliminaries

Let us consider the Riemannian manifold $(\mathcal{M}_N, g)$, where $\mathcal{M}_N$ is N-dimensional manifold and g is a symmetric metric tensor. The Christoffel symbols of the first kind $\Gamma_{i\cdot jk}$ and the Christoffel symbols of the second kind Γ^i_{jk} of Riemannian manifold are defined as

$$\Gamma_{i\cdot jk} = \frac{1}{2}\left(g_{ij,k} - g_{jk,i} + g_{ki,j}\right), \tag{2.1}$$

$$\Gamma^i_{jk} = g^{ip}\Gamma_{p\cdot jk} = \frac{1}{2}g^{ip}\left(g_{pj,k} - g_{jk,p} + g_{kp,j}\right), \tag{2.2}$$

where g_{ij} and g^{ij} is the covariant and contravariant metric tensor, respectively. Hereinafter, the coma $(,)$ denotes partial derivative.

In the general case, the partial derivative of a tensor is not always a tensor, and therefore the term covariant derivative is introduced. We will use the semicolon $(;)$ for a covariant derivative in a Riemannian manifold. The covariant derivative with respect to the Christoffel symbols Γ^i_{jk} is defined as

$$t^{i_1\ldots i_A}_{j_1\ldots j_B;k} = t^{i_1\ldots i_A}_{j_1\ldots j_B,k} + \sum_{p=1}^{A} t^{i_1\ldots i_{\alpha-1}p i_{\alpha+1}\ldots i_A}_{j_1\ldots j_B}\Gamma^{i_\alpha}_{pk} - \sum_{p=1}^{B} t^{i_1\ldots i_A}_{j_1\ldots j_{\alpha-1}p j_{\alpha+1}\ldots j_B}\Gamma^p_{j_\alpha k}, \tag{2.3}$$

where $t^{i_1\ldots i_A}_{j_1\ldots j_B}$ is an arbitrary tensor. The Riemannian curvature tensor R^i_{jmn} of a Riemannian manifold is obtained based on Ricci identity

$$t^{i_1\ldots i_A}_{j_1\ldots j_B;mn} - t^{i_1\ldots i_A}_{j_1\ldots j_B;nm} = \sum_{p=1}^{A} t^{i_1\ldots i_{\alpha-1}p i_{\alpha+1}\ldots i_A}_{j_1\ldots j_B} R^{i_\alpha}_{pmn} - \sum_{p=1}^{B} t^{i_1\ldots i_A}_{j_1\ldots j_{\alpha-1}p j_{\alpha+1}\ldots j_B} R^p_{j_\alpha mn}, \tag{2.4}$$

where

$$R^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}. \tag{2.5}$$

Also, the Riemannian curvature tensor can be expressed in the form

$$R^i_{jmn} = \Gamma^i_{j[m,n]} + \Gamma^p_{j[m}\Gamma^i_{n]p}, \tag{2.6}$$

where $[ij]$ denotes alternation without division with respect to the indices i and j (for example, $a_{[ij]} = a_{ij} - a_{ji}$). For Ricci identity, we will use the notation below

$$t^{i_1\dots i_A}_{j_1\dots j_B;mn} - t^{i_1\dots i_A}_{j_1\dots j_B;nm} = t^{i_1\dots i_A}_{j_1\dots j_B;[mn]}. \tag{2.7}$$

The Riemannian curvature tensor has the following properties

1. $R^i_{jmn} = -R^i_{jnm}$, (anti-symmetry)
2. $\underset{jmn}{Cycl}\, R^i_{jmn} = 0$, (the first Bianchi identity)
3. $\underset{mnu}{Cycl}\, R^i_{jmn;u} = 0$, (the second Bianchi identity)

where $\underset{jmn}{Cycl}$ is the cyclic sum by indices j, m, n.

The covariant curvature tensor of a Riemannian manifold is defined as

$$R_{ijmn} = g_{ip} R^p_{jmn}, \tag{2.8}$$

and has the following properties:

1. $R_{ijmn} = -R_{jimn} = -R_{ijnm}$,
2. $R_{ijmn} = R_{mnij}$,
3. $\underset{\alpha\beta\gamma}{Cycl}\, R_{ijmn} = 0, \ \ \{\alpha, \beta, \gamma\} \subset \{i, j, m, n\}$,
4. $\underset{mnu}{Cycl}\, R_{ijmn;u} = 0$.

Oswald Veblen showed that the following identity

$$R^i_{jmn;u} - R^i_{mju;n} + R^i_{unm;j} - R^i_{nuj;m} = 0, \tag{2.9}$$

is correct [21].

Theorem 2.1. *[6] For the curvature tensor R^i_{jmn} the identity*

$$\underset{mnuv}{Cycl}\, R^i_{jmn;uv} = \underset{mnuv}{Cycl}\, R^i_{jpm} R^p_{nuv} - R^i_{pmu} R^p_{jnv} + R^i_{pnv} R^p_{jmu}. \tag{2.10}$$

is valid.

By contracting by indices i and v in equation (2.10), one obtains the Lovelock differential identity (see [6])

$$\underset{mnu}{Cycl}\, R^p_{jmn;pu} = -\underset{mnu}{Cycl}\, R^p_{jmn} R_{pu}, \tag{2.11}$$

where R_{jm} is the Ricci curvature tensor, i.e. $R_{jm} = R^p_{jmp}$.

Theorem 2.2. *[22] The covariant curvature tensor of a Riemannian manifold satisfies the identity*

$$R_{ijmn;[uv]} + R_{mnuv;[ij]} + R_{uvij;[mn]} = 0. \tag{2.12}$$

Definition 2.1. The Riemannian manifold $(\mathcal{M}_N, g)$ is symmetric Riemannian manifold if a curvature tensor satisfies

$$R^i_{jmn;u} = 0. \tag{2.13}$$

The Riemannian manifold $(\mathcal{M}_N, g)$ is semi-symmetric if a curvature tensor satisfies

$$R^i_{jmn;[uv]} = 0. \tag{2.14}$$

3. Results

In this section, we will present new results for the cyclic sum of the second covariant derivatives of Riemannian curvature tensor.

Let us consider the second Bianchi identity

$$\underset{mnu}{Cycl}\, R^i_{jmn;u} = 0. \tag{3.1}$$

By covariant derivative of this equation by index v we get the equation

$$\underset{mnu}{Cycl}\, R^i_{jmn;uv} = 0. \tag{3.2}$$

In the same way, we have the following identities

$$\underset{muv}{Cycl}\, R^i_{jmu;vn} = 0, \quad \underset{mvn}{Cycl}\, R^i_{jmv;nu} = 0. \tag{3.3}$$

Summing the obtained expressions (3.2) and (3.3), we have equation

$$\begin{aligned} 0 &= \underset{mnu}{Cycl}\, R^i_{jmn;uv} + \underset{muv}{Cycl}\, R^i_{jmu;vn} + \underset{mvn}{Cycl}\, R^i_{jmv;nu} \\ &= R^i_{jmn;uv} + R^i_{jnu;mv} + R^i_{jum;nv} + R^i_{jmu;vn} + R^i_{juv;mn} + R^i_{jvm;un} \\ &+ R^i_{jmv;nu} + R^i_{jvn;mu} + R^i_{jnm;vu}. \end{aligned} \tag{3.4}$$

From here, using every third addend from the previous equation, we get the identity

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} + \underset{nuv}{Cycl}\, R^i_{jnu;mv} - \underset{nuv}{Cycl}\, R^i_{jmn;vu} = 0, \tag{3.5}$$

i.e.

$$\underset{nuv}{Cycl}\left(R^i_{jmn;[uv]} + R^i_{jnu;mv}\right) = 0. \tag{3.6}$$

If we consider the Ricci identity (2.4) for $R^i_{jmn;[uv]}$, from equation (3.6) we obtain

$$\underset{nuv}{Cycl}\left(R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv} - R^i_{jmp}R^p_{nuv} + R^i_{jnu;mv}\right) = 0. \tag{3.7}$$

Since that $\underset{nuv}{Cycl}\, R^p_{nuv} = 0$ (the first Bianchi identity), it follows

$$\underset{nuv}{Cycl}\, R^i_{jnu;mv} = -\underset{nuv}{Cycl}\left(R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv}\right), \tag{3.8}$$

i.e.

$$\underset{nuv}{Cycl}\, R^i_{jnu;mv} = \underset{nuv}{Cycl}\left(R^i_{pmn}R^p_{juv} + R^i_{jpn}R^p_{muv} - R^p_{jmn}R^i_{puv}\right). \tag{3.9}$$

After changing the indices $n \to m$, $u \to n$, $m \to u$, we obtain

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right) \tag{3.10}$$

and with this we have proved the following theorem.

Theorem 3.1. *Let $(\mathcal{M}_N, g)$ be a Riemannian manifold. The Riemannian curvature tensor satisfies the identity*

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right), \tag{3.11}$$

where $\underset{mnv}{Cycl}$ is the cyclic sum with respect to the indices m, n, v.

Corollary 3.1. *Contraction by indices i and u in equation (3.11) gives the Lovelock differential identity (2.11).*

Proof.

$$\begin{aligned}
\underset{mnv}{Cycl}\, R^p_{jmn;pv} &= \underset{mnv}{Cycl}\left(R^p_{spm}R^s_{jnv} + R^p_{jsm}R^s_{pnv} - R^s_{jpm}R^p_{snv}\right)\\
&= \underset{mnv}{Cycl}\left(-R^p_{smp}R^s_{jnv}\right) + \underset{mnv}{Cycl}\left(R^p_{jsm}R^s_{pnv} - R^s_{jpm}R^p_{snv}\right)\\
&= -\underset{mnv}{Cycl}\, R_{sm}R^s_{jnv} + \underset{mnv}{Cycl}\left(R^p_{jsm}R^s_{pnv} - R^p_{jsm}R^s_{pnv}\right)\\
&= -\underset{mnv}{Cycl}\, R_{sm}R^s_{jnv},
\end{aligned} \tag{3.12}$$

i.e.

$$\underset{mnv}{Cycl}\, R^p_{jmn;pv} = -\underset{mnv}{Cycl}\, R^p_{jmn}R_{pv}. \tag{3.13}$$

□

If we add an expression $-\underset{nuv}{Cycl}\, R^i_{jnu;vm} = 0$ to the equation (3.6), then we have the following consequence.

Corollary 3.2. *The Riemannian curvature tensor satisfy the identity*

$$\underset{nuv}{Cycl}\left(R^i_{jmn;[uv]} + R^i_{jnu;[mv]}\right) = 0, \tag{3.14}$$

where $[ij]$ denotes alternation without division with respect to the indices i and j.

After applying Ricci identity, the previous equation takes the form

$$\begin{aligned}\underset{mnv}{Cycl}(R^p_{jmn}R^i_{puv} - R^i_{pmn}R^p_{juv} - R^i_{jpn}R^p_{muv} + R^p_{jnu}R^i_{pmv}\\ - R^i_{pnu}R^p_{jmv} - R^i_{jpu}R^p_{nmv} - R^i_{jnp}R^p_{umv}) = 0.\end{aligned} \tag{3.15}$$

Based on Theorem (3.1) we have the consequence.

Corollary 3.3. *In a semi-symmetric Riemannian manifold the following identity*

$$\underset{mnv}{Cycl}\left(R^i_{pum}R^p_{jnv} + R^i_{jpm}R^p_{unv} - R^p_{jum}R^i_{pnv}\right) = 0. \tag{3.16}$$

holds.

Proof. Given the fact that in semi-symmetric Riemannian manifold the following is valid

$$R^i_{jmn;uv} = R^i_{jmn;vu}, \tag{3.17}$$

i.e.

$$\underset{mnv}{Cycl}\, R^i_{jmn;uv} = \underset{mnv}{Cycl}\, R^i_{jmn;vu}, \tag{3.18}$$

and since $\underset{mnv}{Cycl}\, R^i_{jmn;vu} = 0$ (the second Bianchi identity), it follows that the left hand side of equation (3.11) is equal to zero, thus completing the proof. □

Corollary 3.4. *The equation (3.16) is valid in symmetric Riemannian manifold.*

Below we present the result obtained by cyclic sum of the second covariant derivatives of curvature tensor, when one curvature tensor index and two covariant derivative indices participate in the cyclic sum.

Theorem 3.2. *Let $(\mathcal{M}_N, g)$ be a Riemannian manifold. The Riemannian curvature tensor satisfy the following identity*

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl}\Bigg(C^i_{jmnuv} - R^i_{jmn,uv} + R^i_{jmn,p}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p} - R^p_{jmn}R^i_{uvp} + R^i_{psn}B^{sp}_{muvj} + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm} + \sum_{\beta=1}^{3}\Big(R^i_{j_1pj_3}A^p_{j_\beta uv} - R^p_{j_1sj_3}B^{si}_{j_\beta uvp}\Big)\Bigg), \tag{3.19}$$

where

$$A^i_{jmn} = -\Gamma^i_{jm,n} + \Gamma^p_{jn}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{pj},$$

$$B^{pi}_{jmnu} = \Gamma^p_{jm}\Gamma^i_{nu} + \Gamma^p_{jn}\Gamma^i_{mu},$$

$$C^i_{jmnuv} = C^i_{jmnu,v} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv} - C^i_{jmnp}\Gamma^p_{uv},$$

$$C^i_{jmnu} = R^i_{jmn,u} + R^i_{jmu,n},\ j_1 = j,\ j_2 = m,\ j_3 = n,$$

and $\underset{nuv}{Cycl}$ is the cyclic sum with respect to the indices n, u, v.

Proof. First, we have identity

$$\underset{nuv}{Cycl}\, R^i_{jmn;uv} = R^i_{jmn;uv} + R^i_{jmu;vn} + R^i_{jmv;nu} = \big(R^i_{jmn;u}\big)_{;v} + \big(R^i_{jmu;v}\big)_{;n} + \big(R^i_{jmv;n}\big)_{;u}.$$

Further, we get the following equation

$$\begin{aligned}\big(R^i_{jmn;u}\big)_{;v} &+ \big(R^i_{jmu;v}\big)_{;n} + \big(R^i_{jmv;n}\big)_{;u} =\\ &= \big(R^i_{jmn;u}\big)_{,v} + R^p_{jmn;u}\Gamma^i_{pv} - R^i_{pmn;u}\Gamma^p_{jv} - R^i_{jpn;u}\Gamma^p_{mv} - R^i_{jmp;u}\Gamma^p_{nv} - R^i_{jmn;p}\Gamma^p_{uv}\\ &+ \big(R^i_{jmu;v}\big)_{,n} + R^p_{jmu;v}\Gamma^i_{pn} - R^i_{pmu;v}\Gamma^p_{jn} - R^i_{jpu;v}\Gamma^p_{mn} - R^i_{jmp;v}\Gamma^p_{un} - R^i_{jmu;p}\Gamma^p_{vn}\\ &+ \big(R^i_{jmv;n}\big)_{,u} + R^p_{jmv;n}\Gamma^i_{pu} - R^i_{pmv;n}\Gamma^p_{ju} - R^i_{jpv;n}\Gamma^p_{mu} - R^i_{jmp;n}\Gamma^p_{vu} - R^i_{jmv;p}\Gamma^p_{nu}.\end{aligned}$$

After developing the remaining covariant derivatives on the right hand side of equality and grouping expressions using basic operations for the Ricci calculus, we get

$$\begin{aligned}\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl}\Bigg(& R^i_{jmn,uv} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv}\\ & - R^i_{jmp,n}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p} - R^p_{jmn}R^i_{uvp} + R^i_{pmn}A^p_{juv} + R^i_{jpn}A^p_{muv}\\ & + R^i_{jmp}A^p_{nuv} - R^p_{smn}B^{si}_{juvp} - R^p_{jsn}B^{si}_{muvp} - R^p_{jms}B^{si}_{nuvp} + R^i_{psn}B^{sp}_{muvj}\\ & + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm}\Bigg),\end{aligned}$$

where

$$A^i_{jmn} = -\Gamma^i_{jm,n} + \Gamma^p_{jn}\Gamma^i_{pm} + \Gamma^p_{mn}\Gamma^i_{pj}, \quad B^{pi}_{jmnu} = \Gamma^p_{jm}\Gamma^i_{nu} + \Gamma^p_{jn}\Gamma^i_{mu},$$
$$C^i_{jmnu} = R^i_{jmn,u} + R^i_{jmu,n}.$$

If we introduce notation

$$C^i_{jmnuv} = C^i_{jmnu,v} + C^p_{jmnu}\Gamma^i_{pv} - C^i_{pmnu}\Gamma^p_{jv} - C^i_{jpnu}\Gamma^p_{mv} - C^i_{jmpu}\Gamma^p_{nv} - C^i_{jmnp}\Gamma^p_{uv},$$

the previous equation takes the form

$$\begin{aligned}\underset{nuv}{Cycl}\, R^i_{jmn;uv} = \underset{nuv}{Cycl} \Big(& C^i_{jmnuv} - R^i_{jmu,nv} + C^i_{jmnp}\Gamma^p_{uv} - R^i_{jmp,n}\Gamma^p_{uv} + R^p_{jmn}\Gamma^i_{uv,p} \\ & - R^p_{jmn}R^i_{uvp} + R^i_{pmn}A^p_{juv} + R^i_{jpn}A^p_{muv} + R^i_{jmp}A^p_{nuv} - R^p_{smn}B^{si}_{juvp} \\ & - R^p_{jsn}B^{si}_{muvp} - R^p_{jms}B^{si}_{nuvp} + R^i_{psn}B^{sp}_{muvj} + R^i_{pms}B^{sp}_{nuvj} + R^i_{jps}B^{sp}_{nuvm}\Big)\end{aligned}$$

and, from here, after rearranging, we obtain identity (3.19). This ends the proof. □

4. Conclusion

The first part of the Results section was devoted to the result we obtained by cyclic sum with respect to two indices of curvature tensor and one index of covariant derivative, i.e. $\underset{mnv}{Cycl}\, R^i_{jmn;uv}$. Due to anti-symmetry property of Riemannian curvature tensor R^i_{jmn}, the result we got has a simple form. Following the identity (3.11) obtained, we also listed three consequences implied by Theorem 3.1. In the second part of Results section, we present the cyclic sum $\underset{nuv}{Cycl}\, R^i_{jmn;uv}$ over known quantities, i.e. Riemannian curvature tensor and Christoffel symbols of the second kind.

For further research, one can observe cyclic sum of the second covariant derivatives in other manifolds, as the curvature tensor is an interesting geometric object in other manifolds [25], as well as in studying various mappings and transformations in other manifolds (see [5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 24, 26, 27]).

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η-RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON δ-LORENTZIAN TRANS-SASAKIAN MANIFOLDS

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Abstract. The objective of the present research article is to study the δ-Lorentzian trans-Sasakian manifolds concidering the η-Ricci solitons and gradient Ricci soliton. We have shown that a symmetric second order covariant tensor in a δ-Lorentzian trans-Sasakian manifold is a constant multiple of metric tensor. Also, we have provided an example of η-Ricci soliton on 3-diemsional δ-Lorentzian trans-Sasakian manifold in the region where δ-Lorentzian trans-Sasakian manifold is expanding. Furthermore, we have discussed the results based on gradient Ricci solitons on 3-dimensional δ- Lorentzian trans-Sasakian manifold.
Keywords: η-Ricci Soliton, Gradient Ricci Soliton, δ-Lotentzian trans-Sasakian manifolds, Einstein manifolds

1. Introduction

In geometrical analysis, a differentiable manifolds endowed Lorentzian metric having signature $(-, +, +, \cdots, +)$ is a absolutely fascinating topic in Lorentzian geometry. Matsumoto [19] popularized the study of Lorentzian para-contact manifolds with Lorentzian metric. Ikawa and Erdogan [16] discussed Lorentzian Sasakian manifold. In [38], Yildiz et al. studied Lorentzian α-Sasakian manifold and Lorentzian β-Kenmotsu manifold studied by Funda et al. in [37]. After that, Pujar and Khairnar

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[22] have initiated the notion of Lorentzian trans-Sasakian manifolds and studied some basic results with some of its properties. Before that, Pujar had initiated the study of δ-Lorentzian $\alpha-$Sasakian manifolds and δ-Lorentzian $\beta-$Kenmotsu manifolds ([22], [23]). In [11], De also studied properties of curvatures in Lorentzian trans-Sasakian manifolds which is closely related to this subject.

The interplay between manifolds and indefinite metrics is of interest from the overview of physics and relativity. In 1969, Takahashi [32] introduced the notion of almost contact metric manifolds equipped with pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are known as (ϵ)-almost contact metric manifolds [36]. The concept of (ϵ)-Sasakian manifolds was initiated by Bejancu and Duggal [3]. De and Sarkar [9] studied the notion of (ϵ)-Kenmotsu manifolds. Shukla and Singh [25] extended the study to (ϵ)-trans-Sasakian manifolds with indefinite metric. The semi-Riemannian manifolds has the index 1 and the structure vector field ξ is always a timelike. This motivated the Tripathi et al. [33] to introduce (ϵ)-almost para contact structure where the vector field ξ is spacelike or timelike according to $(\epsilon) = 1$ or $(\epsilon) = -1$.

If M has a Lorentzian metric g, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1-dimensional distribution. Since odd dimensional manifold is able to have a Lorentzian metric. Inspired from the previous results, Bhati [1] developed the notion of δ-Lorentzian trans-Sasakian manifolds.

On the other hand, in 1982, Hamilton [14] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, which means they are stationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2S(g). \tag{1.1}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$L_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function ψ , then g is called a gradient Ricci soliton and equation 1.2 assumes the form $\nabla\nabla\psi = S + \lambda g$.

The roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory.

In 1925, Levy [17] obtained the necessary and sufficient conditions for the existence of such tensors. Later on, R. Sharma [24] initiated the study of Ricci solitons in contact Riemannian geometry. Bagewadi et al. [15] extensively studied Ricci soliton in almost (ϵ, δ)-trans-Sasakian manifolds. In 2009, Cho and Kimura [8] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. In addition, η-Ricci solitons with various structures have been studied by various geometers such as Calin and Crasmareanu [7] and Blaga ([4], [5]). Recently, Venu at al. [35] study the η-Ricci soliton in trans-Sasakian manifold. The first author of the paper also studied some properties of η-Ricci solitons on (ϵ, δ)-trans-Sasakian manifold and normal almost contact manifolds which is merely connected to this topic (for more details see [27], [28], [29], [30], [31]). Therefore, it is natural and interesting to study η-Ricci soliton on δ-Lorentzian trans-Sasakian manifolds. In this paper, we derive the condition for a 3 dimensional δ-Lorentzian trans-Sasakian manifold whose metric as an η-Ricci soliton and derive expression for the scalar curvature.

2. Preliminaries

Let M be an δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \delta\eta(X)\eta(Y), \quad \eta(X) = \delta g(X, \xi), \quad g(\xi,\xi) = -\delta, \tag{2.2}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the $\delta-$Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [34] on M, the vector field ξ is the timelike that is M contains a timelike vector field. In [34], Tanno classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tanno [34].

Gray and Harvella [13] introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kahler manifolds. The class $C_6 \oplus C_5$ [13] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely.

An almost contact metric structure on M is called a trans-Sasakian [21] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4, where J is almost complex structure on $M \times \mathbb{R}$ defined by

$$J\left(X, \psi\frac{d}{dt}\right) = \left(\phi(X) - \psi\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions ψ on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature now we define the δ-Lorentzian trans-Sasakian manifolds as follows.

Definition 2.1. A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X,Y)\xi - \delta\eta(Y)\phi X) \tag{2.4}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans-Sasakian manifold is the usual Lorentzian trans-Sasakian manifold of type (α, β) [11]. δ-Lorentzian trans-Sasakian manifold of type $(0,0)$, $(0,\beta)$ $(\alpha,0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans-Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively.

From (2.4), we have

$$\nabla_X \xi = \delta \left\{-\alpha\phi(X) - \beta(X + \eta(X)\xi\right\}, \tag{2.5}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X,Y) + \delta\eta(X)\eta(Y)]. \tag{2.6}$$

In a δ-Lorentzian trans-Sasakian manifold M, we have the following relations:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \end{aligned} \tag{2.7}$$

$$S(X,\xi) = [((n-1)(\alpha^2+\beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.8}$$

$$Q\xi = \delta(n-1)(\alpha^2+\beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.9}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Further in an δ-Lorentzian trans-Sasakian manifold, we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta) \tag{2.10}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.11}$$

By using (2.7) and (2.10), for constants α and β, we have

$$R(\xi, X)Y = (\alpha^2 + \beta^2)[\delta g(X, Y)\xi - \eta(Y)X], \tag{2.12}$$

$$R(X, Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.13}$$

$$\eta(R(X, Y)Z) = \delta(\alpha^2 + \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.14}$$

$$S(X, \xi) = [((n-1)(\alpha^2 + \beta^2) - \delta(\xi\beta)]\eta(X), \tag{2.15}$$

$$Q\xi = [(n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\xi. \tag{2.16}$$

An important consequence of (2.5) is that ξ is a geodesic vector field

$$\nabla_\xi \xi = 0. \tag{2.17}$$

For arbitrary X vector field, we have that

$$d\eta(\xi, X) = 0. \tag{2.18}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.13), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 + \beta^2) - \delta(\xi\beta). \tag{2.19}$$

It follows from (2.19) that ξ-sectional curvature does not depend on X.

3. η-Ricci solitons on $(M, \phi, \xi, \eta, g, \delta)$

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [12],

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and using (2.7) and also use the symmetry of h, we have

$$\begin{aligned} &2(\alpha^2 + \beta^2)[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] + 2\delta[(Y\alpha)h(\phi X, \xi) - (X\alpha)h(\phi Y, \xi)] \\ &+2\delta[(Y\beta)h(\phi^2 X, \xi) - (X\beta)h(\phi^2 Y, \xi)] + 4\alpha\beta[\eta(Y)h(\phi X, \xi) - \eta(X)h(\phi Y, \xi)]. \end{aligned} \tag{3.3}$$

Adopting $X = \xi$ in (3.3) and by virtue of (2.1), we turn up

$$-2[(\delta\xi\alpha - 2\alpha\beta]h(\phi Y, \xi) + 2[(\alpha^2 + \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0. \tag{3.4}$$

By adopting (2.11) in (3.4), we have

$$[(\alpha^2+\beta^2)-\delta(\xi\beta)][\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{3.5}$$

Suppose $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, it results

$$h(Y,\xi)=\eta(Y)h(\xi,\xi). \tag{3.6}$$

Now, we call a regular δ-Lorentzian trans-Sasakian manifold with $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of δ-Lorentzian trans-Sasakian manifolds.

Differentiating (3.6) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi)+h(\nabla_X Y,\xi)+h(Y,\nabla_X\xi) &= [\delta g(\nabla_X Y,\xi)+\delta g(Y,\nabla_X\xi)]h(\xi,\xi)\\ &\quad+\eta(Y)[(\nabla_X h)(Y,\xi)+2h((\nabla_X\xi,\xi)].\end{aligned} \tag{3.7}$$

By adopting the parallel condition $\nabla h=0$, $\eta(\nabla_X\xi)=0$ and by the virtue of (3.6) in (3.7), we get

$$h(Y,\nabla_X\xi)=\delta g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now adopting (2.5) in the above equation, we turn up

$$-\alpha h(Y,\phi X)+\beta\delta h(Y,X)=-\alpha g(Y,\phi X)h(\xi,\xi)+\beta\delta g(Y,X)h(\xi,\xi). \tag{3.8}$$

Replacing $X=\phi X$ in (3.8) and after simplification, we turn up

$$h(X,Y)=\delta g(X,Y)h(\xi,\xi), \tag{3.9}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.6). Now by considering the above equations, we can gives the conclusion:

Theorem 3.1. *Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-Lorentzian trans-Sasakian manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h\in\Gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

Definition 3.1. Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-almost contact metric manifold. Consider the equation

$$L_\xi g+2S+2\lambda g+2\mu\eta\otimes\eta=0, \tag{3.10}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y)=-g(\nabla_X\xi,Y)-g(X,\nabla_X\xi)-2\lambda g(X,Y)-2\mu\eta(X)\eta(Y), \tag{3.11}$$

for any $X,Y\in\chi(M)$.

The data (g,ξ,λ,μ) which satisfies the equation (3.10) is said to be *η-Ricci soliton* on M [5]; in particular if $\mu=0$ then (g,ξ,λ) is Ricci soliton [5] and its called shrinking, steady or expanding according as $\lambda<0$, $\lambda=0$ or $\lambda>0$ respectively [5].

Now, from (2.5), the equation (3.10) becomes:

$$S(X,Y) = -(\lambda+\beta\delta)g(X,Y) + (\beta\delta-\mu)\eta(X)\eta(Y). \tag{3.12}$$

The above equations yields

$$S(X,\xi) = -(\lambda+\mu)\eta(X) \tag{3.13}$$

$$QX = -(\lambda+\beta\delta)X + (\beta\delta-\mu)\xi \tag{3.14}$$

$$Q\xi = -(\lambda+\mu)\xi \tag{3.15}$$

$$r = -\lambda n - (n-1)\beta\delta - \mu, \tag{3.16}$$

where r is the scalar curvature. Of the two natural situations regarding the vector field V such that $V \in span\{\xi\}$ and $V \perp \xi$, we investigate only the case for $V = \xi$.

Our interest is in the expression for $L_\xi g + 2S + 2\mu\eta\otimes\eta$. A direct computation gives

$$L_\xi g(X,Y) = 2\beta\delta[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.17}$$

In 3-dimensional δ-Lorentzian trans-Sasakian manifold the Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\tfrac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned}$$

Putting $Z = \xi$ in (3.18) and using (2.7) and (2.8) for 3-dimensional δ-Lorentzian trans-Sasakian manifold, we get

$$\begin{aligned} &(\alpha^2+\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ &= [(\alpha^2+\beta^2) - (\xi\beta)][\eta(Y)X - \eta(X)Y] \\ &+\delta\eta(Y)QX - \delta\eta(X)QY - \delta[((\phi Y)\alpha)X + (Y\beta)X] \\ &+\delta[((\phi X)\alpha)Y + (X\beta)Y]. \end{aligned} \tag{3.18}$$

Again, putting $Y = \xi$ in the (3.19) and using (2.1) and (2.11), we turn up

$$QX = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2+\beta^2)\right]X + \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2+\beta^2)\right]\eta(X)\xi. \tag{3.19}$$

From (3.20), we have

$$S(X,Y) = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2+\beta^2)\right]g(X,Y) \tag{3.20}$$

$$+\left[\frac{r}{2}+(\xi\beta)-3(\alpha^2+\beta^2)\right]\delta\eta(X)\eta(Y).$$

Equation (3.21) shows that a 3-dimensional δ-Lorentzian trans-Sasakian manifold is η-Einstein.
Next, we consider the equation

$$h(X,Y)=(L_\xi g)(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y). \tag{3.21}$$

By using (3.17) and (3.21) in (3.22), we have

$$h(X,Y)=\left[r-4(\alpha^2+\beta^2)+2\beta\delta\right]g(X,Y) \tag{3.22}$$

$$+\left[8(\alpha^2+\beta^2)-2\beta\delta-r\right]\delta\eta(X)\eta(Y)+2\mu\eta(X)\eta(Y).$$

Setting $X=Y=\xi$ in (2.3), we turn up

$$h(\xi,\xi)=2[2\delta(\alpha^2+\beta^2)-2\mu]. \tag{3.23}$$

Now, (3.9) becomes

$$h(X,Y)=2[2\delta(\alpha^2+\beta^2)-2\mu]\delta g(X,Y). \tag{3.24}$$

From (3.22) and (3.25), it follows that g is an η-Ricci soliton.
Therefore, we can state as:

Theorem 3.2. *Let $(M,\phi,\xi,\eta,g,\delta)$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold, then (g,ξ,μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V=b\xi$, where b is a function on the 3-dimensional δ-Lorentzian trans-Sasakian manifold. Then

$$g(\nabla_X b\xi,Y)+g(\nabla_Y b\xi,X)+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0$$

or

$$bg((\nabla_X\xi,Y)+(Xb)\eta(Y)+bg(\nabla_Y\xi,X)+(Yb)\eta(X)$$

$$+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0.$$

By using (2.5), we obtain

$$bg(-\delta\alpha\phi X-\beta\delta(X+\eta(X)\xi,Y)+(Xb)\eta(Y)+bg(-\delta\alpha\phi Y-\beta\delta(Y+\eta(Y)\xi,X)$$

$$+(Yb)\eta(X)+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0$$

which yields

$$-2b\beta\delta g(X,Y)-2b\beta\delta\eta(X)\eta(Y)+(Xb)\eta(Y) \tag{3.25}$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (3.26), we get

$$(Xb) + (\xi b)\eta(X) + 2[2(\alpha^2 + \beta^2) - (\xi\beta) + \lambda + \mu - 2b\beta\delta]\eta(X). \tag{3.26}$$

Again putting $X = \xi$ in (3.27), we obtain

$$\xi b = -2(\alpha^2 + \beta^2) + (\xi\beta) - \lambda - \mu + 2b\beta\delta.$$

Plugging this in (3.27), we get

$$(Xb) + 2[2(\alpha^2 + \beta^2) - (\xi\beta) + \lambda + \mu - 2b\beta\delta]\eta(X) = 0$$

or

$$db = -\left\{\lambda + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta\right\}\eta. \tag{3.27}$$

Applying d on (3.28), we get $\left\{\lambda + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta\right\} d\eta$. Since $d\eta \neq 0$ we have

$$\lambda + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta = 0. \tag{3.28}$$

Equation (3.29) in (3.28) yields b as a constant. Therefore, from (3.26), it follows that

$$S(X,Y) = -(\lambda + 2b\beta\delta)g(X,Y) + (2b\beta\delta - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for constant $2\beta\delta$. This leads to the following:

Theorem 3.3. *If in a 3-dimensional δ-Lorentzian trans-Sasakian manifold the metric g is an η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $\beta\delta$ is a constant.*

Ranking $X = Y = \xi$ in (3.9) and (3.21) and comparing, we get

$$\lambda = -2(\alpha^2 + \beta^2) - (\xi\beta) + \mu - 2b\beta\delta = -2K_\xi - \mu. \tag{3.29}$$

From (3.16) and (3.30), we obtain

$$r = 6(\alpha^2 + \beta^2) - 3(\xi\beta) - 2\beta\delta + 2\mu. \tag{3.30}$$

Since λ is a constant, it follows from (3.30) that K_ξ is a constant.

Theorem 3.4. *Let (g, ξ, μ) be an η-Ricci soliton in $(M, \phi, \xi, \eta, g, \delta)$ a 3-dimensional δ-Lorentzian trans Sasakian manifold. Then the scalar $\lambda + \mu = -2K_\xi$, $r = 6K_\xi + 2\mu - 3(\xi\beta) - 2b\beta\delta$.*

Remark 3.1. For $\mu = 0$, (3.30) reduces to $\lambda = -2K_\xi$, so Ricci soliton in 3-dimensional δ-Lorentzian trans-Sasaakian manifold is shrinking.

Example 3.1. Consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0|\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the vector fields are

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \qquad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \qquad e_3 = \frac{-(\delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -\delta$, $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$,
where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \delta g(X, \xi)$ for any vector field X on M, let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (δ)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad -g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is know as Koszul's formula. Now we have

$$\nabla_{e_1} e_3 = -\frac{(\delta)}{z} e_1, \quad \nabla_{e_2} e_3 = -\frac{(\delta)}{z} e_2, \quad \nabla_{e_1} e_2 = 0,$$

by using the above relation, for any vector X on M, we have

$$\nabla_X \xi = \delta[-\alpha\phi X - \beta(X + \eta(X)\xi)],$$

where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence $(\phi, \xi, \eta, g, \delta)$ structure defines the δ-Lorentzian trans-Sasakian structure in $\mathbb{R}^3$.

Here ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{(\delta)}{z} e_1, \quad [e_2, e_3] = -\frac{(\delta)}{z} e_2.$$

due to $g(e_1, e_2) = 0$. Thus we have

$$\nabla_{e_1} e_3 = -\frac{(\delta)}{z} e_1 + e_2, \quad \nabla_{e_1} e_2 = 0$$

$$\nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = -\frac{(\delta)}{z} e_2, \qquad \nabla_{e_2} e_3 = -\frac{(\delta)}{z} e_2 - e_1 \qquad \nabla_{e_3} e_1 = 0$$

$$\nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = -\frac{(\delta)}{z} e_1 + e_2.$$

The manifold M satisfies (2.5) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is an δ-Lorentzian trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1,e_3)e_3 = \frac{(\delta)}{z^2}e_1, \qquad R(e_3,e_1)e_3 = -\frac{(\delta)}{z^2}e_1,$$

$$R(e_2,e_3)e_3 = \frac{(\delta)}{z^2}e_1, \quad R(e_3,e_2)e_3 = -\frac{(\delta)}{z^2}e_1.$$

From the above expression of the curvature tensor we can also obtain

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = \frac{(\delta^2)}{z^2}$$

since $g(e_1,e_3) = g(e_1,e_2) = 0$.
Therefore, we have

$$S(e_i,e_i) = \frac{(\delta)}{z^2}g(e_i,e_i),$$

for $i = 1,2,3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (3.11), we find $\lambda = \frac{(1+z\delta)}{z^2}$ and $\mu = \frac{(\delta)^2}{z}$, the data (g,ξ,λ,μ) is an expanding η-Ricci soliton on (M,ϕ,ξ,η,g).

4. Gradient Ricci Solitons in 3-dimensional δ-Lorentzian trans-Sasakian manifold

If the vector field V is the gradient of a potential function ψ then g is called a gradient Ricci soliton and (1.2) assume the form

$$\nabla\nabla\psi = S + \lambda g. \tag{4.1}$$

This reduces to

$$\nabla_Y D\psi = QY + \lambda Y, \tag{4.2}$$

where D denoted the gradient operator of g. From (4.2) it follows

$$R(X,Y)D\psi = (\bar{\nabla}_X Q)Y - (\bar{\nabla}_Y Q)X. \tag{4.3}$$

Differentiating (3.20) we get

$$\begin{aligned}(\nabla_W Q)X \quad = \quad & \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3(\alpha^2+\beta^2))(\alpha(g(\phi W,X) \\ & +\beta\delta g(W,X) - \delta\beta\eta(X)\eta(W)) + \eta(X)\nabla_W\xi.\end{aligned} \tag{4.4}$$

In (4.4) replacing $W = \xi$, we obtain

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \tag{4.5}$$

Then we have

$$g(\nabla_\xi Q)X - (\bar{\nabla}_X Q)(\xi,\xi) = g(\frac{dr(\xi)}{2}(X-\eta(X)\xi,\xi)) = \frac{dr(\xi)}{2}(g(X,\xi)-\eta(X))) = 0. \tag{4.6}$$

Using (4.6) and (4.5), we obtain

$$g(R(\xi,X)D\psi,\xi) = 0. \tag{4.7}$$

From (2.12), we find

$$g(\bar{R}(\xi,Y)D\psi,\xi) = (\alpha^2+\beta^2)(g(Y,D\psi)-\eta(Y)\eta(D\psi)).$$

Using (11.7), we get

$$(\alpha^2+\beta^2)(g(Y,D\psi)-\eta(Y)\eta(D\psi)) = 0$$

$$(\alpha^2+\beta^2)(g(Y,D\psi)-\eta(Y)g(D\psi,\xi)) = 0,$$

or

$$(g(Y,D\psi)-g(Y,\xi)g(D\psi,\xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \quad \text{since} \quad \alpha^2+\beta^2 \neq -\delta(\xi\beta). \tag{4.8}$$

Now, using (4.8) and (4.2), we get

$$S(X,Y)+\lambda g(X,Y) = g(\nabla_Y D\psi, X) = g(\nabla_Y(\xi\psi)\xi, X)$$

$$= (\xi\psi)g(\bar{\nabla}_Y\xi, X) + Y(\xi\psi)\eta(X)$$

$$= (\xi\psi)g(-\delta\alpha\phi Y - \delta\beta Y - \delta\beta\eta(Y)\xi, X) + Y(\xi\psi)\eta(X)$$

$$S(X,Y)+\lambda g(X,Y) = -\delta\alpha(\xi\psi)g(\phi Y,X) - \delta\beta(\xi\psi)g(Y,X) -\delta\beta(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X). \tag{4.9}$$

Putting $X=\xi$ in (4.9) and using (2.15) we get

$$\bar{S}(Y,\xi)+\lambda\eta(Y) = Y(\xi\psi) = [\lambda+2\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(Y). \tag{4.10}$$

Interchanging X and Y in (4.9), we get

$$S(X,Y)+\lambda g(X,Y) = -\delta\alpha(\xi\psi)g(Y,\phi X)-\delta\beta(\xi\psi)g(X,Y) -\delta\beta(\xi\psi)\eta(Y)\eta(X)+X(\xi\psi)\eta(Y). \tag{4.11}$$

Adding (4.9) and (4.11) we get

$$2S(X,Y)+2\lambda g(X,Y) = -2\delta\beta(\xi\psi)g(X,Y)+Y(\xi\psi)\eta(X) -2\delta\beta(\xi\psi)\eta(X)\eta(Y)+X(\xi\psi)\eta(Y). \tag{4.12}$$

Using (4.10) in (4.12) we have

$$S(X,Y)+\lambda g(X,Y) = -\delta\beta(\xi\psi)[g(X,Y)-\eta(X)\eta(Y)] +[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(X)\eta(Y). \tag{4.13}$$

Then using (4.2) we have

$$\nabla_Y D\psi = -\delta\beta(\xi\psi)(Y-\eta(Y)\xi) +[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(Y)\xi. \tag{4.14}$$

Using (11.14) we calculate

$$\begin{aligned} R(X,Y)D\psi &= \nabla_X\nabla_Y D\psi-\nabla_Y\nabla_X D\psi-\nabla_{[X,Y]}D\psi \\ &= -\delta\beta X(\xi\psi)Y+\delta\beta Y(\xi\psi)X \\ &\quad -\delta\beta Y(\xi\psi)\eta(X)\xi+\delta\beta X(\xi\psi)\eta(Y)\xi \\ &\quad +[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]((\nabla_X\eta)(Y)\xi-(\nabla_Y\eta)(X)\xi) \\ &\quad +[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]((\nabla_X\xi)\eta(Y)\xi-(\nabla_Y\xi)\eta(X)). \end{aligned} \tag{4.15}$$

Taking inner product with ξ in (4.15), we get

$$0 = g((X,Y)D\psi,\xi) = 2\delta\alpha[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]g(\phi Y,X). \tag{4.16}$$

Thus we have $2\delta\alpha[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$.

Now we consider the following cases:

Case (i) $\delta\alpha=0$, or
Case (ii) $[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$,
Case (iii) $\alpha=0$ and $[\lambda+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$.

In this case, we have the following;

Case (i) If $\alpha = 0$, the manifold reduces to a δ-Lorentzian β-Kenmotsu manifold.

Case (ii) Let $[\lambda + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$. If we use this in (4.10) we get $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Substitute this value in (11.12) we obtain

$$S(X,Y) + \lambda g(X,Y) = -\delta\beta(\xi\psi)g(X,Y) - 2\delta\beta\eta(X)\eta(Y). \tag{4.17}$$

Now, contracting (4.17), we get

$$r + 3\lambda = -3\delta\beta(\xi\psi) - 2\delta\beta, \tag{4.18}$$

which implies

$$(\xi\psi) = \frac{r}{-3\delta\beta} + \frac{\lambda}{-\delta\beta} + \frac{2}{-3}. \tag{4.19}$$

If $r = constant$, then $(\xi\psi) = constant = k(say)$. Therefore from (4.8) we have $D\psi = (\xi\psi)\xi = k\xi$. This we can write this equation as

$$g(D\psi, X) = k\eta(X), \tag{4.20}$$

which means that $d\psi(X) = k\eta(X)$. Applying d this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$ Therefore equation (11.1) reduces to

$$S(X,Y) = 2(\alpha^2 + \beta^2 - \delta(\xi\beta))g(X,Y),$$

that is M is an *Einstein* manifold.
Case (iii) Using $\alpha = 0$ and $[\lambda + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$. in (4.10) we obtain $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.

