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[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

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Original Scientific Paper

CONVERGENCE OF S-ITERATIVE METHOD TO A SOLUTION OF FREDHOLM INTEGRAL EQUATION AND DATA DEPENDENCY

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Abstract. The convergence of normal S-iterative method to solution of a nonlinear Fredholm integral equation with modified argument is established. The corresponding data dependence result has also been proved. An example in support of the established results is included in our analysis.
Key words: Fredholm equation, data dependency, Fixed-point theorem.

1. Introduction and Preliminaries

The past few decades have witnessed substantial developments in the field of integral equations and their applications have arisen in many areas, ranging from economics to engineering. Now it is an unquestionable fact that the theory of iterative approximation of fixed points plays a significant role in recent progress of integral equations and their applications. In this context, fixed point iterative methods for solving integral equations have already gained a splendid boost over the past few years (see, for example [1],[2],[4],[5],[7],[8],[16],[17],[19],[20]).

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In 2011, Sahu [23] introduced a normal S-iterative method as follows:

$$\text{(1.1)} \qquad \begin{cases} x_0 \in X, \\ x_{n+1} = Ty_n, \\ y_n = (1-\xi_n)\, x_n + \xi_n T x_n, \quad n \in \mathbb{N} \end{cases}$$

where X is an ambient space, T is a self-map of X and $\{\xi_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$ satisfying certain control condition(s).

It has been shown both analytically and numerically in [23] and [12] that the iterative method (1.1) converges faster than Picard [22], Mann [21], and Ishikawa [10] iterative processes in the sense of Berinde [3] for the class of contraction mappings.

This iterative method, due to its simplicity and fastness, has attracted the attention of many researchers and has been examined in various settings (see [9],[11],[13],[14],[15],[18],[24]).

In this paper, inspired by the above mentioned achievements of normal S-iterative method (1.1), we will use it to show that normal S-iterative method (1.1) converges strongly to the solution of the following integral equation which has been considered in [6]:

$$\text{(1.2)} \qquad x(t) = \int_a^b K(t,s) \cdot h(s, x(s), x(a), x(b))\, ds + f(t), \quad t \in [a,b],$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$, $h : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ and f, $x : [a,b] \to \mathbb{R}$.

Also we give a data dependence result for the solution of integral equation (1.2) with the help of normal S-iterative method (1.1).

We need the following pair of known results:

Theorem 1.1. [6] *Assume that the following conditions are satisfied:*

(A_1) $K \in C([a,b] \times [a,b])$;

(A_2) $h \in C\left([a,b] \times \mathbb{R}^3\right)$;

(A_3) f, $x \in C[a,b]$;

(A_4) there exist constants α, β, $\gamma > 0$ *such that*

$$|h(s, u_1, u_2, u_3) - h(s, v_1, v_2, v_3)| \leq \alpha |u_1 - v_1| + \beta |u_2 - v_2| + \gamma |u_3 - v_3|,$$

for all $s \in [a,b]$, u_i, $v_i \in \mathbb{R}$, $i = 1,2,3$;

(A_5) $M_K(\alpha+\beta+\gamma)(b-a) < 1$,

where M_K *denotes a positive constant such that for all* $t, s \in [a,b]$

$$|K(t,s)| \leq M_K.$$

Then the equation (1.2) *has a unique solution* $x^* \in C[a,b]$, *which can be obtained by the successive approximations method starting with any element* $x_0 \in C[a,b]$. *Moreover, if* x_n *is the n-th successive approximation, then one has:*

$$|x_n - x^*| \leq \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^n}{1 - M_K(\alpha+\beta+\gamma)(b-a)} \cdot |x_0 - x_1|.$$

Lemma 1.1. [25] *Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of non negative numbers for which one assumes there exists $n_0 \in \mathbb{N}$ (set of natural numbers), such that for all $n \geq n_0$*

$$\beta_{n+1} \leq (1-\mu_n)\,\beta_n + \mu_n\gamma_n,$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n.$$

2. Main Results

Theorem 2.1. *Assume that all the conditions $(A_1)-(A_5)$ in Theorem 1.1 are fulfilled. Let $\{\xi_n\}_{n=0}^{\infty}$ be a real sequence in $[0,1]$ satisfying $\sum\limits_{n=0}^{\infty} \xi_n = \infty$. Then equation (1.2) has a unique solution $x^* \in C[a,b]$ and normal S-iterative method (1.1) converges to x^* with the following estimate:*

$$\|x_{n+1} - x^*\| \leq \frac{[M_K(\alpha+\beta+\gamma)(b-a)]^{n+1}}{e^{(1-M_K(\alpha+\beta+\gamma)(b-a))\sum_{k=0}^{n}\xi_k}}\,\|x_0 - x^*\|.$$

Proof. We consider the Banach space $B = (C[a,b], \|\cdot\|_C)$, where $\|\cdot\|_C$ is the Chebyshev's norm on $C[a,b]$, defined by $\|\cdot\|_C = \{\sup|x(t)| : t \in [a,b]\}$. Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by Normal-S iteration method (1.1) for the operator $T : B \to B$ defined by

$$T(x(t)) = \int_a^b K(t,s)\cdot h(s, x(s), x(a), x(b))\,ds + f(t),\ t \in [a,b]. \tag{2.1}$$

We will show that $x_n \to x^*$ as $n \to \infty$.

From (1.1), (2.1), and assumptions $(A_1)-(A_4)$, we have that

$$\begin{aligned}
|x_{n+1}(t) - x^*(t)| &= |T(y_n(t)) - T(x^*(t))| \\
&= \left|\int_a^b K(t,s)\cdot\left[\begin{array}{c} h(s, y_n(s), y_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{array}\right] ds\right| \\
&\leq \int_a^b |K(t,s)|\cdot\left|\begin{array}{c} h(s, y_n(s), y_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{array}\right| ds \\
&\leq M_K \int_a^b \left[\begin{array}{c} \alpha|y_n(s) - x^*(s)| + \beta|y_n(a) - x^*(a)| \\ +\gamma|y_n(b) - x^*(b)| \end{array}\right] ds,
\end{aligned}$$

$$
\begin{aligned}
|y_n(t) - x^*(t)| &\leq (1-\xi_n)|x_n(t) - x^*(t)| + \xi_n |T(x_n)(t) - T(x^*)(t)| \\
&= (1-\xi_n)|x_n(t) - x^*(t)| \\
&\quad + \xi_n \left| \int_a^b K(t,s) \cdot \left[\begin{array}{c} h(s, x_n(s), x_n(a), y_n(b)) \\ -h(s, x^*(s), x^*(a), x^*(b)) \end{array} \right] ds \right| \\
&\leq (1-\xi_n)|x_n(t) - x^*(t)| \\
&\quad + \xi_n M_K \int_a^b \left[\begin{array}{c} \alpha |x_n(s) - x^*(s)| + \beta |x_n(a) - x^*(a)| \\ +\gamma |x_n(b) - x^*(b)| \end{array} \right] ds
\end{aligned}
$$

Now, by taking supremum in the above inequalities, we get

$$\|x_{n+1} - x^*\| \leq M_K(\alpha+\beta+\gamma)(b-a)\|y_n - x^*\|, \tag{2.2}$$

and

$$\|y_n - x^*\| \leq [1 - \xi_n(1 - M_K(\alpha+\beta+\gamma)(b-a))]\|x_n - x^*\|, \tag{2.3}$$

respectively.

Combining (2.2) with (2.3), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq M_K(\alpha+\beta+\gamma)(b-a)[1 - \xi_n(1 - M_K(\alpha+\beta+\gamma)(b-a))]\|x_n - x^*\|.
\end{aligned} \tag{2.4}
$$

Thus, by induction, we get

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\| [M_K(\alpha+\beta+\gamma)(b-a)]^{n+1} \\
&\quad \times \prod_{k=0}^{n} [1 - \xi_k(1 - M_K(\alpha+\beta+\gamma)(b-a))].
\end{aligned} \tag{2.5}
$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}$, the assumption (A_5) yields

$$\xi_k(1 - M_K(\alpha+\beta+\gamma)(b-a)) < 1. \tag{2.6}$$

From the classical analysis, we know that $1 - x \leq e^{-x}$ for all $x \in [0,1]$. Hence by utilizing this fact with (2.6) in (2.5), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\| [M_K(\alpha+\beta+\gamma)(b-a)]^{n+1} \\
&\quad \times e^{-(1 - M_K(\alpha+\beta+\gamma)(b-a)) \sum_{k=0}^{n} \xi_k},
\end{aligned} \tag{2.7}
$$

which yields $\lim_{n\to\infty} \|x_n - x^*\| = 0$. $\square$

We now prove a closeness of solutions of integral equation (1.2) with the help of the normal-S iterative method (1.1).

We consider the following equation:

$$\widetilde{T}(\widetilde{x}(t)) = \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{x}(s), \widetilde{x}(a), \widetilde{x}(b))\, ds + g(t), t \in [a,b], \tag{2.8}$$

where $K : [a, b] \times [a, b] \to \mathbb{R}$, $\widetilde{h} : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$.

Now, we define the following normal-S iterative methods associated with T in (2.1) and $\widetilde{T}$ in (2.8), respectively:

$$\begin{cases} x_0 \in C[a,b], \\ x_{n+1} = \int_a^b K(t,s) \cdot h(s, y_n(s), y_n(a), y_n(b))\, ds + f(t), \\ y_n = (1 - \xi_n) x_n \\ \qquad + \xi_n \int_a^b K(t,s) \cdot h(s, x_n(s), x_n(a), x_n(b))\, ds + f(t), t \in [a,b], n \in \mathbb{N}, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \widetilde{x}_0 \in C[a,b], \\ \widetilde{x}_{n+1} = \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{y}_n(s), \widetilde{y}_n(a), \widetilde{y}_n(b))\, ds + g(t), \\ \widetilde{y}_n = (1 - \xi_n) \widetilde{x}_n \\ \qquad + \xi_n \int_a^b K(t,s) \cdot \widetilde{h}(s, \widetilde{x}_n(s), \widetilde{x}_n(a), \widetilde{x}_n(b))\, ds + g(t), t \in [a,b], n \in \mathbb{N}, \end{cases} \tag{2.10}$$

where $\{\xi_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1]$, $K : [a, b] \times [a, b] \to \mathbb{R}$, h, $\widetilde{h} : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ and f,$g : [a, b] \to \mathbb{R}$.

Theorem 2.2. *Consider the sequences* $\{x_n\}_{n=0}$ *and* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *generated by* (2.9) *and* (2.10)*, respectively, with the real sequence* $\{\xi_n\}_{n=0}^{\infty}$*in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_n$ *for all* $n \in \mathbb{N}$*. Assume that:*

(i) all the conditions of Theorem 2.1 hold and x^* *and* $\widetilde{x}^*$ *are solutions of equations* (2.1) *and* (2.8)*, respectively;*

(ii) there exist non negative constants ε_1 *and* ε_2 *such that* $\left| h(s, u, v, w) - \widetilde{h}(s, u, v, w) \right| \leq \varepsilon_1$ *and* $|f(t) - g(t)| \leq \varepsilon_2$*, for all* $t, s \in [a, b]$*,* $u, v, w \in \mathbb{R}$.