Thus we have the following :

Theorem 4.1. *If a 3-dimensional δ-Lorentzian trans-Sasakian manifold with constant scalar curvature admits gradient Ricci soliton, then the manifold is either a δ-Lorentzian β-Kenmotsu manifold or an Einstein manifold provided $\alpha, \beta = constant$.*

In [9], it was proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 we state the following:

Corollary 4.1. *If a compact 3-dimensional δ-Lorentzian trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either δ-Lorentzian α-Sasakian or δ-Lorentzian β-Kenmotsu.*

Also in [13], authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence, from Theorem 3 we obtain the following:

Corollary 4.2. *If a locally ϕ-symmetric 3-dimensional connected δ-Lorentzian trans-Sasakian manifold its admits gradient Ricci soliton, then manifold is either δ-Lorentzian β-Kenmotsu or Einstein manifold provided $\alpha, \beta = constant$.*

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D_a-HOMOTHETIC DEFORMATION AND RICCI SOLITIONS IN THREE DIMENSIONAL QUASI-SASAKIAN MANIFOLDS

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Abstract. In the present paper, we have studied curvature tensors of a quasi-Sasakian manifold with respect to the D_a-homothetic deformation. We have deduced the Ricci soliton in quasi-Sasakian manifold with respect to the D_a-homothetic deformation. We have also proved that the quasi-Sasakian manifold is not $\bar{\xi}$-projectively flat under D_a-homothetic deformation. Also, we give an example to prove the existance of quasi-Sasakian manifold.
Key words: Quasi-Sasakian manifold, D_a-homothetic deformation, Ricci soliton, Weyl projective curvature tensor.

1. Introduction

In 1967, D. E. Blair [1] introduced the notion of quasi-Sasakian structure to unify Sasakian and cosymplectic structures. The Riemannian curvature tensor of three dimensional quasi-Sasakian manifold is given by [10]

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)[(\frac{r}{2}-\beta^2)X+(3\beta^2-\frac{r}{2})\eta(X)\xi+\eta(X)(\phi grad\beta) \\ &- d\beta(\phi X)\xi]-g(X,Z)[(\frac{r}{2}-\beta^2)Y+(3\beta^2-\frac{r}{2})\eta(Y)\xi \\ &+ \eta(Y)(\phi grad\beta)-d\beta(\phi Y)\xi]+[(\frac{r}{2}-\beta^2)g(Y,Z) \\ &+ (3\beta^2-\frac{r}{2})\eta(Y)\eta(Z)-\eta(Y)d\beta(\phi Z)-\eta(Z)d\beta(\phi Y)]X \end{aligned}$$

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$$- \quad [(\frac{r}{2} - \beta^2)g(X,Z) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Z) - \eta(X)d\beta(\phi Z)$$
$$- \quad \eta(Z)d\beta(\phi X)]Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y],$$

where β is a function on the manifold. In the paper [2], U. C. De and A. K. Mondal have proved that $\xi\beta = 0$. In a quasi-Sasakian manifold, if we consider β is a non-zero constant, then the manifold becomes β-Sasakian and if $\beta = 1$, the manifold becomes a Sasakian manifold.

The notion of D_a-homothetic deformation was introduced by Tanno [11] in 1968. In paper [8], H. G. Nagaraja, D. L. Kiran Kumar and D. G. Prakasha have studied D_a-homothetic deformation of (κ,μ)-contact metric manifolds. Nagaraja and Premalatha have studied D_a-homothetic deformation of K-contact manifolds in the paper [9].

Ricci soliton was introduced by Hamilton [4] which is the generalization of the Einstein metrics and is defined by

$$(L_X g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0,$$

where, L_X denotes the Lie-derivatives of Riemannian metric g along the vector field X, λ is a constant, S the Ricci tensor of type $(0,2)$ and Y, Z are arbitrary vector fields on the manifold. A Ricci soliton is called shrinking or steady or expanding according as λ is negative or zero or positive. Ricci solitons on three dimensional almost contact manifolds have been studied by several authors. For instance, U. C. De and A. K. Mondal studied Ricci solitons on three dimensional quasi-Sasakian manifolds [2]. S. K. Hui and colaborators have investigated Ricci solitons and their generalizations on some classes of almost contact manifolds. For details see [5], [6], [7].

In this paper we would like to study some properties of quasi-Sasakian manifold with D_a-homothetic deformation.

The paper is organized as follows: In Section 2, we have discussed some preliminaries. In Section 3, we give an example of quasi-Sasakian manifold to prove the existance of the said manifold. In Section 4, we deduced some curvature properties of quasi-Sasakian manifold with respect to the D_a-homothetic deformation. In Session 5, we study the Ricci soliton in quasi-Sasakian manifold with respect to the D_a-homothetic deformation. In the last Section, we have derived the $\bar{\xi}$-projective curvature tensor under D_a-homothetic deformation.

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that [10]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1.$$

As a consequence, we get the following:

$$\phi\xi = 0, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$
$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

for all vector fields X, Y on (M)

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X, \phi Y),$$

for all X, Y on M. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η, g) is normal and the fundamental 2-form Φ is closed i.e., $d\Phi = 0$ [1]. The normality condition gives that the induced almost complex structure of $M \times \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M, where $[X, Y]$ is the Lie bracket. The rank of a quasi-Sasakian structure is always an odd integer [1] which is equal to 1 if the structure is cosympletic and it is equal to $(2n + 1)$ if the structure is Sasakian.

For a three-dimensional quasi-Sasakian manifold, we have [2]

$$\nabla_X \xi = -\beta\phi X, \tag{2.1}$$

$$(\nabla_X \phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \tag{2.2}$$

$$(\nabla_X \eta)(Y) = -\beta g(\phi X, Y), \tag{2.3}$$

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.4}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \\ &- \eta(X)d\beta(\phi Y) - \eta(Y)d\beta(\phi X), \end{aligned}$$

$$\begin{aligned} QX &= (\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi \\ &- \eta(X)(\phi grad\beta) - d\beta(\phi X)\xi, \end{aligned}$$

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X). \tag{2.5}$$

The Weyl projective curvature tensor P of type $(1,3)$ on a Riemannian manifold (M, g) of dimension $(2n + 1)$ is defined by [3]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y],$$

for all X, Y, $Z \in \chi(M)$.

3. Example of quasi-Sasakian manifold of dimension three

This example is constructed by following U. C. De and A. K. Mondal in the paper [2].

Let us consider the manifold $M = \{x_1, x_2, x_3 \in \mathbb{R}^3 : x_3 \neq 0\}$ of dimension 3, where $\{x_1, x_2, x_3\}$ are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, \quad e_3 = \frac{\partial}{\partial x_3},$$

which are linearly independent at each point of M, we get

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta ij$, for all $i, j = 1, 2, 3$. Let ∇ be the Riemannian connection and R the curvature tensor of g. The 1-form η is defined by $\eta(X) = g(X, e_3)$, for any X on M, which is a contact form because $\eta \wedge d\eta \neq 0$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then we find

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X) e_3,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Hence (ϕ, e_3, η, g) defines an almost contact metric structure on M.

Using Koszul's formula, we obtain

$$\nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2} e_1,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_3, \quad \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_3} e_2 = \frac{1}{2} e_1$$

and the remaining $\nabla_{e_i} e_j = 0$, for all $i, j = 1, 2, 3$. Thus we see that the structure (ϕ, e_3, η, g) satisfies the formula $\nabla_X e_3 = -\beta \phi X$ for $\beta = -\frac{1}{2}$.

Hence the manifold is a three dimensional quasi-Sasakian manifold with the constant structure function β.

Also, from the definition of curvature tensor, the expressions curvature tensor are given by

$$R(e_1,e_2)e_1=\frac{3}{4}e_2,\quad R(e_1,e_2)e_2=-\frac{3}{4}e_1,\quad R(e_1,e_3)e_1=-\frac{1}{4}e_3,$$

$$R(e_2,e_3)e_3=\frac{1}{4}e_2,\quad R(e_1,e_3)e_3=\frac{1}{4}e_1,\quad R(e_2,e_3)e_2=-\frac{1}{4}e_3$$

and the remaining $R(e_i,e_j)e_k=0$, for all $i,j,k=1,2,3$.

4. D_a-homothetic deformation

Let (M,ϕ,ξ,η,g) be a 3-dimensional quasi-Sasakian manifold. A D_a-homothetic deformation is defind by

$$\bar{\phi}=\phi,\quad \bar{\xi}=\frac{1}{a}\xi,\quad \bar{\eta}=a\eta,\quad \bar{g}=ag+a(a-1)\eta\otimes\eta, \tag{4.1}$$

with a being a positive constant [8].

If $M(\phi,\xi,\eta,g)$ is a quasi-Sasakian manifold with Riemannian connection ∇ and $\bar{\nabla}$ be the connection of the D_a-homothetic deformed quasi-Sasakian manifold $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ which is calculated from ∇ and g. Then the relation between the connections ∇ and $\bar{\nabla}$ is given by

$$\bar{\nabla}_XY=\nabla_XY+(1-a)[\eta(Y)\phi X+\eta(X)\phi Y], \tag{4.2}$$

for any vector fields X, Y, on M.

The Riemannian curvature tensor $\bar{R}$ of $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is given by

$$\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{[X,Y]}Z. \tag{4.3}$$

Using (2.1), (2.2), (4.1) and (4.2) in (4.3), we get

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z+(1-a)\beta[g(X,Z)\eta(Y)\xi\\ &- g(Y,Z)\eta(X)\xi-2g(\phi X,Y)\phi Z-g(\phi X,Z)\phi Y\\ &+ g(\phi Y,Z)\phi X-2\eta(Y)\eta(Z)X+2\eta(X)\eta(Z)Y]\\ &- (1-a)^2[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X].\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W)+(1-a)\beta[g(X,Z)\eta(Y)\eta(W)\\ &- g(Y,Z)\eta(X)\eta(W)-2g(\phi X,Y)g(\phi Z,W)\\ &- g(\phi X,Z)g(\phi Y,W)+g(\phi Y,Z)g(\phi X,W)\\ &- 2g(X,W)\eta(Y)\eta(Z)+2g(Y,W)\eta(X)\eta(Z)]\\ &- (1-a)^2[g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)],\end{aligned} \tag{4.5}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Let, $\{e_i\}, (i = 1,2,3)$ be the orthonormal basis of the tangent space of the manifold. Putting $X = W = e_i$, in (4.5) and summing over i, we get

$$\begin{aligned} \bar{S}(Y,Z) &= S(Y,Z) + 2(1-a)\beta[g(Y,Z) - 3\eta(Y)\eta(Z)] \\ &+ 2(1-a)^2\eta(Y)\eta(Z). \end{aligned} \tag{4.6}$$

From which,

$$\bar{Q}Y = QY + 2(1-a)\beta(Y - 3\eta(Y)\xi) + 2(1-a)^2\eta(Y)\xi,$$

From (2.2) and (4.2), we get

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - (1-a)\eta(Y)\phi^2X.$$

Also, from (2.1) and (4.2), we obtain

$$\bar{\nabla}_X\bar{\xi} = \frac{1-a-\beta}{a}\phi X. \tag{4.7}$$

Thus, from (2.3), (4.1), (4.2) and (4.7), we get

$$(\bar{\nabla}_X\bar{\eta})Y = a^2(1-a-\beta)g(\phi X,Y).$$

Thus, we can state the following

Theorem 4.1. For a D_a-homothetically deformed quasi-Sasakian manifold $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$, the followings hold

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + (1-a)\beta[g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi - 2g(\phi X,Y)\phi Z - g(\phi X,Z)\phi Y \\ &+ g(\phi Y,Z)\phi X - 2\eta(Y)\eta(Z)X + 2\eta(X)\eta(Z)Y] \\ &- (1-a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]. \end{aligned}$$

$$\begin{aligned} \bar{S}(Y,Z) &= S(Y,Z) + 2(1-a)\beta[g(Y,Z) - 3\eta(Y)\eta(Z)] \\ &+ 2(1-a)^2\eta(Y)\eta(Z). \end{aligned}$$

$$\bar{Q}Y = QY + 2(1-a)\beta(Y - 3\eta(Y)\xi) + 2(1-a)^2\eta(Y)\xi,$$

$$(\bar{\nabla}_X\phi)Y = (\nabla_X\phi)Y - (1-a)\eta(Y)\phi^2X.$$

$$\bar{\nabla}_X\bar{\xi} = \frac{1-a-\beta}{a}\phi X.$$

$$(\bar{\nabla}_X\bar{\eta})Y = a^2(1-a-\beta)g(\phi X,Y).$$

5. Ricci soliton in three dimensional quasi-Sasakian manifold with respect to the D_a-homothetic deformation

Let $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a D_a-homothetically deformed quasi-Sasakian manifold of dimension 3. A Ricci soliton $(\bar{g}, V, \lambda)$ is defined on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ as

$$(\bar{L}_V \bar{g})(X, Y) + 2\bar{S}(X, Y) + 2\lambda \bar{g}(X, Y) = 0, \tag{5.1}$$

where $\bar{L}_V \bar{g}$ denotes the Lie derivative of Riemannian metric $\bar{g}$ along a vector field V, $\bar{S}$ is the Ricci tensor of type $(0, 2)$ on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$.

Let us suppose that the vector field V is the Reeb vector field $\bar{\xi}$ on $M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. Then from (5.1), we have

$$(\bar{L}_{\bar{\xi}} \bar{g})(X, Y) + 2\bar{S}(X, Y) + 2\lambda \bar{g}(X, Y) = 0. \tag{5.2}$$

Now, from (2.1) and (4.2), we have

$$\begin{aligned} (\bar{L}_{\bar{\xi}} \bar{g})(X, Y) &= \bar{g}(\bar{\nabla}_X \bar{\xi}, Y) + \bar{g}(X, \bar{\nabla}_Y \bar{\xi}) \\ &= 0. \end{aligned} \tag{5.3}$$

Therefore, from (4.1), (5.2) and (5.3), we get

$$\bar{S}(X, Y) = -\lambda \bar{g}(X, Y). \tag{5.4}$$

Putting $Y = Z = \xi$ in (4.6), we get

$$\bar{S}(\xi, \xi) = 2(\beta + a - 1)^2. \tag{5.5}$$

Putting $X = Y = \bar{\xi}$ in (5.4) and using (5.5), we get

$$\lambda = -\frac{2(\beta + a - 1)^2}{a^2}.$$

Thus we can state the following

Theorem 5.1. If a D_a-homothetically deformed quasi-Sasakian manifold of dimension three admits Ricci soliton, then the Ricci soliton is shrinking.

6. $\bar{\xi}$-projective curvature tensor on quasi-Sasakian manifold with respect to D_a-homothetic deformation

Definition 6.1. The $\bar{\xi}$-projective curvature tensor of type $(1, 3)$ on a quasi-Sasakian manifold of dimension $(2n+1)$ with respect to D_a-homothetic deformation is given by [8]

$$\bar{P}(X, Y)\bar{\xi} = \bar{R}(X, Y)\bar{\xi} - \frac{1}{2n}[S(Y, \bar{\xi})X - S(X, \bar{\xi})Y],$$

for any X, Y on M.
A quasi-Sasakian manifold of dimension n is said to be $\bar{\xi}$-projectively flat with respect to the D_a-homothetic deformation if $\bar{P}(X,Y)\bar{\xi} = 0$.

The Weyl projective curvature tensor $\bar{P}$ of a three dimensional quasi-Sasakian manifold under D_a-homothetic deformation is defined by [8]

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{6.1}$$

Interchanging X and Y,we get

$$\bar{P}(Y,X)Z = \bar{R}(Y,X)Z - \frac{1}{2}[\bar{S}(X,Z)Y - \bar{S}(Y,Z)X]. \tag{6.2}$$

Adding (6.1) and (6.2), we get by using the property $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$

$$\bar{P}(X,Y)Z + \bar{P}(Y,X)Z = 0.$$

Also, from (6.1) by using first Bianchi identity $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$, we get

$$\bar{P}(X,Y)Z + \bar{P}(Y,Z)X + \bar{P}(Z,X)Y = 0.$$

Thus the Weyl projective curvature tensor under D_a-homothetic deformation in a quasi-Sasakian manifold is skew-symmetric and cyclic.

Using (4.1), (4.4) and (4.6) in (6.1) , we get

$$\begin{aligned}
\bar{P}(X,Y)Z &= R(X,Y)Z + (1-a)\beta[g(X,Z)\eta(Y)\xi \\
&- g(Y,Z)\eta(X)\xi - 2g(\phi X,Y)\phi Z - g(\phi X,Z)\phi Y \\
&+ g(\phi Y,Z)\phi X - 2\eta(Y)\eta(Z)X + 2\eta(X)\eta(Z)Y] \\
&- (1-a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\
&- \frac{1}{2}[S(Y,Z)X + 2(1-a)\beta\{g(Y,Z)X - 3\eta(Y)\eta(Z)X\} \\
&+ 2(1-a)^2\eta(Y)\eta(Z)X - S(X,Z)Y \\
&- 2(1-a)\beta\{g(X,Z)Y - 3\eta(X)\eta(Z)Y\} \\
&- 2(1-a)^2\eta(X)\eta(Z)Y].
\end{aligned} \tag{6.3}$$

Replacing Z by $\bar{\xi}$ in (6.3), using (2.4), (2.5) and (4.1), we get

$$\bar{P}(X,Y)\bar{\xi} = \frac{1}{a}[(-(X\beta)\phi Y + (Y\beta)\phi X) + \frac{1}{2}(-(\phi X\beta)Y + (\phi Y\beta)X)].$$

Thus we can say
Theorem 6.1. The $\bar{\xi}$-projective curvature tensor on a D_a-homothetically deformed quasi-Sasakian manifold of dimension 3 is given by

$$\bar{P}(X,Y)\bar{\xi} = \frac{1}{a}[(-(X\beta)\phi Y + (Y\beta)\phi X) + \frac{1}{2}(-(\phi X\beta)Y + (\phi Y\beta)X)].$$

and the manifold is not $\bar{\xi}$-projectively flat with respect to the D_a-homothetic deformation.

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LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS IN ARROWHEAD FORM

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Abstract. This paper deals with different approaches for solving linear systems of the first order differential equations with the system matrix in the symmetric arrowhead form. Some needed algebraic properties of the symmetric arrowhead matrix are proposed. We investigate the form of invariant factors of the arrowhead matrix. Also the entries of the adjugate matrix of the characteristic matrix of the arrowhead matrix are considered. Some reductions techniques for linear systems of differential equations with the system matrix in the arrowhead form are presented.
Keywords: Arrowhead matrices, Linear systems of differential equations, Partial and total reductions of non-homogeneous linear systems of first order operator equations

1. Introduction

Arrowhead matrices are an important type of matrices occurring in wide area of applications. They are popular subject of research related with mathematics, physics and engineering. Some important problems like computing eigenvalues and eigenvectors of arrowhead matrices [9, 2, 21], solving inverse eigenvalue problems [15, 28, 25, 24], computing the inverse of arrowhead matrices [7, 26, 4], and solving symmetric arrowhead systems [5] have been considered by various authors over the last four decades. Arrowhead matrices are often an essential tool for the computation of the eigenvalue problems for large and sparse or tridiagonal matrices [22, 6, 8, 29, 23]. Arrowhead matrices arise in the description of modelling of radiationless transitions in isolated molecules [1], oscillators vibrationally coupled with a Fermi liquid [3]

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and quantum optics [19]. One can also find arrowhead matrices in the models of telecommunication systems (MIMO) [27, 14] and neural networks [18], as well as in robotics and in modern control theory. Motivated by wide applications of arrowhead matrices we are interested in solving linear system of differential equations with the system matrix in the symmetric arrowhead form.

In our previous papers we have considered a partial and a total reduction of non-homogenous linear systems of the first order operator equations with system matrix in an arbitrary form. In [16] the idea was to use the rational canonical form to reduce such a system to an equivalent partially reduced one. The partially reduced system obtained in this fashion consists of higher-order linear operator equations in one variable and first-order linear operator equations in two variables. Another method for solving a linear systems of operator equations, which does not require a change of basis, is discussed in [17]. Obtained totally reduced system consists of higher order operator equations which only differ in the variables and in the non-homogeneous terms. In [12] and [11] we have considered a partial and a total reduction of linear systems of operator equations with the system matrix in the companion form. Papers [12, 11, 10] and [13] expand our research to non-homogeneous linear systems of operator equations involving more than one operator.

This paper deals with both types of reductions, a partial and a total, of linear systems of the first order differential equations with the system matrix in the arrowhead form. We will look more closely at the form of invariant factors of the arrowhead matrix, which we will use for partial reduction. The adjugate matrix of characteristic matrix of the arrowhead matrix presented as polynomial with matrix coefficients will be used to establish the form for the totally reduced system.

In what follows we propose some important properties of arrowhead matrices, and we will start with definition of the arrowhead matrix.

2. Some properties of symmetric arrowhead matrices

A matrix $B \in \mathbb{R}^{n\times n}$ is called a symmetric arrowhead matrix if it has a form

$$\begin{bmatrix} a_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ b_2 & a_2 & 0 & \dots & 0 & 0 \\ b_3 & 0 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & 0 & 0 & \dots & a_{n-1} & 0 \\ b_n & 0 & 0 & \dots & 0 & a_n \end{bmatrix}. \tag{2.1}$$

It is a symmetric matrix obtained by bordering the diagonal matrix with a row and a column with the same elements. The characteristic polynomial of the arrowhead matrix B is

$$\Delta_B(\lambda) = \det(\lambda I - B) = \prod_{i=1}^{n}(\lambda - a_i) - \sum_{i=2}^{n} b_i^2 \prod_{\substack{j=2 \\ j\neq i}}^{n} (\lambda - a_j). \tag{2.2}$$

This formula can be easily derived by expanding the determinant of the matrix $\lambda I - B$ by the first row. The proof of this can be found in [20] and therefore it is omitted here. We denote by d_k, $0 \le k \le n$, the coefficient of the term of degree $n-k$ of the characteristic polynomial $\Delta_B(\lambda)$. Therefore, we have

$$d_0 = 1, \quad d_1 = -\sum_{i=1}^{n} a_i, \quad d_2 = \sum_{1\le i<j\le n} a_i\, a_j - \sum_{i=2}^{n} b_i^2 \ \text{ and}$$

$$d_k = (-1)^k \left(\sum_{1\le i_1<i_2<\ldots<i_k\le n} a_{i_1}\, a_{i_2} \ldots a_{i_k} - \sum_{j=2}^{n} \; b_j^2 \sum_{\substack{2\,\le\, i_1\,<\, i_2\,<\,\ldots\,<\, i_{k-2}\,\le\, n\\ i_1,\, i_2,\, \ldots,\, i_{k-1}\,\ne\, j}} a_{i_1}\, a_{i_2} \ldots a_{i_{k-2}} \right)$$

for $3 \le k \le n$.

Suppose that $a_2 > a_3 > \ldots > a_n$ and $b_i \ne 0$, for $2 \le i \le n$. Then by Cauchy's Interlacing Theorem the eigenvalues λ_i of the matrix B, $1 \le i \le n$, are distinct. Moreover, if $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, then $\lambda_1 > a_2 > \lambda_2 > a_3 > \ldots > a_n > \lambda_n$. For more details, we refer the reader to [24]. If for some i, $2 \le i \le n$, $b_i = 0$, then a_i is eigenvalue of the matrix B. If the number of repetition of the element a_i along the diagonal except on the position $(1,1)$ of the matrix B is k_i, then the element a_i is an eigenvalue of the matrix B with algebraic multiplicity at least $k_i - 1$. The result follows directly from the equation (2.2), since $(\lambda - a_i)^{k_i-1}$ is a factor of $\Delta_B(\lambda)$. If the matrix B is of the form

$$\begin{bmatrix}
a_1 & b_2 & \ldots & b_{i-1} & b_{i_1} & \ldots & b_{i_{k_i}} & b_{i+k_i} & \ldots & b_{n-1} & b_n \\
b_2 & a_2 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{i-1} & 0 & \ldots & a_{i-1} & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
b_{i_1} & 0 & \ldots & 0 & a_i & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{i_{k_i}} & 0 & \ldots & 0 & 0 & \ldots & a_i & 0 & \ldots & 0 & 0 \\
b_{i+k_i} & 0 & \ldots & 0 & 0 & \ldots & 0 & a_{i+k_i} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{n-1} & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & a_{n-1} & 0 \\
b_n & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & a_n
\end{bmatrix},$$

for $a_i \ne a_j$, $2 \le j \le i-1$, $i + k_i \le j \le n$, we will say that elements $b_{i_1}, \ldots, b_{i_{k_i}}$ correspond to the diagonal element a_i. According to Corollary 4 in [27] we have

$\Delta_B(\lambda) = (\lambda - a_i)^{k_i - 1}\Delta_{\widetilde{B}}(\lambda)$, where

$$\widetilde{B} = \begin{bmatrix} a_1 & b_2 & \dots & b_{i-1} & \sqrt{\sum_{j=1}^{k_i} b_{i_j}^2} & b_{i+k_i} & \dots & b_{n-1} & b_n \\ b_2 & a_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{i-1} & 0 & \dots & a_{i-1} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{\sum_{j=1}^{k_i} b_{i_j}^2} & 0 & \dots & 0 & a_i & 0 & \dots & 0 & 0 \\ b_{i+k_i} & 0 & \dots & 0 & 0 & a_{i+k_i} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & 0 & \dots & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ b_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_n \end{bmatrix}.$$

Characteristic polynomial of the matrix $\widetilde{B}$ is polynomial

$$\Delta_{\widetilde{B}}(\lambda) = \prod_{j=1}^{i}(\lambda - a_j)\prod_{j=i+k_i}^{n}(\lambda - a_j) - \sum_{j=2}^{n+1-k_i}\widetilde{b}_j^2 \prod_{\substack{k=2\\k\neq j}}^{i}(\lambda - a_k)\prod_{\substack{k=i+k_i\\k\neq j}}^{n}(\lambda - a_k),$$

where $\widetilde{b}_j = b_j$ for $2 \le j \le i-1$, $\widetilde{b}_i = \sqrt{\sum_{j=1}^{k_i} b_{i_j}^2}$ and $\widetilde{b}_{j-k_i+1} = b_j$ for $i + k_i \le j \le n$. We would like to investigate under what condition a_i is an eigenvalue of the matrix $\widetilde{B}$. We have $\Delta_{\widetilde{B}}(a_i) = -\widetilde{b}_i^2 \prod_{k=2}^{i-1}(a_i - a_k)\prod_{k=i+k_i}^{n}(a_i - a_k)$, and since $a_i \neq a_j$ for $2 \le j \le i-1$ and $i + k_i \le j \le n$ we deduce that a_i is an eigenvalue of $\widetilde{B}$ if and only if $\widetilde{b}_i^2 = \sqrt{\sum_{j=1}^{k_i} b_{i_j}^2} = 0$, i.e., if and only if $b_{i_j} = 0$ for all j, $1 \le j \le k_i$. Therefore, a_i is an eigenvalue of $\widetilde{B}$ if and only if all corresponding elements to the diagonal element a_i in B are zeros. So, in this case algebraic multiplicity of the element a_i in the matrix B is k_i. If there is at least one non-zero corresponding element to a_i, algebraic multiplicity of a_i is $k_i - 1$. Let $a_{i_1}, a_{i_2}, \dots, a_{i_p}$ be different elements along the diagonal with corresponding elements all equal to zero and let $a_{j_1}, a_{j_2}, \dots, a_{j_q}$ be different elements along the diagonal with at least one corresponding element different from zero. Let k_{i_t} and k_{j_s}, $1 \le t \le p$, $1 \le s \le q$, be the numbers of repetition of the elements a_{i_t} and a_{j_s} along the diagonal except on the position $(1,1)$ and define m_{j_s} by $m_{j_s} = \begin{cases} 1, & k_{j_s} > 1 \\ 0, & k_{j_s} = 1. \end{cases}$ Then the minimal polynomial of the matrix B is of the form

$$\mu_B(\lambda) = \prod_{s=1}^{p}(\lambda - a_{i_s})\prod_{s=1}^{q}(\lambda - a_{j_s})^{m_{j_s}}\Delta_{\widetilde{B}}(\lambda),$$

where $\widetilde{B}$ is completely reduced arrowhead matrix, i.e., it is a matrix of the form

$$\begin{bmatrix} a_1 & \widetilde{b}_{j_1} & \dots & \widetilde{b}_{j_q} \\ \widetilde{b}_{j_1} & a_{j_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{b}_{j_q} & 0 & \dots & a_{j_q} \end{bmatrix},$$

$\widetilde{b}_{j_s} = \sqrt{\sum_{t=1}^{k_{j_s}} b_{j_{s_t}}^2}$ and $b_{j_{s_t}}$ are corresponding elements of the element a_{j_s} in the matrix B, $1 \le s \le q$. If the elements $a_2, a_3, \dots, a_n$ are all different, then the minimal and characteristic polynomials of the matrix B are the same. Otherwise, since matrix B is symmetric the number of its invariant factors is equal to $k = max\{k_{i_1}, \dots, k_{i_p}, k_{j_1} - 1, \dots, k_{j_q} - 1\}$. The k-th invariant factor of the matrix B is $\mu_B(\lambda)$. The m-th invariant factor of the matrix B, $1 \le m \le k-1$ is the polynomial

$$\tau_m(\lambda) = \prod_{s=1}^{p} (\lambda - a_{i_s})^{g_{i_s}} \prod_{s=1}^{q} (\lambda - a_{j_s})^{g_{j_s}},$$

where $g_{i_s} = \begin{cases} 1, & k_{i_s} - (k-m) > 0 \\ 0, & \text{otherwise} \end{cases}$ and $g_{j_s} = \begin{cases} 1, & k_{j_s} - (k-m) > 1 \\ 0, & \text{otherwise.} \end{cases}$

From now on we will be concern with the coefficients of the adjugate matrix of the characteristic matrix of the symmetric arrowhead matrix B. Suppose that the adjugate matrix of the characteristic matrix $\lambda I - B$ is written in the form

$$\mathrm{adj}(\lambda\mathrm{I} - \mathrm{B}) = \lambda^{\mathrm{n}-1}\mathrm{B}_0 + \lambda^{\mathrm{n}-2}\mathrm{B}_1 + \dots + \lambda\mathrm{B}_{\mathrm{n}-2} + \mathrm{B}_{\mathrm{n}-1}.$$

Let us determine the coefficients B_k using recurrences $B_k = B \cdot B_{k-1} + d_k I$, for $1 \le k \le n-1$, and $B_0 = I$. The recurrences are obtained by equating coefficients at the same powers of λ on both sides of the equality $\mathrm{adj}(\lambda\mathrm{I} - \mathrm{B}) \cdot (\lambda\mathrm{I} - \mathrm{B}) = \Delta_{\mathrm{B}}(\lambda)\mathrm{I}$.

Lemma 2.1. *The coefficient* $B_k = [b_{ij}^k]_{n\times n}$, $2 \le k \le n-1$, *of the matrix* $\mathrm{adj}(\lambda\mathrm{I} - \mathrm{B})$ *is matrix with entries*

$$b_{11}^k = (-1)^k \sum_{2 \le i_1 < i_2 < \dots < i_k \le n} a_{i_1} a_{i_2} \dots a_{i_k} \qquad b_{1j}^k = (-1)^{k-1} b_j \sum_{\substack{2 \le i_1 < i_2 < \dots < i_{k-1} \le n \\ i_1, i_2, \dots, i_{k-1} \ne j}} a_{i_1} a_{i_2} \dots a_{i_{k-1}}$$

$$b_{j1}^k = (-1)^{k-1} b_j \sum_{\substack{2 \le i_1 < i_2 < \dots < i_{k-1} \le n \\ i_1, i_2, \dots, i_{k-1} \ne j}} a_{i_1} a_{i_2} \dots a_{i_{k-1}} \qquad b_{ij}^k = (-1)^{k-2} b_i b_j \sum_{\substack{2 \le i_1 < i_2 < \dots < i_{k-2} \le n \\ i_1, i_2, \dots, i_{k-2} \ne i, j}} a_{i_1} a_{i_2} \dots a_{i_{k-2}}$$

$$b_{jj}^k = (-1)^k \left(\sum_{\substack{1 \le i_1 < i_2 < \dots < i_k \le n \\ i_1, i_2, \dots, i_k \ne j}} a_{i_1} a_{i_2} \dots a_{i_k} - \sum_{\substack{i=2 \\ i \ne j}}^{n} b_i^2 \sum_{\substack{2 \le i_1 < i_2 < \dots < i_{k-2} \le n \\ i_1, i_2, \dots, i_{k-2} \ne i, j}} a_{i_1} a_{i_2} \dots a_{i_{k-2}} \right)$$

for $2 \le i, j \le n$ *and* $i \ne j$.

Proof. The proof proceeds by induction on k. We have $B_0 = I$. For coefficient B_1 holds $B_1 = B \cdot I + d_1 I$, i.e.,

$$B_1 = \begin{bmatrix} -\sum_{i=2}^{n} a_i & b_2 & \dots & b_{n-1} & b_n \\ b_2 & -\sum_{i=1, i\neq 2}^{n} a_i & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & 0 & \dots & -\sum_{i=1,\neq n-1}^{n} a_i & 0 \\ b_n & 0 & \dots & 0 & -\sum_{i=1}^{n-1} a_i \end{bmatrix}.$$

Coefficient B_1 is also arrowhead matrix. Let $(B)_{\to j}$ stand for the j-th row of the matrix B, and let $(B_{k-1})_{\downarrow j}$ denote the j-th column of the matrix B_{k-1}, $1 \leq j \leq n$. Assume that coefficients of the matrix B_{k-1} satisfy required form. Then we have

$$\begin{aligned} b_{11}^{k} &= (B)_{\to 1} \cdot (B_{k-1})_{\downarrow 1} + d_k = a_1\, b_{11}^{k-1} + \sum_{j=2}^{n} b_j\, b_{j1}^{k-1} + d_k \\ &= (-1)^{k-1} a_1 \sum_{2 \leq i_1 < i_2 < \dots < i_{k-1} \leq n} a_{i_1}\, a_{i_2} \dots a_{i_{k-1}} + (-1)^{k-2} \sum_{j=2}^{n} b_j^2 \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_1, i_2, \dots, i_{k-2} \neq j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-2}} \\ &+ (-1)^{k} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1}\, a_{i_2} \dots a_{i_k} + (-1)^{k-1} \sum_{j=2}^{n} b_j^2 \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_1, i_2, \dots, i_{k-2} \neq j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-2}} \\ &= (-1)^{k} \sum_{2 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1}\, a_{i_2} \dots a_{i_k} \end{aligned}$$

$$\begin{aligned} b_{j1}^{k} &= (B)_{\to j} \cdot (B_{k-1})_{\downarrow 1} = b_j\, b_{11}^{k-1} + a_j\, b_{j1}^{k-1} \\ &= (-1)^{k-1} b_j \sum_{2 \leq i_1 < i_2 < \dots < i_{k-1} \leq n} a_{i_1}\, a_{i_2} \dots a_{i_{k-1}} + (-1)^{k-2} a_j\, b_j \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_1, i_2, \dots, i_{k-2} \neq j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-2}} \\ &= (-1)^{k-1} b_j \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-1} \leq n \\ i_1, i_2, \dots, i_{k-1} \neq j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-1}} \end{aligned}$$

$$\begin{aligned} b_{ij}^{k} &= (B)_{\to i} \cdot (B_{k-1})_{\downarrow j} = b_i\, b_{1j}^{k-1} + a_i\, b_{ij}^{k-1} \\ &= (-1)^{k-2} b_i\, b_j \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_1, i_2, \dots, i_{k-2} \neq j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-2}} + (-1)^{k-3} a_i\, b_i\, b_j \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-3} \leq n \\ i_1, i_2, \dots, i_{k-3} \neq i, j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-3}} \\ &= (-1)^{k-2} b_i\, b_j \sum_{\substack{2 \leq i_1 < i_2 < \dots < i_{k-2} \leq n \\ i_1, i_2, \dots, i_{k-2} \neq i, j}} a_{i_1}\, a_{i_2} \dots a_{i_{k-2}} \end{aligned}$$

$$\begin{aligned}
b_{1j}^{k} &= (B)_{\to 1}\cdot (B_{k-1})_{\downarrow j} = a_1 b_{1j}^{k-1} + \sum_{\substack{i=2\\ i\neq j}}^{n} b_i b_{ij}^{k-1} + b_j b_{jj}^{k-1}\\
&= (-1)^{k-2} a_1\, b_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-2}\le n\\ i_1,i_2,\ldots,i_{k-2}\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-2}} + (-1)^{k-3}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2\, b_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-3}\le n\\ i_1,i_2,\ldots,i_{k-3}\neq i,j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-3}}\\
&+ (-1)^{k-1} b_j \sum_{\substack{1\le i_1<i_2<\ldots<i_{k-1}\le n\\ i_1,i_2,\ldots,i_{k-1}\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-1}} + (-1)^{k}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2\, b_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-3}\le n\\ i_1,i_2,\ldots,i_{k-3}\neq i,j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-3}}\\
&= (-1)^{k-1} b_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-1}\le n\\ i_1,i_2,\ldots,i_{k-1}\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-1}}
\end{aligned}$$

$$\begin{aligned}
b_{jj}^{k} &= (B)_{\to j}\cdot (B_{k-1})_{\downarrow j} + d_k = b_j b_{1j}^{k-1} + a_j b_{jj}^{k-1} + d_k\\
&= (-1)^{k-2} b_j^2 \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-2}\le n\\ i_1,i_2,\ldots,i_{k-2}\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-2}} + (-1)^{k-1} a_j \sum_{\substack{1\le i_1<i_2<\ldots<i_{k-1}\le n\\ i_1,i_2,\ldots,i_{k-1}\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-1}}\\
&+ (-1)^{k-2}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2\, a_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-3}\le n\\ i_1,i_2,\ldots,i_{k-3}\neq i,j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-3}} + (-1)^{k}\sum_{1\le i_1<i_2<\ldots<i_k\le n} a_{i_1}\,a_{i_2}\ldots a_{i_k}\\
&+ (-1)^{k-1}\sum_{i=2}^{n} b_i^2 \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-2}\le n\\ i_1,i_2,\ldots,i_{k-2}\neq i}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-2}}\\
&= (-1)^{k-1}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2 \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-2}\le n\\ i_1,i_2,\ldots,i_{k-2}\neq i}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-2}} + (-1)^{k}\sum_{\substack{1\le i_1<i_2<\ldots<i_k\le n\\ i_1,i_2,\ldots,i_k\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_k}\\
&+ (-1)^{k-2}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2\, a_j \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-3}\le n\\ i_1,i_2,\ldots,i_{k-3}\neq i,j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-3}}\\
&= (-1)^{k-1}\sum_{\substack{i=2\\ i\neq j}}^{n} b_i^2 \sum_{\substack{2\le i_1<i_2<\ldots<i_{k-2}\le n\\ i_1,i_2,\ldots,i_{k-2}\neq i,j}} a_{i_1}\,a_{i_2}\ldots a_{i_{k-2}} + (-1)^{k}\sum_{\substack{1\le i_1<i_2<\ldots<i_k\le n\\ i_1,i_2,\ldots,i_k\neq j}} a_{i_1}\,a_{i_2}\ldots a_{i_k}
\end{aligned}$$

Therefore, we have shown that coefficients of the matrix B_k are of the required form. □

3. The reduction formulas for linear systems of differential equations with the system matrix in the arrowhead form

Let $C^\infty(\mathbb{R})$ be a vector space of all infinitely differentiable functions and let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be a differential operator on the vector space $C^\infty(\mathbb{R})$. We will consider non-homogeneous linear system of differential equations with the system matrix in the symmetric arrowhead form

$$(3.1)\quad \begin{aligned} D(x_1) &= a_1x_1 + b_2x_2 + b_3x_3 + \ldots + b_{n-1}x_{n-1} + b_nx_n + \varphi_1 \\ D(x_2) &= b_2x_1 + a_2x_2 + \varphi_2 \\ D(x_3) &= b_3x_1 + a_3x_3 + \varphi_3 \\ &\vdots \\ D(x_{n-1}) &= b_{n-1}x_1 + a_{n-1}x_{n-1} + \varphi_{n-1} \\ D(x_n) &= b_nx_1 + a_nx_n + \varphi_n, \end{aligned}$$

for $a_i, b_i \in \mathbb{R}$, $\varphi_i \in C^\infty(\mathbb{R})$, $1 \leq i \leq n$.

Since symmetric arrowhead matrix is diagonalizable, we can find a general solution of our system by rewriting it in a basis formed by eigenvectors. The obtained system is completely decoupled, so we get a system of n linear differential equations of the first order in one variable. This method is very convenient theoretically, but in actual calculations usually requires quite a few steps. Furthermore, while there are some approaches for finding eigenvalues and eigenvectors of arrowhead matrix it can be a difficult job.

Applying Theorem 3.7 from the paper [16] and taking into consideration the form of the invariant factors of the symmetric arrowhead matrix we obtain the partially reduced system. Partially reduced system consists of k subsystems, where k is a number of invariant factors of system matrix. Every subsystem corresponds to one invariant factor. The first equation of subsystems is non-homogeneous linear differential equation in one unknown, with the characteristic polynomial equal to the invariant factor and with the non-homogenous term equal to the sum of principal minors of some doubly companion matrices obtained by replacing the first column of the companion matrix of the invariant factor by a column of the first and higher order derivatives of non-homogeneous terms involved in subsystem. Remaining equations are linear differential equations of the first order in two variables. This method also requires the change of basis.

The simple form of our system matrix inspire us to try to derive partial reduction formulas directly. In this manner we state following theorem, a direct method for transforming system (3.1) into partially reduced system.

Theorem 3.1. *The linear system of the first order differential equations (3.1) can* *be transformed into the partially reduced system*

$$\begin{array}{lcl}\Delta_B(D)(x_1) & = & \prod_{j=2}^{n}(D-a_j)(\varphi_1)+\sum_{i=2}^{n} b_i \prod_{\substack{j=2\\ j\neq i}}^{n}(D-a_j)(\varphi_i)\\ (D-a_2)(x_2) & = & b_2x_1+\varphi_2\\ (D-a_3)(x_3) & = & b_3x_1+\varphi_3\\ \vdots & & \\ (D-a_{n-1})(x_{n-1}) & = & b_{n-1}x_1+\varphi_{n-1}\\ (D-a_n)(x_n) & = & b_nx_1+\varphi_n,\end{array} \tag{3.2}$$

where the linear operator $\Delta_B(D)$ is define by replacing λ by D in (2.2).