If the sequence $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *converge to* $\widetilde{x}^*$*, then we have*

$$\|x^* - \widetilde{x}^*\| \leq \frac{3\left[M_K (b-a)\varepsilon_1 + \varepsilon_2\right]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}. \tag{2.11}$$

Proof. Using (1.1), (2.1), (2.8)-(2.10), and assumptions (A_1)-(A_4) and (ii), we obtain

$$\begin{aligned} |x_{n+1}(t) - \widetilde{x}_{n+1}(t)| &= \left| T(y_n)(t) - \widetilde{T}(\widetilde{y}_n)(t) \right| \\ &= \left| \int_a^b K(t,s) \cdot h(s, y_n(s), y_n(a), y_n(b))\, ds + f(t) \right. \end{aligned}$$

$$
\begin{aligned}
& \left. -\int_a^b K(t,s)\cdot \widetilde{h}(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b))\,ds - g(t)\right| \\
\leq\ & \left|\int_a^b K(t,s)\cdot \left[\begin{array}{c} h(s,y_n(s),y_n(a),y_n(b)) \\ -\widetilde{h}(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)) \end{array}\right] ds\right| \\
& +|f(t)-g(t)| \\
\leq\ & M_K\int_a^b \left(\begin{array}{c} \left|\begin{array}{c} h(s,y_n(s),y_n(a),y_n(b)) \\ -h(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)) \end{array}\right| \\ +\left|\begin{array}{c} h(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)) \\ -\widetilde{h}(s,\widetilde{y}_n(s),\widetilde{y}_n(a),\widetilde{y}_n(b)) \end{array}\right| \end{array}\right) ds \\
& +|f(t)-g(t)| \\
\leq\ & M_K\int_a^b \left(\begin{array}{c} \alpha|y_n(s)-\widetilde{y}_n(s)| \\ +\beta|y_n(a)-\widetilde{y}_n(a)|+\gamma|y_n(b)-\widetilde{y}_n(b)|+\varepsilon_1 \end{array}\right) ds \\
& +\varepsilon_2 \\
\leq\ & M_K\int_a^b \left(\begin{array}{c} \alpha|y_n(s)-\widetilde{y}_n(s)| \\ +\beta|y_n(a)-\widetilde{y}_n(a)|+\gamma|y_n(b)-\widetilde{y}_n(b)| \end{array}\right) ds \\
& +M_K\int_a^b \varepsilon_1 ds+\varepsilon_2,
\end{aligned}
$$

$$
\begin{aligned}
|y_n(t)-\widetilde{y}_n(t)| \leq\ & (1-\xi_n)|x_n(t)-\widetilde{x}_n(t)|+\xi_n\left|T(x_n)(t)-\widetilde{T}(\widetilde{x}_n)(t)\right| \\
\leq\ & (1-\xi_n)|x_n(t)-\widetilde{x}_n(t)| \\
& +\xi_n M_K\int_a^b \left(\begin{array}{c} \left|\begin{array}{c} h(s,x_n(s),x_n(a),x_n(b)) \\ -h(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)) \end{array}\right| \\ +\left|\begin{array}{c} h(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)) \\ -\widetilde{h}(s,\widetilde{x}_n(s),\widetilde{x}_n(a),\widetilde{x}_n(b)) \end{array}\right| \end{array}\right) ds \\
& +\xi_n|f(t)-g(t)| \\
\leq\ & (1-\xi_n)|x_n(t)-\widetilde{x}_n(t)| \\
& +\xi_n M_K\int_a^b \left(\begin{array}{c} \alpha|x_n(s)-\widetilde{x}_n(s)| \\ +\beta|x_n(a)-\widetilde{x}_n(a)|+\gamma|x_n(b)-\widetilde{x}_n(b)|+\varepsilon_1 \end{array}\right) ds \\
& +\xi_n\varepsilon_2.
\end{aligned}
$$

Now, by taking supremum in the above inequalities, we get

$$
\begin{aligned}
\|x_{n+1}-\widetilde{x}_{n+1}\| \leq\ & M_K(\alpha+\beta+\gamma)(b-a)\|y_n-\widetilde{y}_n\| \\
& +M_K(b-a)\varepsilon_1+\varepsilon_2,
\end{aligned}
\tag{2.12}
$$

and

$$\|y_n - \widetilde{y}_n\| \leq [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| + \xi_n M_K (b - a) \varepsilon_1 + \xi_n \varepsilon_2, \tag{2.13}$$

respectively.

Combining (2.12) with (2.13) and using assumptions (A_5) and $\frac{1}{2} \leq \xi_n$ for all $n \in \mathbb{N}$ in the resulting inequality, we get

$$\begin{aligned} \|x_{n+1} - \widetilde{x}_{n+1}\| &\leq [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| \\ &\quad + \xi_n M_K (b - a) \varepsilon_1 + \xi_n \varepsilon_2 + 2\xi_n M_K (b - a) \varepsilon_1 + 2\xi_n \varepsilon_2 \\ &= [1 - \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a))] \|x_n - \widetilde{x}_n\| \\ &\quad + \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a)) \\ &\quad \times \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}. \end{aligned} \tag{2.14}$$

Denote by

$$\begin{aligned} \beta_n &= \|x_n - \widetilde{x}_n\|, \\ \mu_n &= \xi_n (1 - M_K (\alpha + \beta + \gamma)(b - a)) \in (0, 1), \\ \gamma_n &= \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)} \geq 0. \end{aligned}$$

The assumption $\frac{1}{2} \leq \xi_n$ for all $n \in \mathbb{N}$ implies $\sum_{n=0}^{\infty} \xi_n = \infty$. Now it can be easily seen that (2.14) satisfies all the conditions of Lemma 1.1. Hence it follows by its conclusion that

$$0 \leq \limsup_{n \to \infty} \|x_n - \widetilde{x}_n\| \leq \limsup_{n \to \infty} \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}.$$