Proof. Let us denote by $\prod_{i=2}^{n}(D-a_i)$ composition of operators $D-a_i$, for $2 \le i \le n$. The partially reduced system (3.2) is obtained by acting $\prod_{i=2}^{n}(D-a_i)$ on the first equation of the system (3.1) and by substituting expressions $(D-a_i)(x_i)$ appearing on the right sides of equality with $b_ix_1+\varphi_i$, for $2 \le i \le n$. Mind that operators $D-a_i$ and $D-a_j$ commute, for every i and j such that $2 \le i, j \le n$. Thus we have

$$\begin{aligned}\prod_{j=1}^{n}(D-a_j)(x_1) &= \sum_{i=2}^{n} b_i \prod_{j=2}^{n}(D-a_j)(x_i)+\prod_{j=2}^{n}(D-a_j)(\varphi_1)=\\ &= \sum_{i=2}^{n} b_i \prod_{\substack{j=2\\ j\neq i}}^{n}(D-a_j)(b_ix_1+\varphi_i)+\prod_{j=2}^{n}(D-a_j)(\varphi_1)=\\ &= \sum_{i=2}^{n} b_i^2 \prod_{\substack{j=2\\ j\neq i}}^{n}(D-a_j)(x_1)\\ &+ \sum_{i=2}^{n} b_i \prod_{\substack{j=2\\ j\neq i}}^{n}(D-a_j)(\varphi_i)+\prod_{j=2}^{n}(D-a_j)(\varphi_1).\end{aligned}$$

Rearranging the equation, we get the first equation from (3.2), i.e., we obtain

$$\Delta_B(D)(x_1)=\prod_{j=2}^{n}(D-a_j)(\varphi_1)+\sum_{i=2}^{n} b_i \prod_{\substack{j=2\\ j\neq i}}^{n}(D-a_j)(\varphi_i).$$

□

Finally we are considering total reduction of our arrowhead form system. As an immediate consequence of Theorems 4.1 from the paper [17] we can transform the system (3.1) into the totally reduced system

$$
(3.3) \quad
\begin{aligned}
\Delta_B(D)(x_1) &= \sum_{k=1}^{n}\sum_{j=1}^{n} b_{1j}^{(k)} D^{n-k}(\varphi_j)\\
\Delta_B(D)(x_2) &= \sum_{k=1}^{n}\sum_{j=1}^{n} b_{2j}^{(k)} D^{n-k}(\varphi_j)\\
&\vdots\\
\Delta_B(D)(x_n) &= \sum_{k=1}^{n}\sum_{j=1}^{n} b_{nj}^{(k)} D^{n-k}(\varphi_j),
\end{aligned}
$$

where the linear operator $\Delta_B(D)$ is define by replacing λ by D in (2.2) and coefficients $b_{ij}^{(k)}$ are calculated in Lemma 2.1.

4. An example

We will illustrate the previous results by the example. Consider the system of the differential equations

$$
(4.1) \quad
\begin{aligned}
D(x_1) &= x_1 + x_2 + 2x_3 + 2x_4 + 18e^t\\
D(x_2) &= x_1 + x_2\\
D(x_3) &= 2x_1 + x_3\\
D(x_4) &= 2x_1 + x_4.
\end{aligned}
$$

The vector form of the system (4.1) is $D(\vec{x}) = B\vec{x} + \vec{\varphi}$, where $\vec{x} = [x_1\ x_2\ x_3\ x_4]^T$ and $\vec{\varphi} = [18e^t\ 0\ 0\ 0]^T$. The system matrix is arrowhead matrix $B = \begin{bmatrix} 1 & 1 & 2 & 2\\ 1 & 1 & 0 & 0\\ 2 & 0 & 1 & 0\\ 2 & 0 & 0 & 1 \end{bmatrix}$, the reduced form of the matrix B is $\widetilde{B} = \begin{bmatrix} 1 & 3\\ 3 & 1 \end{bmatrix}$, the characteristic polynomial of the matrix $\widetilde{B}$ is $\Delta_{\widetilde{B}}(\lambda) = \begin{vmatrix} \lambda-1 & 3\\ 3 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 9 = (\lambda-4)(\lambda+2)$, and for the characteristic polynomial of the matrix B

$$\Delta_B(\lambda) = (\lambda-1)^2\Delta_{\widetilde{B}}(\lambda) = (\lambda-1)^2(\lambda-4)(\lambda+2) = \lambda^4 - 4\lambda^3 - 3\lambda^2 + 14\lambda - 8$$

holds. Coefficients of the characteristic polynomial of the matrix B are $d_0 = 1$, $d_1 = -4$, $d_2 = -3$, $d_3 = 14$ and $d_4 = -8$. The eigenvalues of the matrix B are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$ and $\lambda_4 = 4$. Corresponding eigenvectors are $v_1 = [0\ -2\ 1\ 0]^T$, $v_2 = [0\ \ -2\ 0\ 1]^T$, $v_3 = [-3\ 1\ 2\ 2]^T$ and $v_4 = [3\ 1\ 2\ 2]^T$. The Jordan normal

form of the matrix B is $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, the transformation matrix is $P = \begin{bmatrix} 0 & 0 & -3 & 3 \\ -2 & -2 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$ and its inverse is matrix $P^{-1} = \frac{1}{18}\begin{bmatrix} 0 & -4 & 10 & -8 \\ 0 & -4 & -8 & 10 \\ -3 & 1 & 2 & 2 \\ 3 & 1 & 2 & 2 \end{bmatrix}$.
The system (4.1) can be transformed to equivalent system $D(\vec{y}) = J\vec{y} + \vec{\psi}$, where $\vec{y} = [y_1\ y_2\ y_3\ y_4]^T = P^{-1}\vec{x}$ and $\vec{\psi} = P^{-1}\vec{\varphi}$, i.e., we have

$$\begin{aligned} D(y_1) &= y_1 \\ D(y_2) &= y_2 \\ D(y_3) &= -2y_3 - 3e^t \\ D(y_4) &= 4y_4 + 3e^t. \end{aligned}$$

Solution of the previous system is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} C_1e^t \\ C_2e^t \\ C_3e^{-2t} - e^t \\ C_4e^{4t} - e^t \end{bmatrix}.$$

and the solution of the system (4.1) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3C_3e^{-2t} + 3C_4e^{4t} \\ -2(C_1 + C_2 + 1)e^t + C_3e^{-2t} + C_4e^{4t} \\ (C_1 - 4)e^t + 2C_3e^{-2t} + 2C_4e^{4t} \\ (C_2 - 4)e^t + 2C_3e^{-2t} + 2C_4e^{4t} \end{bmatrix}.$$

The arrowhead matrix B has two invariant factors $\tau_1(\lambda) = \lambda - 1$ and $\tau_2(\lambda) = \mu_B(\lambda) = (\lambda - 1)(\lambda + 2)(\lambda - 4) = \lambda^3 - 3\lambda^2 - 6\lambda + 8$. The rational normal form of the matrix B is $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -8 & 6 & 3 \end{bmatrix}$, the transformation matrix is $T = \frac{1}{9}\begin{bmatrix} 0 & -3 & 6 & -3 \\ -18 & -13 & -7 & 2 \\ 0 & 14 & -4 & -1 \\ 9 & 6 & -6 & 0 \end{bmatrix}$ and its inverse is matrix $T^{-1} = \frac{1}{9}\begin{bmatrix} 0 & -2 & -4 & 5 \\ -3 & -2 & 5 & -4 \\ -3 & -5 & -1 & -10 \\ -30 & -8 & -7 & -16 \end{bmatrix}$.
The system (4.1) can be transformed to equivalent system $D(\vec{z}) = C\vec{z} + \vec{\nu}$, where

$\vec{z} = [z_1\ z_2\ z_3\ z_4]^T = T^{-1}\vec{x}$ and $\vec{\nu} = T^{-1}\vec{\varphi}$, i.e., we have

$$\begin{aligned} D(z_1) &= z_1 \\ D(z_2) &= z_3 - 6e^t \\ D(z_3) &= z_4 - 6e^t \\ D(z_4) &= -8z_2 + 6z_3 + 3z_4 - 60e^t. \end{aligned}$$

Previous system can be transformed into equivalent partially reduced system

$$\begin{aligned} D(z_1) - z_1 &= 0 \\ D^3(z_2) - 3D^2(z_2) - 6D(z_2) + 8z_2 &= 0 \\ z_3 &= D(z_2) + 6e^t \\ z_4 &= D(z_3) + 6e^t. \end{aligned}$$

Solution of the previous system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} C_1e^t \\ C_2e^t + C_3e^{-2t} + C_4e^{4t} \\ C_2e^t - 2C_3e^{-2t} + 4C_4e^{4t} + 6e^t \\ C_2e^t + 4C_3e^{-2t} + 16C_4e^{4t} + 12e^t \end{bmatrix}.$$

and the solution of the system (4.1) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3C_3e^{-2t} - 3C_4e^{4t} \\ -2(C_1 + C_2 + 1)e^t + C_3e^{-2t} - C_4e^{4t} \\ (C_2 - 4)e^t + 2C_3e^{-2t} - 2C_4e^{4t} \\ (C_1 - 4)e^t + 2C_3e^{-2t} - 2C_4e^{4t} \end{bmatrix}.$$

By Theorem 3.1 system (4.1) can be transformed into the system

$$\begin{aligned} D^4(x_1) - 4D^3(x_1) - 3D^2(x_1) + 14D(x_1) - 8x_1 &= 0 \\ D(x_2) - x_2 &= x_1 \\ D(x_3) - x_3 &= 2x_1 \\ D(x_4) - x_4 &= 2x_1. \end{aligned}$$

Solution of the reduced system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t} \\ C_5e^t + C_1e^t + \frac{C_2}{2}te^t - \frac{C_3}{3}e^{-2t} + \frac{C_4}{3}e^{4t} \\ C_6e^t + 2C_1e^t + C_2te^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \\ C_7e^t + 2C_1e^t + C_2te^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \end{bmatrix}. \tag{4.2}$$

To obtain the solution of the system (4.1) we need to find connection between constants C_i, $1 \leq i \leq 7$. Substituting (4.2) into (4.1) we obtain

$$
\begin{aligned}
&(C_1+C_2)e^t + C_2te^t - 2C_3e^{-2t} + 4C_4e^{4t} = \\
&C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t} + (C_1+C_5)e^t + \tfrac{C_2}{2}te^t - \tfrac{C_3}{3}e^{-2t} + \tfrac{C_4}{3}e^{4t} + \\
&(4C_1+2C_6)e^t + 2C_2te^t - \tfrac{4C_3}{3}e^{-2t} + \tfrac{4C_4}{3}e^{4t} + \\
&(4C_1+2C_7)e^t + 2C_2te^t - \tfrac{4C_3}{3}e^{-2t} + \tfrac{4C_4}{3}e^{4t} + 18e^t
\end{aligned}
$$

$$
\begin{aligned}
&(C_1+C_5+\tfrac{C_2}{2})e^t + \tfrac{C_2}{2}te^t + \tfrac{2C_3}{3}e^{-2t} + \tfrac{4C_4}{3}e^{4t} = \\
&(C_1+C_5)e^t + \tfrac{C_2}{2}te^t - \tfrac{C_3}{3}e^{-2t} + \tfrac{C_4}{3}e^{4t} + C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t}
\end{aligned}
$$

$$
\begin{aligned}
&(2C_1+C_2+C_6)e^t + C_2te^t + \tfrac{4C_3}{3}e^{-2t} + \tfrac{8C_4}{3}e^{4t} = \\
&(2C_1+C_6)e^t + C_2te^t - \tfrac{2C_3}{3}e^{-2t} + \tfrac{2C_4}{3}e^{4t} + 2C_1e^t + 2C_2te^t + 2C_3e^{-2t} + 2C_4e^{4t}
\end{aligned}
$$

$$
\begin{aligned}
&(2C_1+C_2+C_7)e^t + C_2te^t + \tfrac{4C_3}{3}e^{-2t} + \tfrac{8C_4}{3}e^{4t} = \\
&(2C_1+C_7)e^t + C_2te^t - \tfrac{2C_3}{3}e^{-2t} + \tfrac{2C_4}{3}e^{4t} + 2C_1e^t + 2C_2te^t + 2C_3e^{-2t} + 2C_4e^{4t}.
\end{aligned}
$$

Comparing both sides of the equalities, we have $C_1 = 0$, $C_2 = 0$ and $C_2 = 9C_1 + C_5 + 2C_6 + 2C_7 + 18$, i.e., we get the solution of the system (4.1)

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} C_3e^{-2t} + C_4e^{4t} \\ -2(C_6+C_7+9)e^t - \frac{C_3}{3}e^{-2t} + \frac{C_4}{3}e^{4t} \\ C_6e^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \\ C_7e^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \end{bmatrix}.
$$

From now on, we will focus on the total reduction method. We will start with calculation of the coefficients of the matrix adj(λI − B): $B_0 = I$, $B_1 = B + d_1I = \begin{bmatrix} -3 & 1 & 2 & 2 \\ 1 & -3 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix}$, $B_2 = B \cdot B_1 + d_2I = \begin{bmatrix} 3 & -2 & -4 & -4 \\ -2 & -5 & 2 & 2 \\ -4 & 2 & -2 & 4 \\ -4 & 2 & 4 & -2 \end{bmatrix}$ and $B_3 = B \cdot B_2 + d_3I = \begin{bmatrix} -1 & 1 & 2 & 2 \\ 1 & 7 & -2 & -2 \\ 2 & -2 & 4 & -4 \\ 2 & -2 & -4 & 4 \end{bmatrix}$. Totally reduced system obtained from the system (4.1) is completely decoupled homogenous system of four fourth order differential equations which differ only in variables

$$\begin{aligned} D^4(x_1) - 4D^3(x_1) - 3D^2(x_1) + 14D(x_1) - 8x_1 &= 0 \\ D^4(x_2) - 4D^3(x_2) - 3D^2(x_2) + 14D(x_2) - 8x_2 &= 0 \\ D^4(x_3) - 4D^3(x_3) - 3D^2(x_3) + 14D(x_3) - 8x_3 &= 0 \\ D^4(x_4) - 4D^3(x_4) - 3D^2(x_4) + 14D(x_4) - 8x_4 &= 0. \end{aligned}$$

Solution of the totally reduced system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t} \\ C_5e^t + C_6te^t + C_7e^{-2t} + C_8e^{4t} \\ C_9e^t + C_{10}te^t + C_{11}e^{-2t} + C_{12}e^{4t} \\ C_{13}e^t + C_{14}te^t + C_{15}e^{-2t} + C_{16}e^{4t} \end{bmatrix}.$$

Our last task is to find relations between constants C_i for $1 \leq i \leq 16$. As we have seen in the previous consideration, we can do that by plugging the solution of the totally reduced system into the original system (4.1). We obtain

$$\begin{aligned} &(C_1 + C_2)e^t + C_2te^t - 2C_3e^{-2t} + 4C_4e^{4t} = \\ &C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t} + C_5e^t + C_6te^t + C_7e^{-2t} + C_8e^{4t}+ \\ &2C_9e^t + 2C_{10}te^t + 2C_{11}e^{-2t} + 2C_{12}e^{4t}+ \\ &2C_{13}e^t + 2C_{14}te^t + 2C_{15}e^{-2t} + 2C_{16}e^{4t} + 18e^t \end{aligned}$$

$$\begin{aligned} &(C_5 + C_6)e^t + C_6te^t - 2C_7e^{-2t} + 4C_8e^{4t} = \\ &C_1e^t + C_2te^t + C_3e^{-2t} + C_4e^{4t} + C_5e^t + C_6te^t + C_7e^{-2t} + C_8e^{4t} \end{aligned}$$

$$\begin{aligned} &(C_9 + C_{10})e^t + C_{10}te^t - 2C_{11}e^{-2t} + 4C_{12}e^{4t} = \\ &2C_1e^t + 2C_2te^t + 2C_3e^{-2t} + 2C_4e^{4t} + C_9e^t + C_{10}te^t + C_{11}e^{-2t} + C_{12}e^{4t} \end{aligned}$$

$$\begin{aligned} &(C_{13} + C_{14})e^t + C_{14}te^t - 2C_{15}e^{-2t} + 4C_{16}e^{4t} = \\ &2C_1e^t + 2C_2te^t + 2C_3e^{-2t} + 2C_4e^{4t} + C_{13}e^t + C_{14}te^t + C_{15}e^{-2t} + C_{16}e^{4t}. \end{aligned}$$

Combining like terms for each equation yields

$$\begin{aligned} C_2 - C_1 &= C_5 + 2C_9 + 2C_{13} + 18 & 0 &= C_6 + 2C_{10} + 2C_{14} \\ -3C_3 &= C_7 + 2C_{11} + 2C_{15} & 3C_4 &= C_8 + 2C_{12} + 2C_{16} \end{aligned}$$

$$\begin{aligned} C_6 &= C_1 & C_2 &= 0 & -3C_7 &= C_3 & 3C_8 &= C_4 \\ C_{10} &= 2C_1 & C_2 &= 0 & -3C_{11} &= 2C_3 & 3C_{12} &= 2C_4 \\ C_{14} &= 2C_1 & C_2 &= 0 & -3C_{15} &= 2C_3 & 3C_{16} &= 2C_4. \end{aligned}$$

By substituting $C_6 = C_1$, $C_{10} = 2C_1$ and $C_{14} = 2C_1$ into $C_6 + 2C_{10} + 2C_{14} = 0$, we obtain that $C_1 = 0$. Together with $C_2 = 0$ the first equation becomes $C_5 + 2C_9 + 2C_{13} + 18 = 0$. The equation $-3C_3 = C_7 + 2C_{11} + 2C_{15}$ is direct consequence of equations $-3C_7 = C_3$, $-3C_{11} = 2C_3$ and $-3C_{15} = 2C_3$. Same holds for equations $3C_4 = C_8 + 2C_{12} + 2C_{16}$, $3C_8 = C_4$, $3C_{12} = 2C_4$ and $3C_{16} = 2C_4$. Therefore, we get $C_1 = C_2 = C_6 = C_{10} = C_{14} = 0$, $C_5 = -2(C_9 + C_{13} + 9)$, $C_7 = -\frac{C_3}{3}$, $C_{11} = C_{15} = -\frac{2C_3}{3}$, $C_8 = \frac{C_4}{3}$ and $C_{12} = C_{16} = \frac{2C_4}{3}$. Hence, solution of the system (4.1) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} C_3e^{-2t} + C_4e^{4t} \\ -2(C_9 + C_{13} + 9)e^t - \frac{C_3}{3}e^{-2t} + \frac{C_4}{3}e^{4t} \\ C_9e^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \\ C_{13}e^t - \frac{2C_3}{3}e^{-2t} + \frac{2C_4}{3}e^{4t} \end{bmatrix}.$$

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WEYL TYPE THEOREMS FOR ALGEBRAICALLY CLASS p-$wA(s,t)$ OPERATORS

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Abstract. In this paper, we study Weyl type theorems for $f(T)$, where T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ and f is an analytic function defined on an open neighborhood of the spectrum of T. Also we show that if $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, then generalized Weyl's theorem , a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB})$, where d_{AB} denote the generalized derivation $\delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\delta_{AB}(X) = AX - XB$ or the elementary operator $\Delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\Delta_{AB}(X) = AXB - X$.
Keywords: class p-$wA(s,t)$ operator, polaroid operator, Bishop's property (beta), Weyl type theorems, elementary operator.

1. Introduction and Preliminaries

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $\mathcal{H}$. Throughout this paper $R(T)$, $\ker(T)$, $\sigma(T)$ denotes range, null space and spectrum of $T \in B(\mathcal{H})$ respectively. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel

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condition $\ker U = \ker |T|$, then this decomposition is called the polar decomposition, which is one of the most important results in operator theory. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker U = \ker |T|$. An operator $T \in B(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. The Aluthge transformation introduced by Aluthge[5] is defined by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ where $T = U|T|$ be the polar decomposition of $T \in B(H)$. The generalized Aluthge transformation $T(s,t)$ ($s,t > 0$) is given by $T(s,t) = |T|^s U|T|^t$. Recall that an operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ $(0 < p \leq 1)$, *w-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$, *class A* if $|T^2| \geq |T|^2$, *class* $A(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ ([13]) and *class* $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([16]). Prasad and Tanahashi [19] introduced *class p-*$wA(s,t)$ operators as follows:

Definition 1.1. ([19]) Let $T = U|T|$ be the polar decomposition of T and let $s,t > 0$ and $0 < p \leq 1$. T is called class p-$wA(s,t)$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \quad \text{and} \quad (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}.$$

In general the following inclusions hold:
p-hyponormal $\subseteq$ w-hyponormal $\subseteq$ class $wA(s,t)$ $\subseteq$ class p-$wA(s,t)$.

Many interesting results for class p-$wA(s,t)$ has been studied in [10, 11, 19, 20, 21, 22, 24].

Let $\alpha(T)$ and $\beta(T)$ denote the nullity and the deficiency of $T \in B(\mathcal{H})$, defined by $\alpha(T)=\dim(\ker(T))$ and $\beta(T)=\dim(\ker(T^*)$. An operator T is said to be *upper semi-Fredholm* (resp.,*lower semi- Fredholm*) if $R(T)$ of $T \in B(\mathcal{H})$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). Let $SF_+(\mathcal{H})$ (resp., $SF_-(\mathcal{H})$) denote the semigroup of upper semi-Fredholm (resp., lower semi-Fredholm) operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *semi-Fredhom*, $T \in SF(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cup SF_-(\mathcal{H})$ and *Fredholm*, $T \in F(\mathcal{H})$, if $T \in SF_+(\mathcal{H}) \cap SF_-(\mathcal{H})$. The index of semi-Fredholm operator T is defined by ind $(T) = \alpha(T) - \beta(T)$. Recall[14], the *ascent* of an operator $T \in B(\mathcal{H})$, $a(T)$, is the smallest non negative integer $\mathtt{p}$ such that $\ker(T^{\mathtt{p}}) = \ker(T^{(\mathtt{p}+1)})$. Such $\mathtt{p}$ does not exist, then $\mathtt{p}(T) = \infty$. The *descent* of $T \in B(\mathcal{H})$, $d(T)$, is defined as the smallest non negative integer $\mathtt{q}$ such that $R(T^{\mathtt{q}}) = R(T^{(\mathtt{q}+1)})$. An operator $T \in B(\mathcal{H})$ is *Weyl*, $T \in W(\mathcal{H})$ it is Fredholm of index zero and *Browder* if T is Fredholm of finite ascent and descent. The Weyl spectrum of T, denoted by $\sigma_W(T)$, is given by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathcal{H})\}.$$

We say that $T \in B(\mathcal{H})$ satisfies *Weyl's theorem* if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T).$$

where $E_0(T)$ denote the set of eigenvalues of T of finite geometric multiplicity isolated in $\sigma(T)$. Let $SF_+^-(\mathcal{H}) = \{T \in SF_+(\mathcal{H}) : \text{ind(T)} \leq 0\}$. essential approximate point spectrum $\sigma_{SF_+^-}(T)$ of T is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{H})\}.$$

Let $\sigma_a(T)$ denote the approximate point spectrum of $T \in B(\mathcal{H})$. An operator $T \in B(\mathcal{H})$ holds *a-Weyl's theorem* if,

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T),$$

where $E_0^a(T) = \{\lambda \in \mathbb{C} : \lambda \in \text{iso } \sigma_a(T) \text{ and } 0 < \alpha(T - \lambda) < \infty\}$. We say that an operator $T \in B(\mathcal{H})$ satisfies *a-Browder's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$, where $\Pi_0^a(T)$ denote the set the left poles of T of finite rank. An operator $T \in B(\mathcal{H})$ is called *B-Fredholm*, $T \in BF(\mathcal{H})$ if there exist a non negative integer n for which the induced operator

$$T_{[n]} : R(T_{[n]}) \to R(T_{[n]})(\text{ in particular } T_{[0]} = T).$$

is Fredholm in the usual sense (see [7]). An operator $T \in B(\mathcal{H})$ is called *B-Weyl*, $T \in BW(\mathcal{H})$, if it is B-Fredholm with $\text{ind}(T_{[n]}) = 0$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin BW(\mathcal{H})\}$ (see [7]). Let $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. We say that T satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$. A bounded operator $T \in B(\mathcal{H})$ is said to satisfy *generalized Browders's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$, where $\Pi(T)$ is the set of poles of T (See [8]). We refer the readers to [1], where Weyl type theorems are extensively treated.

Recall that an operator $T \in B(\mathcal{H})$ is said to have the *single-valued extension property* (SVEP) if for every open subset U of $\mathbb{C}$ and any analytic function $f : U \to H$ such that $(T - z)f(z) \equiv 0$ on U, we have $f(z) \equiv 0$ on U. A Hilbert space operator $T \in B(\mathcal{H})$ satisfies *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and every sequence $f_n : U \longrightarrow \mathcal{H}$ of analytic functions with $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of U, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of U. For $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, the local resolvent set of T at x $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, which verifies $(T - z)f(z) = x$. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x. For each subset F of $\mathbb{C}$, the local spectral subspace of T, $\mathcal{H}_T(F)$, is given by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

Bishop's property $(\beta) \Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

See [1, 17] for more details.

Weyl's theorem for class p-$wA(s,t)$ has been studied in [22]. In this paper, we focus Weyl type theorems for algebraically class p-$wA(s,t)$ operators and elementary operator with class p-$wA(s,t)$ operator entries.

2. algebraically class p-$wA(s,t)$ operators and Weyl type theorem

We say that $T \in B(\mathcal{H})$ is *algebraically class p-$wA(s,t)$ operator* with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ if there exists a non- constant complex polynomial q such that $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$.

In general, the following inclusions hold:

p-hyponormal $\subset$ class p-$wA(s,t)$ $\subset$ algebraically class p-$wA(s,t)$.

Lemma 2.1. *[20] Let $T \in B(\mathcal{H})$ be a class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is nilpotent.*

Proof. Suppose $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then there exists a non- constant complex polynomial q such that $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Since $\sigma(q(T)) = q(\sigma(T))$ and $\sigma(T) = \{0\}$, the operator $q(T) - q(0)$ is quasinilpotent. By Lemma 2.1, $\sigma(q(T) - q(0)) = \{0\}$ implies that $q(T) - q(0) = 0$. Hence it follows that,

$$0 = q(T) - q(0) = cT^m(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)$$

where $m \geq 1$. Since $T - \lambda_i I$ is invertible for every $\lambda_i \neq 0$, we must have $T^m = 0$. □

It is well known that if both ascent and descent of T are finite then they are equal (see, [14, Proposition 38.3]). Moreover, $0 < a(T - \mu I) = d(T - \mu I) < \infty$ precisely when μ is a pole of the resolvent of T (see, [14, Proposition 50.2]).

An operator $T \in B(H)$ is *polaroid* if the isolated points of the spectrum of T are poles of the resolvent T. Evidently, T is polaroid implies T is isoloid (ie., every isolated point of $\sigma(T)$ is an eigenvalue of T).

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is polaroid.*

Proof. Assume that $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ and let μ be an isolated point of $\sigma(T)$. To prove that T is polaroid, it is enough to show that $a(T - \mu I) < \infty$ and $d(T - \mu I) < \infty$. Let E_μ denote the spectral projection associated with λ. Then the Riesz idempotent E of T with respect to z is defined by

$$E_\mu := \frac{1}{2\pi i}\int_{\partial D}(zI - T)^{-1}dz,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. We can represent T on $\mathcal{H} = R(E_\mu) \oplus \ker(E_\mu)$ as follows

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $\sigma(A) = \{\mu\}$ and $\sigma(B) = \sigma(T) \setminus \{\mu\}$.

Since $T \in B(\mathcal{H})$ is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, $q(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ for some non constant complex polynomial q. Thus, $\sigma(q(A)) = q(\sigma(A)) = q(\mu)$. Therefore, $q(A) - q(\mu)$ is quasi nilpotent. Then by Lemma 2.1, $q(A) - q(\mu) = 0$. Put $r(z) = q(A) - q(\mu)$, then $r(A) = 0$ and so A is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Since $\sigma(A) = \{\mu\}$, it follows from Theorem 2.1 that $A - \mu I$ is nilpotent and so $a(A - \mu I) < \infty$ and $d(A - \mu I) < \infty$. Also, $a(B - \mu I) < \infty$ and $d(B - \mu I) < \infty$ follows from the invertibility of $B - \mu I$. Consequently, $T - \mu I$ has finite ascent and descent. This completes the proof. □

Theorem 2.3. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T satisfies generalized Weyl's theorem.*

Proof. Suppose that T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. From Theorem 2.2, T is polaroid . Since T is algebraically class p-$wA(s,t)$ with $s,t \leq 1$, $p(T)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$ for some nonconstant polynomial q, it follows that $q(T)$ has Bishop's property (β) by [24, Theorem 2.4] or [22]. Therefore, $q(T)$ has SVEP. Then by [17, Theorem 3.3.9] T has SVEP . Hence the required result follows from [3, Theorem 4.1]. □

Corollary 2.1. *Let $T \in B(\mathcal{H})$ be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T satisfies Weyl's theorem.*

According to Berkani and Koliha [8], an operator $T \in B(\mathcal{H})$ is said to be Drazin invertible if T has finite ascent and descent. The Drazin spectrum of $T \in B(\mathcal{H})$, denoted by $\sigma_D(T)$, is defined $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}$ (See, [7]). Let $H(\sigma(T))$ denote the set of analytic functions which are defined on an open neighborhood of $\sigma(T)$.

Theorem 2.4. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then the equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in H(\sigma(T))$.*

Proof. Since T is algebrically class p-$wA(s,t)$ with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, T has SVEP. Hence, $f(T)$ satisfies generalized Browder's theorem. Then by [12, Theorem 2.1] we have

$$\sigma_{BW}(f(T)) = \sigma_D(f(T)).$$

By [12, Theorem 2.7]), $\sigma_D(f(T)) = f(\sigma_D(T))$ and hence $\sigma_{BW}(f(T)) = f(\sigma_D(T))$. Since T is algebraically class p-$wA(s,t)$ with $0 < s,t,s+t \leq 1$, T satisfies generalized Weyl's theorem. Thus, T satisfies generalized Browder's theorem and so $f(\sigma_D(T)) = f(\sigma_{BW}(T))$. Therefore, $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. This completes the proof. □

Theorem 2.5. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ s. Since the equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in H(\sigma(T))$ by Theorem 2.4, it follows that $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$. □

Theorem 2.6. *Let T^* be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem holds for T.*

Proof. Since T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, $q(T^*)$ is class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$ for some nonconstant polynomial q. It follows from [22] that $q(T^*)$ has SVEP. Therefore, T^* has SVEP by [17, Theorem 3.3.9]. By Theorem 2.2, T^* is polaroid. Since T^* is polaroid, T is polaroid. By applying [4, Theorem 3.10], it follows that a-Weyl's theorem holds for T. □

Theorem 2.7. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. Recall that for every $T \in B(\mathcal{H})$, the following inclusion

$$\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$$

is always true. Now it suffices to show that $\sigma_{SF_+^-}(f(T)) \supseteq f(\sigma_{SF_+^-}(T))$. Let $\lambda \notin \sigma_{SF_+^-}(f(T))$. Then $f(T) - \lambda I \in SF_-^+(\mathcal{H})$. Let

$$f(T) - \lambda I = c(T - \mu_1)(T - \mu_2)....., (T - \mu_n)g(T), \tag{2.1}$$

where $c, \mu_1, \mu_2 ...., \mu_n \in \mathbb{C}$ and $g(T)$ is invertible. Since T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$, T has SVEP. It follows from [1, Corollary 3.19] that $\text{ind}(T - \mu) \leq 0$ for all μ for which $T - \mu$ is Fredholm, $T - \mu_i$ is Fredholm of index zero for each $i = 1, 2, .., n$. Therefore, $\mu_i \notin \sigma_{SF_+^-}(T)$ for all $1 \leq i \leq n$. Hence,

$$\lambda = f(\mu_i) \notin f(\sigma_{SF_+^-}(T)).$$

This completes the theorem. □

Recall that an operator $T \in B(\mathcal{H})$ is said to be a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Evidently, if T is a-isoloid, then it is isoloid.

Theorem 2.8. *Let T^* be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. From Theorem 2.6, a-Weyl's theorem holds for T. Hence, T satisfies a-Browder's theorem. Since T^* is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, T^* has SVEP. If $f \in H(\sigma(T))$, then by [17, Theorem 3.3.9], $f(T)$, or $f(T)$ satisfies the SVEP. Applying [18, Theorem 2.4], it follows that $f(T)$ satisfies a- Browder's theorem. To prove a-Weyl's theorem holds for $f(T)$ it is enough to show that $E_0^a(f(T)) = \Pi_0^a(f(T))$. The inclusion $\Pi_0^a(f(T)) \subseteq E_0^a(f(T))$ is trivial. To prove the reverse inclusion let $\lambda \in E_0^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $\alpha(f(T)-\lambda I) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\mu_i \in \sigma_a(T)$, then μ_i is an isolated point of $\sigma_a(T)$ by (2.1). That is, T is a-isoloid. Thus, $0 < \alpha(f(T) - \mu_i I) < \infty$ for each $i = 1, 2, ..., n$. Since T satisfies a-Weyl's theorem , $T - \mu_i I \in SF_+^-(\mathcal{H})$for each $i = 1, 2, ..., n$. Therefore $f(T) - \lambda I \in SF_+(\mathcal{H})$ and

$$\text{ind}(f(T) - \lambda I) = \sum_{i=1}^{n} \text{ind}(f(T) - \mu_i I) \leq 0.$$

Hence, $\lambda \notin \sigma_{SF_+^-}(f(T))$. Since $f(T)$ satisfies a-Browder's theorem, $\lambda \in \Pi_0^a(f(T))$. This completes the proof. □

Theorem 2.9. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then Weyl's theorem holds for $T + R$ for any finite rank operator $F \in B(\mathcal{H})$ commuting with T.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then from Theorem 2.2, isolated point of spectrum of T are eigenvalues. By Theorem 2.1, T satisfies Weyl's theorem. Then it follows that Weyl's theorem holds for $T + R$ for any finite rank operator $R \in B(\mathcal{H})$ by [15, Theorem 3.3],. □

Theorem 2.10. *Let T be an algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then for any function $f \in H(\sigma(T))$ and any finite rank operator $R \in B(\mathcal{H})$ commuting with T, Weyl's theorem holds for $f(T) + R$.*

Proof. Suppose T is algebraically class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then T is polaroid by Theorem 2.2 and hence T is isoloid. Therefore, $f(T)$ is isoloid for any function f analytic on a neighborhood of $\sigma(T)$ by [15, Lemma 3.6]. Then $f(T)$ obeys generalized Weyl theorem for any function $f \in H(\sigma(T))$ by Theorem 2.5. Then from [15, Theorem 3.3], it follows that Weyl's theorem holds for $f(T) + R$ for any finite rank operator R. □

3. elementary operator d_{AB} and Weyl type theorem

Let d_{AB} denote the *generalized derivation* $\delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\delta_{AB}(X) = AX - XB$ or the *elementary operator* $\Delta_{AB} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\Delta_{AB}(X) =$

$AXB - X$. In this section we show that if $A, B^* \in B(H)$ are class p-$wA(s,t)$ operator with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then generalized Weyl's theorem , a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem holds for $f(d_{AB})$ for every $f \in H\sigma(d_{AB})$. Recall that an operator $T \in B(H)$ is said to have the property (δ) if for every open covering (U, V) of $\mathbb{C}$, we have $\mathcal{H} = \mathcal{H}_T(\bar{U}) + \mathcal{H}_T(\bar{V})$.

Lemma 3.1. *Let $A, B \in B(\mathcal{H})$. If A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then d_{AB} has SVEP.*

Proof. Suppose that A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$. Then A and B^* satisfies Bishop's property (β) by [24, Theorem 2.4] or [22]. Hence B satisfies property (δ) by [17, Theorem 2.5.5]. Since both AX and XB satisfies property (C) by Corollary 3.6.11of [17]. Then SVEP holds for both $AX - XB$ and $AXB - X$ by [17, Theorem 3.6.3] and [17, Note 3.6.19]. Then, d_{AB} has SVEP.

□

Lemma 3.2. *Let $A, B \in B(\mathcal{H})$. If A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then d_{AB} is polaroid.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, A and B^* are polaroid by Proposition 2.2. It is known that if B^* is polaroid then B is polaroid. Hence the required result follows by [26, Lemma 4.1] □

Theorem 3.1. *If $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then generalized Weyl's theorem holds for d_{AB}.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, d_{AB} has SVEP by Lemma 3.1. By Lemma 3.2, d_{AB} is polaroid. Then by applying [4, theorem 3.10], it follows that generalized Weyl's theorem holds for d_{AB} □

Theorem 3.2. *If $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, then generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H(\sigma(d_{AB}))$.*

Proof. Since A and B^* are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$, d_{AB} has SVEP by Lemma 3.1. By Lemma 3.2 the operator d_{AB} is polaroid and so d_{AB} is isoloid. Then by applying [25, theorem 2.2], it follows that generalized Weyl's theorem holds for $f(d_{AB})$ for every $f \in H\sigma(d_{AB})$. □

We say that $T \in B(\mathcal{H})$ possesses property (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ [23]. In Theorem 2.8 of [2], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in B(\mathcal{H})$ possesses property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [6]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [6] that an operator possessing property (gw) possesses property (w) but the converse is not true in general.

Theorem 3.3. *Let $A, B^* \in B(\mathcal{H})$ are class p-$wA(s,t)$ operators with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. Then a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for every $f \in H(\sigma(d_{AB}))$.*

Proof. By Lemma 3.1, the operator d_{AB} has SVEP. The operator d_{AB} is polaroid by Lemma 3.2,. Then by applying [4, theorem 3.12], it follows that a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for every $f \in H(\sigma(d_{AB}))$. □

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ON SOME EQUIVALENCE RELATION ON NON-ABELIAN CA-GROUPS

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Abstract. A non-abelian group G is called a CA-group (CC-group) if $C_G(x)$ is abelian (cyclic) for all $x \in G \setminus Z(G)$. We say $x \sim y$ if and only if $C_G(x) = C_G(y)$. We denote the equivalence class including x by$[x]_\sim$. In this paper, we prove that if G is a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$, then $2^{r-1} \leq |G'| \leq 2^{\binom{r}{2}}$. where $\frac{|G|}{|Z(G)|} = 2^r, 2 \leq r$ and characterize all groups whose $[x]_\sim = xZ(G)$ for all $x \in G$ and $|G| \leq 100$. Also, we will show that if G is a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$, then $G \cong C_m \times Q_8$ where C_m is a cyclic group of odd order m and if G is a CC-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$, then $G \cong Q_8$.
Keywords: CA-group, CC-group, centralizer of a group, derived subgroup.

1. Introduction

Throughout this paper all groups are assumed to be finite. We denote by $Z(G)$, $C_G(x), \mathrm{Cent}(G), |\mathrm{Cent}(G)|, x^G, G'$ and $k(G)$ the center of the group G, the centralizer of $x \in G$, the set of centralizers of the group G, the number of centralizers of the group G, the conjugacy class of $x \in G$, the derived subgroup of the group G, the number of conjugacy classes of the group G, respectively. The authors in [8], denoted by $[m, n]$ the GAP ID of a group which is a label that uniquely identifies a group in GAP. The first number in $[m, n]$ is the order of the group, and the second number simply enumerates different groups of the same order. We will use usual notation, for example C_n, D_{2n} and Q_{2^n} denote the cyclic group of order n, the

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dihedral group of order $2n$ and the generalized quaternion group of order 2^n respectively. The *non-commuting graph* $\Gamma(G)$ with respect to G is a graph with vertex set $G \setminus Z(G)$ and two distinct vertices x and y, are adjacent whenever $[x, y] \neq 1$. A non-abelian group G is called a CA-group (CC-group) if $C_G(x)$ is abelian (cyclic) for all $x \in G \setminus Z(G)$. We say $x \sim y$ if and only if $C_G(x) = C_G(y)$, and $x \sim_1 y$ if and only if $xZ(G) = yZ(G)$. We denote the equivalence class including x under $\sim$ by $[x]_\sim$. The number of equivalence classes of $\sim$ and $\sim_1$ on the group G are equal with $|\mathrm{Cent}(G)|$ and $\frac{|G|}{|Z(G)|}$ respectively. The influence of $|\mathrm{Cent}(G)|$ on the group G has been investigated in [3, 2, 4]. In [5], CA-groups whose $[x]_\sim = xZ(G)$ for all $x \in G$ has been investigated. In this paper we have investigated the equivalency of above relations. We will use the following lemmas to prove the main theorems.

Lemma 1.1. *[1, Lemma 3.6] Let G be a non-abelian group. Then the following are equivalent:*

1) *G is a* CA-*group.*
2) *If $[x, y] = 1$ then $C_G(x) = C_G(y)$, where $x, y \in G \setminus Z(G)$.*
3) *If $[x, y] = [x, z] = 1$ then $[y, z] = 1$, where $x \in G \setminus Z(G)$.*
4) *If $A, B \leq G, Z(G) \lneqq C_G(A) \leqslant C_G(B) \lneqq G$, then $C_G(A) = C_G(B)$.*

Lemma 1.2. *[1, Proposition 2.6] Let G be a finite non-abelian group and $\Gamma(G)$ be a regular graph. Then G is nilpotent of class at most 3 and $G = A \times P$, where A is an abelian group and P is a p-group (p is a prime) and furthermore $\Gamma(P)$ is a regular graph.*

Lemma 1.3. *[5, Lemma 11] Let G be a non-abelian group. Then $xZ(G) \subseteq [x]_\sim$, for all $x \in G$. Also the equality happens if and only if $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$.*

Lemma 1.4. *[5, Lemma 12] Let G be a finite non-abelian group. Then the following are equivalent:*

1) *If $[x, y] = 1$, then $xZ(G) = yZ(G)$, where $x, y \in G \setminus Z(G)$.*
2) *G is a* CA-*group and $[x]_\sim = xZ(G)$, for all $x \in G$.*
3) *$[x, y] = 1$ and $[x, w] = 1$ imply that $yZ(G) = wZ(G)$, where $x, y, w \in G \setminus Z(G)$.*

Lemma 1.5. *[5, Theorem 3] Let G be a non-abelian group. The following are equivalent:*

1) *G is a* CA-*group and $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$.*
2) *$G = A \times P$, where A is an abelian group, P is a 2-group, P is a* CA-*group and $|\mathrm{Cent}(P)| = \frac{|P|}{|Z(P)|}$.*

3) *$G = A \times P$, where A is an abelian group and $C_P(x) = Z(P) \cup xZ(P)$, for all $x \in P \setminus Z(P)$.*

Lemma 1.6. *[5, Lemma 13] Let G be a non-abelian group. Let $[x]_\sim$ and $[y]_\sim$ be two different classes of $\sim$. If $[x_0, y_0] \neq 1$ for some $x_0 \in [x]_\sim$ and $y_0 \in [y]_\sim$, then $[u, v] \neq 1$ for all $u \in [x]_\sim$ and $v \in [y]_\sim$.*

Lemma 1.7. *[5, Lemma 20] Let G_1 and G_2 be two groups. Let $[g_1]_\sim = g_1Z(G_1)$, for all $g_1 \in G_1$ and $[g_2]_\sim = g_2Z(G_2)$, for all $g_2 \in G_2$. Then $[X]_\sim = XZ(G_1 \times G_2)$, for all $X \in G_1 \times G_2$.*

Lemma 1.8. *[6, Theorem 2.1] Let G be a non-abelian group and $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$. Then $\frac{G}{Z(G)}$ is an elementary abelian 2-group.*

Lemma 1.9. *[7, Corollary 2.3] Let G be a non-abelian nilpotent group. Then G is a* CC*-group if and only if $G \cong C_m \times Q_{2^n}$, where m and n are positive integers and m is odd.*

In Section 2 we will provide some results about the equivalency of relations.

2. Proof of the main theorems

In this section we prove the main theorems. For doing this we first prove some lemmas.