By (i), we have that $\lim_{n \to \infty} x_n = x^*$. Using this fact and the assumption $\lim_{n \to \infty} \widetilde{x}_n = \widetilde{x}^*$, we get

$$\|x^* - \widetilde{x}^*\| \leq \frac{3 [M_K (b - a) \varepsilon_1 + \varepsilon_2]}{1 - M_K (\alpha + \beta + \gamma)(b - a)}.$$

□

Remark 2.1. The result given in Theorem 2.2 relate the solutions of equations (2.1) and (2.8) in the sense that if f is close to g and h is close to $\widetilde{h}$, then not only the solutions of equations (2.1) and (2.8) are close to each other, but also depend continuously on the functions involved therein. Further, if $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, then the solution x^* of equation (2.1) tends the solution $\widetilde{x}^*$ of the equation (2.8).

Example 2.1. Consider the following integral equation

$$x(t)=\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x(s)}{2}+\frac{x(0)+x(1)}{3}\right]ds+\frac{t+e^{-t}}{3}, t\in[0,1].$$

where $K\in C([0,1]\times[0,1])$, $K(t,s)=\frac{3t-2s}{5}$, $h\in C\left([0,1]\times\mathbb{R}^3\right)$, $h(s,u,v,w)=\frac{s-\sin u}{2}+\frac{v+w}{3}$, $f\in C[0,1]$, $f(t)=\frac{t+e^{-t}}{3}$, $x\in C[0,1]$ and its perturbed integral equation

$$\widetilde{x}(t)=\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\widetilde{x}(s)}{2}+\frac{\widetilde{x}(0)+\widetilde{x}(1)}{3}-s+\frac{1}{7}\right]ds+\frac{t+2e^{-t}}{3}, t\in[0,1],$$

where $K\in C([0,1]\times[0,1])$, $K(t,s)=\frac{3t-2s}{5}$, $k\in C\left([0,1]\times\mathbb{R}^3\right)$, $k(s,u,v,w)=\frac{s-\sin u}{2}+\frac{v+w}{3}-s+\frac{1}{7}$, $g\in C[0,1]$, $g(t)=\frac{t+2e^{-t}}{3}$, $\widetilde{x}\in C[0,1]$.

Define the operator $T:C[0,1]\to C[0,1]$ by

$$T(x(t))=\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x(s)}{2}+\frac{x(0)+x(1)}{3}\right]ds+\frac{t+e^{-t}}{3}, t\in[0,1].$$

We now show that the operator T is a contraction with contractivity factor $\frac{7}{10}$. Indeed,

$$\begin{aligned}
&|T(x_1(t))-T(x_2(t))|\\
=&\left|\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin x_1(s)}{2}+\frac{x_1(0)+x_1(1)}{3}-\frac{s-\sin x_2(s)}{2}-\frac{x_2(0)+x_2(1)}{3}\right]ds\right|\\
\leq&\left|\int_0^1 \left|\frac{3t-2s}{5}\right|\left|\frac{s-\sin x_1(s)}{2}+\frac{x_1(0)+x_1(1)}{3}-\frac{s-\sin x_2(s)}{2}-\frac{x_2(0)+x_2(1)}{3}\right|ds\right|\\
\leq&\left|\int_0^1 \left|\frac{3t-2s}{5}\right|\left[\frac{1}{2}|\sin x_1(s)-\sin x_2(s)|+\frac{1}{3}|x_1(0)-x_2(0)|+\frac{1}{3}|x_1(1)-x_2(1)|\right]ds\right|.
\end{aligned}$$

Now using the Chebyshev norm, we obtain

$$\begin{aligned}
\|Tx_1-Tx_2\| &\leq \sup_{t,s\in[0,1]}\left|\frac{3t-2s}{5}\right|\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right)(1-0)\|x_1-x_2\|\\
&= \frac{7}{10}\|x_1-x_2\|.
\end{aligned}$$

One can easily show on the same lines as above that the mapping $\widetilde{T}:C[0,1]\to C[0,1]$ defined by

$$\widetilde{T}(\widetilde{x}(t))=\int_0^1 \frac{3t-2s}{5}\left[\frac{s-\sin\widetilde{x}(s)}{2}+\frac{\widetilde{x}(0)+\widetilde{x}(1)}{3}-s+\frac{1}{7}\right]ds+\frac{t+2e^{-t}}{3}, t\in[0,1],$$

is also a contraction with contractivity factor $\frac{7}{10}$.

Since all the conditions of Theorem 2.1 are satisfied by the integral equations (2.1) and (2.8) so by its conclusion, normal S-iterative method (1.1) converges to unique solution x^* and $\widetilde{x}^*$, respectively in $C\,[0,1]$.

Now we have the following estimates:

$$|K\,(t,s)| = \left|\frac{3t-2s}{5}\right| \le \frac{3}{5} = M_K, t, s \in [0,1]\,,$$

$$|h\,(s,u,v,w) - k\,(s,u,v,w)| = \left|s - \frac{1}{7}\right| \le \frac{1}{7} = \varepsilon_1, foralls \in [0,1]\,, u, v, w \in \mathbb{R},$$

$$|f\,(t) - g\,(t)| = \left|\frac{t+e^{-t}-t-2e^{-t}}{3}\right| = \frac{e^{-t}}{3} \le \frac{1}{3} = \varepsilon_2, s \in [0,1]\,.$$

In view of the above estimates, all the conditions of Theorem 2.2 are satisfied and hence from (2.11), we have

$$\|x^* - \widetilde{x}^*\| \le \frac{88}{21}.$$

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DOMINATION PARAMETERS AND DIAMETER OF ABELIAN CAYLEY GRAPHS

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Abstract. Using the domination parameters of Cayley graphs constructed out of $\mathbb{Z}_p \times \mathbb{Z}_m$, where $m \in \{p^\alpha, p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}, p, q, r$ are distinct prime numbers and α, β, γ are positive integers, in this paper we have discussed the total and connected domination number and diameter of these Cayley graphs.
Key words: Cayley graph, total dominating set, connected dominating set, total domination number, connected domination number

1. Introduction and Preliminaries

Let $(G, \cdot)$ be a group and $S = S^{-1}$ be a non empty subset of G not containing the identity element e of G. The simple graph Γ whose vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{v, vs\} | v \in V(\Gamma), s \in S\}$ is called the Cayley graph of G corresponding to the set S and is denoted by $\mathrm{Cay}(G, S)$. By $\mathbb{Z}_n$ we denote the cyclic group of order n. For any vertex $v \in V(\Gamma)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(\Gamma) | \{u, v\} \in E(\Gamma)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(\Gamma)$, the open neighborhood of X is $N(X) = \bigcup_{v \in X} N(v)$ and the closed neighborhood of X is $N[X] = N(X) \cup X$ [6]. A set $D \subseteq V(\Gamma)$ is said to be a *dominating set* if $N[D] = V(\Gamma)$ or equivalently, every vertex in $V(\Gamma) \backslash D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(\Gamma)$ is the minimum cardinality of a dominating set in Γ. A dominating set with cardinality $\gamma(\Gamma)$ is called a γ*-set*. A set $T \subseteq V(\Gamma)$ is said to be a *total*

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dominating set if $N(T) = V(\Gamma)$ or equivalently, every vertex in $V(\Gamma)$ is adjacent to a vertex in T. The *total domination number* $\gamma_t(\Gamma)$ is the minimum cardinality of a total dominating set in Γ. A total dominating set with cardinality $\gamma_t(\Gamma)$ is called a γ_t*-set*. A graph Γ is said to be connected graph if there is at least one path between every pair of vertices in Γ. The connected components of a graph are its maximal connected subgraphs. A dominating set D of Γ is said to be a connected dominating set if the induced subgraph generated by D is connected. The minimum cardinality of a connected dominating set of Γ is called the connected domination number of Γ and is denoted by $\gamma_c(\Gamma)$, and the corresponding set is denoted by γ_c*-set* of Γ. Let λ be the length of the longest sequence of consecutive integers in $\mathbb{Z}_m$, each of which shares a prime factor with m. Dominating sets were defined by Berge and Ore [1, 16]. The concept of total domination in graphs was initiated by E.J. Cockayne and R.W. Dows and S.T. Hedetniemi [4]. S.T Hedetniemi, R.C. Laskar[7] introduced the connected domination number in graphs. Madhavi [10] present the concept of Euler totient Cayley graphs and their domination parameters studied by Uma Maheswary and B. Maheswary [11]. Also some properties of direct product graphs of Cayley graphs with arithmetic graphs discussed by Uma Maheswary and B. Maheswary [13], and their domination parameters studied by Uma Maheswary and B. Maheswary and M. Manjuri [12, 14, 15].

A walk is a sequence of pairwise adjacent vertices of a graph. A path is a walk in which no vertex is repeated. The distance between two vertices of a graph is the number of edges of the shortest path between them. The diameter of a connected graph is the maximum distance between any two vertices of the graph. According to this definition, the diameter of a disconnected graph is infinite, but if we consider the diameter as the maximum finite shortest path length in the graph, this is the same as the largest of diameters of the graph's connected components. So in this paper by diameter of a disconnected graph we mean the largest diameter of its connected components. Let $v, w \in V(\Gamma)$ then the distance between v, w is denoted by $d(v, w)$ and the diameter of Γ is denoted by $\mathrm{diam}(\Gamma)$ [2, 3].

Here we study the total and connected dominating sets and diameter of Cayley graphs constructed out of $\mathbb{Z}_p \times \mathbb{Z}_m$ where $m \in \{p^\alpha, p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}, p, q, r$ are distinct prime numbers and α, β, γ are positive integers. The domination number of these graphs are presented in [8] and we present some of the results without proofs .

Theorem 1.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha}$. *Then*

1) $\gamma(\Gamma) = 2$ *where* $p = 2$ *and* $\alpha = 1$.

2) $\gamma(\Gamma) = 4$ *where* $p = 2$ *and* $\alpha \geq 2$.

3) $\gamma(\Gamma) = 3$ *where* $p \geq 3$ *and* $\alpha \geq 1$.

Theorem 1.2. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta}$, $p, q \geq 2$ *and* $\alpha, \beta \geq 1$. *Then* $\gamma(\Gamma)$ *is given by Table 1.1.*

Theorem 1.3. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ *where* $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta r^\gamma}$, $p, q, r \geq 2$ *and* $\alpha, \beta, \gamma \geq 1$. *Then* $\gamma(\Gamma)$ *is given by Table 1.2.*

Table 1.1: $\gamma(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$

Γ	$\gamma(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{pq}, \Phi)$	4	
$\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$	8	$(\alpha, \beta) \neq (1,1)$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$	6	$(\alpha, \beta) \neq (1,1)$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	5	$(\alpha, \beta) \neq (1,1)$ $p = 3, q \geq 5$ *or* $q = 3, p \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	4	$(\alpha, \beta) \neq (1,1)$ $p, q \geq 5$

Table 1.2: $\gamma(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$

Γ	$\gamma(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2qr}, \Phi)$	8	
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2pr}, \Phi)$	8	
$\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$	12	$\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$	10	$\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$ $p = 3, r \geq 5$ *or* $r = 3, p \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$	8	$\alpha \neq 1$ *or* $\beta \neq 1$ *or* $\gamma \neq 1$ $p, r \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	$6 \leq \gamma(\Gamma) \leq 8$	$\alpha, \beta, \gamma \geq 1$ one of the prime factors is 3
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	5	$p, q, r \geq 5$ and $\alpha, \beta, \gamma \geq 1$

Let $p_1, p_2, \ldots, p_k$ be consecutive prime numbers, $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers and $\Phi = \varphi_2 \times \varphi_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}$.