Lemma 2.1. *Let G be a* CA*-group. Then $C_G(x) = Z(G) \cup [x]_\sim$, for all $x \in G \setminus Z(G)$.*

Proof. Since $Z(G) \subseteq C_G(x)$ and $[x]_\sim \subseteq C_G(x)$ we have $Z(G) \cup [x]_\sim \subseteq C_G(x)$. Suppose $g \in C_G(x) \setminus Z(G)$. Then $[g, x] = 1$. By Lemma 1.1, $C_G(x) = C_G(g)$ which implies that $[x]_\sim = [g]_\sim$. Hence $g \in [x]_\sim$ and we have $C_G(x) \subseteq Z(G) \cup [x]_\sim$. Therefore $C_G(x) = Z(G) \cup [x]_\sim$, for all $x \in G \setminus Z(G)$. $\square$

Lemma 2.2. *Let G be a non-abelian group. Then G is a* CA*-group and $[x]_\sim = xZ(G)$, for all $x \in G$ if and only if $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k(G))}$.*

Proof. Let G be a CA-group and $[x]_\sim = [x]_{\sim_1}$, for all $x \in G$. Let $xZ(G) \neq yZ(G)$ for some $x, y \in G \setminus Z(G)$. Since $XY \neq YX$ for all $X \in xZ(G)$ and $Y \in yZ(G)$, therefore there exists an edge between X and Y. Hence there are $|Z(G)|^2$ edges between elements of $xZ(G)$ and $yZ(G)$. Also there are $\frac{|G|}{|Z(G)|} - 1$ different classes of $xZ(G)$ for $x \in G \setminus Z(G)$. Thus $|E(\Gamma(G))| = \binom{\frac{|G|}{|Z(G)|}-1}{2}|Z(G)|^2$. Note that by [1, Lemma 3.27], $|E(\Gamma(G))| = \frac{|G|^2-k(G)|G|}{2}$. Hence $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$.

Conversely, suppose $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$. So $|G| = |Z(G)|+(k(G)-|Z(G)|)\frac{|G|}{2|Z(G)|}$. Since for all $x \in G \setminus Z(G)$, $|x^G| \leq \frac{|G|}{2|Z(G)}$ we have $|x^G| = \frac{|G|}{2|Z(G)|}$, for all $x \in G \setminus Z(G)$.

So $|C_G(x)| = 2|Z(G)|$, for all $x \in G \setminus Z(G)$. Now by [5, Lemma 15] G is a CA-group and $[x]_\sim = [x]_{\sim_1}$. $\square$

Example 2.1. Let G be a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$ and $|G| \leq 100$. Then G is one of the group with GAP ID in Table 2.1.

Table 2.1: The GAP ID of group G where $|G| = \frac{2|Z(G)|^2}{3|Z(G)|-k(G)}$ and $|G| \leq 100$.

[8, 3]	[8, 4]						
[16, 3]	[16, 4]	[16, 6]	[16, 11]	[16, 12]	[16, 13]		
[24, 10]	[24, 11]						
[32, 2]	[32, 4]	[32, 5]	[32, 12]	[32, 17]	[32, 22]	[32, 23]	[32, 24]
[32, 25]	[32, 26]	[32, 37]	[32, 38]	[32, 46]	[32, 47]	[32, 48]	
[40, 11]	[40, 12]						
[48, 21]	[48, 22]	[48, 24]	[48, 45]	[48, 46]	[48, 47]		
[56, 9]	[56, 10]						
[64, 3]	[64, 17]	[64, 27]	[64, 29]	[64, 44]	[64, 51]	[64, 56]	[64, 57]
[64, 58]	[64, 59]	[64, 73]	[64, 74]	[64, 75]	[64, 76]	[64, 77]	[64, 78]
[64, 79]	[64, 80]	[64, 81]	[64, 82]	[64, 84]	[64, 85]	[64, 86]	[64, 87]
[64, 103]	[64, 112]	[64, 115]	[64, 126]	[64, 184]	[64, 185]	[64, 193]	[64, 194]
[64, 195]	[64, 196]	[64, 197]	[64, 198]	[64, 247]	[64, 248]	[64, 261]	[64, 262]
[64, 263]							
[72, 10]	[72, 11]	[72, 37]	[72, 38]				
[80, 21]	[80, 22]	[80, 24]	[80, 46]	[80, 47]	[80, 48]		
[88, 9]	[88, 10]						
[96, 45]	[96, 47]	[96, 48]	[96, 52]	[96, 54]	[96, 55]	[96, 60]	[96, 162]
[96, 163]	[96, 165]	[96, 166]	[96, 167]	[96, 221]	[96, 222]	[96, 223]	

Theorem 2.1. *Let G be a* CA-*group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $2^{r-1} \leq |G'| \leq 2^{\binom{r}{2}}$, where $\frac{|G|}{|Z(G)|} = 2^r, 2 \leq r$.*

Proof. Let G be a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$. First we show that $|G'| \leq 2^{\binom{r}{2}}$. Since $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemmas 1.8 and 1.3, we find that $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G), g^2 \in Z(G)$, for all $g \in G$ and G' is an elementary abelian 2-group. Since G is a non-abelian group, there exist $x, y \in G$ such that $[x, y] = z \neq 1$ and $[x, xy] \neq 1$ and $[y, xy] \neq 1$. By Lemma 1.4, $xZ(G) \neq yZ(G)$, $xZ(G) \neq xyZ(G)$ and $yZ(G) \neq xyZ(G)$. Let $H_1 = Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G)$. Since $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $H_1 \leq G$. By Lemma 1.6, none of the elements of $xZ(G)$ are commute with elements of $yZ(G)$ and $xyZ(G)$. Also none of the elements of $yZ(G)$ are commute with elements of $xyZ(G)$. Therefore $Z(H_1) = Z(G)$. Since $G' \leq Z(G)$ and $t^2 = 1$, for all $t \in G^{'}$, we have the following:

$$[x, y]^{-1} = [y, x] = [x, y] = [x, xy] = [y, yx] = z, [eu, fw] = [e, f],$$

for all $e, f \in \{x, y, xy\}$ and for all $u, w \in Z(G)$. Hence

$$\begin{aligned} H_1' &= \langle [g_1, h_1] | g_1, h_1 \in H_1 \rangle = \langle [eu, fw] | e, f \in \{x, y, xy\}, u, w \in Z(G) \rangle \\ &= \langle [e, f] | e, f \in \{x, y, xy\} \rangle = \langle [x, y] \rangle = \langle z \rangle = \{1, z\}. \end{aligned}$$

Thus $|H_1'| = 2 \leq 2^{\binom{2}{2}}$ and $\frac{|H_1|}{|Z(H_1)|} = \frac{4|Z(G)|}{|Z(G)|} = 2^2$. If $G = H_1$ then proof is complete, so assume that $G \neq H_1$. Hence there exists $a \in G \setminus H_1$. Let $H_2 = H_1\langle a \rangle$. Since $a^2 \in Z(G)$ we have

$$\begin{aligned} H_2 = \quad & H_1\langle a \rangle = H_1 \cup aH_1 = Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G) \\ & \cup aZ(G) \cup axZ(G) \cup ayZ(G) \cup axyZ(G) \end{aligned}$$

and since $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $H_2 \leq G$. By Lemma 1.6 $Z(H_2) = Z(G)$. Let $[a, x] = t_1, [a, y] = t_2$. Therefore $1 \neq [a, xy] = [a, x][a, y] = t_1 t_2$. In above we had $[x, y] = [x, xy] = [y, xy] = z$. On the other hand $[e_1 u, f_1 w] = [e_1, f_1]$, for all $u, w \in Z(G)$ and for all $e_1, f_1 \in \{x, y, xy, a, ax, ay, axy\}$. Also $[g_2, h_2 k_2] = [g_2, h_2][g_2, k_2]$, for all $g_2, h_2, k_2 \in H_2$. Hence

$$\begin{aligned} H_2' = \quad & \langle [g_2, h_2] | g_2, h_2 \in H_2 \rangle = \langle [e_1 u, f_1 w] | e_1, f_1 \in \{x, y, xy, a, ax, ay, axy\} \rangle \\ & = \langle [x, y], [a, x], [a, y] \rangle = \langle z, t_1, t_2 \rangle. \end{aligned}$$

Therefore $|H_2'| \leq 2^{\binom{3}{2}}$ and $\frac{|H_2|}{|Z(H_2)|} = \frac{8|Z(G)|}{|Z(G)|} = 2^3$. If $G = H_2$, then the proof is complete. Let $G \neq H_2$. Therefore there exists $b \in G \setminus H_2$. Let $H_3 = H_2\langle b \rangle$. Let $[b, x] = l_1, [b, y] = l_2, [b, a] = l_3$. By a Similar calculation we have, $Z(H_3) = Z(G)$ and $H_3' = \langle z, t_1, t_2, l_1, l_2, l_3 \rangle$. Hence $|H_3'| \leq 2^6 = 2^{\binom{4}{2}}$ and $\frac{|H_3|}{|Z(H_3)|} = \frac{16|Z(G)|}{|Z(G)|} = 2^4$. By continuing this process, we have the following subgroups: $Z(G) \leq H_1 \leq H_2 \leq ... \leq H_i \leq ... \leq G$, such that $Z(H_i) = Z(G)$, $|H_i'| \leq 2^{\binom{i+1}{2}}$, $\frac{|H_i|}{|Z(H_i)|} = 2^{i+1}$. Since G is finite, there exists $2 \leq r$, such that $G = H_{r-1}, |G'| \leq 2^{\binom{r}{2}}$ and $\frac{|G|}{|Z(G)|} = \frac{|H_{r-1}|}{|Z(H_{r-1})|} = 2^r$. Since $[w]_\sim = wZ(G)$, for all $w \in G \setminus Z(G)$, so by Lemma 2.1, $|w^G| = \frac{|G|}{|C_G(w)|} = \frac{|G|}{2|Z(G)|}$, for all $w \in G \setminus Z(G)$. Consequently, as $w^G \subseteq wG'$, we have $\frac{|G|}{2|Z(G)|} = 2^{r-1} \leq |G'|$. □

Theorem 2.2. *Let G be a non-abelian* CC*-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $G \cong C_m \times Q_8$ where C_m is a cyclic group of odd order m.*

Proof. Let G be a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Therefore G is a CA-group. By lemma 1.3, $|\text{Cent}(G)| = \frac{|G|}{|Z(G)|}$ and by lemma 1.5, $G \cong A \times P$ where A is an abelian group and P is a 2-group. Hence G is a nilpotent group. By lemma

1.9, $G \cong C_m \times Q_{2^n}$ where C_m is a cyclic group of order odd m. Since $[x]_\sim = xZ(G)$ for all $x \in G$, we have by lemma 1.3, that $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$ and by Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group which implies that $G' \leq Z(G)$. Hence $(C_m \times Q_{2^n})' \subseteq Z(C_m \times Q_{2^n})$ and $1 \times Q'_{2^n} \subseteq C_m \times Z(Q_{2^n}) \cong C_m \times C_2$. Therefore $Q'_{2^n} \cong C_2$ and $|Q'_{2^n}| = 2$. Since $|Q'_{2^n}| = 2^{n-2}$, we have $n = 3$ and $G \cong C_m \times Q_8$.

Conversely Q_8 is a CC-group and $[x]_\sim = xZ(G)$, for all $x \in G$. Therefore $C_m \times Q_8$ is also a CC-group and by Lemma 1.7, $[x]_\sim = xZ(G)$ for all $x \in G \cong C_m \times Q_8$. □

Proposition 2.1. *Let G be a non-abelian group and $G' \leq Z(G)$. Then if $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ then $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$.*

Proof. Let $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Since $G' \leq Z(G)$, so $xG' \leq xZ(G)$. By Lemma 1.3, $xZ(G) \subseteq [x]_\sim$, for all $x \in G$. Hence $xZ(G) \subseteq [x]_\sim = x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. This implies that $[x]_\sim = x^G = xG' = xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|xG'| = |xZ(G)|$ we have $G' = Z(G)$ and the proof is complete. □

Example 2.2. Let G be an extra especial group of order 32. Then $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$.

Theorem 2.3. *Let G be a* CA*-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Then G is a 2-group, $\frac{G}{Z(G)}$ is an elementary abelian 2-group, $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$.*

Proof. Since G is a CA-group, by Lemma 2.1, $C_G(x) = [x]_\sim \cup Z(G)$, for all $x \in G \setminus Z(G)$. Therefore $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{|Z(G)|+|[x]_\sim|} = \frac{|G|}{|Z(G)|+|x^G|}$ which implies that $|x^G|^2 + |Z(G)||x^G| - |G| = 0$. So $|x^G|$ is a constant and $\Gamma(G)$ is a regular graph. By Lemma 1.2, $G = A \times P$ where A is an abelian group and P is a p-group (p is a prime) and by Lemma 1.3, $xZ(G) \subseteq [x]_\sim$, for all $x \in G \setminus Z(G)$. Therefore $xZ(G) \subseteq [x]_\sim = x^G \subseteq xG'$ which implies that $xZ(G) \subseteq xG'$. Thus $Z(G) \leq G'$ and $Z(G) = A \times Z(P) \leq G' = 1 \times P'$. Hence $A \cong 1$ and $Z(P) \leq P'$. So G is a p-group and $G \cong P$ and there exist positive integers m, n, t so that $|P| = p^n, |Z(P)| = p^t, |x^P| = p^m$ and $p^m = \frac{p^n}{(p^t+p^m)}$. This implies that $p^{2m} + p^{t+m} = p^n$ and $p^{m-t} + 1 = p^{n-m-t}$. Since p is a prime, by discussing the different states of the prime numbers, we obtain $p = 2$ and $m = t$. Since $xZ(P) \subseteq [x]_\sim = x^P$ and $|x^P| = |Z(P)|$, so $[x]_\sim = x^P = xZ(P)$, for all $x \in P \setminus Z(P)$. By Lemma 1.3, $|\mathrm{Cent}(P)| = \frac{|P|}{|Z(P)|}$. This implies by Lemma 1.8, that $\frac{P}{Z(P)}$ is an elementary abelian 2-group and $P' \leq Z(P)$. Hence $Z(P) = P'$. □

Corollary 2.1. *Let G be a* CC*-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. Then $G \cong Q_8$.*

Proof. By Theorem 2.3, $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and $G' = Z(G)$. and by Theorem 2.2, $G \cong C_m \times Q_8$ where m is an odd positive integer. Since $G' = Z(G)$, so $1 \times Q_8' \cong C_m \times Z(Q_8)$. Therefore $C_m \cong 1$. Hence $G \cong Q_8$. □

Lemma 2.3. *A group G is a* CA-*group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ if and only if $|G| = 2|Z(G)|^2$ and $k(G) = 3|Z(G)| - 1$.*

Proof. Let G be a CA-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. By Theorem 2.3, $[x]_\sim = xZ(G)$ for all $x \in G \setminus Z(G)$ and by Lemma 2.1, $C_G(x) = [x]_\sim \cup Z(G)$, for all $x \in G \setminus Z(G)$. Hence $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{|Z(G)|+|[x]_\sim|} = \frac{|G|}{2|Z(G)|}$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |xZ(G)|$, for all $x \in G \setminus Z(G)$ we have $|Z(G)| = \frac{|G|}{2|Z(G)|}$ which implies that

$$|G| = 2|Z(G)|^2. \tag{2.1}$$

Since $[x]_\sim = xZ(G)$, for all $x \in G \setminus Z(G)$, by Lemma 2.2,

$$|G| = \frac{2|Z(G)|^2}{(3|Z(G)| - k(G))}. \tag{2.2}$$

From Equations 2.1 and 2.2 we have $k(G) = 3|Z(G)| - 1$.

Conversely suppose $|G| = 2|Z(G)|^2$ and $k(G) = 3|Z(G)| - 1$. This implies that $|G| = \frac{2|Z(G)|^2}{(3|Z(G)|-k(G))}$ and by Lemma 2.2, G is a CA-group and $[x]_\sim = xZ(G)$ for all $x \in G \setminus Z(G)$. Also by Lemma 2.1, $|C_G(x)| = 2|Z(G)|$. This implies that $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{2|Z(G)|^2}{2|Z(G)|} = |Z(G)|$. Since $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemma 1.3, $|\mathrm{Cent}(G)| = \frac{|G|}{|Z(G)|}$. Hence by Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G)$ and $x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |Z(G)|$, for all $x \in G \setminus Z(G)$, we have $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Hence we conclude that $[x]_\sim = x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. □

Lemma 2.4. *Let G be a* CA-*group and $[x]_\sim = xZ(G)$, for all $x \in G$. Then $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$ if and only if $|G| = 2|Z(G)|^2$.*

Proof. Let G be a CA-group and $[x]_\sim = x^G$, for all $x \in G \setminus Z(G)$. By Lemma 2.3, $|G| = 2|Z(G)|^2$. Conversely let $|G| = 2|Z(G)|^2$. Since G is a CA-group and $[x]_\sim = xZ(G)$, for all $x \in G$, by Lemma 2.1, $C_G(x) = Z(G) \cup [x]_\sim = Z(G) \cup xZ(G)$, for all $x \in G \setminus Z(G)$. Therefore $|C_G(x)| = 2|Z(G)|$, for all $x \in G \setminus Z(G)$. This implies that $|x^G| = \frac{|G|}{|C_G(x)|} = \frac{|G|}{2|Z(G)|} = \frac{2|Z(G)|^2}{2|Z(G)|} = |Z(G)|$, for all $x \in G \setminus Z(G)$. Since $[x]_\sim = xZ(G)$, for all $x \in G \setminus Z(G)$, by Lemma 1.3 and Lemma 1.8, $\frac{G}{Z(G)}$ is an elementary abelian 2-group. Therefore $G' \leq Z(G)$. Hence $x^G \subseteq xG' \subseteq xZ(G)$, for all $x \in G \setminus Z(G)$. Since $|x^G| = |Z(G)| = |xZ(G)|$, we have $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ and finally $[x]_\sim = x^G = xZ(G)$ for all $x \in G \setminus Z(G)$. □

Example 2.3. Let G be a non-abelian CA-group and assume that $[x]_\sim = x^G$ for all $x \in G \setminus Z(G)$ and $|G| \leq 100$. Then $G \cong Q_8$ or D_8.

Lemma 2.5. *Let G be a non-abelian group. Then $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$ if and only if $G' = Z(G)$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$.*

Proof. Let $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Since $x^G \subseteq xG'$, so $Z(G) \leq G'$. Now we show that $G' \leq Z(G)$. Let $1 \neq t \in G'$. Then there exist $x, y \in G$ so that $[x, y] = t$. Hence $t = y^{-1}x^{-1}yx = y^{-1}y^x$. Since $y^G = yZ(G)$, there exists $z \in Z(G)$ such that $y^x = yz$. Therefore $t = y^{-1}y^x = y^{-1}yz = z$. This implies that $t \in Z(G)$. Thus $G' \leq Z(G)$ and we have $G' = Z(G)$. Moreover $|G| = |Z(G)| + (k(G) - |Z(G)|)|x^G|$ because $|x^G| = |xZ(G)|$ for all $x \in G \setminus Z(G)$. Hence $\frac{|G|}{|Z(G)|} = k(G) - |Z(G)| + 1$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$.

Conversely, suppose $G' = Z(G)$ and $k(G) = \frac{|G|}{|Z(G)|} + |Z(G)| - 1$. Then $x^G \subseteq xG' = xZ(G)$, for all $x \in G \setminus Z(G)$. Hence $|x^G| \leq |xZ(G)|$, for all $x \in G \setminus Z(G)$. Since $k(G) - |Z(G)| = \frac{|G|}{|Z(G)|} - 1$ we have $|x^G| = |xZ(G)|$, for all $x \in G \setminus Z(G)$. Therefore $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. □

Lemma 2.6. *Let G be a non-abelian group and $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Then G is a p-group where p is a prime.*

Proof. Since $|x^G| = |Z(G)|$, for all $x \in G \setminus Z(G)$, so $\Gamma(G)$ is a regular graph. By Lemma 1.2, $G \cong A \times P$ where A is an abelian group and P is a p-group (p is a prime). By Lemma 2.5, $G' = Z(G)$ which implies that $A \cong 1$ and G is a p-group. □

Theorem 2.4. *Let G be a* CC*-group and $x^G = xZ(G)$, for all $x \in G \setminus Z(G)$. Then $G \cong Q_8$.*

Proof. By Lemma 2.6, G is a p-group. So G is a nilpotent group. By Lemma 1.9, $G \cong C_m \times Q_{2^n}$ where n is positive integer and m is an odd positive integer. By Lemma 2.5, $G' = Z(G)$, so $1 \times Q'_{2^n} \cong C_m \times C_2$. Hence $Q'_{2^n} \cong C_2$ and $|Q'_{2^n}| = 2$. Since $|Q'_{2^n}| = 2^{n-2}$ we have $n = 3$. Hence $G \cong Q_8$ and the proof is complete. □

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ON $\mathcal{I}-$CONVERGENCE OF SEQUENCES IN GRADUAL NORMED LINEAR SPACES

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Abstract. In this paper, we introduce the concepts of $\mathcal{I}$ and $\mathcal{I}^*-$convergence of sequences in gradual normed linear spaces. We study some basic properties and implication relations of the newly defined convergence concepts. Also, we introduce the notions of $\mathcal{I}$ and $\mathcal{I}^*-$Cauchy sequences in the gradual normed linear space and investigate the relations between them.
Keywords: Gradual number; gradual normed linear space; ideal; filter; ideal convergence.

1. Introduction

The idea of fuzzy sets [20] was first introduced by Zadeh in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays, it has wide applicability in different branches of science and engineering. The term "fuzzy number" plays a crucial role in the study of fuzzy set theory. Fuzzy numbers were basically the generalization of intervals, not numbers. Even fuzzy numbers do not obey a few algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many authors due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, Fortin et.al. [8] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are

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mainly known by their respective assignment function which is defined in the interval $(0,1]$. So in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have uses in computation and optimization problems.

In 2011, Sadeqi and Azari [15] first introduced the concept of gradual normed linear space. They studied various properties of the space from both the algebraic and topological point of view. Further progress in this direction has been occurred due to Ettefagh, Azari, and Etemad (see [6],[7]) and many others. For extensive study on gradual real numbers one may refer to ([1],[5][12],[18],[21],[22]).

On the other hand in 2001, the idea of ideal convergence was first introduced by Kostyrko et al. [11] mainly as an extension of statistical convergence. They also showed that ideal convergence was also a generalized form of some other known convergence concepts. Later on, several investigations in this direction have been occurred due to Debnath and Rakshit [2], Demirci [3], Gogola et al. [9], Mursaleen and Mohiuddine [13], Savas and Das[17] and many others. For an extensive view of this article, we refer to [4, 10, 14, 16, 19].

Research on the convergence of sequences in gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in gradual normed linear spaces (for details one may refer to [6], [7],[15]).

Recently, the convergence of sequences in gradual normed linear spaces was introduced by Ettefagh et. al. [7]. Also, they have investigated some properties from the topological point of view [6]. Therefore, the study of ideal convergence of sequences in gradual normed linear spaces is very natural.

2. Definitions and Preliminaries

Definition 2.1. [8] A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual number is said to be non-negative if for every $\xi \in (0,1]$, $A_{\tilde{r}}(\xi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [8], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 2.2. Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by $A_{\tilde{r}_1 * \tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) * A_{\tilde{r}_2}(\xi)$, $\forall \xi \in (0,1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$A_{\tilde{r}_1+\tilde{r}_2}(\xi) = A_{\tilde{r}_1}(\xi) + A_{\tilde{r}_2}(\xi) \quad \textit{and} \quad A_{c\tilde{r}}(\xi) = cA_{\tilde{r}}(\xi), \ \forall \xi \in (0,1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $A_{\tilde{p}}(\xi) = p$ for any $\xi \in (0, 1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\xi) = 0$ and $A_{\tilde{1}}(\xi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with the gradual addition and multiplication forms a real vector space [8].

Definition 2.3. [15] Let X be a real vector space. The function $||\cdot||_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\xi \in (0, 1]$, following three conditions are true for any $x, y \in X$
(G_1) $A_{||x||_G}(\xi) = A_{\tilde{0}}(\xi)$ iff $x = 0$;
(G_2) $A_{||\lambda x||_G}(\xi) = |\lambda| A_{||x||_G}(\xi)$ for any $\lambda \in \mathbb{R}$;
(G_3) $A_{||x+y||_G}(\xi) \le A_{||x||_G}(\xi) + A_{||y||_G}(\xi)$.

The pair $(X, ||\cdot||_G)$ is called a gradual normed linear space (GNLS).

Example 2.1. [15] Let $X = \mathbb{R}^n$ and for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \xi \in (0, 1]$, define $||\cdot||_G$ by $A_{||x||_G}(\xi) = e^{\xi} \sum_{i=1}^{n} |x_i|$. Then $||\cdot||_G$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, ||\cdot||_G)$ is a GNLS.

Definition 2.4. [15] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual convergent to $x \in X$, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x||_G}(\xi) < \varepsilon$, $\forall n \ge N$.

Definition 2.5. [15] Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradual Cauchy, if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_k - x_j||_G}(\xi) < \varepsilon$, $\forall k, j \ge N$.

Theorem 2.1. *([15], Theorem 3.6) Let $(X, ||\cdot||_G)$ be a GNLS, then every gradual convergent sequence in X is also a gradual Cauchy sequence.*

Definition 2.6. [11] Let X is a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(iii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Some standard examples of ideal are given below:
(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$. Here $\mathbb{N}$ denotes the set of all natural numbers.
(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.
(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.
(iv) Suppose $\mathbb{N} = \bigcup_{p=1}^{\infty} D_p$ be a decomposition of $\mathbb{N}$ (for $i \neq j$, $D_i \cap D_j = \emptyset$). Then the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [9] and [10].

Definition 2.7. [11] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.8. [11] Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A real sequence (x_k) is said to be $\mathcal{I}$-convergent to l if for each $\varepsilon > 0$, the set $C(\varepsilon) = \{k \in \mathbb{N} :\mid x_k - l \mid\geq \varepsilon\}$ belongs to $\mathcal{I}$. l is called the $\mathcal{I}$-limit of the sequence (x_k) and is written as $\mathcal{I}\text{-}lim_{k\to\infty} x_k = l$.

Definition 2.9. [11] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}^*-$convergent to l, if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\}$ in the associated filter $\mathcal{F}(\mathcal{I})$ such that $\lim\limits_{k\in M} x_k = l$.

Definition 2.10. [14] A sequence (x_k) of real numbers is said to be $\mathcal{I}-$Cauchy, if for every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_N| \geq \varepsilon\} \in \mathcal{I}$.

Definition 2.11. [14] A sequence (x_k) of real numbers is said to be $\mathcal{I}^*-$Cauchy, if there exists a set $M = \{m_1 < m_2 < ... < m_i < ...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is a Cauchy sequence i.e. $\lim\limits_{i,j\to\infty} \mid x_{m_i} - x_{m_j} \mid= 0$.

Definition 2.12. [11] An admissible ideal $\mathcal{I}$ is said to satisfy the condition AP, if for every countable family of mutually disjoint sets $\{C_n\}_{n\in\mathbb{N}}$ from $\mathcal{I}$, there exists a countable family of sets $\{B_n\}_{n\in\mathbb{N}}$ such that the symmetric difference $C_j \triangle B_j$ is finite for every $j \in \mathbb{N}$ and $\bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.

Throughout the article $\mathcal{I}$ will denote the non-trivial admissible ideal of $\mathbb{N}$.

3. Main Results

Definition 3.1. Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}-$convergent to $x \in X$ if for every $\xi \in (0, 1]$ and $\varepsilon > 0$, the set $C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$. Symbolically we write, $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$.

Example 3.1. Let $X = \mathbb{R}^n$ and $||\cdot||_G$ be the norm defined in Example 2.1. Consider the ideal $\mathcal{I}$ consisting of all subsets of $\mathbb{N}$ which intersects finitely many D_p's where $D_p = \{2^{p-1}(2s - 1) : s \in \mathbb{N}\}, p \in \mathbb{N}$ is the decomposition of $\mathbb{N}$ into disjoint subsets i.e $\mathbb{N} =$

$\bigcup_{p=1}^{\infty} D_p$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. Consider the sequence (x_k) in $\mathbb{R}^n$ defined by $x_k = (0,0,...,0,\frac{1}{p}),\ if\, k \in D_p$. Then $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} \mathbf{0}$ where $\mathbf{0}$ denotes the vector $(0,0,...,0) \in \mathbb{R}^n$.

Justification. It is obvious that $A_{||x_k-\mathbf{0}||_G}(\xi) = \frac{1}{p}e^{\xi}$ for $k \in D_p$. Let $\varepsilon > 0$ be given. Then by Archimedean property, there exists $m \in \mathbb{N}$ such that $\frac{1}{m}e^{\xi} < \varepsilon$ and consequently, the following inclusion is true,

$$\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \geq \frac{1}{m}e^{\xi}\} \tag{3.1}$$

and as $A_{||x_k-\mathbf{0}||_G}(\xi) = \frac{1}{p}e^{\xi}$ for $k \in D_p$, we have

$$\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \geq \frac{1}{m}e^{\xi}\} = \bigcup_{p=1}^{m} D_p \in \mathcal{I}. \tag{3.2}$$

From (3.1) and (3.2), we obtain $\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}$. Hence $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} \mathbf{0}$.

Theorem 3.1. *Let $(X, ||\cdot||_G)$ be a GNLS. If a sequence (x_k) is gradual convergent to $x \in X$, then (x_k) is gradually $\mathcal{I}-$convergent to $x \in X$.*

Proof. Proof follows directly from the fact that $\mathcal{I}_f \subset \mathcal{I}$. □

But the converse of Theorem 3.1 is not true. Example 3.2 illustrates the fact.

Example 3.2. Let $X = \mathbb{R}^n$ and $||\cdot||_G$ be the norm defined in Example 2.1. Consider the sequence (x_k) in $\mathbb{R}^n$ defined as

$$x_k = \begin{cases} (0,0,...,0,n) & if\ k = p^2, p \in \mathbb{N} \\ (0,0,....0,0) & otherwise. \end{cases}$$

Let $\mathbf{0}$ denotes the vector $(0,0,....0,0) \in \mathbb{R}^n$. Then for any $\varepsilon > 0$ and $\xi \in (0,1]$, $\{k \in \mathbb{N} : A_{||x_k-\mathbf{0}||_G}(\xi) \geq \varepsilon\} \subseteq \{1,4,9,..\} \in \mathcal{I}_d$. Hence $x_k \xrightarrow{\mathcal{I}_d-||\cdot||_G} \mathbf{0}$ in $\mathbb{R}^n$.

Theorem 3.2. *Let (x_k) be any sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$ in X. Then x is uniquely determined.*

Proof. If possible suppose $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$ and $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} y$ for some $x \neq y$ in X. Let $\varepsilon > 0$ be arbitrary. Then, for any $\varepsilon > 0$ and $\xi \in (0,1]$, we have, $B_1(\xi,\varepsilon), B_2(\xi,\varepsilon) \in \mathcal{F}(\mathcal{I})$ where $B_1(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k-x||_G}(\xi) < \varepsilon\}$ and $B_2(\xi,\varepsilon) = \{k \in \mathbb{N} : A_{||x_k-y||_G}(\xi) < \varepsilon\}$. Clearly $B_1(\xi,\varepsilon) \cap B_2(\xi,\varepsilon) \in \mathcal{F}(\mathcal{I})$ and is non-empty. Choose $m \in B_1(\xi,\varepsilon) \cap B_2(\xi,\varepsilon)$, then $A_{||x_m-x||_G}(\xi) < \varepsilon$ and $A_{||x_m-y||_G}(\xi) < \varepsilon$. Hence $A_{||x-y||_G}(\xi) \leq A_{||x_m-x||_G}(\xi) + A_{||x_m-y||_G}(\xi) < \varepsilon + \varepsilon = 2\varepsilon$. Since ε is arbitrary, so $A_{||x-y||_G}(\xi) = A_{\tilde{0}}(\xi)$, which gives $x = y$. □

Theorem 3.3. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$ and $y_k \xrightarrow{\mathcal{I}-||\cdot||_G} y$. Then*
(i) $x_k + y_k \xrightarrow{\mathcal{I}-||\cdot||_G} x + y$ and (ii) $cx_k \xrightarrow{\mathcal{I}-||\cdot||_G} cx$.

Proof. (i) Suppose $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$ and $y_k \xrightarrow{\mathcal{I}-||\cdot||_G} y$. Then, for given $\varepsilon > 0$, we have, $C_1(\xi, \varepsilon), C_2(\xi, \varepsilon) \in \mathcal{I}$ where $C_1(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{2}\}$ and $C_2(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \frac{\varepsilon}{2}\}$. Now as the inclusion $(\mathbb{N} \setminus C_1(\xi, \varepsilon)) \cap (\mathbb{N} \setminus C_2(\xi, \varepsilon)) \subseteq \{k \in \mathbb{N} : A_{||x_k + y_k - x - y||_G}(\xi) < \varepsilon\}$ holds, so we must have

$$\{k \in \mathbb{N} : A_{||x_k + y_k - x - y||_G}(\xi) \geq \varepsilon\} \subseteq C_1(\xi, \varepsilon) \cup C_2(\xi, \varepsilon) \in \mathcal{I};$$

and consequently, $x_k + y_k \xrightarrow{\mathcal{I}-||\cdot||_G} x + y$.
(ii) If $c = 0$, then there is nothing to prove. So let us assume $c \neq 0$. Then since $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$, we have for given $\varepsilon > 0$, $C_1(\xi, \varepsilon) \in \mathcal{I}$ where $C_1(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \frac{\varepsilon}{|c|}\}$. Now since $A_{||cx_k - cx||_G}(\xi) = |c| A_{||x_k - x||_G}(\xi)$ holds for any $c \in \mathbb{R}$, we must have $C_2(\xi, \varepsilon) \subseteq C_1(\xi, \varepsilon)$ where $C_2(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||cx_k - cx||_G}(\xi) \geq \varepsilon\}$, which as a consequence implies $C_2(\xi, \varepsilon) \in \mathcal{I}$. This completes the proof. □

Theorem 3.4. *Let (x_k) be any sequence in the GNLS $(X, ||\cdot||_G)$. If every subsequence of (x_k) is gradually $\mathcal{I}-$convergent to x, then (x_k) is also gradually $\mathcal{I}-$convergent to x.*

Proof. If possible suppose (x_k) is not gradually $\mathcal{I}-$convergent to x. Then there exists some $\varepsilon > 0$ and $\xi \in (0, 1]$ such that $C(\xi, \varepsilon) \notin \mathcal{I}$, where $C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\}$. So $C(\xi, \varepsilon)$ must be an infinite set. Let $C(\xi, \varepsilon) = \{k_1 < k_2 < ... < k_j < ...\}$. Now define a sequence (y_j) as $y_j = x_{k_j}$ for $j \in \mathbb{N}$. Then (y_j) is a subsequence of (x_k) which is not gradually $\mathcal{I}-$convergent to x, a contradiction. □

Remark 3.1. Converse of the above theorem is not true.

Proof. Easy so omitted. One can verify it by considering Example 3.2 also. □

Theorem 3.5. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, ||\cdot||_G)$ such that (y_k) is gradual convergent and $\{k \in \mathbb{N} : x_k \neq y_k\} \in \mathcal{I}$. Then (x_k) is gradually $\mathcal{I}-$convergent.*

Proof. Suppose $\{k \in \mathbb{N} : x_k \neq y_k\} \in \mathcal{I}$ holds and $y_k \xrightarrow{||\cdot||_G} y$. Then by definition for every $\varepsilon > 0$ and $\xi \in (0, 1]$, $\{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\}$ is a finite set and therefore

$$\{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}. \tag{3.3}$$

Now since the inclusion

$$\{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : A_{||y_k - y||_G}(\xi) \geq \varepsilon\} \cap \{k \in \mathbb{N} : x_k \neq y_k\}$$

holds, so using Equation (3.3) and the hypothesis we get,

$$\{k \in \mathbb{N} : A_{||x_k - y||_G}(\xi) \geq \varepsilon\} \in \mathcal{I}.$$

Hence $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} y$ and the proof is complete. $\square$

Definition 3.2. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*-$convergent to $x \in X$ if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradual convergent to x. Symbolically we write, $x_k \xrightarrow{\mathcal{I}^*-||\cdot||_G} x$.

Theorem 3.6. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{\mathcal{I}^*-||\cdot||_G} x$. Then $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$.*

Proof. Let us assume that $x_k \xrightarrow{\mathcal{I}^*-||\cdot||_G} x$. Then, there exists $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that for every $\varepsilon > 0$ and $\xi \in (0, 1]$, there exists $N(= N_\varepsilon(\xi)) \in \mathbb{N}$ such that $A_{||x_{m_k} - x||_G}(\xi) < \varepsilon \, \forall k > N$. Since $\mathcal{I}$ is admissible, we must have $C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \varepsilon\} \subseteq (\mathbb{N} \setminus M) \cup \{m_1, m_2, ..., m_N\} \in \mathcal{I}$. Hence $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$. $\square$

Remark 3.2. Converse of the above theorem is not true in general. Consider Example 3.1. It was shown that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} \mathbf{0}$. But the same sequence is not gradually $\mathcal{I}^*-$convergent to $\mathbf{0}$. Beacuse for any $H \in \mathcal{I}$ there exists $p \in \mathbb{N}$ such that $H \subseteq \bigcup_{j=1}^{p} D_j$ and as a consequence $D_{p+1} \subseteq \mathbb{N} \setminus H$. Let M denote the set $\mathbb{N} \setminus H$, then $M \in \mathcal{F}(\mathcal{I})$ and (x_{m_k}) is gradual convergent to $(0, 0, .., 0, \frac{1}{p+1})$, not to $\mathbf{0}$. Hence x_k is not gradually $\mathcal{I}^*-$convergent to $\mathbf{0}$.

Theorem 3.7. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ which satisfies the condition AP and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$ such that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$. Then $x_k \xrightarrow{\mathcal{I}^*-||\cdot||_G} x$.*

Proof. Let us assume that $x_k \xrightarrow{\mathcal{I}-||\cdot||_G} x$. Then, for every $\xi \in (0, 1]$ and $\eta > 0$, the set $C(\xi, \eta) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq \eta\} \in \mathcal{I}$. This enables us to construct a countable family of mutually disjoint sets $\{C_m(\xi)\}_{m \in \mathbb{N}}$ in $\mathcal{I}$ by considering

$$C_1(\xi) = \{k \in \mathbb{N} : A_{||x_k - x||_G}(\xi) \geq 1\}$$

and

$$C_m(\xi) = \{k \in \mathbb{N} : \frac{1}{m} \leq A_{||x_k - x||_G}(\xi) < \frac{1}{m-1}\} = C(\xi, \frac{1}{m}) \backslash C(\xi, \frac{1}{m-1}), \text{for } m \geq 2.$$

Now since $\mathcal{I}$ satisfies the condition AP, so for the above countable collection $\{C_m(\xi)\}_{m \in \mathbb{N}}$, there exists another countable family of subsets $\{B_m(\xi)\}_{m \in \mathbb{N}}$ of $\mathbb{N}$ satisfying

$$C_j(\xi)\triangle B_j(\xi) \text{ is finite } \forall j\in\mathbb{N} \text{ and } B(\xi)=\bigcup_{j=1}^{\infty} B_j(\xi)\in\mathcal{I}. \tag{3.4}$$

Let $\varepsilon>0$ be arbitrary. By Archimedean property we can choose $m\in\mathbb{N}$ such that $\frac{1}{m+1}<\varepsilon$ and hence the following inclusion holds

$$\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}\subseteq\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\frac{1}{m+1}\}=\bigcup_{j=1}^{m+1}C_j(\xi)\in\mathcal{I}.$$

Using (3.4) we can say that there exists an integer $k_0\in\mathbb{N}$, such that

$$\bigcup_{j=1}^{m+1}B_j(\xi)\cap(k_0,\infty)=\bigcup_{j=1}^{m+1}C_j(\xi)\cap(k_0,\infty).$$

Choose $k\in\mathbb{N}\backslash B(\xi)\in\mathcal{F}(\mathcal{I})$ such that $k>k_0$. Then we must have $k\notin\bigcup_{j=1}^{m+1}B_j(\xi)$ and hence $k\notin\bigcup_{j=1}^{m+1}C_j(\xi)$. Thus we have, $A_{||x_k-x||_G}(\xi)<\frac{1}{m+1}<\varepsilon$. Hence we have $x_k\xrightarrow{\mathcal{I}^*-||\cdot||_G}x$. □

Definition 3.3. Let (x_k) be a sequence in the GNLS $(X,||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}-$Cauchy if for every $\varepsilon>0$ and $\xi\in(0,1]$, there exists a natural number $N(=N_\varepsilon(\xi))$ such that the set $C(\xi,\varepsilon)=\{k\in\mathbb{N}: A_{||x_k-x_N||_G}(\xi)\geq\varepsilon\}\in\mathcal{I}$.

Theorem 3.8. *Let $(X,||\cdot||_G)$ be a GNLS. Then every gradually $\mathcal{I}-$convergent sequence in X is gradually $\mathcal{I}-$Cauchy sequence.*

Proof. Let (x_k) be a sequence in X such that $x_k\xrightarrow{\mathcal{I}-||\cdot||_G}x$. Then, for every $\varepsilon>0$ and $\xi\in(0,1]$,

$$C(\xi,\varepsilon)\in\mathcal{I}, \text{ where } C(\xi,\varepsilon)=\{k\in\mathbb{N}: A_{||x_k-x||_G}(\xi)\geq\varepsilon\}.$$

Clearly, $\mathbb{N}\setminus C(\xi,\varepsilon)\in\mathcal{F}(\mathcal{I})$ and therefore, is non-empty. Choose $N(=N_\varepsilon(\xi))\in\mathbb{N}\setminus C(\xi,\varepsilon)$. Then we have $A_{||x_k-x_N||_G}(\xi)<\varepsilon$.

Let $B(\xi,\varepsilon)=\{k\in\mathbb{N}: A_{||x_k-x_N||_G}(\xi)\geq 2\varepsilon\}$. Now we prove that the following inclusion is true

$$B(\xi,\varepsilon)\subseteq C(\xi,\varepsilon).$$

For if $p\in B(\xi,\varepsilon)$ we have

$$2\varepsilon\leq A_{||x_p-x_N||_G}(\xi)\leq A_{||x_p-x||_G}(\xi)+A_{||x-x_N||_G}(\xi)<A_{||x_p-x||_G}(\xi)+\varepsilon,$$

which implies $p\in C(\xi,\varepsilon)$. Thus we conclude that $B(\xi,\varepsilon)\in\mathcal{I}$, which means (x_k) is gradually $\mathcal{I}-$Cauchy sequence. □

Definition 3.4. Let (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. Then (x_k) is said to be gradually $\mathcal{I}^*-$Cauchy if there exists a set $M = \{m_1 < m_2 < ... < m_k < ..\} \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is gradual Cauchy sequence.

Theorem 3.9. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and (x_k) be a sequence in the GNLS $(X, ||\cdot||_G)$. If (x_k) is gradually $\mathcal{I}^*-$Cauchy then it is gradually $\mathcal{I}-$Cauchy.*

Proof. Suppose (x_k) is gradually $\mathcal{I}^*-$Cauchy. Then, there exists a set $M = \{m_1 < m_2 < ... < m_k < ..\} \in \mathcal{F}(\mathcal{I})$ such that for every $\varepsilon > 0$, there exists $i_0(= i_0(\xi, \varepsilon)) \in \mathbb{N}$ such that $A_{||x_{m_i} - x_{m_j}||_G}(\xi) < \varepsilon$ holds for any $i, j > i_0$. Let $N(= N_\varepsilon(\xi)) = m_{i_0+1}$. Then we have for any $\varepsilon > 0$,

$$C(\xi, \varepsilon) = \{k \in \mathbb{N} : A_{||x_k - x_N||_G}(\xi) \geq \varepsilon\} \subseteq (\mathbb{N} \setminus M) \cup \{m_1, m_2, .., m_{i_0}\} \in \mathcal{I}.$$

Hence (x_k) is gradually $\mathcal{I}-$Cauchy. □

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IDEAL CONVERGENCE OF DOUBLE SEQUENCES OF CLOSED SETS

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Abstract. In the present paper, we introduce the concepts of ideal inner and ideal outer limits which always exist even if empty sets for double sequences of closed sets in Pringsheim's sense. Next, we give some formulas for finding ideal inner and outer limits in a metric space. After then, we define Kuratowski ideal convergence of double sequences of closed sets by means of the ideal inner and ideal outer limits of a double sequence of closed sets. Additionally, we give some examples that our result is more general than the results obtained before.
Keywords: Double sequence of sets, ideal convergence, Kuratowski convergence.