Theorem 1.4. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)$*, where* $p_1 = 3$ *and* $\alpha \geq 2$*. Then* $\gamma(\Gamma) \geq 4k + 4$.

For $p = 2$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$, where $\Phi = \varphi_p \times \varphi_m$ and m is a multiple of 2, is a disconnected graph with two connected components, say Γ_1 and Γ_2, where $V(\Gamma_1) = \{(1, v) | v \textit{ is odd}\} \cup \{(0, v) | v \textit{ is even}\}$ and $V(\Gamma_2) = \{(0, v) | v \textit{ is odd}\} \cup \{(1, v) | v \textit{ is even}\}$. Since every Cayley graph $\mathrm{Cay}(G, S)$ is $|S|$-regular (see for example [5]), we find that Γ is $|\Phi|$-regular.

Let X be a set of consecutive integers in $\mathbb{Z}_m$ such that for every $x \in X$, we have $\gcd(x, m) > 1$. In this case we call X_i a consecutive set. We use X_i^k to show that the consecutive set X_i has k elements.

Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$. In Section 2. we calculate $\gamma_t(\Gamma)$ and $\gamma_c(\Gamma)$ and $\mathrm{diam}(\Gamma)$ where $m = p^\alpha$. We consider the case $m = p^\alpha q^\beta$ in Section 3. and the case $m = p^\alpha q^\beta r^\gamma$ is considered in Section 4.

2. Total and connected domination number and diameter of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$

Let p be a prime number, α a positive integer and $\Phi = \varphi_p \times \varphi_{p^\alpha}$. In this section, we obtain the total and connected domination number and diameter of $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$.

Theorem 2.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$*. Then*

1) $\mathrm{diam}(\Gamma) = 1$ *where* $p = 2$ *and* $\alpha = 1$*.*

2) $\mathrm{diam}(\Gamma) = 2$ *where* $p = 2, \alpha \geq 2$ *or* $p \geq 3, \alpha \geq 1$*.*

Proof. 1) In this case $\Gamma \cong 2K_2$, and clearly the diameter of Γ is 1.

2) Let $p = 2$ and $\alpha \geq 2$. Then Γ is a disconnected graph with two connected components, say Γ_1 and Γ_2, where $V(\Gamma_1) = \{(1, v)| v \textit{ is odd}\} \cup \{(0, v)| v \textit{ is even}\}$ and $V(\Gamma_2) = \{(0, v)| v \textit{ is odd}\} \cup \{(1, v)| v \textit{ is even}\}$.

Let $(u, v), (u', v') \in V(\Gamma_1)$. Then we have the following two possibilities:

i) $u = u'$ and $v \neq v'$. Obviously (u, v) and (u, v') are not adjacent. This implies that $d((u, v), (u', v')) \geq 2$. On the other hand the vertex $(u-1, v-1)$ is adjacent to both vertices. So $d((u, v), (u', v')) = 2$.

ii) $u \neq u'$ and $v \neq v'$. We know that $u - u' \in \varphi_2$ and $v - v'$ is an odd integer. Since all of the odd integers in $\mathbb{Z}_{2^\alpha}$ to be included into a φ_{2^α}, hence $v - v' \in \varphi_{2^\alpha}$. Thus (u, v) is adjacent to (u', v'). So $d((u, v), (u', v')) = 1$.

Since (u, v) and (u', v') are arbitrary vertices of Γ_1, hence the diameter of Γ_1 is 2. Similarly the diameter of Γ_2 is 2.

Let $p \geq 3$ and $\alpha \geq 1$. Then Γ is connected graph where

$$V(\Gamma) = \{(0, 0), \ldots, (0, p^\alpha - 1), \ldots, (p-1, 0), \ldots, (p-1, p^\alpha - 1)\}.$$

Assume that (u, v) and (u', v') are arbitrary vertices of Γ. Now we have the following three possibilities:

i) $u = u'$ and $v \neq v'$. Since (u, v) and (u', v') are not adjacent $d((u, v), (u', v')) \geq 2$. Let v and v' be multiple of p. Note that 0 is multiple of p. Then $(u-1, p-1)$ is adjacent to both (u, v) and (u', v'). Let v and v' be non-multiple of p. Then $(u-1, p)$ is common neighbor of (u, v) and (u', v'). Now let one of either v or v' is multiple of p. Without loss of generality let v is multiple of p and v' is non-multiple of p. Suppose that v and v' are both even or odd. Then $(u-1, \frac{v+v'}{2})$ is adjacent to both (u, v) and (u', v'). Since $v - v'$ is even so $v - v'$ is divisible by 2. Hence $v - \frac{v+v'}{2} = \frac{2v-v-v'}{2} = \frac{v-v'}{2} \in \varphi_{p^\alpha}$ and also $v' - \frac{v+v'}{2} = \frac{v'-v}{2} \in \varphi_{p^\alpha}$. Now assume that one of either v or v' is even. Then $(u-1, 2v')$ is common neighbor of (u, v) and (u', v'). Therefore $d((u, v), (u', v')) = 2$.

ii) $u \neq u'$ and $v = v'$. In this case vertex $(u'', v-1)$ where $u'' \neq u, u'$ is adjacent to both (u, v) and (u', v'). Thus $d((u, v), (u', v')) = 2$.

iii) $u \neq u^{'}$ and $v \neq v^{'}$. If (u,v) and $(u^{'},v^{'})$ be adjacent then $d((u,v),(u^{'},v^{'})) = 1$. If (u,v) and $(u^{'},v^{'})$ be non-adjacent then similar to $i)$ and $ii)$, $d((u,v),(u^{'},v^{'})) = 2$. Therefore in this case $\operatorname{diam}(\Gamma) = 2$. $\square$

Theorem 2.2. *Let* $\Gamma = \operatorname{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha}, \Phi)$. *Then*

1) $\gamma_t(\Gamma) = 4$ *and* $\gamma_c(\Gamma)$ *does not exist where* $p = 2$ *and* $\alpha \geq 1$.