1. Introduction

Convergence is one of the most vital concept in mathematics. In the analysis, there are different approaches at the limit of the function sequences due to the requirements. At the first pointwise convergence are studied. After that several types of convergence of sequences of functions were studied according to the need. The modes of convergence used in different areas of mathematics are uniform convergence, almost everywhere convergence, continuous convergence, convergence in measure, L_p convergence, etc.

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In variational analysis pointwise limits are inadequate for mathematical purposes. A different approach to convergence is required in which, on the geometric level, limits of sequences of sets have the leading role. Motivation for the development of this geometric approach has come from optimization, stochastic processes, control systems and many other subjects. The theory of set convergence will provide ways of approximating set-valued mappings through convergence of graphs and epigraphs. The concepts of inner and outer limits for a sequence of sets are due to the French mathematician-politician Painlevé, who introduced them in 1902 in his lectures on analysis at the University of Paris; set convergence was defined as the equality of these two limits. Hausdorff [9] and Kuratowski [15] popularized such convergence by including it in their books, and that's how Kuratowski's name ended up to be associated with it. Recent years have witnessed a rapid development on applications of set-valued and variational analysis. For more information about inner and outer limits of sequences of sets, we refer to [1, 2, 5, 16, 19, 21, 22, 24, 25, 26, 27].

In contrast to ordinary sequences, various types of convergence for double sequences can be defined due to order of elements of $\mathbb{N}^2$. The best known and well-studied convergence notion for double sequence is Pringsheim [20] convergence. Therefore, throughout the paper by the usual convergence of a double sequence we refer to the convergence in Pringsheim's sense.

Statistical convergence of sequences was introduced by Fast [7] and was extended to the double sequences by Mursaleen and Edely [18] and Tripathy [28] independently. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [12] as a generalization of statistical convergence [7, 23], which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. This approach is much more general as most of the known convergence methods become special cases, but there are many ambiguities about this convergence. So this type of convergence is studied actively in summability in last several decades. These two types of convergence are extended to double sequences(see [3, 4, 6, 8, 10, 11, 13, 14, 17, 18, 29, 30]).

In this paper we will study ideal inner and outer limits of a double sequence of sets and give some characterization for them.

2. Definition and Preliminaries

A real double sequence (x_{ij}) is said to be convergent to the limit p in Pringsheim's sense, written $\lim_{i,j\to\infty} x_{ij} = p$, if for every $\varepsilon > 0$, there exists an integer n_0 such that $|x_{ij} - p| < \varepsilon$ whenever $i, j > n_0$. In case of this convergence the row-index i and the column-index j tend to infinity independently from each other.

Let $E \subseteq \mathbb{N}^2$ and $E(m,n) = \{(i,j) : i \leq m, j \leq n\}$. Then, the double natural density of E is defined by

$$\delta_2(E) = \lim_{m,n\to\infty} \frac{|E(m,n)|}{mn}$$

if the limit on the right hand-side exists, where the vertical bars denote the cardinality of the set $E(m,n)$.

A real double sequence $x = (x_{ij})$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, the set $\{(i,j) : |x_{ij} - L| > \varepsilon\}$ has double natural density zero.

The limit as $k, l \to \infty$ with $(k,l) \in K \subseteq \mathbb{N}^2$ will be indicated by $\lim_{(k,l)\in K}$.

Let S be a non-empty set. A class $\mathcal{I}$ of subsets of S is said to be an ideal on S if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, and for each $A \in \mathcal{I}$ and each $B \subset A$, we have $B \in \mathcal{I}$. An ideal $\mathcal{I}$ on S is called non-trivial if $\mathcal{I} \neq \emptyset$ and $S \notin \mathcal{I}$. If the ideal $\mathcal{I}$ of S further satisfies $\{s\} \in \mathcal{I}$ for each $s \in S$, then it is an admissible ideal. A non-empty class $\mathcal{F}$ of subsets of S is said to be a filter on S if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $A \subset B$, we have $B \in \mathcal{F}$. It is obvious that $\mathcal{I}$ on S is non-trivial if and only if $\mathcal{F}(\mathcal{I}) = \{S \setminus A \colon A \in \mathcal{I}\}$ is a filter on S.

Let $S = \mathbb{N}^2$ and let $\mathcal{I}_2$ be a ideal of subsets of $\mathbb{N}^2$. Then a nontrivial ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is called strongly admissible if $\{n\} \times \mathbb{N}$ and $\mathbb{N} \times \{n\}$ belong to $\mathcal{I}_2$ for each $n \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Let

$$\mathcal{I}_2(f) = \{A \subset \mathbb{N}\times\mathbb{N} \colon (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i,j) \notin A)\}.$$

Then $\mathcal{I}_2(f)$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2(f) \subset \mathcal{I}_2$.

Let (X, d) be a metric space. A double sequence (x_{ij}) in X is said to be $\mathcal{I}_2$-convergent to $\xi \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} \colon d(x_{ij}, \xi) \geq \varepsilon\} \in \mathcal{I}_2$$

and written $\mathcal{I}_2\text{-}\lim_{i,j\to\infty} x_{ij} = \xi$.

If $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal, then Pringsheim convergence implies $\mathcal{I}_2$-convergence of double sequences.

An ideal is said to be an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property $(AP2)$, if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j \in \mathcal{I}_2(f)$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$.

A double sequence (x_{ij}) of elements of X is said to be $\mathcal{I}_2^*$-convergent to $\xi \in X$ if there exists a set $K = \{(i,j) : i, j = 1, 2, 3 \ldots\}$ in $\mathcal{F}(\mathcal{I}_2)$ such that $\lim_{(i,j)\in K} d(x_{ij}, \xi) = 0$. It is denoted by $\mathcal{I}_2^*\text{-}\lim_{i,j\to\infty} x_{ij} = \xi$.

Lemma 2.1. *[3, Theorem 1] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.*

$$\text{If} \quad \mathcal{I}_2^*\text{-}\lim_{i,j\to\infty} x_{ij} = \xi, \quad \text{then} \quad \mathcal{I}_2\text{-}\lim_{i,j\to\infty} x_{ij} = \xi.$$

Lemma 2.2. *[3, Theorem 3] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal with property $(AP2)$, then $\mathcal{I}_2\text{-}\lim_{i,j\to\infty} x_{ij} = \xi$ implies $\mathcal{I}_2^*\text{-}\lim_{i,j\to\infty} x_{ij} = \xi$.*

A point $\lambda \in X$ is called a $\mathcal{I}_2$-limit point of (x_{ij}) in a metric space (X, d) if and only if there exist a set $K = \{(k_i, l_j) \colon i, j \in \mathbb{N}\} \subset \mathbb{N}^2$ such that $K \notin \mathcal{I}_2$ and

$\lim_{i,j\to\infty} x_{k_i,l_j} = \lambda$. A point $\gamma \in X$ is called a $\mathcal{I}_2$-cluster point of (x_{ij}) in a metric space (X,d) if and only if for each $\varepsilon > 0$ the set $\{(i,j) \in \mathbb{N}^2 \colon d(x_{ij},\gamma) < \varepsilon\} \notin \mathcal{I}_2$. The set of all $\mathcal{I}_2$-limits points and $\mathcal{I}_2$-cluster points of (x_{ij}) will be denoted by $\mathcal{I}_2(\Lambda_x)$ and $\mathcal{I}_2(\Gamma_x)$, respectively. Obviously, for a strongly admissible ideal $\mathcal{I}_2$ we have $\mathcal{I}_2(\Lambda_x) \subseteq \mathcal{I}_2(\Gamma_x)$.

From now on $\mathcal{I}_2$ will be considered as a nontrivial strongly admissible ideal in $\mathbb{N}^2$.

The concepts of ideal limit superior and inferior of double sequences of real numbers were introduced in [4, 8], as follows:

Definition 2.1. Define the sets A_x and B_x by

$$A_x = \{a \in \mathbb{R} \colon \{(i,j) \colon x_{ij} > a\} \notin \mathcal{I}_2\} \quad \textit{and} \quad B_x = \{b \in \mathbb{R} \colon \{(i,j) \colon x_{ij} < b\} \notin \mathcal{I}_2\}.$$

Then, $\mathcal{I}_2$-limit superior and inferior of a real double sequence x are defined by

$$\mathcal{I}_2\text{-}\limsup x = \begin{cases} \sup A_x & , \quad \text{if } A_x \neq \emptyset, \\ -\infty & , \quad \text{if } A_x = \emptyset \end{cases}$$

and

$$\mathcal{I}_2\text{-}\liminf x = \begin{cases} \inf B_x & , \quad \text{if } B_x \neq \emptyset, \\ \infty & , \quad \text{if } B_x = \emptyset. \end{cases}$$

Lemma 2.3. *Let $x = (x_{ij})$ be a double sequence of real numbers. Then, the following statements hold:*

(a) $\mathcal{I}_2\text{-}\limsup x = \beta \Leftrightarrow$ for any $\varepsilon > 0$, $\{(i,j) \colon x_{ij} > \beta - \varepsilon\} \notin \mathcal{I}_2$ and $\{(i,j) \colon x_{ij} > \beta + \varepsilon\} \in \mathcal{I}_2$.

(b) $\mathcal{I}_2\text{-}\liminf x = \alpha \Leftrightarrow$ for any $\varepsilon > 0$, $\{(i,j) \colon x_{ij} < \alpha + \varepsilon\} \notin \mathcal{I}_2$ and $\{(i,j) \colon x_{ij} < \alpha - \varepsilon\} \in \mathcal{I}_2$.

Let (X,d) be a metric space and $A \subset X$, $x \in X$. Then the distance from x to A with respect to d is given by $d(x,A) := \inf_{a\in A} d(x,a)$, where we set $d(x,\emptyset) := \infty$. The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x,\varepsilon)$, i.e.,

$$B(x,\varepsilon) = \{y \in X \colon d(x,y) < \varepsilon\}.$$

3. Main Results

In this section, we introduce Kuratowski ideal convergence of double sequences of closed sets. For this purpose, we define the set

$$\mathcal{I}_2^+ := \{N \subseteq \mathbb{N}^2 \colon N \notin \mathcal{I}_2\}.$$

We now define ideal outer and inner limits of a double sequence of closed sets, as follows.

Definition 3.1. Let (X, d) be a metric space and let (C_{kl}) be double a sequence of closed subsets of X. The ideal outer limit and the inner limit of a double sequence (C_{kl}) are defined as

$$\mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} \quad := \quad \Big\{x\colon \forall \varepsilon > 0,\ \exists N \in \mathcal{I}_2^+,\ \forall\ (k,l) \in N\colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\Big\},$$

and

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} \quad := \quad \Big\{x\colon \forall \varepsilon > 0,\ \exists N \in \mathcal{F}(\mathcal{I}_2),\ \forall\ (k,l) \in N\colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\Big\}$$

respectively. When the ideal outer and inner limits are equal to the same set C, this set is called to the ideal limit of double sequence (C_{kl}). In this case, we say that the double sequence (C_{kl}) is Kuratowski ideal convergent to the set C and we denote

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} = \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} = \mathcal{I}_2\text{-}\lim_{k,l\to\infty} C_{kl} = C.$$

Furthermore, the inclusion

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} \subseteq \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl}$$

always holds. Hence, $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} C_{kl}$ is equal to the set C if and only if the inclusion

$$\mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} \subseteq C \subseteq \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$$

holds.

Remark 3.1. $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} C_{kl} = C$ if and only if the following conditions are satisfied:

(i) for every $x \in C$ and for every $\varepsilon > 0$ the set $\{(k,l) \in \mathbb{N}^2 : B(x,\varepsilon) \cap C_{kl} \neq \emptyset\}$ belongs to $\mathcal{F}(\mathcal{I}_2)$;

(ii) for every $x \in X \setminus C$ there exists $\varepsilon > 0$ such that $\{(k,l) \in \mathbb{N}^2 : B(x,\varepsilon) \cap C_{kl} = \emptyset\}$ belongs to $\mathcal{F}(\mathcal{I}_2)$.

We will give two examples showing that our study is generalization of previously studied works by means of the choice of the ideal.

(I) If $\mathcal{I}_2 = \mathcal{I}_2(f)$, then

$$\begin{aligned} \mathcal{I}_2(f)\text{-}\liminf_{k,l\to\infty} C_{kl} &= \liminf_{k,l\to\infty} C_{kl}, \\ \mathcal{I}_2(f)\text{-}\limsup_{k,l\to\infty} C_{kl} &= \limsup_{k,l\to\infty} C_{kl} \end{aligned}$$

and Kuratowski $\mathcal{I}_2(f)$-convergence coincides with the usual Kuratowski convergence studied in [24].

(II) If $\mathcal{I}_2 = \mathcal{I}_2(\delta) = \{A \subset \mathbb{N}^2 : \delta_2(A) = 0\}$, then

$$\begin{aligned} \mathcal{I}_2(\delta)\text{-}\liminf_{k,l\to\infty} C_{kl} &= st\text{-}\liminf_{k,l\to\infty} C_{kl}, \\ \mathcal{I}_2(\delta)\text{-}\limsup_{k,l\to\infty} C_{kl} &= st\text{-}\limsup_{k,l\to\infty} C_{kl} \end{aligned}$$

and Kuratowski $\mathcal{I}_2(\delta)$-convergence coincides with the Kuratowski statistical convergence studied in [25].

Note that if $\mathcal{I}_2$ is a strongly admissible ideal, then $\mathcal{I}_2(f) \subseteq \mathcal{I}_2$. It is obvious that the followings inclusion holds.

$$\liminf_{k,l\to\infty} C_{kl} \subseteq \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} \subseteq \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} \subseteq \limsup_{k,l\to\infty} C_{kl}.$$

Therefore, each Kuratowski convergent sequence is Kuratowski $\mathcal{I}_2-$convergent, i.e.

$$\lim_{k,l\to\infty} C_{kl} = C \Rightarrow \mathcal{I}_2\text{-}\lim_{k,l\to\infty} C_{kl} = C.$$

However, the converse of this claim does not hold in general. The following example illustrate this claim.

Example 3.1. Let A and B be two different nonempty closed sets in X. For any strongly admissible ideal $\mathcal{I}_2 \neq \mathcal{I}_2(f)$ we may take $N \in \mathcal{I}_2 \setminus \mathcal{I}_2(f)$ and put $C_{kl} = A$ for $k, l \in N$ and $C_{kl} = B$ otherwise. Then $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} C_{kl} = B$. However $\limsup_{k,l\to\infty} C_{kl} = A \cup B$ and $\liminf_{k,l\to\infty} C_{kl} = A \cap B$.

The following theorems give us characterization of ideal inner and outer limits for double sequences of closed sets.

Theorem 3.1. *Let (X, d) be a metric space and (C_{kl}) be a double sequence of closed subsets of X. Then*

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} = \bigcap_{N\in\mathcal{I}_2^+} \operatorname{cl} \bigcup_{(k,l)\in N} C_{kl} \quad \textit{and} \quad \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} = \bigcap_{N\in\mathcal{F}(\mathcal{I}_2)} \operatorname{cl} \bigcup_{(k,l)\in N} C_{kl}$$

Proof. We shall prove only the first statement, the proof of second one being analogous. Let $x \in \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$ and $N \in \mathcal{I}_2^+$ be arbitrary. For each $\varepsilon > 0$, there exists $N_1 \in \mathcal{F}(\mathcal{I}_2)$ such that for every $(k, l) \in N_1$

$$C_{kl} \cap B(x, \varepsilon) \neq \emptyset.$$

Since $N \cap N_1 \neq \emptyset$, there exists $(k_0, l_0) \in N \cap N_1$ such that $C_{k_0 l_0} \cap B(x, \varepsilon) \neq \emptyset$. Therefore,

$$\Big(\bigcup_{(k,l)\in N} C_{kl} \Big) \cap B(x, \varepsilon) \neq \emptyset.$$

This gives us $x \in \operatorname{cl}\bigcup_{(k,l)\in N} C_{kl}$. This holds for any $N \in \mathcal{I}_2^+$. Consequently,

$$x \in \bigcap_{N\in\mathcal{I}_2^+} \operatorname{cl} \bigcup_{(k,l)\in N} C_{kl}.$$

For the reverse inclusion, suppose that $x \notin \mathcal{I}_2 - \liminf_{k,l\to\infty} C_{kl}$. Then, there exists $\varepsilon > 0$ such that

$$N = \big\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\big\} \notin \mathcal{F}(\mathcal{I}_2)$$

and so, the set

$$N = \big\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) = \emptyset\big\} \in \mathcal{I}_2^+.$$

Thus

$$\Big(\bigcup_{(k,l)\in N} C_{kl}\Big) \cap B(x,\varepsilon) = \emptyset.$$

This implies that $x \notin \operatorname{cl}\bigcup_{(k,l)\in N} C_{kl}$ which achieves the proof. □

According to Theorem 3.1, we conclude that both ideal outer and inner limits of a double sequence (C_{kl}) are closed sets.

Theorem 3.2. *Let (X,d) be a metric space and (C_{kl}) be a double sequence of closed subsets of X. Then, we have*

$$\begin{aligned} \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} &= \Big\{x \colon \mathcal{I}_2\text{-}\lim_{k,l\to\infty} d(x,C_{kl}) = 0\Big\}, \\ \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} &= \Big\{x \colon \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} d(x,C_{kl}) = 0\Big\}. \end{aligned}$$

Proof. Assume that C be any closed set in X. Then we can write

$$d(x,C) \geq \varepsilon \quad \Leftrightarrow \quad C \cap B(x,\varepsilon) = \emptyset. \tag{3.1}$$

Suppose that $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} d(x,C_{kl}) = 0$. Then, for each $\varepsilon > 0$ we get the set

$$\big\{(k,l) \in \mathbb{N}^2 \colon d(x,C_{kl}) \geq \varepsilon\big\}$$

belongs to $\mathcal{I}_2$. Taking into account (3.1), we have the set

$$\big\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) = \emptyset\big\}$$

belongs to $\mathcal{I}_2$. This implies that

$$\big\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\big\}$$

belongs to $\mathcal{F}(\mathcal{I}_2)$. Thus we have $x \in \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$.

Conversely, suppose that $x \in \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$, then for each $\varepsilon > 0$ there exists $N \in \mathcal{F}(\mathcal{I}_2)$ such that $C_{kl} \cap B(x,\varepsilon) \neq \emptyset$ for every $(k,l) \in N$. Since

$$\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) = \emptyset\} \subseteq \mathbb{N}^2 \setminus N,$$

we have

$$\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) = \emptyset\} \in \mathcal{I}_2.$$

By virtue of (3.1), the set

$$\{(k,l) \in \mathbb{N}^2 \colon d(x, C_{kl}) \geq \varepsilon\}$$

belongs to $\mathcal{I}_2$. This implies that $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} d(x, C_{kl}) = 0$.

Similarly, for any closed set C we have

$$d(x,C) < \varepsilon \Leftrightarrow C \cap B(x,\varepsilon) \neq \emptyset. \tag{3.2}$$

Assume that $\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} d(x, C_{kl}) = 0$. Then, for each $\varepsilon > 0$ we can write

$$\{(k,l) \in \mathbb{N}^2 \colon d(x, C_{kl}) < \varepsilon\} \notin \mathcal{I}_2.$$

By relation (3.2) for each $\varepsilon > 0$ we obtain

$$\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\} \notin \mathcal{I}_2.$$

This gives us $x \in \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl}$. Now, we show the reverse inclusion. Let $x \in \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl}$. Then, for every $\varepsilon > 0$

$$\{(k,l) \in \mathbb{N}^2 \colon C_{kl} \cap B(x,\varepsilon) \neq \emptyset\} \notin \mathcal{I}_2.$$

We have from (3.2) and Lemma 2.3(b), $\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} d(x, C_{kl}) = 0$. □

Theorem 3.3. *Let (X,d) be a metric space and (C_{kl}) be a double sequence of closed subsets of X. Then*

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} = \Big\{x \colon \forall\ (k,l) \in \mathbb{N}^2,\ \exists y_{kl} \in C_{kl} \colon \mathcal{I}_2\text{-}\lim_{k,l\to\infty} y_{kl} = x\Big\}. \tag{3.3}$$

Proof. Let $x \in \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$ be an arbitrary. By Theorem 3.2, we obtain $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} d(x, C_{kl}) = 0$. Given an arbitrary $\varepsilon > 0$,

$$\{(k,l) \in \mathbb{N}^2 \colon d(x, C_{kl}) \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2.$$

Considering that C_{kl} is a closed set, for $(k,l) \in \mathbb{N}^2$ there exists $y_{kl} \in C_{kl}$ such that $d(x, y_{kl}) \leq 2d(x, C_{kl})$. Then, we have $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} y_{kl} = x$.

Conversely, if x is an element of the set given by the right side of the equality (3.3). Then, there exist $\{y_{kl} \mid y_{kl} \in A_{kl}, k, l \in \mathbb{N}\}$ such that $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} y_{kl} = x$. Then for every $\varepsilon > 0$

$$\{(k,l) \in \mathbb{N}^2 \colon d(x, y_{kl}) \geq \varepsilon\} \in \mathcal{I}_2.$$

The inequality $d(x, y_{kl}) \geq d(x, C_{kl})$ yields the inclusion

$$\{(k,l) \in \mathbb{N}^2 \colon d(x, C_{kl}) \geq \varepsilon\} \subseteq \{(k,l) \in \mathbb{N}^2 \colon d(x, y_{kl}) \geq \varepsilon\}.$$

This implies that $\mathcal{I}_2\text{-}\lim_{k,l\to\infty} d(x, C_{kl}) = 0$. By Theorem 3.2, we have

$$x \in \mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}.$$

□

Theorem 3.4. *Let (X, d) be a metric space and (C_{kl}) be a double sequence of closed subsets of X. If $\mathcal{I}_2$ is a strongly admissible ideal of $\mathbb{N}^2$ having the property (AP2). Then*

(3.4) $$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl} = \left\{x : \exists N \in \mathcal{F}(\mathcal{I}_2), \forall (k,l) \in N, \exists y_{kl} \in C_{kl} : \lim_{(k,l)\in N} y_{kl} = x\right\}.$$

Proof. Assume that $\mathcal{I}_2$ is a strongly admissible ideal with the property (AP2). By Lemma 2.2, $\mathcal{I}_2^*$ convergence is equivalent to $\mathcal{I}_2$ convergence. By Theorem 3.3 the proof is straightforward. □

We note that the property (AP2) in Theorem 3.4 can not be dropped. The following example shows this fact.

Example 3.2. Let $X = \mathbb{R}$ equipped with the usual Euclidean metric and let the sets $(N_j)_{j\in\mathbb{N}}$ be a decomposition of $\mathbb{N}$. We define

$$\triangle_j = \{(m,n) : \min\{m,n\} \in N_j\} \quad j = 1, 2, 3 \ldots$$

Then $\{\triangle_j\}_{j\in\mathbb{N}}$ is a decomposition of $\mathbb{N}^2$ and the ideal

$$\mathcal{I}_2 = \{A \subset \mathbb{N}^2 : A \, \text{is included in a finite union of } \triangle_j{}'s\}$$

a strongly admissible ideal (see [3, Theorem 2]). Put $A_{kl} = \{\frac{1}{j}\}$ if and only if $(k,l) \in \triangle_j$. Then the sequence $\{y_{kl} \colon y_{kl} \in A_{kl}, (k,l) \in \mathbb{N}^2\}$ can be defined by $y_{kl} = \frac{1}{j}$ for $(k,l) \in \triangle_j$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then

$$\left\{(k,l) \in \mathbb{N}^2 \colon y_{kl} \geq \delta\right\} \subseteq \triangle_1 \cup \triangle_2 \cup \ldots \cup \triangle_q.$$

So $\mathcal{I}_2 - \lim_{k,l\to\infty} y_{kl} = 0$ and $\mathcal{I}_2 - \liminf_{k,l\to\infty} A_{kl} = \{0\}$.

Suppose in contrary that 0 belongs to the right-hand side set of the equality (3.4). Then there is a set $M \in \mathcal{F}(\mathcal{I}_2)$ such that for $(m,n) \in M$, there exists $y_{mn} \in A_{mn}$ and

(3.5) $$\lim_{(m,n)\in M} y_{mn} = 0.$$

By the definition of $\mathcal{F}(\mathcal{I}_2)$ we have $M = \mathbb{N}^2 \setminus H$, where $H \in \mathcal{I}_2$. By the definition of $\mathcal{I}_2$ there is a $p \in \mathbb{N}$ such that

$$H \subseteq \bigcup_{j=1}^{p} \triangle_j.$$

But then $\triangle_{p+1} \subset \mathbb{N}^2 \setminus H = M$. But from the construction of $\triangle_{p+1}$ it follows that for any $n_0 \in \mathbb{N}$, $y_{kl} = \frac{1}{p+1} > 0$ hold for infinitely many $(k,l)'s$ with $(k,l) \in M$ and $k,l \geq n_0$. This contradicts (3.5).

Corollary 3.1. *Let X be a normed linear space and (C_{kl}) be a double sequence of closed subsets of X. If the ideal $\mathcal{I}_2$ has property (AP2) and there is a set $K \in \mathcal{F}(\mathcal{I}_2)$ such that C_{kl} is convex for each $(k,l) \in K$, then $\mathcal{I}_2$-$\liminf_{k,l\to\infty} C_{kl}$ is convex and so, when it exist, is $\mathcal{I}_2$-$\lim_{k,l\to\infty} C_{kl}$.*

Proof. Suppose that $\mathcal{I}_2$-$\liminf_{k,l\to\infty} C_{kl} = C$. If x_1 and x_2 belong to C, by Theorem 3.4, we can find for all $(k,l) \in N$ in some set $N \in \mathcal{F}(\mathcal{I}_2)$ points y_{kl}^1 and y_{kl}^2 in C_{kl} such that $\lim_{(k,l)\in N} y_{kl}^1 = x_1$ and $\lim_{(k,l)\in N} y_{kl}^2 = x_2$. Since $K \in \mathcal{F}(\mathcal{I}_2)$, we get $M \in \mathcal{F}(\mathcal{I}_2)$ with $M = N \cap K$. Then, for arbitrary $\mu \in [0,1]$ and $(k,l) \in M$, let us define

$$y_{kl}^{\mu} := (1-\mu)y_{kl}^1 + \mu y_{kl}^2 \quad \text{and} \quad x_{\mu} := (1-\mu)x_1 + \mu x_2.$$

Therefore, $\lim_{(k,l)\in M} y_{kl}^{\mu} = x_{\mu}$ is obtained. By Theorem 3.4, we have $x_{\mu} \in C$. This implies that the set C is convex. □

Theorem 3.5. *Let (X,d) be a metric space and (C_{kl}) be a double sequence of closed subsets of X. Then, we have*

$$\mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl} = \Big\{x\colon\ \forall\ (k,l) \in \mathbb{N}^2,\ \exists y_{kl} \in C_{kl}\colon x \in \mathcal{I}_2(\Gamma_y)\Big\}. \tag{3.6}$$

Proof. Let x be an arbitrary point in $\mathcal{I}_2$-$\limsup_{k,l\to\infty} C_{kl}$. By Theorem 3.2, we have

$$\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} d(x, C_{kl}) = 0.$$

By Lemma 2.3, for every $\varepsilon > 0$ the set

$$\Big\{(k,l) \in \mathbb{N}^2 \colon d(x, C_{kl}) < \frac{\varepsilon}{2}\Big\} \notin \mathcal{I}_2.$$

Since C_{kl} is closed for $(k,l) \in \mathbb{N}^2$ there exists $y_{kl} \in C_{kl}$ such that $d(x, y_{kl}) \leq 2d(x, C_{kl})$. It is clear that x is an ideal cluster point of (y_{kl}). That is, $x \in \mathcal{I}_2(\Gamma_y)$.

On the other hand, if x is an element of the set given by the right side of the equality (3.6), then there exists a sequence $\{y_{kl}\colon y_{kl} \in C_{kl}, (k,l) \in \mathbb{N}^2\}$ such that $x \in \mathcal{I}_2(\Gamma_y)$. That is, for every $\varepsilon > 0$

$$\Big\{(k,l) \in \mathbb{N}^2 : d(x, y_{kl}) < \varepsilon\Big\} \notin \mathcal{I}_2.$$

The inequality $d(x, y_{kl}) \geq d(x, C_{kl})$ yields the inclusion

$$\{(k,l) \in \mathbb{N}^2 : d(x, y_{kl}) < \varepsilon\} \subseteq \{(k,l) \in \mathbb{N}^2 : d(x, C_{kl}) < \varepsilon\}.$$

So, the set $N' = \{(k,l) \in \mathbb{N}^2 : d(x, C_{kl}) < \varepsilon\} \notin \mathcal{I}_2$. That is, $N' \in \mathcal{I}_2^+$. By (3.2), for every $(k,l) \in N'$ we obtain $C_{kl} \cap B(x, \varepsilon) \neq \emptyset$. This means that $x \in \mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl}$. $\square$

From Theorem 3.3 and Theorem 3.5, we conclude that, when $C_{kl} \neq \emptyset$ for all $k, l \in \mathbb{N}$, ideal outer and inner limit sets can be characterized in terms of the sequences $(y_{kl})_{k,l\in\mathbb{N}}$ by selecting a $y_{kl} \in C_{kl}$ for each $(k,l) \in \mathbb{N}^2$: the set of all $\mathcal{I}_2$-cluster points of such sequences is $\mathcal{I}_2\text{-}\limsup_{k,l\to\infty} C_{kl}$, while the set of all $\mathcal{I}_2$-limits of such sequences is $\mathcal{I}_2\text{-}\liminf_{k,l\to\infty} C_{kl}$.

In Theorem 3.5 the set of $\mathcal{I}_2$-cluster points can not be replaced by the set of $\mathcal{I}_2$−limit points, which is shown by the next example.

Example 3.3. Consider ideal $\mathcal{I}_2(\delta)$ and the sets

$$N_j = \left\{2^{j-1}(2k-1) : k \in \mathbb{N}\right\} \qquad (j = 1, 2, 3 \ldots).$$

Now we define $D_{ij} = N_i \times N_j$. Then $D_{ij} \cap D_{pq} = \emptyset$ for $(i,j) \neq (p,q)$ and

$$\delta_2(D_{ij}) = \frac{1}{2^i 2^j}\,(i, j = 1, 2, 3 \ldots).$$

Now we define a double sequence (A_{kl}) as follows

$$A_{kl} = \left\{1 - \frac{1}{ij}\right\},\ (k,l) \in D_{ij}\ (i, j = 1, 2, 3 \ldots).$$

then

$$\mathcal{I}_2(\delta)\text{-}\limsup_{k,l\to\infty} A_{kl} = \left\{1 - \frac{1}{ij} : i, j = 1, 2, 3 \ldots\right\} \cup \{1\}.$$

If a sequence (y_{kl}) is formed by selecting a $y_{kl} \in A_{kl}$, then $y_{kl} = 1 - \frac{1}{ij}$ for $(k,l) \in D_{ij}$ and 1 is not a $\mathcal{I}_2(\delta)$−limit point of (y_{kl}) (see [4, Example 2]) .

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Original Scientific Paper

ROUGH CONTINUOUS CONVERGENCE OF SEQUENCES OF SETS

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Abstract. In this paper, we define a new type of convergence of sequences of sets by using the continuous convergence (or α-convergence) of the sequence of distance functions. Then we proved in which case it is equivalent to rough Wijsman convergence by considering the different values of the roughness degrees.
Keywords: Wijsman convergence, rough convergence, α-convergence, equicontinuity.

1. Introduction

Wijsman [11] has introduced a new type of convergence, which is considered as one of the most important contributions to the theory of convergence of sequences of sets and it is called by his name. He used pointwise convergence of distance functions to define this type of convergence. He [12] also proved a necessary and sufficient condition related to the pointwise limit and limit inferior of the sequences of distance functions under various constraints in order for a sequence of sets to be Wijsman convergent.

In the 2000s, after Phu [8] put forward the idea of rough convergence in the normed spaces, Phu's work was extended to statistical convergence by Aytar [1], and to ideal convergence by Dündar and Çakan [4]. Phu's [8] idea showed that a sequence

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which is not convergent in the usual sense might be convergent to a point, with a certain degree of roughness. In 2016, by combining the two concepts (Wijsman convergence and rough convergence), the idea of rough Wijsman convergence of a sequence of sets was defined by Ölmez and Aytar [7]. Then, Subramanian and Esi [10] defined the concept of rough Wijsman convergence for a triple sequences of sets. Recently, Babaarslan and Tuncer [2] applied the theory of rough convergence to the fuzzy set theory using the double sequences.

Continuous convergence, which is a stronger type of convergence than pointwise convergence (see [6], [9]), has been referred to as α-convergence in recent years (see [3], [5]). Pointwise convergence is equivalent to α-convergence on sequences or nets of functions that are equicontinuous. Das and Papanastassiou [3] defined the concepts of α-equal convergence, α-uniform equal convergence and α-strong uniform equal convergence on the sequences of real-valued functions. Gregoriades and Papanastassiou [5] defined the concept of exhaustive, which is a property weaker than equicontinuity for sequences and nets of functions on metric spaces, and using this property, they investigated the relationships between α-convergence, pointwise convergence and uniform convergence. They also gave a generalization of Ascoli's theorem using the concept of exhaustive.

The main purpose of this article is to observe the results using α-convergence instead of pointwise convergence of distance functions. In this context, first we define the concept of rough continuous convergence. Then we examined the relations between the new definitions obtained with different roughness degrees r_1 and r_2 (see Propositions 3.1 and 3.2). As the main results of this paper, we show that in which cases the new definition coincides with the rough Wijsman convergence (see Theorem 3.1). By giving illustrative examples, the similarity (see Example 3.1) and difference (see Example 3.2) between definitions are obtained.

2. Preliminaries

Throughout this paper, we assume that X is a nonempty set and ρ_X is a metric on X and that A, A_n are nonempty closed subsets of X for each $n \in \mathbb{N}$.

Let (x_n) be a sequence in the metric space X, and r be a nonnegative real number, the sequence (x_n) is said to be rough convergent to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $\rho_X(x_n, x) < r + \varepsilon$ for each $n \geq n(\varepsilon)$ [8].

The *distance function* $d(\cdot, A) : X \to [0, \infty)$ is defined by the formula

$$d(x, A) = \inf\{\rho_X(x, y) : y \in A\}$$

[6, 11].

We say that the sequence (A_n) is *Wijsman convergent* to the set A if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \xrightarrow{W} A$, as $n \to \infty$ [11].

Given $r \geq 0$, we say that a sequence (A_n) is *rough Wijsman convergent* to the set A if for every $\varepsilon > 0$ and each $x \in X$ there exists an $N(x, \varepsilon) \in \mathbb{N}$ such that

$$|d(x, A_n) - d(x, A)| < r + \varepsilon \text{ for all } n \geq N(x, \varepsilon)$$

and we write $d(x, A_n) \xrightarrow{r} d(x, A)$ or $A_n \overset{r-W}{\rightarrow} A$ as $n \to \infty$ [7].

Let (Y, ρ_Y) be another metric space and D be a subset of X. Assume the f, f_n functions from X to Y for each $n \in \mathbb{N}$. The sequence (f_n) *α-converges* to f iff for every $x \in X$ and for every sequence (x_n) of points of X converging to x, the sequence $(f_n(x_n))$ converges to $f(x)$. We shall write $f_n \xrightarrow{\alpha} f$ to denote that the sequence (f_n) α-converges to f (see [5, 6, 9]).

The open ball with centre $x \in X$ and radius $\delta > 0$ is the set

$$S(x, \delta) = \{y \in X : \rho_X(x, y) < \delta\}.$$

The sequence (f_n) is called *equicontinuous* at x if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho_Y(f_n(y), f_n(x)) < \varepsilon$ whenever $y \in S(x, \delta)$, $n \in \mathbb{N}$ [6].

3. Main Results

Definition 3.1. Let $r_1 \geq 0$ and $r_2 \geq 0$. The sequence (A_n) is said to be rough α-convergent (or continuous convergent) to the set A with the roughness degree $r_1 \wedge r_2$ if for every sequence (x_n) which is $x_n \xrightarrow{r_1} x$, the condition $d(x_n, A_n) \xrightarrow{r_2} d(x, A)$ holds at each $x \in X$. In this case, we use the notation $A_n \xrightarrow{r_1 \wedge r_2 - \alpha} A$.
If take $r_1 = 0$ and use the notation r instead of r_2, the sequence (A_n) is said to be rough α-convergent to the set A, and we write $A_n \xrightarrow{r-\alpha} A$.

Let us give an illustrative example to explain the Definition 3.1 to the readers.

Example 3.1. Define

$$A_n := \begin{cases} [-3, -1] \times [-1, 1] & , \text{ if } n \text{ is an odd integer} \\ [1, 3] \times [-1, 1] & , \text{ if } n \text{ is an even integer} \end{cases}$$

and $A = \{0\} \times [-1, 1]$ in the space $\mathbb{R}^2$ equipped with the Euclid metric.

First we show that the sequence (A_n) is rough Wijsman convergent to the set A. Let $\varepsilon > 0$ and $(x^*, y^*) \in \mathbb{R}^2$. Then we calculate

$$d((x^*, y^*), A) = \begin{cases} \sqrt{(x^* - 0)^2 + (y^* - 1)^2} & , \text{ if } x^* \in \mathbb{R} \text{ and } y^* > 1 \\ \sqrt{(x^* - 0)^2 + (y^* + 1)^2} & , \text{ if } x^* \in \mathbb{R} \text{ and } y^* < -1 \\ |x^*| & , \text{ if } x^* \in \mathbb{R} \text{ and } -1 \leq y^* \leq 1 \end{cases}.$$

Similarly, $d((x^*, y^*), A_n)$ can be easily calculated. Then there exists an $n_1 = n_1((x^*, y^*), \varepsilon)$ such that it can be easily obtained

$$|d((x^*, y^*), A_n) - d((x^*, y^*), A)| \leq 3 + \varepsilon$$

for each $n \geq n_1$ using the inequality $\sqrt{(x^* - x)^2 + (y^* - y)^2} \leq |x^* - x| + |y^* - y|$. Hence, it is proved that $A_n \overset{r-W}{\longrightarrow} A$, for every $r \geq 3$.

Now we show that the sequence (A_n) is rough α-convergent to the set A. Assume that the sequence (x_n, y_n) converges to the point (x^*, y^*). Hence there exists an $n_2 = n_2((x^*, y^*), \varepsilon)$ such that it can be easily calculated

$$|d((x_n, y_n), A_n) - d((x^*, y^*), A)| \leq 3 + \varepsilon$$

for each $n \geq n_2$. This proves that $A_n \overset{r-\alpha}{\longrightarrow} A$ for each $r \geq 3$.

Lastly we show that $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$. Let $(x_n, y_n) \overset{r_1}{\longrightarrow} (x^*, y^*)$. Then there exists an $n_3 = n_3((x^*, y^*), \varepsilon)$ such that $|x_n - x^*| < r_1 + \varepsilon$ and $|y_n - y^*| < r_1 + \varepsilon$ for every $n \geq n_3$. Hence the inequality

$$|d((x_n, y_n), A_n) - d((x^*, y^*), A)| \leq 3 + r_1 + \varepsilon$$

is obvious for every $n \geq n_3$. If we take $r_2 = r_1 + 3$, then we get $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$.

Proposition 3.1. *If the sequence (A_n) is rough α-convergent to the set A with the roughness degree $r_1 \wedge r_2$ then it rough α-converges to the set A with the roughness degree r_2.*

Proof. Assume $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$. Take $x \in X$. Let (x_n) be a sequence such that $x_n \longrightarrow x$. We also have $x_n \overset{r_1}{\longrightarrow} x$. Since $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$, we get

$$d(x_n, A_n) \overset{r_2}{\longrightarrow} d(x, A). \tag{3.1}$$

Then (3.1) holds for each sequence (x_n) such that $x_n \longrightarrow x$. Hence we have $A_n \overset{r_2-\alpha}{\longrightarrow} A$, which completes the proof. □

As can be seen following example, the converse implication of Proposition 3.1 doesn't hold in general.

Example 3.2. Define

$$A_n := \begin{cases} \left\{-2 + \dfrac{1}{n}\right\} & \text{, if } n \text{ is an odd integer} \\ \left\{2 - \dfrac{1}{n}\right\} & \text{, if } n \text{ is an even integer} \end{cases}$$

and $A = [-2, 2]$.

First we show that the sequence (A_n) is rough Wijsman convergent to the set A. We have

$$d(x, A_n) = \begin{cases} \left|x + 2 - \frac{1}{n}\right| & \text{, if } n \text{ is an odd integer} \\ \left|x - 2 + \frac{1}{n}\right| & \text{, if } n \text{ is an even integer} \end{cases}$$

and

$$d(x, A) = \begin{cases} |x + 2| & \text{, if } x < -2 \\ 0 & \text{, if } -2 \leq x \leq 2 \\ |x - 2| & \text{, if } x > 2 \end{cases}$$

for each $x \in \mathbb{R}$. Hence, for each $\varepsilon > 0$ and each x, there exists an $n_1 = n_1(x, \varepsilon)$ such that $n \geq n_1$ we have

$$|d(x, A_n) - d(x, A)| \leq 4 + \varepsilon.$$

Therefore, we get $A_n \overset{r-W}{\longrightarrow} A$ for each $r \geq 4$.

Now we show that the sequence (A_n) is rough α-convergent to the set A. Assume $x_n \longrightarrow x$. Since

$$d(x_n, A_n) = \begin{cases} \left|x_n + 2 - \frac{1}{n}\right| & , \text{ if } n \text{ is an odd integer} \\ \left|x_n - 2 + \frac{1}{n}\right| & , \text{ if } n \text{ is an even integer} \end{cases},$$

for each $\varepsilon > 0$ there exists an $n_2 = n_2(x, \varepsilon)$ such that $n \geq n_2$ we have

$$|d(x_n, A_n) - d(x, A)| \leq 4 + \varepsilon.$$

This is desired result, i.e., $A_n \overset{r-\alpha}{\longrightarrow} A$ for every $r \geq 4$.

Lastly we show that $A_n \overset{r_1 \wedge r_2 - \alpha}{\not\rightarrow} A$. Take $r_1 = r_2 = 4$. Define $x_n = 6$ for each n and $x = 2$. Then the sequence (x_n) is rough α-convergent to the point x with the roughness degree $r_1 = 4$. On the other hand, we have

$$d(x_n, A_n) = \begin{cases} \left|8 - \frac{1}{n}\right| & , \text{ if } n \text{ is an odd integer} \\ \left|4 + \frac{1}{n}\right| & , \text{ if } n \text{ is an even integer} \end{cases}.$$

If we take $\varepsilon = 1$, then we have

$$|d(x_n, A_n) - d(x, A)| = 8 \nleq 5 = r_2 + \varepsilon$$

for every odd terms. Hence we get $A_n \overset{r_1 \wedge r_2 - \alpha}{\not\rightarrow} A$.

The question may come to mind: Could the converse implication of Proposition 3.1 be obtained based on a particular selection of r_1 and r_2? Before answering this question as Proposition 3.2, we will give a simple inequality:

Lemma 3.1. *If the set A is a nonempty closed subset of X, then we have*

$$|d(x, A) - d(y, A)| \leq \rho_X(x, y)$$

for each $x, y \in X$.

The proof of Lemma 3.1 is obvious from the Lipschitz continuity of distance functions.

Proposition 3.2. *If the sequence (A_n) is α-convergent to the set A with the roughness degree r, then it is α-convergent to the set A with the roughness degree $r_1 \wedge r_2$ for each r_1 and r_2 such that $r_2 \geq r_1 + r$.*

Proof. Let $\varepsilon > 0$ and $x \in X$. If we assume that $x_n \overset{r_1}{\longrightarrow} x$, then it is clear that there exists a sequence $(y_n) \subset X$ such that $y_n \to x$ and $\rho_X(x_n, y_n) \leq r_1$. Since the sequence (A_n) is α-convergent to the set A with the roughness degree r, there exists an $n_1(x,\varepsilon) \in \mathbb{N}$ such that $n \geq n_1$ we have

$$|d(y_n, A_n) - d(x, A)| < r + \varepsilon.$$

By Lemma 3.1, we get

$$|d(x_n, A_n) - d(y_n, A_n)| \leq \rho_X(x_n, y_n) \leq r_1$$

for each $n \in \mathbb{N}$. Then we have

$$\begin{aligned} |d(x_n, A_n) - d(x, A)| &\leq |d(x_n, A_n) - d(y_n, A_n)| + |d(y_n, A_n) - d(x, A)| \\ &< r_1 + r + \varepsilon \end{aligned}$$

for each $n \geq n_1$. If we take $r_2 = r_1 + r$, then we say that the sequence (A_n) is α-convergent to the set A with the roughness degree $r_1 \wedge r_2$, which completes the proof. □

Before giving the main result of the paper, let's give a lemma. It will be used in the proof of Theorem 3.1.

Lemma 3.2. *The sequence $(d(\cdot, A_n))$ of distance functions is equicontinuous.*

Proof. Let $x \in X$, $\varepsilon > 0$ and $z \in S(x, \varepsilon)$. We have

$$\begin{aligned} \rho_X(y,z) &\leq \rho_X(y,x) + \rho_X(x,z) \\ \rho_X(x,y) &\leq \rho_X(x,z) + \rho_X(z,y) \end{aligned}$$

for $y \in A_n$, where n fixed. Since

$$\begin{aligned} d(z, A_n) &= \inf_{y\in A_n} \rho_X(y,z) \leq \inf_{y\in A_n} (\rho_X(y,x) + \rho_X(x,z)) \\ &= \inf_{y\in A_n} \rho_X(y,x) + \rho_X(x,z) < d(x, A_n) + \varepsilon \\ d(x, A_n) &= \inf_{y\in A_n} \rho_X(x,y) \leq \inf_{y\in A_n} (\rho_X(x,z) + \rho_X(z,y)) \\ &= \inf_{y\in A_n} (\rho_X(z,y)) + \rho_X(x,z) < d(z, A_n) + \varepsilon, \end{aligned}$$

we get

$$-\varepsilon < d(z, A_n) - d(x, A_n) < \varepsilon.$$

Therefore, if we take $\delta = \varepsilon > 0$, then we get

$$|d(z, A_n) - d(x, A_n)| < \varepsilon$$

for each $n \in \mathbb{N}$ and each $z \in S(x, \varepsilon)$. Since the point x is arbitrary, the sequence $(d(\cdot, A_n))$ of functions is equicontinuous. □

Theorem 3.1. *The concepts of rough Wijsman convergence and rough α-convergence are equivalent to each other with the same roughness degree.*

Proof. First we assume that the sequence (A_n) is rough α-convergent to the set A. Let $\varepsilon > 0$ and $x \in X$. Define $x_n = x$ for each $n \in \mathbb{N}$. Since $A_n \overset{r-\alpha}{\longrightarrow} A$, there exists an $n_1(x,\varepsilon) \in \mathbb{N}$ such that $n \geq n_1$, we have

$$|d\left(x_n, A_n\right) - d\left(x, A\right)| < r + \varepsilon.$$

Then we get

$$\begin{aligned} |d\left(x, A_n\right) - d\left(x, A\right)| &= |d\left(x_n, A_n\right) - d\left(x, A\right)| \\ &< r + \varepsilon \end{aligned}$$

for each $n \geq n_1$. Therefore the sequence (A_n) is rough Wijsman convergent to the set A.

On the other hand, now we assume that the sequence (A_n) is rough Wijsman convergent to the set A with the roughness degree r. Then the sequence $(d\left(\cdot, A_n\right))$ of functions is rough convergent to the function $d\left(\cdot, A\right)$ on X with the same roughness degree r. Let $x \in X$ and $\varepsilon > 0$. Hence there exists an $n_1\left(x, \varepsilon\right) \in \mathbb{N}$ such that $n \geq n_1$ we have

$$|d\left(x, A_n\right) - d\left(x, A\right)| < r + \frac{\varepsilon}{2}.$$

By Lemma 3.2, there exists $\delta\left(x, \varepsilon\right) > 0$ such that

$$|d\left(y, A_n\right) - d\left(x, A_n\right)| < \frac{\varepsilon}{2} \tag{3.2}$$

for each $n \in \mathbb{N}$ and each $y \in S\left(x, \delta\right)$. Take a sequence (x_n) such that $x_n \longrightarrow x$. In this case, there exists an $n_2\left(x, \delta\right) \in \mathbb{N}$ such that $\rho_X\left(x_n, x\right) < \delta$ for each $n \geq n_2$. Hence by the inequality (3.2), we get

$$|d\left(x_n, A_n\right) - d\left(x, A_n\right)| < \frac{\varepsilon}{2}$$

for each $n \geq n_2$. Define $n_0 = \max\left\{n_1, n_2\right\}$. Therefore we have

$$\begin{aligned} |d\left(x_n, A_n\right) - d\left(x, A\right)| &\leq |d\left(x_n, A_n\right) - d\left(x, A_n\right)| + |d\left(x, A_n\right) - d\left(x, A\right)| \\ &< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon \end{aligned}$$

for each $n \geq n_0$. Since x is an arbitrary point, we say that the sequence (A_n) is rough α-convergent to the set A. □

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AN EXAMINATION OF THE CONDITION UNDER WHICH A CONCHOIDAL SURFACE IS A BONNET SURFACE IN THE EUCLIDEAN 3-SPACE

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Abstract. In this study, we examine the condition of the conchoidal surface to be a Bonnet surface in Euclidean 3-space. Especially, we consider the Bonnet conchoidal surfaces which admit an infinite number of isometries. In addition, we study the necessary conditions which have to be fulfilled by the surface of revolution with the rotating curve $c(t)$ and its conchoid curve $c_d(t)$ to be the Bonnet surface in Euclidean 3-space.
Keywords. Conchoidal surface, Bonnet surface, Euclidean 3-space.

1. Introduction

The conchoid of Nicomedes, which is called by the Greek geometer Nicomedes's name, was originally contrived around 200 BC to trisect an angle and duplicate the cube. For any curve and a fixed point, let a straight line, which meets the curve at the point Q, is drawn through the fixed point. If P and R are points on this line such that $RQ = QP = const.$, then the conchoid of curve with respect to the fixed point is the locus of P and R [12].

The conchoids play an important role in many applications as the construction of buildings, astronomy [9], optics [2], physics [19]. Although the

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conchoidal constructions were extensively mentioned by the ancient Greeks in the seventeenth century, they have been recently addressed by different authors, too. One of these has been put forward by Odehnal. He obtained a generalized conchoid transformation considering a construction with the help of cross ratios [13]. Moreover, Peternel, etc. presented the conchoidal surface of rational ruled surfaces, the conchoidal surfaces of spheres, the conchoids and the pedal surfaces [15, 16, 17].

Surfaces, which admit a one-parameter family of isometries preserving the mean curvature, have been proposed by Bonnet and although Bonnet raised these surfaces [3], the term “Bonnet surface” was firstly used by Lalan [11]. Bonnet showed that all surfaces with the constant mean curvature can be isometrically mapped to each other and the deformable surfaces with the non-constant mean curvature are the isothermic Weingarten surfaces which can be deformable to the revolution surfaces. After that, many mathematicians have contributed these surfaces [18, 10, 7, 1].

Bonnet surfaces may be broken up into three types which is described as follows:

(i) Surfaces of the constant mean curvature other than the plane or the sphere.

(ii) Isothermic Weingarten surfaces of the non-constant mean curvature which admit a one parameter family of geometrically distinct non-trivial isometries.

(iii) Surfaces of the non-constant mean curvature that admit a single non-trivial isometry [10].

In [4], the authors studied the conchoidal surfaces, the surfaces of revolution given with the conchoid curve and their geometrical properties in Euclidean 3-space. In our work, using the geometric properties obtained for conchoidal surfaces in reference [4], we have examined the conditions under which the conchoidal surface and the surface of revolution given with conchoid curve is a Bonnet surface in Euclidean 3- space. According to that, we get the following results:

(1) If a regular surface M and a conchoidal surface M_d are minimal, then they are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvatures.

(2) The surfaces M with the radius function $r(u_0, v)$ or $r(u, v_0)$ are the surfaces of the type (ii) which admit an infinite number of isometries. Also, the result is similar for the conchoidal surfaces M_d.

(3) If a regular surface M and a conchoidal surface M_d, which are the surfaces of revolution generated by the rotating curve and its conchoid curve, are minimal, then they are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvatures.

(4) If a regular surface M and a conchoidal surface M_d, which are the surfaces of revolution generated by the rotating curve and its conchoid curve with the radius function $r(u_0, v)$ or $r(u, v_0)$, are the surfaces of the type (ii) which admit an infinite number of isometries.

2. Preliminaries

Let M be a smooth surface in $\mathbb{E}^3$ given with the patch $X(u, v)$ for

$(u,v) \in D \subset E^3$. The tangent space to M at an arbitrary point p of M is spanned by $\{X_u, X_v\}$. Let N be the unit normal vector field of the surface M defined by $N = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$. The first fundamental form I and the second fundamental form II of the surface M are

$$I = edu^2 + 2fdudv + gdv^2, \quad II = ldu^2 + 2mdudv + ndv^2, \tag{2.1}$$

respectively, where

$$e = \langle X_u, X_u \rangle\,,\ f = \langle X_u, X_v \rangle\,,\ g = \langle X_v, X_v \rangle\,, \tag{2.2}$$

and

$$l = \langle X_{uu}, N \rangle\,,\ m = \langle X_{uv}, N \rangle\,,\ n = \langle X_{vv}, N \rangle\,. \tag{2.3}$$

In [8], the Gaussian curvature K and the mean curvature H are

$$K = \frac{ln - m^2}{eg - f^2}, \quad H = \frac{en - 2fm + gl}{2(eg - f^2)}. \tag{2.4}$$

A surface M in $\mathbb{E}^3$ is called Weingarten surface if there exists a non-trivial functional relation

$$\Omega(K, H) = 0 \tag{2.5}$$

with respect to its Gaussian curvature K and its mean curvature H, where Ω is the Jakobian determinant [14].

If a surface M in $\mathbb{E}^3$ has the coefficients of first fundamental form which satisfy the conditions $e = g,\ f = 0,$ then it is called isothermic [5]. According to [18], the isothermic surface provides the condition

$$\frac{\partial^2}{\partial u \partial v}\left(log\frac{g}{e}\right) = 0. \tag{2.6}$$

We assume a smooth surface $M \subset E^3$ and a fixed reference point O which can be considered as the origin of a cartesian coordinate system. Let M is described by a polar representation

$$X(u,v) = r(u,v)s(u,v) \tag{2.7}$$

with $\|s(u,v)\| = 1$. Considering $s(u,v) = (\cos u \cos v, \sin u \cos v, \sin v)$ of the unit sphere S^2, so $s(u,v)$ and $r(u,v)$ are called spherical part and radius function of $X(u,v)$, respectively.

In [17, 15], the one-sided conchoidal surface M_d of M is derived by adding $d \in \mathbb{R}$ to the radius function $r(u,v)$ and thus M_d admits the polar representation

$$M_d(u,v) = (r(u,v) + d)s(u,v). \tag{2.8}$$

Let M be a regular surface given with the parametrization (2.7). Then the coefficients of the first fundamental form of the surface M are

$$\begin{aligned} e &= r^2\cos^2 v + r_u^2, \\ f &= r_u r_v, \\ g &= r^2 + r_v^2. \end{aligned} \tag{2.9}$$

Additionally, its Gaussian curvature and its mean curvature are
(2.10)

$$\begin{aligned} K = & -\frac{1}{r^2A^2}\left[rr_{uv}\cos v - 2r_u r_v\cos v + rr_u\sin v\right)^2 \\ & -\cos^2 v\left(2r_u^2 + rr_v\sin v\cos v + r^2\cos^2 v - rr_{uu}\right)\left(2r_v^2 + r^2 - rr_{vv}\right)\Big], \end{aligned}$$

and
(2.11)

$$\begin{aligned} H &= -\frac{1}{2r^2A^{3/2}}\left[\cos v\left(2r_u^2 + rr_v\sin v\cos v + r^2\cos^2 v - rr_{uu}\right)\left(r^2 + r_v^2\right)\right. \\ & +\cos v\left(2r_v^2 + r^2 - rr_{vv}\right)\left(r^2\cos^2 v + r_u^2\right) \\ & \left. +2r_u r_v\left(rr_{uv}\cos v - 2r_u r_v\cos v + rr_u\sin v\right)\right], \end{aligned}$$

where $A = (r^2 + r_v^2)\cos^2 v + r_u^2$. Also, if M_d is a conchoidal surface given with the parametrization (2.8), its Gaussian curvature and its mean curvature are

$$\begin{aligned} \widetilde{K} &= -\frac{1}{(r\pm d)^2A^2}\left[((r\pm d)r_{uv}\cos v - 2r_u r_v\cos v + (r\pm d)r_u\sin v)^2\right. \\ & -\cos^2 v\left(2r_u^2 + (r\pm d)r_v\sin v\cos v\right. \\ & \left.\left.+(r\pm d)^2\cos^2 v - (r\pm d)r_{uu}\right)\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)\right], \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \widetilde{H} &= -\frac{1}{2(r\pm d)^2A^{3/2}}\left[\cos v\left(2r_u^2 + (r\pm d)r_v\sin v\cos v\right.\right. \\ & \left.+(r\pm d)^2\cos^2 v - (r\pm d)r_{uu}\right)\left((r\pm d)^2 + r_v^2\right) \\ & +\cos v\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)\left((r\pm d)^2\cos^2 v + r_u^2\right) \\ & \left.+2r_u r_v\left((r\pm d)r_{uv}\cos v - 2r_u r_v\cos v + (r\pm d)r_u\sin v\right)\right], \end{aligned} \tag{2.13}$$

where $A = ((r\pm d)^2 + r_v^2)\cos^2 v + r_u^2$ [4].

Let M be a surface of revolution generated by the rotating curve $c(t)$. The surface is given with the surface patch

$$X(t,s) = (r(t)\cos t, r(t)\sin t\cos s, r(t)\sin t\sin s), \tag{2.14}$$

where $c(t) = r(t)(\cos t, \sin t)$. The coefficients of the first fundamental form of the surface M hold:

$$\begin{aligned} e &= r^2 + (r')^2, \\ f &= 0, \\ g &= r^2 \sin^2 t. \end{aligned} \tag{2.15}$$

The Gaussian and mean curvatures of the surface M are as follows:

$$K = \frac{(r' \cos t - r \sin t)(rr'' - 2(r')^2 - r^2)}{r \sin t (r^2 + (r')^2))^3}, \tag{2.16}$$

and

$$H = \frac{r \sin t(rr'' - 2(r')^2 - r^2) + (r^2 + (r')^2)(r' \cos t - r \sin t)}{2r \sin t (r^2 + (r')^2)^{3/2}}, \tag{2.17}$$

respectively. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t)$. The surface is parametrized by

$$\widetilde{X}(t, s) = ((r(t) \pm d) \cos t, (r(t) \pm d) \sin t \cos s, (r(t) \pm d) \sin t \sin s), \tag{2.18}$$

where $c_d(t) = (r(t) \pm d)(\cos t, \sin t)$. The coefficients of the first fundamental form of the surface M_d are calculated as

$$\begin{aligned} \widetilde{e} &= (r(t) \pm d)^2 + (r')^2, \\ \widetilde{f} &= 0, \\ \widetilde{g} &= (r(t) \pm d)^2 \sin^2 t. \end{aligned} \tag{2.19}$$

The Gaussian and mean curvatures of the surface M_d become

$$\widetilde{K} = \frac{(r' \cos t - (r(t) \pm d) \sin t)((r(t) \pm d)r'' - 2(r')^2 - (r(t) \pm d)^2)}{(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2))^3}, \tag{2.20}$$

$$\begin{aligned} \widetilde{H} = &\frac{(r(t) \pm d) \sin t((r(t) \pm d)r'' - 2(r')^2 - (r(t) \pm d)^2)}{2(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2)^{3/2}} \\ &+ \frac{((r(t) \pm d)^2 + (r')^2)(r' \cos t - (r(t) \pm d) \sin t)}{2(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2)^{3/2}}, \end{aligned} \tag{2.21}$$

respectively [4].

3. Discussion and Conclusion

3.1. An examination of the condition of the conchoidal surface to be a Bonnet surface in $\mathbf{E}^3$

In this section, we will examine condition which is the conchoidal surface to be a Bonnet surface in Euclidean 3-space. Especially, we will deal with the conchoidal surfaces admitting an infinite number of isometries. Thus, it will be sufficient to determine: (a) the conchoidal surfaces of the constant mean curvature and (b) the isothermic Weingarten conchoidal surfaces.

(a) The conchoidal surfaces of the constant mean curvature

Let M be a regular surface given with the parametrization (2.7). It is possible that the mean curvature H given by (2.11) is equal to a non-zero constant when the radius function $r(u, v)$ is a constant. This means that the surface M is a sphere.

Example 3.1. Let the radius function be a constant. For $r(u, v) = 3$ and $d = 1$, the conchoidal surface M_d is given by the parametrization

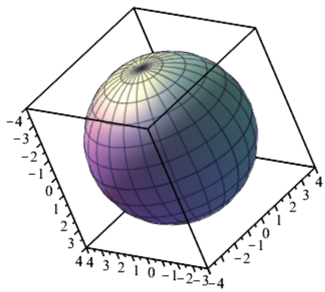

Figure 3.1: Conchoidal surface with $r(u, v) = 3$ and $d = 1$

$$X_d(u, v) = (4 \cos u \cos v, 4 \sin u \cos v, 4 \sin v). \tag{3.1}$$

It denotes a sphere as given in Figure 3.1.

The mean curvature is a constant when the surface M is minimal, except that the radius function is a constant. In this case, considering [4], if u-parameter radius function is

$$r(u) = \pm \frac{\sqrt{\cos v}}{\sqrt{c_1 \sin(2u \cos v) - c_2 \cos(2u \cos v)}} \tag{3.2}$$

or if v-parameter radius function is

$$r(v) = \frac{1}{c_1 \sin v}, \tag{3.3}$$

where c_1, c_2 are constants, then M is the minimal surface. So, the surfaces M determined by (3.2) and (3.3) are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvature.

Similar results for conchoidal surface M_d are obtained as follows:

If the radius function is a constant, the mean curvature $\widetilde{H}$ of the conchoidal surface is equal to $\dfrac{1}{r \pm d}$. This means that the surface M_d is a sphere. If u-parameter radius function is

$$r(u) = \pm \frac{\sqrt{\cos v}}{\sqrt{c_1 \sin(2u\cos v) - c_2 \cos(2u\cos v)}} \pm d \tag{3.4}$$

or if v-parameter radius function is

$$r(v) = \mp d + \frac{1}{c_1 \sin v}, \tag{3.5}$$

where c_1, c_2 are constants, then the surface M_d is minimal. So, the conchoidal surfaces M_d determined by (3.4) and (3.5) are the conchoidal surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvature.

Example 3.2. Let the radius function is given by

$$r(u) = \frac{\sqrt{\cos v}}{\sqrt{\sin(2u\cos v) - \cos(2u\cos v)}} \tag{3.6}$$

and $d = -1$. Then, the conchoidal surface M_d is parametrized by

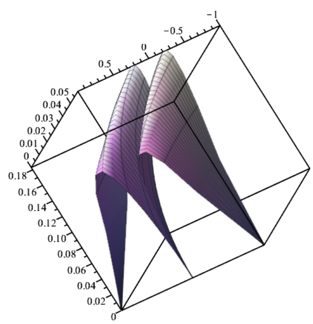

Figure 3.2: Conchoidal surface with $r(u)$ and $d = -1$

$$X_d(u, v) = (r(u) - 1)(\cos u \cos v, \sin u \cos v, \sin v). \tag{3.7}$$

It is shown as given in Figure 3.2.

Example 3.3. Let the radius function is given by $r(v) = \dfrac{1}{2\sin v}$ and $d = -1$. Then, the conchoidal surface M_d is parametrized by

$$X_d(u, v) = \left(\frac{1}{2\sin v} - 1\right)(\cos u \cos v, \sin u \cos v, \sin v). \tag{3.8}$$

It is shown as given in Figure 3.3.

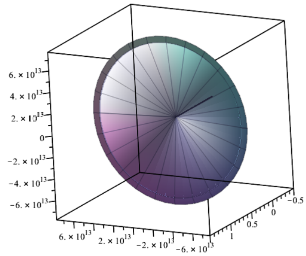

Figure 3.3: Conchoidal surface with $r(v)$ and $d = -1$

(b) The isothermic Weingarten conchoidal surfaces of the non-constant mean curvature

Firstly, let's calculate the condition which is satified by the surface M to be an isothermal surface. When the curves of an orthogonal system have the constant geodesic curvature, the system is an isothermal [6]. For this, we assume that the parameter curves of the surface M constitute the orthogonal system, namely, $\langle X_u, X_v\rangle = 0$. When the surface is assigned by these parametric curves and the linear element is written $ds^2 = edu^2 + gdv^2$, from [6], the condition that the geodesic curvature is a constant becomes $\dfrac{\partial^2}{\partial u \partial v}\left(log\dfrac{g}{e}\right) = 0$.

When the parameter curves are orthogonal, $\langle X_u, X_v\rangle = r_u r_v = 0$. This means that $r_u = 0$ or $r_v = 0$. Therefore the parametric curves of the conchoidal surface M_d are orthogonal. Thus, when the surface M is isothermal, the obtained cases are valid for the conchoidal surface M_d. So, we have the following cases:

Case 1: We assume that $r_u = 0$ and $r_v \neq 0$. In order to examine whether the surface M with the radius function $r(u_0, v)$ is a Bonnet surface, we will work the isothermic Weingarten surfaces.

Using (2.9) into (2.6), then we obtain as follows:

$$\frac{\partial^2}{\partial u \partial v}\left(log\frac{r^2 + r_v^2}{r^2 \cos^2 v}\right) = 0. \tag{3.9}$$

From (3.9), we conclude that the surface M with the radius function $r(u_0, v)$ is the isothermal surface.

Secondly, we investigate the necessary conditions for the surface M to be a Weingarten surface. Differentiating (2.10) and (2.11) with respect to u and considering $r_u = 0$, then we find $\dfrac{\partial K}{\partial u} = 0$ and $\dfrac{\partial H}{\partial u} = 0$. Hence, the surface M with the radius function $r(u_0, v)$ is the Weingarten surface. Additionally, from (2.11), we see that the mean curvature of the surface M with the radius function $r(u_0, v)$ is the non-constant.

As a result, since the surface M is both the isothermal and Weingarten surface with the non-constant mean curvature, then it has an infinite number of the Bonnet nets. Thus, the following theorem is given.

Theorem 3.1. *The surface M with the radius function $r(u_0, v)$ is a surface of the type (ii) which admits an infinite number of isometries. So, this surface is a Bonnet surface.*

Let M_d be a conchoidal surface of M given with the parametrization (2.8). If the radius function $r(u, v)$ is a $v-$parameter function, then the coefficients of the first fundamental form of the surface M_d are

$$\begin{aligned} \widetilde{e} &= (r \pm d)^2 \cos^2 v, \\ \widetilde{f} &= 0, \\ \widetilde{g} &= (r \pm d)^2 + r_v^2. \end{aligned} \tag{3.10}$$

Considering these coefficients, the conchoidal surface M_d of M with the radius function $r(u_0, v)$ is the isothermic surface, since we get

$$\frac{\partial^2}{\partial u \partial v}\left(log \frac{(r \pm d)^2 + r_v^2}{(r \pm d)^2 \cos^2 v} \right) = 0. \tag{3.11}$$

To determine the necessary condition to be a Weingarten surface of M_d, we have (2.12) and (2.13) for $r_u = 0$. From $\dfrac{\partial \widetilde{K}}{\partial u} = 0$ and $\dfrac{\partial \widetilde{H}}{\partial u} = 0$, the conchoidal surface M_d of M with the radius function $r(u_0, v)$ is the Weingarten surface. From (2.13), it is easily seen that $\widetilde{H} \neq const$. Therefore, the following theorem is given for the conchoidal surface M_d.

Theorem 3.2. *The conchoidal surface M_d with the radius function $r(u_0, v)$ is a surface of the type (ii) which admits an infinite number of isometries. So, this surface is a Bonnet surface.*

Corollary 3.1. *There is no surfaces M and M_d that admits a single non-trivial isometry with the non-constant mean curvature.*

Example 3.4. Let the radius function is given by $r(v) = \dfrac{1}{\cos v}$ and $d = 2$. Then, the conchoidal surface M_d is parametrized by

$$X_d(u, v) = \left(\frac{1}{\cos v} + 2 \right) (\cos u \cos v, \sin u \cos v, \sin v). \tag{3.12}$$

It is a Bonnet surface and shown as given in Figure 3.4.

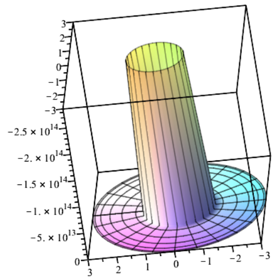

Figure 3.4: Conchoidal surface with $r(v) = \dfrac{1}{\cos v}$ and $d = 2$

Case 2: We assume that $r_v = 0$ and $r_u \neq 0$. In order to examine whether the surface M with the radius function $r(u, v_0)$ is a Bonnet surface, we will study this kind of surface to be the isothermic Weingarten surface.
Using (2.9) into (2.6), then we obtain as follows:

$$\frac{\partial^2}{\partial u \partial v}\left(log \frac{r^2}{r^2 \cos^2 v + r_u^2}\right) = \frac{2rr_u \sin 2v(r_u^2 - rr_{uu})}{(r^2 \cos^2 v + r_u^2)^2}. \tag{3.13}$$

For $\frac{\partial^2}{\partial u \partial v}\left(log \frac{g}{e}\right) = 0$, there exists $r_u^2 - rr_{uu} = 0$ from (3.13), that is, the surface M admitting $r_u^2 - rr_{uu} = 0$ is an isothermic surface. When we solve this differential equation, we find $r(u) = e^{c_1 u} c_2$, where c_1, c_2 are constants. Thus, the following theorem can be written.

Theorem 3.3. *The surface M with the radius function $r(u, v_0)$ is an isothermic surface if and only if it is parametrized by*

$$X(u, v) = e^{c_1 u} c_2(\cos u \cos v, \sin u \cos v, \sin v). \tag{3.14}$$

Let M_d be a conchoidal surface of M given with the parametrization (2.8). If the radius function $r(u, v)$ is a $u-$parameter function, then the coefficients of the first fundamental form of the surface M_d are

$$\begin{aligned} \widetilde{e} &= (r \pm d)^2 \cos^2 v + r_u^2, \\ \widetilde{f} &= 0, \\ \widetilde{g} &= (r \pm d)^2. \end{aligned} \tag{3.15}$$

Considering these coefficients for the conchoidal surface M_d of M with the radius function $r(u, v_0)$, we get

$$\frac{\partial^2}{\partial u \partial v}\left(log \frac{(r \pm d)^2}{(r \pm d)^2 \cos^2 v + r_u^2}\right) = \frac{2(r \pm d)r_u \sin 2v(r_u^2 - (r \pm d)r_{uu})}{((r \pm d)^2 \cos^2 v + r_u^2)^2}. \tag{3.16}$$

For $\frac{\partial^2}{\partial u \partial v}\left(log\frac{g}{e}\right) = 0$, there exists $r_u^2 - (r \pm d)r_{uu} = 0$ from (3.16), that is, the surface M admitting $r_u^2 - (r \pm d)r_{uu} = 0$ is an isothermic surface. Solving this differential equation, then we obtain $r(u) = e^{c_1 u}c_2 \mp d$, where c_1, c_2 are constants. Thus, the following theorem can be written.

Theorem 3.4. *The conchoidal surface M_d with the radius function $r(u, v_0)$ is an isothermic surface if and only if it is parametrized by*

$$X_d(u,v) = (e^{c_1 u}c_2 \mp d)(\cos u \cos v, \sin u \cos v, \sin v). \tag{3.17}$$

Secondly, we investigate the necessary condition for the surface M to be a Weingarten surface, namely $\frac{\partial K}{\partial u}\frac{\partial H}{\partial v} - \frac{\partial K}{\partial v}\frac{\partial H}{\partial u} = 0$. Differentiating (2.10), (2.11) and considering $r_v = 0$, then we get

$$\frac{\partial K}{\partial u}\frac{\partial H}{\partial v} - \frac{\partial K}{\partial v}\frac{\partial H}{\partial u} = \frac{2c_1^3 \sin v(-\cos^4 v + 2\cos^2 v + c_1^2)}{c_2^3 e^{3c_1 u}(\cos^2 v + c_1^2)^{7/2}}. \tag{3.18}$$

If (3.18) is equal to zero, then $(\cos^2 v - 1)^2 = c_1^2 + 1$. Thus, $\cos v$ is a constant and this contradicts with M, which is defined (3.14), being a surface. There is no surface M given by (3.14) that is a Weingarten surface and so, the surface M with the radius function $r(u, v_0)$ is not a Bonnet surface. When we examine the conchoidal surface M_d, we get similar results. There is no surface M_d given by (3.17) that is a Weingarten surface and so, the surface M_d with the radius function $r(u, v_0)$ is not a Bonnet surface.

Example 3.5. Let the radius function is given by $r(u) = 2e^u$ and $d = 1$. Then, the conchoidal surface M_d is parametrized by

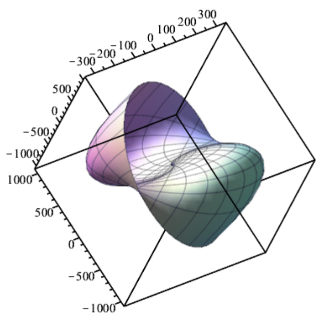

Figure 3.5: Conchoidal surface with $r(u) = 2e^u$ and $d = 1$

$$X_d(u,v) = (2e^u + 1)(\cos u \cos v, \sin u \cos v, \sin v). \tag{3.19}$$

It is the isothermic surface, however it is not the Weingarten surface. Thus, it is not a Bonnet surface and it is shown as given in Figure 3.5.

3.2. An examination of the condition of the surface of revolution given with conchoid curve to be a Bonnet surface in $\mathbb{E}^3$

In this section, we will examine condition which is the surface of revolution given with the rotating curve $c(t)$ and its the conchoid curve $c_d(t)$ to be a Bonnet surface.

(a) The surfaces of revolution of the constant mean curvature

Assume that M and M_d are the surfaces of revolution generated by the rotating curve $c(t)$ and its conchoid curve $c_d(t)$ parametrized by (2.14) and (2.18). It is possible that the mean curvature H given by (2.17) is equal to a non-zero constant when the radius function $r(t)$ is a constant. This means that the surfaces M and M_d are the spheres.

Example 3.6. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = 5$. Then, its parametrization is given by

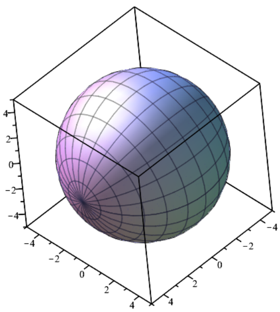

Figure 3.6: Surface of revolution with a constant radius function

$$X_d(t,s) = (5\cos t, 5\sin t\cos s, 5\sin t\sin s). \tag{3.20}$$

It denotes a sphere and it is shown as given in Figure 3.6.

Their mean curvatures are constants when the surfaces M and M_d are the minimal surfaces. According to that, considering [4], if the radius function is $r(t) = \dfrac{c}{\cos t}$, the surface M is a minimal and if the radius function is $r(t) = \pm d + \dfrac{c}{\cos t}$, the surface M_d is a minimal. So, the surfaces M and M_d are the surfaces of the type (i) which can be recognised by an infinite of isometries preserving the principal curvatures where M is determined by (2.14) with $r(t) = \dfrac{c}{\cos t}$ and M_d is determined by (2.18) with $r(t) = \pm d + \dfrac{c}{\cos t}$.

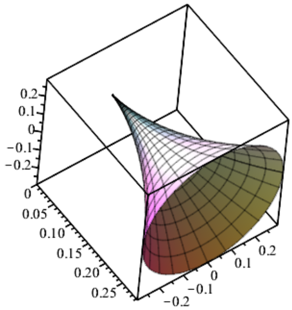

Figure 3.7: Surface of revolution with $c_d(t) = \left(\dfrac{1}{\cos t} - 1\right)(\cos t, \sin t)$

Example 3.7. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = \left(\dfrac{1}{\cos t} - 1\right)(\cos t, \sin t)$. Then, its parametrization is given by

$$X_d(t,s) = \left(\frac{1}{\cos t} - 1\right)(\cos t, \sin t \cos s, \sin t \sin s). \tag{3.21}$$

It is shown as given in Figure 3.7.

(b) The isothermic Weingarten surface of revolution of the non-constant mean curvature

According to (2.15), from $f = 0$, we see that the parameter curves of the surface M constitute the orthogonal system. Similarly, from $\widetilde{f} = 0$, the parameter curves of the surface of revolution M_d are the orthogonal system.

Firstly, we consider the surface providing the condition $\dfrac{\partial^2}{\partial t \partial s}\left(log\dfrac{g}{e}\right) = 0$ since every Bonnet surface is an isothermic surface. For the surface M, using (2.15), then we have $\dfrac{\partial^2}{\partial t \partial s}\left(log\dfrac{r^2 \sin^2 t}{r^2 + (r')^2.}\right) = 0.$

Then, we need to show the necessary condition for the surface of revolution M to be a Weingarten surface. From (2.5), (2.16) and (2.17), we find $\dfrac{\partial K}{\partial s} = 0$ and $\dfrac{\partial H}{\partial s} = 0$. So, the surface of revolution M is the isothermic Weingarten surface.

Using (2.17), we realize that the mean curvature of the surface M is a non-constant. Hence, the surface of revolution M generated by the rotating curve $c(t)$ with the non-constant mean curvature is the Bonnet surface since it is the isothermic Weingarten surface. Also, if we study the surface of revolution M_d generated by the conchoid curve $c_d(t)$ with the help of the above calculations, then we conclude that the surface M_d is the Bonnet surface.

Theorem 3.5. *The surface of revolution M parametrized by (2.14) and the surface of revolution M_d parametrized by (2.18) are the surfaces of the type (ii) which admit an infinite number of isometries. So, the surfaces of revolution M and M_d are the Bonnet surfaces.*

Corollary 3.2. *There is no surface of revolution given with the conchoid curve that permits a single non-trivial isometry with the non-constant mean curvature.*

Example 3.8. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = (2\sin t + 2)(\cos t, \sin t)$. Then, its parametrization is given by

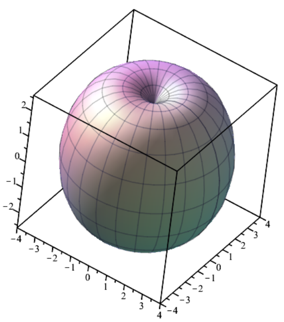

Figure 3.8: Surface of revolution with $c_d(t) = (2\sin t + 2)(\cos t, \sin t)$

$$X_d(t,s) = (2\sin t + 2)(\cos t, \sin t \cos s, \sin t \sin s). \tag{3.22}$$

It is shown as given in Figure 3.8 and it is a Bonnet surface.

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LIFTS OF GOLDEN STRUCTURES ON THE TANGENT BUNDLE

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Abstract. The present paper aims to study the complete lift of golden structure on tangent bundles. Integrability conditions for complete lift and third order tangent bundle are established.
Keywords: Golden structure, Complete lift, Nijenhuis tensor, Projection tensors, Tangent bundle.

1. Introduction

The lift of geometric objects on a differentiable manifold is an important tool in the study of differential geometry of tangent bundle. The study of polynomial structure on differentiable manifold was started by Goldberg and Yano in 1970 [4]. Omran et al [1] studied lifts of various structures such as almost product, almost par-contact, para-cantact structures on manifold and integrability conditions of these structures are established. Khan [8] studied complete and horizontal lifts of metallic structures and discussed the integrability of such structures. Several investigators studied lifts of geometric objects in [2, 3, 9, 5, 11, 12, 17]. This paper aims to study the lifts of a golden structure on the tangent bundle and prolongation of a golden structure in third-order tangent bundle.

Suppose M be n-dimensional differentiable manifold. A tensor field F of type (1,1) is said to be the golden structure on M if F satisfies the equation [8]

$$F^2 - F - I = 0, \tag{1.1}$$

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where I is the unit vector field on M and F is of constant rank r everywhere in M.

If g be a Riemannian metric on M such that

$$g(FX,Y) = g(X,FY), \tag{1.2}$$

for all X and Y are vector fields on M. Then a golden structure is said to be a golden Riemannian structure.

Let us introduce the operators l and m

$$\begin{aligned} (a) \quad & l = F^2 - F \\ (b) \quad & m = I - (F^2 - F) \end{aligned} \tag{1.3}$$

The following identities can be easily obtained:

$$\begin{aligned} l + m &= 0 \\ l^2 = l, \quad m^2 = m, \quad lm &= ml = 0 \\ Fl = lF = F, \quad Fm &= mF = 0. \end{aligned} \tag{1.4}$$

Let D_l and D_m of complementary distributions corresponding to the projection tensors l and m respectively in M. If the rank of F is r, then D_l is r-dimensional and D_m is $(n-r)$-dimensional, where $\dim M = r$.

2. The complete lift of a golden structure F on the tangent bundle $T(M)$

Let M be an n-dimensional differentiable manifold and TM its tangent bundle. The set of function, vector field, 1-form and tensor field of type (1,1) are represented by $\wp_0^0(M), \wp_0^1(M), \wp_1^0(M)$ and $\wp_1^1(M)$ respectively in M and $\wp_0^0(TM), \wp_0^1(TM), \wp_1^0(TM)$ and $\wp_1^1(TM)$ respectively in TM [5].

Let $F, G \in \wp_1^1(M)$. It is well known [19]

$$(FG)^C = F^C G^C. \tag{2.1}$$

Setting $F = G$ in above equation (2.1), then

$$(F^2)^C = (F^C)^2. \tag{2.2}$$

and

$$(F+G)^C = F^C + G^C. \tag{2.3}$$

Taking the complete lifts of both sides of the equation (1.1), then the obtained equation is

$$\begin{aligned} (F^2 - F - I)^C &= 0 \\ (F^2)^C - F^C - I^C &= 0 \end{aligned}$$

Using the equation (2.2) and $I^C = I$, then we have

$$(F^C)^2 - F^C - I = 0 \tag{2.4}$$

By using the equations (1.1), (2.4) and [19], we can easily say that the rank of F^C is $2r$ if and only if the rank of F is r. Therefore, the following theorems have been obtained:

Theorem 2.1. *Let $F \in \wp_1^1(M)$ be a golden structure in M, then its complete lift F^C is also a golden structure in TM.*

Theorem 2.2. *The golden structure F of rank r in M if and only if its complete lift F^C is of rank $2r$ in TM.*

Since F be a golden structure of rank r in M. Then the complete lift l^C of l and m^C of m are complementary projection tensors in TM. Thus, there exists two complementary distributions D_l^C and D_m^C determined by l^C and m^C respectively in TM [2].

3. Some theorems on integrability of golden structure on the tangent bundle

Let N be the Nijenhuis tensor of golden structure F in M and N^C be the Nijenhuis tensor of F^C in TM. Then we have [19]

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]. \tag{3.1}$$

and

$$\begin{aligned} N^C(X^C,Y^C) &= [F^CX^C,F^CY^C] - F^C[F^CX^C,Y^C] \\ &- F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C]. \end{aligned} \tag{3.2}$$

Let X and Y be vector fields and F tensor field of type (1,1) in M, then

$$\begin{aligned} [X^C,Y^C] &= [X,Y]^C \\ (X+Y)^C &= X^C + Y^C \\ F^CX^C &= (FX)^C. \end{aligned} \tag{3.3}$$

Using the equations (1.4) and (3.5), we have

$$\begin{aligned} F^Cl^C &= (Fl)^C = F^C \\ F^Cm^C &= (Fm)^C = 0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *The following identities hold:*

$$N^C(m^CX^C,m^CY^C) = (F^C)^C[m^CX^C,m^CY^C], \tag{3.5}$$

$$m^CN^C(X^C,Y^C) = m^C[F^CX^C,F^CY^C], \tag{3.6}$$

$$m^C(l^CX^C,l^CY^C) = m^C[F^CX^C,F^CY^C], \tag{3.7}$$

$$m^CN^C((F^2-\alpha F)^CX^C,(F^2-F)^CY^C) = m^CN^C(l^CX^C,l^CY^C). \tag{3.8}$$

Proof: The proof of the equations (3.5) to (3.8) follow by using the equations (1.4), (3.4) and (3.1).

Theorem 3.2. *Let X and Y be vector fields and F tensor field of type (1,1) in M, the following conditions are equivalent*

$$(a)\quad m^C N^C(X^C, Y^C) = 0$$
$$(b)\quad m^C N^C(l^C X^C, l^C Y^C) = 0$$
$$(c)\quad m^C N^C((F^2 - F)^C X^C, (F^2 - F)^C Y^C) = 0.$$

Proof: Making use of the equation (3.8), we get

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2 - F)^C X^C, (F^2 - F)^C Y^C) = 0$$

Since the right sides of the the equations (3.6), (3.7) are equal and using the last equation which shows that conditions (a), (b), and (c) are equivalent.

Theorem 3.3. *The complete lift D_m^C in TM of a distribution D_m in M is integral if D_m is integrable in M.*

Proof: The distribution D_m is integral if and only if [19]

$$l[mX, mY] = 0 \tag{3.9}$$

for all $X, Y \in \wp_0^1(M)$, where $l = I - m$.

Taking complete lift of both sides and using (3.5), we have

$$l^C[m^C X^C, m^C Y^C] = 0 \tag{3.10}$$

for all $X, Y \in \wp(M)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to m^C. Thus the condition (3.9) implies (3.10).

Theorem 3.4. *The complete lift D_m^C in TM of a distribution D_m in M is integral if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$, for all $X, Y \in \wp(M)$.*

Proof: The distribution D_m is integral in M if and only if [19]

$$N(mX, mY) = 0$$

for all $X, Y \in \wp(M)$. By virtue of condition (3.5), we have

$$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C)$$

Using the equation (3.10), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.11}$$

and

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.12}$$

Adding the equations (3.11) and (3.12), we have

$$(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$$

Since $l^C + m^C = I^C = I$, we get

$$N^C(m^C X^C, m^C Y^C) = 0.$$

Theorem 3.5. *Let the distribution D_l be integrable in M, that is $mN(X,Y) = 0$ for all $X, Y \in \wp_0^1(M)$. Then the distribution D_l^C is integrable in TM if and only if the one of the conditions of Theorem (3.2) is satisfied.*

Proof: The distribution D_l is integral in M if and only if

$$mN(lX, lY) = 0$$

Thus distribution D_l^C is integrable in TM if and only if

$$m^C N^C(l^C X^C, l^C Y^C) = 0,$$

Hence the theorem follows by using of the equation (3.8).

Theorem 3.6. *Let complete lift F^C of a golden structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: The golden structure F in M is partially integrable if and only if

$$N(lX, lY) = 0, \forall X, Y \in \wp_0^1(M). \tag{3.13}$$

Using the equations (1.4) and (3.1), we have

$$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0$$

and from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0.$$

Theorem 3.7. *The complete lift F^C of a golden structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: A necessary and sufficient condition for a golden structure in M to be integrable is that

$$(N(X,Y)) = 0 \tag{3.14}$$

for all $X, Y \in \wp_0^1(M)$.

Using the equation (3.1), we get

$$N^C(X^C, Y^C) = (N(X,Y))^C.$$

Therefore, using the equation (3.14) we obtain the result.

4. Prolongation of a golden structure in third-order tangent bundle T_3M

Let M be n-dimensional differentiable manifold and T_3M its third order tangent bundle over M. Let F^{III} be the third lift on F in T_3M. If X be vector field and F, G be tensor field of type (1,1), then

$$\begin{aligned}(G^{III}F^{III})X^{III} &= (G^{III}(F^{III}X^{III}) \\ &= (G^{III}(FX)^{III}) \\ &= (G(FX))^{III} \\ &= (GF)^{III}X^{III}\end{aligned} \tag{4.1}$$

Thus,

$$G^{III}F^{III} = (GF)^{III}$$

Theorem 4.1. *Let $F \in \wp_1^1(M)$ be a golden structure in M, then the third lift F^{III} is also a golden structure in T_3M.*

Proof: If $P(t)$ is a polynomial in one variable t, then we get [19]

$$(P(F))^{III} = P(F^{III}) \tag{4.2}$$

for all $F \in \wp_1^1(M)$.

Taking the third lifts of both sides of the equation (1.1), we get

$$(F^2 - F - I)^{III} = 0$$
$$(F^2)^{III} - F^{III} - I^{III} = 0$$

Using the equation (4.2) and $I^{III} = I$, we have

$$(F^{III})^2 - F^{III} - I = 0 \tag{4.3}$$

which shows that F^{III} is a golden structure in T_3M.

Theorem 4.2. *The third lift F^{III} is integrable in T_3M if and only if F is integrable in M.*

Proof: Let N^{III} and N be Nijenhuis tensors of F^{III} and F respectively. Then we have

$$N^{III}(X,Y) = (N(X,Y))^{III}. \tag{4.4}$$

since golden structure is integrable in M if and only if $N(X,Y) = 0$. then from (4.4), we get

$$N^{III}(X,Y) = 0. \tag{4.5}$$

Thus F^{III} is integrable if and only if F is integrable in M.

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Original Scientific Paper

GEOMETRIC INEQUALITIES FOR DOUBLY WARPED PRODUCTS POINTWISE BI-SLANT SUBMANIFOLDS IN CONFORMAL SASAKIAN SPACE FORM

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Abstract. In this paper, we have established some geometric inequalities for the squared mean curvature in terms of warping functions of a doubly warped product pointwise bi-slant submanifold of a conformal Sasakian space form with a quarter symmetric metric connection. The equality cases havve also been considered. Moreover, some applications of obtained results are derived.
Keywords: doubly warped products, bi-slant submanifolds, quarter symmteric metric connection,conformal Sasakian space form.

1. Introduction

In 2000, B. Unal [17] introduced the notion of doubly warped products as a generalization of warped products and it states that: let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively. Further, let us suppose that f_1 & f_2 are positive differentiable functions on N_1 and N_2 respectively. Then, the doubly warped product $N = {}_{f_2}N_1 \times_{f_1} N_2$ is defined as the product manifold $N_1 \times N_2$ equipped with the warped metric $g = f_2^2 g_1 + f_1^2 g_2$. In a meticulous manner, if $t_1 : N_1 \times N_2 \to N_1$ and $t_2 : N_1 \times N_2 \to N_2$ are natural projections, then the metric g is given by [17]

$$g(X,Y) = (f_2 \circ t_2)^2 \, g_1(\iota_1^* X, \iota_1^* Y) + (f_1 \circ t_1)^2 \, g_2(t_2^* X, t_2^* Y), \tag{1.1}$$

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for any vector fields X, Y on N, where $*$ denotes the symbol for tangent maps.

It is important to note that on a doubly warped product manifold $N = {}_{f_2}N_1 \times_{f_1} N_2$ if either f_1 or f_2 is constant on N but not both, then N is a single warped product. Furthermore, if both f_1 and f_2 are constant function on N, then N is locally a Riemannian product. A doubly warped product manifold is said to be proper if both f_1 and f_2 are non-constant functions on N.

On the other hand, the immersibility/non-immersibility of a Riemannian manifold in a space form is one of the most fundamental problems in the theory of submanifold which started with the most celebrated Nash embedding theorem [11]. In this theorem, actually Nash was aiming to take extrinsic help. However, due to the lack of control of the extrinsic properties of the submanifolds by the known intrinsic invariant, the aim cannot be reached. Motivated by this and to overcome the difficulties, Chen introduced new types of Riemannian invariants and established general optimal relationship between extrinsic invariants and intrinsic invariants on the submanifold. Motivated by Chen's result, several inequalities have been obtained by many geometers for warped products and doubly warped products in different setting of the ambient manifolds [4, 5, 8, 9, 10, 12, 13, 15, 16, 19, 20]. In this paper, we have studied doubly warped product pointwise bi-slant submanifolds isometrically immersed into a conformal Sasakian space form with a quarter symmetric metric connection. The inequalities which we shall obtain in this paper are very fascinating because we derive upper bound and lower bound for warping functions in terms of mean curvature, scalar curvature and pointwise constant φ-sectional curvature c. The obtained results generalize some other inequalities as special cases.

2. Preliminaries

Let $\tilde{N}$ be a Riemannian manifold with Riemannian metric g. A linear connection $\bar{\nabla}$ on $\tilde{N}$ is called a quarter-symmetric connection if its torsion tensor T given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

and satisfies

$$T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y,$$

where π is a 1-form and $\mathcal{V}$ is a vector field such that $\pi(X) = g(X, \mathcal{V})$ and φ is a (1,1) tensor field. If $\bar{\nabla} g = 0$, then $\bar{\nabla}$ is known as quarter-symmetric metric connection and $\bar{\nabla} g \neq 0$, then $\bar{\nabla}$ is known as quarter symmetric non-metric connection. In this setting, it is shown in [14], one can easily obtain a special quarter-symmetric connection defined as

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \lambda_1 \pi(Y) X - \lambda_2 g(X,Y)\mathcal{V}. \tag{2.1}$$

This is a general class of connection in the sense of (2.1) can be obtained as:

1. when $\lambda_1 = \lambda_2 = 1$, then the above connection reduces to semi-symmetric metric connection.

2. when $\lambda_1 = 1$ and $\lambda_2 = 0$, then the above connection reduces to semi-symmetric non metric connection.

The curvature tensor with respect to $\bar{\nabla}$ is given by

$$\bar{\mathcal{R}}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{2.2}$$

Similarly, we can define the curvature tensor with respect to $\tilde{\bar{\nabla}}$. Now, using (2.1), the curvature tensor takes the following form [18]

$$\begin{aligned}\bar{\mathcal{R}}(X,Y,Z,W) &= \tilde{\bar{\mathcal{R}}}(X,Y,Z,W) + \lambda_1\alpha(X,Z)g(Y,W) - \lambda_1\alpha(Y,Z)g(X,W)\\ &\quad +\lambda_2\alpha(Y,W)g(X,Z) - \lambda_2\alpha(X,W)g(Y,Z)\\ &\quad +\lambda_2(\lambda_1-\lambda_2)g(X,Z)\beta(Y,W) - \lambda_2(\lambda_1-\lambda_2)g(Y,Z)\beta(X,W).\end{aligned} \tag{2.3}$$

where

$$\alpha(X,Y) = (\tilde{\bar{\nabla}}_X\pi)(Y) - \lambda_1\pi(X)\pi(Y) + \frac{\lambda_2}{2}g(X,Y)\pi(\mathcal{V})$$

and

$$\beta(X,Y) = \frac{\pi(\mathcal{V})}{2}g(X,Y) + \pi(X)\pi(Y)$$

are $(0,2)$ tensors.

For simplicity, we denote by $tr(\alpha) = a$ and $tr(\beta) = b$.

Let N be an m-dimensional submanifold of a Riemannian manifold $\tilde{N}$ and ∇, $\tilde{\nabla}$ be the induced quarter symmetric-metric connection and Levi-Civita connection of N, respectively. Then the corresponding Gauss formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad X,Y \in \Gamma(TN), \tag{2.4}$$

$$\tilde{\bar{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{\sigma}(X,Y), \quad X,Y \in \Gamma(TN), \tag{2.5}$$

where $\tilde{\sigma}$ is the second fundamental form given by $\sigma(X,Y) = \tilde{\sigma}(X,Y) - \lambda_2 g(X,Y)\mathcal{V}^{\perp}$.

Furthermore, the equation of Gauss is given by [18]:

$$\begin{aligned}\bar{\mathcal{R}}(X,Y,Z,W) &= \mathcal{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(Y,W),\sigma(X,Z))\\ &\quad +(\lambda_1-\lambda_2)g(\sigma(Y,Z),\mathcal{V}^{\perp})g(X,W)\\ &\quad +(\lambda_2-\lambda_1)g(\sigma(X,Z),\mathcal{V}^{\perp})g(Y,W).\end{aligned} \tag{2.6}$$

Now, let $\tilde{N}$ be a (2n+1) odd-dimensional Riemannian manifold. Then $\tilde{N}$ is said to be an almost contact metric manifold with structure (φ, ξ, η, g) if there exist a tensor φ of type $(1,1)$, a vector field ξ (structure vector field) and a 1-form η satisfying [3]

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X),$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.7}$$

for any X,Y on $\tilde{N}$. The 2-form Φ is called the fundamental 2-form in $\tilde{N}$ and the manifold is said to be a contact metric manifold if $\Phi = d\eta$. A Sasakian manifold is a normal contact metric manifold. In fact, an almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

A $(2n+1)$-dimensional Riemannian manifold $\tilde{N}$ endowed with the almost contact metric structure (φ, η, ξ, g) is called a conformal Sasakian manifold if for a C^∞ function $f : \tilde{N} \to \mathbb{R}$, there are [1]

$$\tilde{g} = exp(f)g, \tilde{\varphi} = \varphi, \tilde{\eta} = (exp(f))^{\frac{1}{2}}\eta, \tilde{\xi} = (exp(-f))^{\frac{1}{2}}\xi \tag{2.8}$$

is a Sasakian structure on $\tilde{N}$. Using Koszul formula, we derive the following relation between the connections $\tilde{\nabla}$ and ∇

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^{\#}\}, \tag{2.9}$$

for all vector fields X, Y on $\tilde{N}$, where $\omega(X) = X(f)$ and $g(\omega^{\#}, X) = \omega(X)$.

An almost contact metric manifold $(\tilde{N}, \varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\begin{aligned} g(\tilde{\tilde{\mathcal{R}}}(X,Y)Z,W) \;=\; & exp(f)\Big\{\frac{c+3}{4}\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ & + g(X,Z)g(\xi,W)\eta(Y) - g(Y,Z)g(\xi,W)\eta(X) \\ & - g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ & - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} - \frac{1}{2}\big(B(X,Z)g(Y,W) \\ & - B(Y,Z)g(X,W) + B(Y,W)g(Y,Z) - B(X,W)g(Y,Z)\big) \\ & - \frac{1}{4}||\omega^{\#}||^2\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big), \end{aligned} \tag{2.10}$$

for any vector fields X, Y, Z, W tangent to $\tilde{N}$, where $B = \nabla\omega - \frac{1}{2}\omega \otimes \omega$, is said to be a conformal Sasakian space form [1].

From (2.1) and (2.10), we get

$$\begin{aligned} g(\bar{\mathcal{R}}(X,Y)Z,W) &= exp(f)\Big\{\frac{c+3}{4}\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big) \\ &+\frac{c-1}{4}\big(\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &+g(X,Z)g(\xi,W)\eta(Y) - g(Y,Z)g(\xi,W)\eta(X) \\ &-g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ &-2g(\varphi X,Y)g(\varphi Z,W)\Big\} - \frac{1}{2}\big(B(X,Z)g(Y,W) \\ &-B(Y,Z)g(X,W) + B(Y,W)g(Y,Z) - B(X,W)g(Y,Z)\big) \\ &-\frac{1}{4}||\omega^{\#}||^2\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big) \\ &+\lambda_1\alpha(X,Z)g(Y,W) - \lambda_1\alpha(Y,Z)g(X,W) \\ &+\lambda_2 g(X,Z)\alpha(Y,W) - \lambda_2 g(Y,Z)\alpha(X,W) \\ &+\lambda_2(\lambda_1-\lambda_2)g(X,Z)\beta(Y,W) - \lambda_2(\lambda_1-\lambda_2)g(Y,Z)\beta(X,W). \end{aligned} \tag{2.11}$$

The squared norm of T at $p \in N$ is given by

$$||T||^2 = \sum_{i,j=1}^{m} g^2(Je_i, e_j), \tag{2.12}$$

where $\{e_1, \cdots, e_m\}$ is any orthonormal basis of the tangent space TN of N.

It was proved in [6] that a submanifold N of an almost Hermitian manifold $(\tilde{N}, J, g)$ is pointwise slant if and only if

$$T^2 = -\cos^2\theta(p)I, \quad \forall\, p \in N, \tag{2.13}$$

for some real-valued function $\theta(p)$ on N. A pointwise slant submanifold is *proper* if it contains neither totally real points nor complex points.

Clearly, it is easy to check that

$$g(TX, TY) = \cos^2\theta(p)g(X, Y), \tag{2.14}$$

$$g(FX, FY) = \sin^2\theta(p)g(X, Y), \tag{2.15}$$

for any $X, Y \in \Gamma(TN)$.

The following definition is given by Chen and Uddin in [8]:
A submanifold N of dimension m of an almost Hermitian manifold $\tilde{N}^{4n}$ is said to be a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$, such that

(i) $TN = \mathfrak{D}_1 \oplus \mathfrak{D}_2$,

(ii) $J\mathfrak{D}_1 \perp \mathfrak{D}_2$ and $J\mathfrak{D}_2 \perp \mathfrak{D}_1$,

(iii) Each distribution $\mathfrak{D}_i$ is pointwise slant with slant function $\theta_i : TN - \{0\} \to \mathbb{R}$ for $i = 1, 2$.

In fact, pointwise bi-slant submanifold are more general class of submanifolds and bi-slant, pointwise semi-slant, semi-slant and CR-submanifolds are the particular cases of these submanifolds.

Since N is a pointwise bi-slant submanifold, we defined an adapted orthonormal frame as $m = 2d_1 + 2d_2$ follows

$$\{e_1, e_2 = \sec\theta_1 Te_1, \cdots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 Te_{2d_1-1},$$
$$\cdots, e_{2d_1+1}, e_{2d_1+2} = \sec\theta_2 Te_{2d_1+1}, \cdots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 Te_{2d_1+2d_2-1}\}.$$

Thus, we defined it such that $g(e_1, Je_2) = -g(Je_1, e_2) = -g(Je_1, \sec\theta_1 Te_1)$, which implies that $g(e_1, Je_2) = -\sec\theta_1 g(Te_1, Te_2)$.

Following (2.14), we get $g(e_1, Je_2) = cos\theta_1 g(e_1, e_2)$. Therefore, we easily obtained the following relation

$$||T||^2 = \sum_{i,j=1}^{m} g^2(e_i, Je_j) = (m_1\cos^2\theta_1 + m_2\cos^2\theta_2), \tag{2.16}$$

where $m_1 = \dim\mathfrak{D}_1$ and $m_2 = \dim\mathfrak{D}_2$.

Let $\varphi : N = {}_{f_2}N_1 \times_{f_1} N_2 \to \tilde{N}$ be isometric immersion of a doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian manifold of $\tilde{N}$ of constant sectional curvature c. Suppose that m_1, m_2 and m be the dimensions of N_1, N_2 and $N_1 \times_f N_2$, respectively. Then for unit vector fields X and Z tangent to N_1 and N_2 respectively, we have

$$\begin{aligned} \kappa(X \wedge Z) &= g\left(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z\right) \\ &= \frac{1}{f_1}\left\{\left(\nabla^1_X X\right) f_1 - X^2 f_1\right\} + \frac{1}{f_2}\left\{\left(\nabla^2_Z Z\right) f_2 - Z^2 f_2\right\}. \end{aligned} \tag{2.17}$$

If we consider the local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ such that $\{e_1, e_2, \cdots, e_{m_1}\}$ tangent to N_1 and $\{e_{m_1+1}, \cdots, e_m\}$ are tangent to N_2, then the sectional curvatre in terms of doubly warped product is defined by

$$\sum_{1\le i\le m_1}\sum_{m_1+1\le j\le m}\kappa\left(e_i \wedge e_j\right) = \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2}, \tag{2.18}$$

for each $j = m_1 + 1, \cdots, m$.

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\tilde{N}^n$ and denoted by $\tilde{\tau}(T_p\tilde{N}^n)$, which at some p in $\tilde{N}^n$ is given as :

$$\tilde{\tau}(T_p\tilde{N}^n) = \sum_{1\le i<j\le n}\tilde{\kappa}_{ij}, \tag{2.19}$$

where $\tilde{\kappa}_{ij} = \tilde{\kappa}(e_i \wedge e_j)$. It is clear that first equality (2.19) is congruent to the following equation, which will be frequently used in the subsequent proof:

$$2\tilde{\tau}(T_p\tilde{N}^n) = \sum_{1\leq i\neq j\leq n} \tilde{\kappa}_{ij}. \tag{2.20}$$

Similarly, scalar curvature $\tilde{\tau}(L_p)$ of L-plane is given by

$$\tilde{\tau}(L_p) = \sum_{1\leq i<j\leq n} \tilde{\kappa}_{ij}. \tag{2.21}$$

An orthonormal basis of the tangent space T_pN is $\{e_1, \cdots, e_m\}$ such that $e_r = \{e_{m+1}, \cdots, e_{2n+1}\}$ belongs to the normal space $T^{\perp}N$. Then, we have

$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r), \ \ ||\sigma||^2 = \sum_{i,j=1}^m g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m (\sigma^r_{ij})^2,$$
$$||H||^2 = \frac{1}{m^2}\sum_{i=1}^m g(\sigma(e_i, e_i), \sigma(e_i, e_i)),$$

where $\|H\|^2$ is the squared norm of the mean curvature vector H of N.

Let κ_{ij} and $\tilde{\kappa}_{ij}$ be the sectional curvature of the plane section spanned by e_i and e_j at p in a submanifold N^m and a Riemannian manifold $\tilde{N}^n$ respectively. Thus, κ_{ij} and $\tilde{\kappa}_{ij}$ are the intrinsic and the extrinsic sectional curvatures of the span $\{e_i, e_j\}$ at p. Thus from the Gauss Equation, we have

$$\begin{aligned} 2\tau(T_pN^m) &= \kappa_{ij} = 2\tilde{\tau}(T_pN^m) - \sum_{i,j=1}^m \{(\lambda_1 - \lambda_2)g(\sigma(e_j, e_j), \mathcal{Q}^{\perp})g(e_i, e_i) \\ &+ (\lambda_2 - \lambda_1)g(\sigma(e_i, e_j), \mathcal{Q}^{\perp})g(e_j, e_i)\} + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m \left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right) \\ &= \tilde{\kappa}_{ij} - \sum_{i,j=1}^m \{(\lambda_1 - \lambda_2)g(\sigma(e_j, e_j), \mathcal{Q}^{\perp})g(e_i, e_i) \\ &+ (\lambda_2 - \lambda_1)g(\sigma(e_i, e_j), \mathcal{Q}^{\perp})g(e_j, e_i)\} + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m \left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right). \end{aligned} \tag{2.22}$$

The following consequences come from Gauss equation and (2.22)

$$\begin{aligned} \tau(T_pN_1{}^{m_1}) &= \tilde{\tau}(T_pN_1^{m_1}) - \sum_{1\leq j<k\leq m_1} \{(\lambda_1 - \lambda_2)g(\sigma(e_j, e_j), \mathcal{Q}^{\perp})g(e_k, e_k) \\ &+ (\lambda_2 - \lambda_1)g(\sigma(e_j, e_k), \mathcal{Q}^{\perp})g(e_k, e_j)\} + \sum_{r=m+1}^{2n+1}\sum_{1\leq j<k\leq m_1} \left(\sigma^r_{jj}\sigma^r_{kk} - (\sigma^r_{jk})^2\right), \\ \tau(T_pN_2{}^{m_2}) &= \tilde{\tau}(T_pN_2^{m_2}) - \sum_{m_1+1\leq s<t\leq m} \{(\lambda_1 - \lambda_2)g(\sigma(e_t, e_t), \mathcal{Q}^{\perp})g(e_s, e_s) \end{aligned}$$

$$+ \quad (\lambda_2 - \lambda_1) g(\sigma(e_s, e_t), \mathcal{Q}^{\perp}) g(e_t, e_s)\big\} + \sum_{r=m+1}^{2n+1} \sum_{m_1+1 \leq s < t \leq m} m\big(\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2\big). \tag{2.23}$$

3. Main Inequalities

First, we recall the following result of B.-Y. Chen for later use.

Lemma 3.1. *[7] Let $m \geq 2$ and $a_1, \cdots, a_m, b$ be (m+1) real numbers such that*

$$\left(\sum_{i=1}^{m} a_i\right)^2 = (m-1)\left(\sum_{i=1}^{m} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_m$.

Now, we prove the following main result of this section.

Theorem 3.1. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

(i) The relation between warping functions and the squared norm of mean curvature is given by

$$\begin{aligned} \frac{m_2 \Delta_1 f_1}{f_1} + \frac{m_1 \Delta_2 f_2}{f_2} \quad &\leq \quad \frac{m^2}{4}||H||^2 + exp(f)\bigg\{\frac{(c+3)}{4} m_1 m_2 + \frac{1}{2} trB + \frac{1}{4} m_1 m_2 ||\omega^*||^2 \\ &- \quad \frac{(c-1)}{8}[2 + 3m_1 cos^2\theta_1 + 3m_2 cos^2\theta_2]\bigg\} \\ &- \quad \frac{1}{2}\big\{(\lambda_1 + \lambda_2)a + \lambda_2(\lambda_1 - \lambda_2)b + 2m_1 m_2(\lambda_1 - \lambda_2)\pi(H)\big\}, \end{aligned} \tag{3.1}$$

where ∇ and Δ are the gradient and the laplacian operators, repectively and H is the mean curvature vector of N^m.

(ii) The equality case holds in (3.1) if and only if φ is a mixed totally geodesic isometric immersion and the following satisfies $m_1 H_1 = m_2 H_2$, where H_1 and H_2 are partial mean curvature vectors of H along $N_1^{m_1}$ and $N_2^{m_2}$, respectively and $\pi(H) = \frac{1}{m}\sum_{i=1}^{m} \pi(\sigma(e_i, e_j)) = g(\mathcal{Q}, H)$.

Proof. *let $\{e_1, ..., e_m\}$ and $\{e_{m+1}, ..., e_{2n+1}\}$ as orthonormal tangent frame and orthonormal normal frame on N, respectively. Putting $X = W = e_i$, $Y = Z = e_j$,*

$i \neq j$ in (2.21) and using(2.6), we obtain

$$
\begin{aligned}
g(\bar{\mathcal{R}}(e_i, e_j, e_j, e_i) &= exp(f)\Big\{\frac{c+3}{4}\big(g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\big) \\
&+ \frac{c-1}{4}\big(\eta(e_i)\eta(e_j)g(e_j, e_i) - \eta(e_j)\eta(e_j)g(e_i, e_i)\big) \\
&+ g(e_i, e_j)g(\xi, e_i)\eta(e_j) - g(e_j, e_j)g(\xi, e_i)\eta(e_i) \\
&- g(\varphi e_j, e_j)g(\varphi e_i, e_i) - g(\varphi e_i, e_j)g(\varphi e_j, e_i) \\
&- 2g(\varphi e_i, e_j)g(\varphi e_j, e_i)\Big\} - \frac{1}{2}\big(B(e_i, e_j)g(e_j, e_i) \\
&- B(e_j, e_j)g(e_i, e_i) + B(e_j, e_i)g(e_i, e_j) - B(e_i, e_i)g(e_j, e_j)\big) \\
&- \frac{1}{4}||\omega^{\#}||^2\big(g(e_i, e_j)g(e_j, e_i) - g(e_j, e_j)g(e_i, e_i)\big) \\
&+ \Lambda_1\alpha(e_i, e_j)g(e_j, e_i) - \Lambda_1\alpha(e_j, e_j)g(e_i, e_i) \\
&+ \lambda_2 g(e_i, e_j)\alpha(e_j, e_i) - \lambda_2 g(e_j, e_j)\alpha(e_i, e_i) \\
&+ \lambda_2(\lambda_1 - \lambda_2)g(e_i, e_j)\beta(e_j, e_i) - \lambda_2(\lambda_1 - \lambda_2)g(e_j, e_j)\beta(e_i, e_i), \\
&- (\lambda_1 - \lambda_2)g(h(e_j, e_j), \mathcal{P}^{\perp})g(e_i, e_i) \\
&- (\lambda_2 - \lambda_1)g(h(e_i, e_j), \mathcal{P}^{\perp})g(e_j, e_i)
\end{aligned}
\tag{3.2}
$$

By taking summation $1 \leq i, j \leq m$ and using Gauss equation with (3.2), we have

$$
\begin{aligned}
2\tau &= exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2 - 2m + 3||P||^2\Big\} + (m-1)trB \\
&+ \frac{1}{4}m(m-1)||\omega^{\#}||^2 + (\lambda_1 + \lambda_2)(1-m)a + \lambda_2(\lambda_1 - \lambda_2)(1-m)b \\
&+ (\lambda_2 - \lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2 \\
&= exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2 - 2m + 3m_1cos^2\theta_1 + 3m_2cos^2\theta_2\big) \\
&+ (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} + (\lambda_1 + \lambda_2)(1-m)a \\
&+ \lambda_2(\lambda_1 - \lambda_2)(1-m)b + (\lambda_2 - \lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2,
\end{aligned}
\tag{3.3}
$$

where

$$
||P||^2 = \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j) \quad \text{and} \quad \pi(\mathcal{H}) = \frac{1}{m}\sum_{j=1}^{m}\pi(h(e_j, e_j)) = g(\mathcal{V}^{\perp}, \mathcal{H}).
$$

Let us assume that

$$
\begin{aligned}
\delta &= 2\tau - \Big\{exp(f)\big\{\frac{(c+3)}{8}m_1(m_1-1) + \frac{(c-1)}{8}(2 - 2m_1) + \frac{(c-1)}{4}3m_1cos^2\theta_1 \\
&+ \frac{1}{2}(m_1 - 1)trB + \frac{1}{8}m_1(m_1-1)||\omega^{\#}||^2\big\}
\end{aligned}
$$

$$+ \frac{(c+3)}{8} m_2(m_2-1) + \frac{(c-1)}{8}(2-2m_2) + \frac{(c-1)}{4} 3m_2 cos^2\theta_2$$
$$+ \frac{1}{2}(m_2-1)trB + \frac{1}{8}m_2(m_2-1)||\omega^{\#}||^2\Big\} + (\lambda_1+\lambda_2)(1-m)a$$
$$+ \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H})\Big\} - \frac{n^2}{2}||H||^2. \quad (3.4)$$

Then, from (3.3) and (3.4), we have

$$m^2||H||^2 = 2(\delta + ||\sigma||^2). \quad (3.5)$$

Thus, the orthonormal frame $\{e_1, \cdots, e_m\}$ *the proceeding equation takes the following form*

$$\left(\sum_{i=1}^{m} \sigma_{ii}^{m+1}\right)^2 = 2\Big\{\delta + \sum_{i=1}^{m}(\sigma_{ii}^{m+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{m+1})^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^{r})^2\Big\}. \quad (3.6)$$

By using the algebraic Lemma 3.1 and relation (3.6), we have

$$2\sigma_{11}^{m+1}\sigma_{22}^{m+1} \geq \sum_{i\neq j}(\sigma_{ij}^{m+1})^2 + \sum_{i,j=1}^{m}\sum_{r=m+2}^{2n+1}(\sigma_{ij}^{r})^2 + \delta. \quad (3.7)$$

If we substitute $a_1 = \sigma_{11}^{m+1}$, $a_2 = \sum_{i=2}^{m_1}\sigma_{ii}^{m+1}$ *and* $a_3 = \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}$ *in the above equation (3.6), we have*

$$\begin{aligned}\left(\sum_{i=1}^{m} a_i\right)^2 &= 2\Big\{\delta + \sum_{i=1}^{m} a_i^2 + \sum_{i\neq j\leq m}(\sigma_{ij}^{m+1})^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^{r})^2 \\ &- \sum_{2\leq j\neq k_\leq m_1}\sigma_{jj}^{m+1}\sigma_{kk}^{m+1} - \sum_{m_1+1\leq s\neq t_\leq m}\sigma_{ss}^{m+1}\sigma_{tt}^{m+1}\Big\}. \quad (3.8)\end{aligned}$$

Thus a_1, a_2, a_3 *satisfy the Chen's Lemma (for* $m = 3$*), that is*

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

Then $2a_1a_2 \geq b$*, with equality holding if and only if* $a_1 + a_2 = a_3$*. In the case of under considering, this means that*

$$\begin{aligned}\sum_{1\leq j<k_\leq m_1}\sigma_{jj}^{m+1}\sigma_{kk}^{m+1} + \sum_{m_1+1\leq s<t_\leq m}\sigma_{ss}^{m+1}\sigma_{tt}^{m+1} &\geq \tfrac{\delta}{2} + \textstyle\sum_{1\leq\alpha_3<\beta_3\leq m}(\sigma_{\alpha_3\beta_3}^{m+1})^2 \\ &+ \textstyle\sum_{r=m+1}^{2n+1}\sum_{\alpha_3\beta_3=1}^{m}(\sigma_{\alpha_3\beta_3}^{r})^2. \quad (3.9)\end{aligned}$$

Equality holds if and only if

$$\sum_{i=1}^{m_1}\sigma_{ii}^{m+1} = \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}. \quad (3.10)$$

Again, using Gauss equation, we derive

$$m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} = \tau-\sum_{1\le j<k\le m_1}\kappa(e_j\wedge e_k)-\sum_{m_1+1\le s<t\le m}\kappa(e_s\wedge e_t). \tag{3.11}$$

Then, the scalar curvature for the conformal Sasakian space form with quarter-symmetric connection from (2.22), we get

$$\begin{aligned} m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} &= \tau-exp(f)\Big\{\frac{(c+3)}{8}m_1(m_1-1)+\frac{(c-1)}{8}(2-2m_1)\\ &+ \frac{(c-1)}{4}3m_1cos^2\theta_1+\frac{1}{2}(m_1-1)trB+\frac{1}{8}m_1(m_1-1)||\omega^{\#}||^2\Big\}\\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-m_1)a+\lambda_2(\lambda_1-\lambda_2)(1-m_1)b\\ &+ (\lambda_2-\lambda_1)m_1(m_1-1)\pi(H)\big\}-\sum_{r=m+1}^{2n+1}\sum_{m_1+1\le j<k\le m}\left(\sigma^r_{jj}\sigma^r_{kk}-(\sigma^r_{jk})^2\right)\\ &- exp(f)\Big\{\frac{(c+3)}{8}m_2(m_2-1)+\frac{(c-1)}{8}(2-2m_2)\\ &+ \frac{(c-1)}{4}3m_2cos^2\theta_2+\frac{1}{2}(m_2-1)trB+\frac{1}{8}m_2(m_2-1)||\omega^{\#}||^2\Big\}\\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-m_2)a+\lambda_2(\lambda_1-\lambda_2)(1-m_2)b\\ &+ (\lambda_2-\lambda_1)m_2(m_2-1)\pi(H)\big\}-\sum_{r=m+1}^{2n+1}\sum_{m_1+1\le s<t\le m}\left(\sigma^r_{ss}\sigma^r_{tt}-(\sigma^r_{st})^2\right). \end{aligned} \tag{3.12}$$

Now making use of (3.9) and (3.12), we have

$$\begin{aligned} m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} &\le \tau-exp(f)\Big\{\frac{(c+3)}{8}[m(m-1)-2m_1m_2]+\frac{(c-1)}{8}(4-2m)\\ &+ \frac{1}{2}(m-2)trB+\frac{1}{8}[m(m-1)-2m_1m_2]||\omega^{\#}||^2\\ &+ \frac{(c-1)}{4}[3m_1cos^2\theta_1+3m_2cos^2\theta_2]\Big\}\\ &+ \frac{1}{2}\big\{(\lambda_1+\lambda_2)(2-m)a+\lambda_2(\lambda_1-\lambda_2)(2-m)b\\ &+ (\lambda_2-\lambda_1)[m(m-1)-2m_1m_2]\pi(H)\big\}-\frac{\delta}{2}. \end{aligned} \tag{3.13}$$

Using (3.4) in the above equation, we obtain

$$\frac{m_2\Delta_1 f_1}{f_1}+\frac{m_1\Delta_2 f_2}{f_2} \le \frac{m^2}{4}||H||^2+exp(f)\Big\{\frac{(c+3)}{4}m_1m_2+\frac{1}{2}trB+\frac{1}{4}m_1m_2||\omega^*||^2$$

$$
\begin{aligned}
& - \frac{(c-1)}{8}[2+3m_1cos^2\theta_1+3m_2cos^2\theta_2]\Big\} \\
(3.14) \qquad & - \frac{1}{2}\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\},
\end{aligned}
$$

which is inequality (3.1). The equality sign holds in (3.1) if and only if the leaving term in (3.9) and (3.10) imply that

$$
(3.15) \qquad \sum_{r=m+1}^{2n+1}\sum_{i=1}^{m_1}\sigma_{ii}^r = \sum_{r=m+1}^{2n+1}\sum_{t=m_1+1}^{m}\sigma_{tt}^r = 0,
$$

and $m_1H_1 = m_2H_2$.

Moreover from (3.10), we obtain

$$
(3.16) \qquad \sigma_{jt} = 0, \ \forall\, 1 \le j \le m_1, m+1 \le t \le m, m+1 \le r \le 2n+1.
$$

This shows that φ *is a mixed, totally geodesic immersion. The converse part of (3.16) is true for pointwise bi-slant warped product immersion into conformal Sasakian space form. Hence, the proof is complete.* □

Following corollaries are easy consequence of the above theorem.

Corollary 3.1. *Let* $\tilde{N}(c)$ *be a (2n+1)-dimensional conformal Sasakian space form and* $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ *be an isometric immersion of an m-dimensional pointwise semi-slant doubly warped product into* $\tilde{N}(c)$ *equipped with quarter symmetric connection. Then*

$$
\begin{aligned}
\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \le & \ \frac{m^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\
& - \frac{(c-1)}{8}[2+3m_1+3m_2cos^2\theta_2]\Big\} \\
(3.17) \qquad & - \frac{1}{2}\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\}.
\end{aligned}
$$

Similarly, if $\theta_1 = \pi/2$ and $\theta_2 = \theta$, in Theorem 3.1, then we have

Corollary 3.2. *Let* $\tilde{N}(c)$ *be a (2n+1)-dimensional conformal Sasakian space form and* $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ *be an isometric immersion of an m-dimensional pointwise hemi-slant doubly warped product into* $\tilde{N}(c)$ *equipped with quarter symmetric connection. Then*

$$
\begin{aligned}
\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \le & \ exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\
& - \frac{(c-1)}{8}[2+3m_2cos^2\theta]\Big\} \\
(3.18) \qquad & - \frac{1}{2}\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\}.
\end{aligned}
$$

Also, if $\theta_1 = 0$ and $\theta_2 = \pi/2$, in Theorem 3.1, then we have

Corollary 3.3. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional from poitwise CR-doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

$$\begin{aligned} \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} &\leq \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \frac{(c-1)}{8}[2+3m_1]\Big\} \\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)a + \lambda_2(\lambda_1-\lambda_2)b + 2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \end{aligned} \tag{3.19}$$

Furthermore, we have the following corollary of Theorem 3.1

Corollary 3.4. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric minimal immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then the following inequality holds:*

$$\begin{aligned} \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} &\leq exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \frac{(c-1)}{8}[2+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} \\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)a + \lambda_2(\lambda_1-\lambda_2)b + 2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \end{aligned} \tag{3.20}$$

For the semi-symmetric metric connection $\lambda_1 = \lambda_2 = 1$, we have

Theorem 3.2. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space from and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with semi-symmetric connection. Then the following inequality holds:*

$$\begin{aligned} \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} &\leq \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \frac{(c-1)}{8}[2+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} - a. \end{aligned} \tag{3.21}$$

For the semi-symmetric metric nonmetric connection, if we put $\lambda_1 = 1$ and $\lambda_2 = 0$ in Theorem 3.1, then we have

Theorem 3.3. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with semi-symmetric metric non metric connection satisfies the following inequality*

$$\begin{aligned} \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} &\leq \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \frac{(c-1)}{8}[2+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} \\ &- \frac{1}{2}(a+2m_1m_2\pi(H)). \qquad (3.22) \end{aligned}$$

Next, we have the following theorem

Theorem 3.4. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

$$\begin{aligned} (i)\ \ (\frac{\Delta_1 f_1}{m_1 f_1}) + (\frac{\Delta_2 f_2}{m_2 f_2}) &\geq \tau - \frac{m^2(m-2)}{2(m-1)}||H||^2 \\ &- exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2) + \frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1 \\ &+ 3m_2cos^2\theta_2\big) + \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\} \\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-n)a + \lambda_2(\lambda_1-\lambda_2)(1-n)b \\ &+ (\lambda_1-\lambda_2)n(n-1)\pi(H)\big\}, \qquad (3.23) \end{aligned}$$

where $m_i = dimN_i$, i=1,2 and Δ^i is the laplacian operator on N_i, i=1,2.

(ii) If the equality sign holds in (3.23), then the equality sign in (3.36) holds auotomatically.

(iii) If $m = 2$, then equality sign in (3.23) holds identically.

Proof. Let us consider that ${}_{f_2}N_1 \times_{f_1} N_2$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product $\tilde{N}(c)$ with pointwise φ-sectional curvature c endowed with quarter symmetric connection. Then from the equation of Gauss, we obtain

$$2\tau = exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2-2m+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2\big)$$

$$+ \quad (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} + (\lambda_1+\lambda_2)(1-m)a$$

$$(3.24) + \quad \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2.$$

Now, we consider that

$$\delta = 2\tau - exp(f)\Big\{\frac{(c+3)}{4}(m+1)(m-2) + \frac{(c-1)}{4}\big(2-2m+3m_1cos^2\theta_1+3m_2cos^2\theta_2\big)$$
$$+ \quad (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} - (\lambda_1+\lambda_2)(1-m)a$$
$$- \quad \lambda_2(\lambda_1-\lambda_2)(1-m)b - (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H}) - \frac{m^2(m-2)}{m-1}||\mathcal{H}||^2.$$
(3.25)

Then from (3.24) and (3.25), it follows that

$$(3.26) \qquad m^2||H||^2 = (m-1)\big\{||\sigma||^2 + \delta - exp(f)\frac{(c+3)}{2}\big\}.$$

Let $\{e_1,\cdots,e_m\}$ *be an orthonormal frame, the equation takes the following form*

$$\left(\sum_{r=m+1}^{2n+1}\sum_{i=1}^{m}\sigma_{ii}^r\right)^2 = (m-1)\Big\{\delta + \sum_{r=m+1}^{2n+1}\sum_{i=1}^{m}(\sigma_{ii}^r)^2 + \sum_{r=m+1}^{2n+1}\sum_{i<j}(\sigma_{ij}^r)^2$$
$$(3.27) \qquad + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 - exp(f)\frac{(c+3)}{2}\Big\},$$

which implies that

$$\left(\sigma_{11}^{m+1} + \sum_{i=2}^{m_1}\sigma_{ii}^{m+1} + \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}\right)^2 = \delta + (\sigma_{11}^{m+1})^2 + \sum_{i=2}^{m_1}(\sigma_{ii}^{m+1})^2$$
$$+ \sum_{t=m_1+1}(\sigma_{tt}^{m+1})^2 + \sum_{2\le j\ne l\le m_1}\sigma_{jj}^{m+1}\sigma_{ll}^{m+1}$$
$$- \sum_{m_1+1\le t\ne s\le m_1}(\sigma_{jj}^{m+1})(\sigma_{ll}^{m+1}) + \sum_{i<j=1}^{m}(\sigma_{ij}^{m+1})^2$$
$$(3.28) \qquad + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 - exp(f)\frac{(c+3)}{2}.$$

Let us consider that $b_1 = \sigma_{11}^{m+1}$, $b_2 = \sum_{i=2}^{m_1}(\sigma_{ii}^{m+1})^2$ *and* $b_2 = \sum_{t=m_1}^{m}(\sigma_{tt}^{m+1})^2$. *Then from (3.1) and the equation (3.28), we have*

$$\frac{\delta}{2} - exp(f)\frac{(c+3)}{2} + \sum_{i<j=1}^{m}(\sigma_{ij}^{m+1})^2) + \frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 \le \sum_{2\le j\ne+l\le m_1}\sigma_{jj}^{m+1}\sigma_{ll}^{m+1}$$
$$(3.29) \qquad + \sum_{m_1+1\le t\ne s\le m}\sigma_{tt}^{m+1}\sigma_{ss}^{m+1}.$$

Equality holds if and only if

$$\sum_{i=1}^{m_1}\sigma_{ii}^{m+1}=\sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}. \tag{3.30}$$

On the other hand from (3.29) and the definition of scalar curvature, we have

$$\begin{aligned}\kappa(e_1\wedge e_{m_1+1}) \;\geq\; & \sum_{r=m+1}^{2n+1}\sum_{j\in P_{1m_1+1}}(\sigma_{1j}^r)^2+\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{j\in P_{1m_1+1}}^{i\neq j}(\sigma_{ij}^r)^2\\ &+\sum_{r=m+1}^{2n+1}\sum_{j\in P_{1m_1+1}}(\sigma_{m_1+1j}^r)^2+\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{i,j\in P_{1m_1+1}}(\sigma_{1j}^r)^2\\ &+\frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m_1+1}(\sigma_{1j}^r)^2+\frac{\delta}{2},\end{aligned}$$

where $P_{1m_1+1}=\{1,...,m\}-\{1,m_1+1\}$. *Thus, it implies that*

$$\kappa(e_1\wedge e_{m_1+1})=\frac{\delta}{2}, \tag{3.31}$$

Since, $N={}_{f_2}N_1\times_{f_1}N_2$ *is a pointwise bi-slant doubly warped product submanifold, we have* $\nabla_X Z=\nabla_Z X=(X\ln f_1)Z+(Z\ln f_2)X$, *for any unit vector fields* X *and* Z *tangent to* N_1 *and* N_2, *respectively. Then from (2.18),(3.25) and* (3.31), *the scalar curvature derives as;*

$$\begin{aligned}\tau \;\leq\; & \frac{1}{f_1}\left\{(\nabla_{e_1}e_1)f_1-e_1^2f_1\right\}+\frac{1}{f_2}\left\{(\nabla_{e_2}e_2)f_2-e_2^2f_2\right\}\\ &+ exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2)+\frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1+3m_2cos^2\theta_2\big)\\ &+\frac{1}{2}(m-1)trB+\frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\}\\ &+\frac{1}{2}\{(\lambda_1+\lambda_2)(1-m)a+\lambda_2(\lambda_1-\lambda_2)(1-m)b+(\lambda_2-\lambda_1)m(m-1)\pi(H)\}.\end{aligned} \tag{3.32}$$

Let the equality holds in (3.32), *then all leaving terms in (3.29) and* (3.31), *we obtain the follwing conditions, i.e.*

$$\begin{aligned}&\sigma_{1j}^r=0,\quad \sigma_{jm_1+1}^r=0,\quad \sigma_{ij}^r=0,\quad \text{where}\quad i\neq j,\quad \text{and}\quad r\in\{m+1,\cdots,2n+1\}\\ &\sigma_{1j}^r=\sigma_{jm_1+1}^r=\sigma_{ij}^r=0,\quad \text{and}\quad \sigma_{11}^r+\sigma_{m_1+1m_1+1}.\end{aligned} \tag{3.33}$$

Similarly, we extend the relation (3.32) as follows

$$\tau \;\leq\; \frac{1}{f_1}\left\{(\nabla_{e_a}e_\alpha)f_1-e_\alpha^2f_1\right\}+\frac{1}{f_2}\left\{\left(\nabla_{e_\beta}e_\beta\right)f_2-e_\beta^2f_2\right\}$$

$$+ \quad \frac{m^2(m-2)}{2(m-1)}||H||^2 + exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2) + \frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1$$
$$+ \quad 3m_2cos^2\theta_2) + \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\}$$
$$+ \quad \frac{1}{2}\{(\lambda_1+\lambda_2)(1-m)a + \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(H)\}. \tag{3.34}$$

for any $\alpha = 1, \cdots, m_1$ and $\beta = m_1+1, \cdots m$. Taking the summing up α from 1 to m_1 and summing up β from m_1+1 to m_2 repectively, we arrive at

$$m_1m_2\tau \leq \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} + exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2)$$
$$+ \quad \frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1+3m_2cos^2\theta_2)$$
$$+ \quad \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\}$$
$$+ \quad \frac{1}{2}\{(\lambda_1+\lambda_2)(1-m)a + \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(H)\}. \tag{3.35}$$

Similarly, the equality sign holds in (3.35) identically. Thus the equality sign in (3.32) holds for each $\alpha \in \{1, \cdots, n_1\}$ and $\beta \in \{n_1+1, \cdots, n\}$. Then we get

$$\sigma^r_{\alpha j} = 0, \quad \sigma^r_{ij} = 0, \quad \sigma^r_{ij} = 0, \quad \text{where} \quad i \neq j, \quad \text{and} \quad r \in \{n+1, \cdots, 2m+1\}$$
$$\sigma^r_{\alpha j} = \sigma^r_{ij} = \sigma^r_{ij} = 0, \quad \text{and} \quad \sigma^r_{\alpha\alpha} + \sigma^r_{\beta\beta} = 0, \quad i,j \in P_{1n_1+1}, r = n+2, \cdots, 2m+1. \tag{3.36}$$

Moreover, If $m = 2$. Then $m_1 = m_2 = 1$. thus from (2.18), we get $\tau = \Delta_1 f_1 + \Delta_2 f_2$. Hence the equality in (3.23) holds, which proves the theorem completely. □

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SOME VECTOR FIELDS ON THE TANGENT BUNDLE WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. In this paper, we examine some special vector fields, such as incompressible vector fields, harmonic vector fields, concurrent vector fields, conformal vector fields and projective vector fields on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ and obtain some properties related to them.
Key words: Complete lift metric, semi-symmetric metric connection, tangent bundle, vector fields.

1. Introduction

Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [1]. Semi-symmetric metric connections play an important role in the study of Riemannian manifolds. In [2], Hayden introduced the idea of a metric connection with torsion on a Riemannian manifold. Using Hayden's idea, Yano [6] studied a semi-symmetric metric connection on a Riemannian manifold. He proved that a Riemannian manifold endowed with the semi-symmetric metric connection has vanishing curvature tensor if and only if the Riemannian manifold is conformally flat. After that, the generalization of this result for vanishing Ricci tensor of the semi-symmetric metric connection was shown by Imai in [3, 4].

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The geometry of tangent bundle TM is based on the fundamental paper of Sasaki [5] published in 1958. He used a given Riemannian metric g on a differentiable manifold M to construct a metric $\widetilde{g}$ on the tangent bundle TM of M. Today this metric is called the *Sasaki metric*. The well-known Riemannian or pseudo-Riemannian metrics on TM are constructed from the Riemannian metric g given on M by classical lifts, such as

1. The complete lift metric or the metric II;
2. The metric $I+II$;
3. The Sasaki metric or the metric $I+III$;
4. The metric $II+III$; where $I=g_{ij}dx^i dx^j$, $II=2g_{ij}dx^i\delta y^j$, $III=g_{ij}\delta y^i\delta y^j$ are all quadratic differential forms defined globally on the tangent bundle TM over M [8].

In our paper [9], we originally define a semi-symmetric metric connection on the tangent bundle equipped with complete lift metric. We compute all forms of the curvature tensors of the semi-symmetric metric connection and study their properties. Also, we have investigated conditions for the tangent bundle with this connection and the complete lift metric to be locally conformally flat. The goal of the present paper is to characterize some vector fields such as incompressible, harmonic, concurrent, conformal, projective with respect to the semi-symmetric metric connection on the tangent bundle over a Riemannian manifold.

2. Preliminaries

Let M be an $n-$dimensional differentiable manifold and TM be its tangent bundle with the natural projection $\pi: TM \longmapsto M$. Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighborhood and (x^h), $h=1,...,n$ are the coordinate functions. Let $(y^h)=(x^{\overline{h}})$, $\overline{h}=n+1,...,2n$ be the Cartesian coordinates in each tangent space T_pM at $p\in M$ with respect to natural basis $\left\{\frac{\partial}{\partial x^h}\mid_p\right\}$, where p is an arbitrary point in U with local coordinates (x^h). Then we can introduce the local coordinates (x^h, y^h) on the open set $\pi^{-1}(U)\subset TM$. Here, the coordinate system of $(x^h, y^h)=(x^h, x^{\overline{h}})$ is called induced coordinates on $\pi^{-1}(U)$ from (U, x^h). In the paper, we use Einstein's convention on repeated indices.

Let $X=X^h\frac{\partial}{\partial x^h}$ be the local expression in U of a vector field X on M. Let ∇ be a (torsion-free) linear connection on M. The vertical lift ${}^VX$, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are given respectively by

$${}^VX = X^h\partial_{\overline{h}},$$

$${}^HX = X^h\partial_h - y^s\Gamma^h_{sk}X^k\partial_{\overline{h}}$$

and

$${}^CX = X^h\partial_h + y^s\partial_s X^h\partial_{\overline{h}}$$

with respect to the induced coordinates, where $\partial_h = \frac{\partial}{\partial x^h}, \partial_{\overline{h}} = \frac{\partial}{\partial y^h}$ and Γ^h_{jk} are the components of the connection ∇.

Suppose that a (p,q) tensor field S on M, $q > 1$, is given. We then define a $(p,q-1)$ tensor field γS on TM by

$$\gamma S = (y^s S^{j_1 \dots j_p}_{s i_2 \dots i_q})\partial_{\overline{j_1}} \otimes \dots \otimes \partial_{\overline{j_p}} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_q}$$

with respect to the induced coordinates (x^i, y^i) [8]. The tensor field γS determines a global tensor field on TM. We easily see that for any $(1,1)$ tensor field P, γP has components

$$(\gamma P) = \begin{pmatrix} 0 \\ y^j P^i_j \end{pmatrix}$$

and γP is a vertical vector field on TM.

With the connection ∇, the set of the $2n$ linearly independent vector fields on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM which are the following forms:

$$\begin{aligned} E_j &= \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \\ E_{\overline{j}} &= \partial_{\overline{j}}. \end{aligned}$$

is a frame field [8]. We call it the adapted frame and it will be written by $\{E_\beta\} = \left\{E_j, E_{\overline{j}}\right\}$. With respect to adapted frame $\{E_\beta\}$, the vertical lift ${}^V X$, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are respectively expressed by [8]

$$\begin{aligned} {}^V X &= X^j E_{\overline{j}}, \\ {}^H X &= X^j E_j, \\ {}^C X &= X^j E_j + y^s \nabla_s X^j E_{\overline{j}}. \end{aligned} \tag{2.1}$$

The complete lift metric ${}^C g$ on the tangent bundle TM over a (pseudo-)Riemannian manifold (M,g) is defined as follows:

$$\begin{aligned} {}^C g\left({}^H X, {}^H Y\right) &= 0, \\ {}^C g\left({}^H X, {}^V Y\right) &= {}^C g\left({}^V X, {}^H Y\right) = g\left(X, Y\right), \\ {}^C g\left({}^V X, {}^V Y\right) &= 0 \end{aligned}$$

for all vector fields X and Y on M [8]. Note that ${}^C g$ is a pseudo-Riemannian metric on TM. The covariant and contravariant components of the complete lift metric ${}^C g$ on TM are respectively given in the adapted local frame by

$${}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and

$$^{C}g^{\alpha\beta} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & 0 \end{pmatrix}.$$

The semi-symmetric metric connection $\overline{\nabla}$ on TM with respect to the complete lift metric ^{C}g is given as follows.

Proposition 2.1. *[9]The semi-symmetric metric connection $\overline{\nabla}$ on the tangent bundle TM with the complete lift metric ^{C}g over a (pseudo-)Riemannian manifold (M,g) is given by*

$$(2.2)\qquad \begin{cases} \overline{\nabla}_{E_i}E_j = \Gamma_{ij}^{k}E_k + \{y^s R_{sij}{}^{k} + y_j\delta_i^k - y^k g_{ij}\}E_{\overline{k}}, \\ \overline{\nabla}_{E_i}E_{\overline{j}} = \Gamma_{ij}^{k}E_{\overline{k}}, \\ \overline{\nabla}_{E_{\overline{i}}}E_j = 0, \overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ_{ij}^{h} and $R_{hji}{}^{s}$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor field R of the pseudo-Riemannian metric g on M.

3. Some Vector Fields on TM with respect to Semi-symmetric Metric Connection

In this section, we firstly search the properties of being harmonic and incompresible of the lifting vector fields. After that we will find the general forms of concurrent, conformal, projective vector fields with respect to the semi-symmetric metric connection on the tangent bundle TM and give some important results related to them.

3.1. Lifting vector fields being incompressible (divergence-free) and harmonic

Firstly, we shall give the definition of an incompressible vector field on TM with respect to the semi-symmetric metric connection.

Definition 3.1. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. A vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM is called incompressible vector field with respect to the semi-symmetric metric connection if $\widetilde{V}$ satisfies the following condition

$$trace(\overline{\nabla}\widetilde{V}) = \overline{\nabla}_\alpha \widetilde{V}^\alpha = 0.$$

Proposition 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. Then, for any vector field V on M,* *i) The vertical lift* ${}^{V}V$ *is an incompressible vector field on* TM *with respect to the semi-symmetric metric connection* $\overline{\nabla}$*;*

ii) The horizontal lift ${}^{H}V$ *or the complete lift* ${}^{C}V$ *is an incompressible vector field on* TM *with respect to the semi-symmetric metric connection* $\overline{\nabla}$ *if and only if the vector field* V *is incompressible on* M *with respect to the Levi-Civita connection* ∇*.*

Proof. Using (2.1) and (2.2), we calculate

$$\begin{aligned}
trace(\overline{\nabla}^{V}V) &= \overline{\nabla}_{\alpha}{}^{V}V^{\alpha} = \overline{\nabla}_{\overline{h}}v^{h} = 0 \\
trace(\overline{\nabla}^{H}V) &= \overline{\nabla}_{\alpha}{}^{H}V^{\alpha} = \overline{\nabla}_{h}v^{h} \\
&= (\partial_h - y^{s}\Gamma^{m}_{sh}\partial_{\overline{m}})\, v^{h} + \overline{\Gamma}^{\,h}_{hm}v^{m} \\
&= \nabla_{h}v^{h} = trace(\nabla V) \\
trace(\overline{\nabla}^{C}V) &= \overline{\nabla}_{\alpha}{}^{C}V^{\alpha} = \overline{\nabla}_{h}v^{h} + \overline{\nabla}_{\overline{h}}v^{\overline{h}} \\
&= (\partial_h - y^{s}\Gamma^{m}_{sh}\partial_{\overline{m}})\, v^{h} + \overline{\Gamma}^{\,h}_{hm}v^{m} + \partial_{\overline{h}}\left(y^{s}\nabla_{s}v^{h}\right) \\
&= 2\nabla_{h}v^{h} = 2trace(\nabla V)
\end{aligned}$$

from which, it is easy to see that the results *(i)* and *(ii)*. □

Definition 3.2. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. A vector field $\widetilde{V} = v^{h}E_{h} + v^{\overline{h}}E_{\overline{h}}$ on TM is called a harmonic vector field with respect to the semi-symmetric metric connection $\overline{\nabla}$ if $\widetilde{V}$ satisfies the following condition

$$\left(\overline{\nabla}_{i}\widetilde{V}^{\epsilon}\right){}^{C}g_{\epsilon j} - \left(\overline{\nabla}_{j}\widetilde{V}^{\epsilon}\right){}^{C}g_{\epsilon i} = 0,$$

where ${}^{C}g_{ij}$ are the components of the complete lift metric ${}^{C}g$ on TM.

The following lemma comes immediate from standard calculations.

Lemma 3.1. *Let* M *be a (pseudo-)Riemannian manifold and* TM *be its tangent bundle with the semi-symmetric metric connection* $\overline{\nabla}$*. Then*

i) For the vertical lift ${}^{V}V$*, we get*

$$\left(\overline{\nabla}_{\alpha}{}^{V}V^{\epsilon}\right){}^{C}g_{\epsilon\beta} - \left(\overline{\nabla}^{V}_{\beta}V^{\epsilon}\right){}^{C}g_{\epsilon\alpha} = \begin{pmatrix} \nabla_{i}v_{j} - \nabla_{j}v_{i} & 0 \\ 0 & 0 \end{pmatrix}; \tag{3.1}$$

ii) For the horizontal lift ${}^{H}X$*, we get*

$$\begin{aligned}
&\left(\overline{\nabla}_{\alpha}{}^{H}V^{\epsilon}\right){}^{C}g_{\epsilon\beta} - \left(\overline{\nabla}_{\beta}{}^{H}V^{\epsilon}\right){}^{C}g_{\epsilon\alpha} \\
&= \begin{pmatrix} y^{s}\left[R_{siaj} - R_{sjai} + g_{si}g_{ja} - g_{sj}g_{ia}\right]v^{a} & \nabla_{i}v_{j} \\ -\nabla_{j}v_{i} & 0 \end{pmatrix};
\end{aligned} \tag{3.2}$$

iii) For the complete lift ${}^{C}V$, we get

$$(3.3)\qquad \left(\overline{\nabla}_{\alpha}{}^{C}V^{\epsilon}\right){}^{C}g_{\epsilon\beta}-\left(\overline{\nabla}_{\beta}{}^{C}V^{\epsilon}\right){}^{C}g_{\epsilon\alpha} = \begin{pmatrix} y^{s}\left[\nabla_{s}\left(\nabla_{i}v_{j}-\nabla_{j}v_{i}\right)+\left(g_{si}g_{ja}-g_{sj}g_{ia}\right)v^{a}\right] & \nabla_{i}v_{j}-\nabla_{j}v_{i} \\ \nabla_{i}v_{j}-\nabla_{j}v_{i} & 0 \end{pmatrix}.$$

A manifold whose curvature tensor is of the form

$$R_{ijkl}=\kappa(g_{il}g_{jk}-g_{jl}g_{ik})$$

is called a manifold of constant curvature [7]. Here κ is the sectional curvature of the manifold.

From (3.2) and the above definition, we write

$$(3.4)\qquad \begin{aligned} R_{siaj}&=\kappa(g_{sj}g_{ia}-g_{ij}g_{sa}) \\ R_{sjai}&=\kappa(g_{si}g_{ja}-g_{ji}g_{sa}) \\ \Rightarrow\quad R_{siaj}-R_{sjai}&=\kappa(g_{sj}g_{ia}-g_{si}g_{ja}). \end{aligned}$$

When we use the above equation (3.4) on the equation (3.2) and take $\kappa=1$, we obtain

$$\begin{aligned} & y^{s}\left[R_{siaj}-R_{sjai}+g_{si}g_{ja}-g_{sj}g_{ia}\right]v^{a} \\ = \; & y^{s}\left[\kappa(g_{sj}g_{ia}-g_{si}g_{ja})+g_{si}g_{ja}-g_{sj}g_{ia}\right]v^{a} \\ = \; & y^{s}\left[(g_{sj}g_{ia}-g_{si}g_{ja})+g_{si}g_{ja}-g_{sj}g_{ia}\right]v^{a} \\ = \; & 0. \end{aligned}$$

Hence, as a corollary of Lemma 3.1, we obtain

Proposition 3.2. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. Then, for any vector field V on M,*

i) The vertical lift ${}^{V}V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if the vector field V is a harmonic vector field with respect to the Levi-Civita connection ∇;

ii) The complete lift ${}^{C}V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if the vector field V is a harmonic vector field with respect to the Levi-Civita connection ∇ and $g_{si}g_{ja}-g_{sj}g_{ia}=0$;

iii) The horizontal lift ${}^{H}V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if the vector field V is parallel with respect to the Levi-Civita connection ∇ and M has constant sectional curvature 1.

3.2. Concurrent vector fields

Definition 3.3. A vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM is called a concurrent vector field with respect to the semi-symmetric metric connection $\overline{\nabla}$ if it satisfies

$$\overline{\nabla}_\beta \widetilde{V}^\epsilon = \overline{\nabla}_{E_\beta} \widetilde{V}^\epsilon = \widetilde{k}\delta^\epsilon_\beta, \tag{3.5}$$

where $\widetilde{k}$ is a function on TM and δ^ϵ_β is the Kronecker symbol.

Proposition 3.3. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. The vector field $\widetilde{V}$ on TM is concurrent with respect to semi-symmetric metric connection $\overline{\nabla}$ if and only if the vector field $\widetilde{V}$ has the form*

$$\widetilde{V} = \begin{pmatrix} v^h \\ \frac{1}{n}\left[trace\left(\nabla V\right)\right] y^h \end{pmatrix}$$

and the following condition is satisfied

$$\frac{1}{n}\left[\nabla_j\left(trace\left(\nabla V\right)\right) y^h\right] + \left(y^s R_{sja}{}^h + y_a \delta^h_j - y^h g_{ja}\right) v^a = 0.$$

Proof. With respect to the adapted frame, firstly putting $\epsilon = h, \beta = \overline{j}$ in (3.5), it follows that

$$\begin{aligned} \overline{\nabla}_{\overline{j}} v^h &= E_{\overline{j}} v^h + \overline{\Gamma}^h_{\overline{j}a} v^a + \overline{\Gamma}^h_{\overline{j}\overline{a}} v^{\overline{a}} = \widetilde{k}\delta^h_{\overline{j}} \\ &\Rightarrow \partial_{\overline{j}} v^h = 0 \\ &\Rightarrow v^h = v^h\left(x^h\right). \end{aligned}$$

Similarly putting $\epsilon = h, \beta = j$ and $\epsilon = \overline{h}, \beta = \overline{j}$, we respectively get

$$\begin{aligned} \overline{\nabla}_j v^h &= E_j v^h + \overline{\Gamma}^h_{ja} v^a + \overline{\Gamma}^h_{j\overline{a}} v^{\overline{a}} = \widetilde{k}.\delta^h_j \\ &\Rightarrow \partial_j v^h + \Gamma^h_{ja} v^a = \widetilde{k}.\delta^h_j \\ &\Rightarrow \nabla_j v^h = \widetilde{k}.\delta^h_j \qquad (h \to j) \\ &\Rightarrow \frac{1}{n}\nabla_j v^j = \widetilde{k} \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_{\overline{j}} v^{\overline{h}} &= E_{\overline{j}} v^{\overline{h}} + \overline{\Gamma}^{\overline{h}}_{\overline{j}a} v^a + \overline{\Gamma}^{\overline{h}}_{\overline{j}\overline{a}} v^{\overline{a}} = \widetilde{k}.\delta^{\overline{h}}_{\overline{j}} \\ &\Rightarrow \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n}\nabla_j v^j.\delta^{\overline{h}}_{\overline{j}} \\ &\Rightarrow \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n}\left[trace\left(\nabla V\right)\delta^{\overline{h}}_{\overline{j}}\right] \\ &\Rightarrow \partial_{\overline{j}} v^{\overline{h}} = \frac{1}{n}\left[trace\left(\nabla V\right)\left(\partial_{\overline{j}} y^h\right)\right] \end{aligned}$$

$$\Rightarrow \quad \partial_{\bar{j}} v^{\bar{h}} = \partial_{\bar{j}} \left[\frac{1}{n} trace\,(\nabla V)\, y^h \right]$$
$$\Rightarrow \quad v^{\bar{h}} = \frac{1}{n} \left[trace\,(\nabla V) \right] y^h.$$

Finally putting $\epsilon = \bar{h}, \beta = j$, we find

$$\begin{aligned}
\overline{\nabla}_j v^{\bar{h}} &= E_j v^{\bar{h}} + \overline{\Gamma}_{ja}^{\bar{h}} v^a + \overline{\Gamma}_{j\bar{a}}^{\bar{h}} v^{\bar{a}} = \widetilde{k} \delta_j^{\bar{h}} \\
&\Rightarrow \begin{array}{c} E_j \left[\frac{1}{n} \left[trace\,(\nabla V) \right] y^h \right] + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a \\ + \Gamma_{ja}^h \left[\frac{1}{n} \left[trace\,(\nabla V) \right] y^a \right] = 0 \end{array} \\
&\Rightarrow \begin{array}{c} \left(\partial_j - y^s \Gamma_{sj}^m \partial_{\bar{m}} \right) \left[\frac{1}{n} \left[trace\,(\nabla V) \right] y^h \right] \\ + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a + y^a \Gamma_{ja}^h \left[\frac{1}{n} \left[trace\,(\nabla V) \right] \right] = 0 \end{array} \\
&\Rightarrow \begin{array}{c} \frac{1}{n} \left[\partial_j \left(trace\,(\nabla V) \right) y^h \right] - y^s \Gamma_{sj}^h \left[\frac{1}{n} \left[trace\,(\nabla V) \right] \right] \\ + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a + y^a \Gamma_{ja}^h \left[\frac{1}{n} \left[trace\,(\nabla V) \right] \right] = 0 \end{array} \\
&\Rightarrow \frac{1}{n} \left[\partial_j \left(trace\,(\nabla V) \right) y^h \right] + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a = 0 \\
&\Rightarrow \frac{1}{n} \left[\nabla_j \left(trace\,(\nabla V) \right) y^h \right] + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a = 0.
\end{aligned}$$

□

3.3. Conformal vector fields

Let $\widetilde{V}$ be a vector field on TM with components $\left(v^h, v^{\bar{h}} \right)$ with respect to the adapted frame $\{E_\beta\}$. Then $\widetilde{V}$ is a fibre-preserving vector field on TM if and only if v^h depends only on the variables $\left(x^h \right)$.

Definition 3.4. A vector field $\widetilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is called a fibre-preserving conformal vector field with respect to the semi-symmetric metric connection $\overline{\nabla}$ if it satisfies

$$L_{\widetilde{V}}{}^C g_{\alpha\beta} = (\overline{\nabla}_\alpha \widetilde{V}^\in)^C g_{\in\beta} + (\overline{\nabla}_\beta \widetilde{V}^\in)^C g_{\in\alpha} = 2\widetilde{\Omega}^C g_{\alpha\beta}.$$

Putting $(\alpha, \beta) = (i, \bar{j})$, $(\bar{i}, j)$ and (i, j), from the above equation, it can be written the following system

$$(3.6) \quad \begin{cases} i) \left(\nabla_i v^h \right) g_{hj} + \left(E_{\bar{j}} v^{\bar{h}} \right) g_{hi} = 2\widetilde{\Omega} g_{ij}, \\ ii) \left(E_{\bar{i}} v^{\bar{h}} \right) g_{hj} + \left(\nabla_j v^h \right) g_{hi} = 2\widetilde{\Omega} g_{ij}, \\ iii) \begin{array}{c} \left[E_i v^{\bar{h}} + \left(y^s R_{sia}{}^h + y_a \delta_i^h - y^h g_{ia} \right) v^a + \Gamma_{ia}{}^h v^{\bar{a}} \right] g_{hj} \\ + \left[E_j v^{\bar{h}} + \left(y^s R_{sja}{}^h + y_a \delta_j^h - y^h g_{ja} \right) v^a + \Gamma_{ja}{}^h v^{\bar{a}} \right] g_{hi} \end{array} = 0. \end{cases}$$

Proposition 3.4. *The scalar function $\widetilde{\Omega}$ on TM depends only on the variables $\left(x^h \right)$ with respect to the induced coordinates $\left(x^h, y^h \right)$.*

Proof. Applying $E_{\overline{k}}$ to the both sides of the equation *(ii)* in (3.6), we have

$$g_{hj}E_{\overline{k}}E_{\overline{i}}v^{\overline{h}} = 2E_{\overline{k}}\left(\overline{\Omega}\right)g_{ij}$$

from which we get

$$E_{\overline{k}}\left(\overline{\Omega}\right)g_{ij} = E_{\overline{i}}\left(\overline{\Omega}\right)g_{kj}.$$

It follows that

$$(n-1)E_{\overline{k}}\left(\overline{\Omega}\right) = 0.$$

This shows that the scalar function $\widetilde{\Omega}$ on TM depends only on the variables $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, y^h\right)$. Thus we can regard $\widetilde{\Omega}$ as a function on M and in the following we write ρ instead of $\widetilde{\Omega}$. □

From (3.6) and Proposition 3.4, $E_{\overline{i}}\left(v^{\overline{h}}\right)$ depends only the variables $\left(x^h\right)$, thus we can put

(3.7) $$v^{\overline{h}} = y^a A_a^h + B^h,$$

where A_a^h and B^h are certain functions which depend only on the variable $\left(x^h\right)$. Furthermore, we can easily show that A_a^h and B^h are the components of a $(1,1)$ tensor field and a contravariant vector field on M, respectively.

Any vector field V on a (pseudo-)Riemannian manifold (M,g) is a Killing vector field if $L_V g_{ij} = \nabla_i v_j + \nabla_j v_i = 0$.

Proposition 3.5. *If we put*

$$B = B^h \frac{\partial}{\partial x^h},$$

then the vector field B on M is a Killing vector field with respect to the Levi-Civita connection ∇.

Proof. Substituting (3.7) into the equation *(iii)* in (3.6) we have

(3.8) $$\nabla_i B_j + \nabla_j B_i = 0$$

and

(3.9) $$\begin{aligned} &v^a(R_{siaj} + R_{sjai} + g_{sa}g_{ij} - g_{ia}g_{sj} + g_{sa}g_{ji} - g_{ja}g_{si}) \\ &+\nabla_i A_{sj} + \nabla_j A_{si} = 0 \end{aligned}$$

where $B_i = g_{im}B^m$ and $A_{sj} = g_{hj}A_s^h$. Hence by (3.8), it follows

$$L_B g_{ij} = \nabla_i B_j + \nabla_j B_i = 0.$$

This shows B is a Killing vector field on M with respect to the Levi-Civita connection ∇. □

Substituting (3.7) into the equation *(ii)* in (3.6), we have

$$\begin{aligned} & E_{\bar{i}}\left(v^{\bar{h}}\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\rho g_{ij} \\ \Rightarrow\ & \partial_{\bar{i}}\left(y^s A_s^h + B^h\right) g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\rho g_{ij} \\ \Rightarrow\ & A_i^h g_{hj} + \left(\nabla_j v^h\right) g_{hi} = 2\rho g_{ij} \\ \Rightarrow\ & g_{hj} A_i^h = 2\rho g_{ij} - g_{hi}\left(\nabla_j v^h\right). \end{aligned} \tag{3.10}$$

Let ∇ be a linear connection on M. A vector field V on M is said to be a projective vector field if there exists a 1-form θ such that

$$(L_V \nabla)(X, Y) = \theta(X)Y + \theta(Y)X$$

for any vector fields X and Y on M. In this case θ is called the associated 1-form of V. It can locally be expressed in the following form

$$L_V \Gamma_{ij}^h = \theta_i \delta_j^h + \theta_j \delta_i^h.$$

Proposition 3.6. *The vector field V with components $\left(v^h\right)$ is a projective vector field on M with respect to the Levi-Civita connection ∇, if $\delta_a^h g_{ij} - g_{ia}\delta_j^h + \delta_a^h g_{ji} - g_{ja}\delta_i^h = 0$.*

Proof. Applying the covariant derivative ∇_k to the both sides of (3.10), we obtain

$$\begin{aligned} g_{hj}\nabla_k A_i^h &= \nabla_k\left[2\rho g_{ij} - g_{hi}\left(\nabla_j v^h\right)\right] \\ &= 2\left(\nabla_k \rho\right) g_{ij} - g_{hi}\nabla_k \nabla_j v^h \\ &= 2\rho_k g_{ij} - g_{hi}\left(L_V \Gamma_{kj}^{\ h} - R_{akj}^{\ \ \ h} v^a\right) \\ \nabla_k A_{ij} &= 2\rho_k g_{ij} - L_V \Gamma_{kj}^{\ h} g_{hi} - R_{akij} v^a. \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.9), we have

$$v^a(R_{siaj} + R_{sjai} + g_{sa}g_{ij} - g_{ia}g_{sj} + g_{sa}g_{ji} - g_{ja}g_{si}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0$$

$$\begin{array}{c} v^a(R_{siaj} + R_{sjai} + g_{sa}g_{ij} - g_{ia}g_{sj} + g_{sa}g_{ji} - g_{ja}g_{si}) \\ +2\rho_i g_{sj} - L_V \Gamma_{ij}^{\ h} g_{hs} - R_{aisj}v^a + 2\rho_j g_{si} - L_V \Gamma_{ji}^{\ h} g_{hs} - R_{ajsi}v^a \end{array} = 0$$

$$v^a\left(g_{sa}g_{ij} - g_{ia}g_{sj} + g_{sa}g_{ji} - g_{ja}g_{si}\right) + 2\left(\rho_i g_{sj} + \rho_j g_{si}\right) = 2L_V \Gamma_{ij}^{\ h} g_{hs}$$

$$L_V \Gamma_{ij}^{\ h} = \rho_i \delta_j^h + \rho_j \delta_i^h + \frac{1}{2} v^a\left(\delta_a^h g_{ij} - g_{ia}\delta_j^h + \delta_a^h g_{ji} - g_{ja}\delta_i^h\right),$$

where $\rho_i = \nabla_i \rho$. Hence, V is a projective vector field on M with respect to the Levi-Civita connection ∇. □

Now we consider the converse problem, that is, let M admit a projective vector field $V = v^h \frac{\partial}{\partial x^h}$ with respect to the Levi-Civita connection ∇. Then we have the following proposition.

Proposition 3.7. *The vector field $\widetilde{V}$ on TM defined by*

$$\widetilde{V} = v^h E_h + \left(y^s A_s^h + B^h\right) E_{\overline{h}}$$

is a fibre-preserving conformal vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$, where $A_i^h = g^{ha} A_{ai}$, $A_{ij} = 2\rho g_{ij} - \nabla_j v_i$, and $g_{ji} B^j = B_i$, $2p_i g_{sj} - L_V \Gamma_{ij}^h g_{hs} + (g_{sm} g_{ij} - g_{im} g_{sj}) = 0$.

Proof. If B_h, v^h and A_i^h are given so that they satisfy the above assumptions, we see that $\widetilde{V} = v^h E_h + \left(y^s A_s^h + B^h\right) E_{\overline{h}}$ is a fibre-preserving conformal vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$. We omit standard calculations. □

3.4. Projective vector fields

In this section, we study fibre-preserving projective vector fields on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$. We shall first state following lemma which is needed later on.

Lemma 3.2. *The Lie derivations of the adapted frame with respect to the fibre-preserving vector field $\widetilde{V} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ are given as follows*

$$L_{\widetilde{V}} E_h = -(\partial_h v^a) E_a + \left\{ y^b v^c R_{hcb}^a - v^{\overline{b}} \Gamma_{bh}^a - (E_h v^{\overline{a}} \right\} E_{\overline{a}},$$

$$L_{\widetilde{V}} E_{\overline{h}} = \left\{ v^b \Gamma_{bh}^a - (E_{\overline{h}} v^{\overline{a}}) \right\} E_{\overline{a}}.$$

The general form of fibre-preserving vector fields on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ are given by

Theorem 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. Then a vector field $\widetilde{V}$ is a fibre-preserving projective vector field with associated 1-form $\overline{\theta}$ on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if the vector field $\widetilde{V}$ has the following form*

$$\widetilde{V} =^H V +^V B + \gamma A,$$

where the vector fields $V=(v^h)$, $B=(B^h)$, *the* $(1,1)-$*tensor field* $A=(A_i^h)$ *and the associated 1-form* $\overline{\theta}$ *satisfy the following conditions*

$$\begin{gathered}(i)\overline{\theta}=\theta_i dx^i,\\ (ii)\nabla_i\theta_j=(n-1)\left(L_V g_{ij}\right),\\ (iii)\nabla_j A_i^h=\theta_j\delta_i^h-v^c R_{cji}{}^h,\\ (iv)\nabla_i\nabla_j v^h+R_{aij}{}^h v^a=\theta_i\delta_j^h+\theta_j\delta_i^h,\\ (v)\nabla_i\nabla_j B^k+R_{hij}{}^k B^h+B^h g_{hj}\delta_i^k-B^k g_{ij}=0,\\ (vi)L_V\Gamma_{ij}^h=\theta_i\delta_j^h+\theta_j\delta_i^h.\end{gathered}$$

Proof. A vector field $\widetilde{V}=v^h E_h+v^{\overline{h}}E_{\overline{h}}$ on TM is a fibre-preserving projective vector field with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if there exists a 1-form $\widetilde{\theta}$ with components $\left(\widetilde{\theta}_i,\widetilde{\theta}_{\overline{i}}\right)$ on TM such that

$$\begin{aligned}\left(L_{\widetilde{X}}\overline{\nabla}\right)(\widetilde{Y},\widetilde{Z}) &= L_{\widetilde{X}}(\overline{\nabla}_{\widetilde{Y}}\widetilde{Z})-\overline{\nabla}_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z})-\overline{\nabla}_{(L_{\widetilde{X}}\widetilde{Y})}\widetilde{Z}\\ &= \widetilde{\theta}(\widetilde{Y})\widetilde{Z}+\widetilde{\theta}(\widetilde{Z})\widetilde{Y}\end{aligned}$$

for any vector fields $\widetilde{Y}$ and $\widetilde{Z}$ on TM. We compute the following system

$$\begin{aligned}\left(L_{\widetilde{V}}\overline{\nabla}\right)(E_{\overline{i}},E_{\overline{j}}) &= L_{\widetilde{V}}(\overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}})-\overline{\nabla}_{E_{\overline{i}}}(L_{\widetilde{V}}E_{\overline{j}})-\overline{\nabla}_{(L_{\widetilde{V}}E_{\overline{i}})}E_{\overline{j}}\\ &= \widetilde{\theta}(E_{\overline{i}})E_{\overline{j}}+\widetilde{\theta}(E_{\overline{j}})E_{\overline{i}},\end{aligned} \tag{3.12}$$

$$\begin{aligned}\left(L_{\widetilde{V}}\overline{\nabla}\right)(E_{\overline{i}},E_j) &= L_{\widetilde{V}}(\overline{\nabla}_{E_{\overline{i}}}E_j)-\overline{\nabla}_{E_{\overline{i}}}(L_{\widetilde{V}}E_j)-\overline{\nabla}_{(L_{\widetilde{V}}E_{\overline{i}})}E_j\\ &= \widetilde{\theta}(E_{\overline{i}})E_j+\widetilde{\theta}(E_j)E_{\overline{i}},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left(L_{\widetilde{V}}\overline{\nabla}\right)(E_i,E_j) &= L_{\widetilde{V}}(\overline{\nabla}_{E_i}E_j)-\overline{\nabla}_{E_i}(L_{\widetilde{V}}E_j)-\overline{\nabla}_{(L_{\widetilde{V}}E_i)}E_j\\ &= \widetilde{\theta}(E_i)E_j+\widetilde{\theta}(E_j)E_i.\end{aligned} \tag{3.14}$$

From (3.12), by virtue of (2.2) and Lemma 3.2 we obtain

$$\left\{\partial_{\overline{i}}(\partial_{\overline{j}}v^{\overline{a}}\right\}E_{\overline{a}}=\widetilde{\theta}_{\overline{i}}E_{\overline{j}}+\widetilde{\theta}_{\overline{j}}E_{\overline{i}}. \tag{3.15}$$

Similarly, from (3.13) we get

$$\left\{-v^c R_{jci}{}^a+(E_{\overline{i}}v^{\overline{b}})\Gamma_{bj}^a+E_{\overline{i}}(E_j v^{\overline{a}})\right\}E_{\overline{a}}=\widetilde{\theta}_{\overline{i}}E_j+\widetilde{\theta}_j E_{\overline{i}} \tag{3.16}$$

from which, we have

$$\widetilde{\theta}_{\overline{i}}=0. \tag{3.17}$$

Due to $\widetilde{\theta}_{\overline{i}}=0$, (3.15) to

$$\partial_{\overline{i}}(\partial_{\overline{j}}v^{\overline{a}})=0,$$

and we obtain

(3.18) $$v^{\overline{a}} = y^s A_s^a + B^a,$$

where A_s^a and B^a are certain functions which depend only on the variables $\left(x^h\right)$ and the coordinate transformation rule implies that A is a $(1,1)$-tensor field with components (A_s^a) and B is a vector field with components (B^a). Hence, the fibre-preserving projective vector field $\widetilde{V}$ on TM can be expressed in the following form

(3.19) $$\begin{aligned} \widetilde{V} &= v^h E_h + v^{\overline{h}} E_{\overline{h}} = v^h E_h + \{y^s A_s^a + B^a\} E_{\overline{h}} \\ &= {}^H V + {}^V B + \gamma A. \end{aligned}$$

Substituting (3.18) into (3.16), we obtain

(3.20) $$R_{aji}{}^h v^a + \nabla_j A_i^h = \delta_i^h \theta_j.$$

Substituting (3.18) and (3.20) into (3.14), we have

(3.21) $$\begin{aligned} & \{\nabla_i \nabla_j v^h + R_{aij}{}^h v^a\} E_h + \{\nabla_i \nabla_j B^k + R_{hij}{}^k B^h + B^h g_{hj} \delta_i^k \\ & -B^k g_{ij} + y^s(\nabla_i \nabla_j A_s^k + A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + v^h \nabla_h R_{sij}{}^k \\ & -v^h \nabla_i R_{jhs}{}^k + \nabla_j v^h R_{sih}{}^k + \nabla_i v^h R_{sjh}{}^k + \nabla_j v^a g_{sa} \delta_i^k \\ & -\nabla_j v^a \delta_s^k g_{ia} + \nabla_i v^a g_{sj} \delta_a^k - \nabla_i v^a \delta_s^k g_{aj} + A_s^h g_{hj} \delta_i^k - g_{sj} A_i^k)\} E_{\overline{h}} \\ = \; & \widetilde{\theta}_i E_j + \widetilde{\theta}_j E_i. \end{aligned}$$

From (3.21), we have

(3.22) $$\nabla_i \nabla_j v^h + R_{aij}{}^h v^a = \widetilde{\theta}_i \delta_j^h + \widetilde{\theta}_j \delta_i^h,$$

(3.23) $$\nabla_i \nabla_j B^k + R_{hij}{}^k B^h + B^h g_{hj} \delta_i^k - B^k g_{ij} = 0,$$

(3.24) $$\begin{aligned} & \nabla_i \nabla_j A_s^k + A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + v^h \nabla_h R_{sij}{}^k \\ & -v^h \nabla_i R_{jhs}{}^k + \nabla_j v^h R_{sih}{}^k + \nabla_i v^h R_{sjh}{}^k \\ & +\nabla_j v^a g_{sa} \delta_i^k - \nabla_j v^a \delta_s^k g_{ia} + \nabla_i v^a g_{sj} \delta_a^k \\ & -\nabla_i v^a \delta_s^k g_{aj} + A_s^h g_{hj} \delta_i^k - g_{sj} A_i^k \\ = \; & 0. \end{aligned}$$

The equation (3.22) shows that the induced vector field $V = v^h \frac{\partial}{\partial x^h}$ is a projective vector field with respect to the Levi-Civita Connection ∇. Hence we obtain

(3.25) $$L_V R_{ij} = -(n-1) \nabla_i \theta_j.$$

Contracting k and s in (3.24) and using (3.20) and (3.25), we get

$$\nabla_i \theta_j = (n-1) \left(L_V g\right)_{ij}.$$

In the case, (3.24) is reduced to

$$\begin{aligned} & A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + v^h \nabla_h R_{sij}{}^k + \nabla_j v^h R_{sih}{}^k \\ & +\nabla_i v^h R_{shj}{}^k + \nabla_j v_s \delta_i^k + \nabla_i v^k g_{sj} + A_{sj}\delta_i^k - g_{sj} A_i^k \\ = & \ 0. \end{aligned}$$

Conversely, if B^h, v^h, θ_h and A_i^h are given so that they satisfy *(i)-(vi)*, reserving the above steps, we see that $\widetilde{X} =^H V +^V B + \gamma A$ is a fibre-preserving projective vector field on TM with respect to the semi-symmetric metric connection $\overline{\nabla}$. Hence, the proof is complete. □

Let $\widetilde{V}$ be a fibre-preserving vector field on TM with components $(v^h, v^{\overline{h}})$. It is well-known that every fibre-preserving vector field $\widetilde{V}$ on TM induces a vector field V on M with components (v^h). The below result follows immediately from Theorem 3.1 and from its Proof.

Corollary 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. Every fibre-preserving projective vector field $\widetilde{V}$ is of the form (3.19) and it naturally induces a projective vector field V on M.*

Let $\widetilde{V}$ be a vector field on TM with components $(v^h, v^{\overline{h}})$ with respect to the adapted frame $\{E_\beta\}$. Then $\widetilde{V}$ is a vertical vector field on TM if and only if $v^h = 0$. In the present case, the vector field $\widetilde{V}$ in Theorem 3.1 reduces to $\widetilde{V} =^V B + \gamma A$. Hence, from the Theorem 3.1, we obtain the following conclusion.

Corollary 3.2. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\overline{\nabla}$. If TM admits a vertical projective vector field $\widetilde{V}$,then the vector field $\widetilde{V}$ is defined by*

$$\widetilde{V} =^V B + \gamma A,$$

where the vector field $B = \left(B^h\right)$, the $(1,1)$-tensor field $A = \left(A_i^h\right)$ and the associated 1-form $\widetilde{\theta}$ satisfy the following conditions

$$\begin{gathered} (i)\overline{\theta} = \theta_i dx^i, \\ (ii)\nabla_j A_i^h = \theta_j \delta_i^h, \\ (iii)\nabla_i \theta_j = 0, \\ (iv)\nabla_i \nabla_j B^k + R_{hij}{}^k B^h + B^h g_{hj}\delta_i^k - B^k g_{ij} = 0, \\ (v)A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + A_{sj}\delta_i^k - g_{sj} A_i^k = 0. \end{gathered}$$

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