2) $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 3$ *where* $p \geq 3$ *and* $\alpha \geq 1$.

Proof. 1) Let $p = 2$ and $\alpha = 1$. Then $\Gamma \cong 2K_2$, and obviously $\gamma_t(\Gamma) = 4$.

Assume that $p = 2$ and $\alpha \geq 2$. Then by [8, Theorem 2.1], $\gamma(\Gamma) = 4$ and $D = \{(0,0),(0,1),(1,0),(1,1)\}$ is a γ-set for Γ. Since $(0,0)$ and $(0,1)$ are adjacent to $(1,1)$ and $(1,0)$, respectively. Hence D is a γ_t-set for Γ. Thus $\gamma_t(\Gamma) = 4$.

In this case Γ is a disconnected graph. Hence by the definition of connected dominating set, γ_c*-set* does not exist for Γ

2) Let $p \geq 3$ and $\alpha \geq 1$. By [8, Theorem 2.1], we find that $\gamma(\Gamma) = 3$ and $D = \{(0,1),(1,0),(2,2)\}$ is a γ-set for Γ. Vertices of D dominate among themselves. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 3$. $\square$

Example 2.1. Let $\Gamma_1 = \operatorname{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}, \Phi)$ and $\Gamma_2 = \operatorname{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi)$, which are shown in Figures 2.1 and 2.2, respectively. Clearly Γ_1 is a disconnected graph with two connected components. Thus γ_c*-set* does not exist for Γ_1. Also, total dominating set of Γ_1, is $\{(0,0),(0,1),(1,0),(1,1)\}$. Note that total and connected dominating set of Γ_2 is $\{(0,1),(1,0),(2,2)\}$.

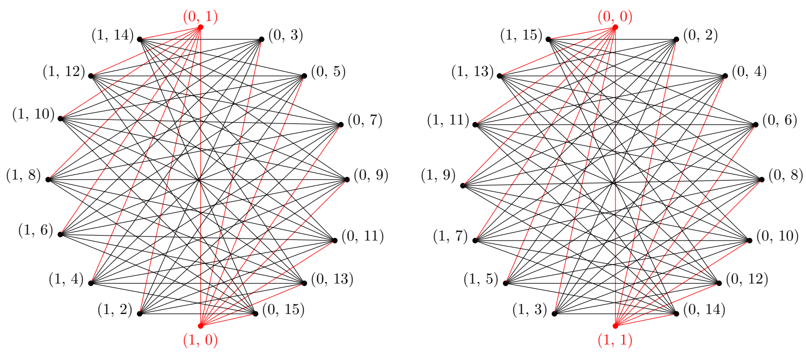

FIG. 2.1: The graph $\Gamma_1 = \operatorname{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^4}, \Phi)$ and its total dominating set.

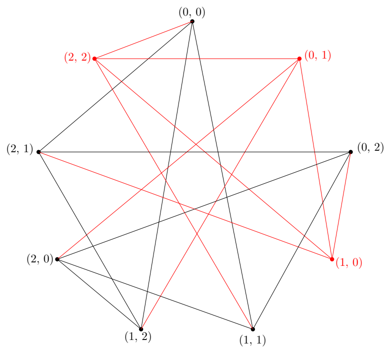

FIG. 2.2: The graph $\Gamma_2 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi)$ and its total and connected dominating set.

3. Total and connected domination number and diameter of $\text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$

Let p, q be prime numbers, α, β positive integers and $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta}$. In this section, we find the total and connected domination number and diameter of $\Gamma = \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$.

Lemma 3.1. *Let* $\Gamma = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$*, where* $\alpha, \beta \geq 1$*. Then* $\text{diam}(\Gamma) = 3$*.*

Proof. Γ is a disconnected graph with two connected components, say Γ_1 and Γ_2, where $V(\Gamma_1) = \{(1, v) | v \text{ is odd}\} \cup \{(0, v) | v \text{ is even}\}$ and $V(\Gamma_2) = \{(0, v) | v \text{ is odd}\} \cup \{(1, v) | v \text{ is even}\}$.

Let $(u, v), (u^{'}, v^{'}) \in V(\Gamma_1)$. Then we have the following two possibilities:

i) $u = u^{'}$ and $v \neq v^{'}$. Clearly $d((u, v), (u^{'}, v^{'})) \geq 2$. Let v and $v^{'}$ be multiple of $2q$. Then $(u - 1, 2q - 1)$ is common neighbor of (u, v) and $(u^{'}, v^{'})$. Also if v and $v^{'}$ be non-multiple of $2q$, then $(u - 1, 2q)$ is adjacent to both (u, v) and $(u^{'}, v^{'})$. Note that a trivial observation shows that v and v' have the same parity. Let v and $v^{'}$ be both multiple of one of the prime factors 2 or q. Then the other prime factor is adjacent to both v and $v^{'}$. Now let one of either v or $v^{'}$ is odd and is multiple of q. Then $(u, v), (u^{'}, v^{'}) \in \{(1, v) | v \text{ is odd}\}$. If $\frac{v+v^{'}}{2}$ be even, then $(u - 1, \frac{v+v^{'}}{2})$ is common neighbor of (u, v) and $(u^{'}, v^{'})$. Also if $\frac{v+v^{'}}{2}$ be odd, then $(u - 1, \frac{v+v^{'}}{2} + q)$ is adjacent to both (u, v) and $(u^{'}, v^{'})$. Let one of either v or $v^{'}$ is multiple of $2q$. So $(u, v), (u^{'}, v^{'}) \in \{(0, v) | v \text{ is even}\}$. If $\frac{v+v^{'}}{2} \in \varphi_{2^\alpha q^\beta}$, then $(u - 1, \frac{v+v^{'}}{2})$ is common neighbor of (u, v) and $(u^{'}, v^{'})$. Moreover if $\frac{v+v^{'}}{2}$ be even, then $(u - 1, \frac{v+v^{'}}{2} + q)$ is adjacent to both (u, v) and $(u^{'}, v^{'})$. Thus in this case $d((u, v), (u^{'}, v^{'})) = 2$.

ii) $u \neq u^{'}$ and $v \neq v^{'}$. If $v - v^{'} \in \varphi_{2^\alpha q^\beta}$, then $d((u, v), (u^{'}, v^{'})) = 1$. Suppose that $v - v^{'} \notin \varphi_{2^\alpha q^\beta}$, since $u \neq u^{'}$ and $u, u^{'} \in \mathbb{Z}_2$, we have no common neighbor between (u, v) and $(u^{'}, v^{'})$. This implies that $d((u, v), (u^{'}, v^{'})) \geq 3$. Without loss of generality assume that $u = 0$ and $u^{'} = 1$. Since $v - v^{'}$ is an odd integer, we find that $v - v^{'} + 2 \in \varphi_{2^{\alpha}q^{\beta}}$. Thus $(0, v)(1, v + 1)(0, v + 2)(1, v^{'})$ is a path of length 3 between $(0, v)$ and $(1, v^{'})$. So $\mathrm{diam}(\Gamma_1) = 3$ and similarly $\mathrm{diam}(\Gamma_2) = 3$. Therefore $\mathrm{diam}(\Gamma) = 3$. □

Lemma 3.2. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha}q^{\beta}}, \Phi)$*, where* $\alpha, \beta \geq 1$*. Then* $\gamma_c(\Gamma)$ *does not exist and* $\gamma_t(\Gamma) = 8$*.*

Proof. Γ is a disconnected graph with exactly two connected components Γ_1 and Γ_2 where $V(\Gamma_1) = \{(1, v)|v \textit{ is odd}\} \cup \{(0, v)|v \textit{ is even}\}$ and $V(\Gamma_2) = \{(0, v)|v \textit{ is odd}\} \cup \{(1, v)|v \textit{ is even}\}$. Hence by the definition of connected dominating set, γ_c-*set* does not exist for Γ.

Assume first that $(\alpha, \beta) = (1, 1)$. Then by [8, Proposition 3.1], $A = \{(0, 0), (1, q)\}$ and $B = \{(0, 1), (1, q + 1)\}$ dominate $V(\Gamma_1)\backslash A$ and $V(\Gamma_2)\backslash B$, respectively. Hence $\gamma(\Gamma) = 4$. Vertices of A are not adjacent to each other and A is not dominated by one vertex. Note that $(1, 1)$ and $(0, q + 1)$ are adjacent to $(0, 0)$ and $(1, q)$, respectively. Hence $T_1 = \{(0, 0), (1, 1), (1, q), (0, q + 1)\}$ is a γ_t-set for Γ_1. Similarly $T_2 = \{(0, 1), (1, 0), (0, q), (1, q + 1)\}$ is a γ_t-set for Γ_2. Therefore $\gamma_t(\Gamma) = 8$.

Next consider the case where $(\alpha, \beta) \neq (1, 1)$. By [8, Lemma 3.2], $\gamma(\Gamma) = 8$ and $D = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\}$ is a γ-set for Γ. Vertices $(0, 1), (0, 0), (0, 3)$,
$(0, 2)$ are adjacent to vertices $(1, 0), (1, 1), (1, 2), (1, 3)$ respectively. Thus D becomes a γ_t-set for Γ. Hence $\gamma_t(\Gamma) = 8$. □

Proposition 3.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^{\alpha}p^{\beta}}, \Phi)$*, where* $\alpha, \beta \geq 1$*. Then* $\mathrm{diam}(\Gamma) = 3$*.*

Proof. Let $(u, v), (u^{'}, v^{'}) \in V(\Gamma)$. Then we have the following three possibilities:

i) $u = u^{'}$ and $v \neq v^{'}$. In this case $d((u, v), (u^{'}, v^{'})) \geq 2$. Suppose that v and $v^{'}$ are both even or odd. Hence by case *i)* of Lemma 3.1, $d((u, v), (u^{'}, v^{'})) = 2$. Since in $\mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha}p^{\beta}}$ we have two connected components, where in each of them, if $u = u^{'}$ then v and $v^{'}$ are both even or odd.

Assume that one of either v or $v^{'}$ is even. Without loss of generality let v is even and $v^{'}$ is odd. Also let $(u^{''}, v^{''})$ where $u^{''} \neq u$, is common neighbor between $(u, v), (u^{'}, v^{'})$. If $v^{''}$ be even then $v - v^{''} \notin \varphi_{2^{\alpha}p^{\beta}}$ and if $v^{''}$ be odd then $v^{'} - v^{''} \notin \varphi_{2^{\alpha}p^{\beta}}$. Thus we have no common neighbor between (u, v) and $(u^{'}, v^{'})$. Hence $d((u, v), (u^{'}, v^{'})) \geq 3$. We consider $u^{''}, u^{'''} \neq u$, if v and $v^{'}$ be multiple of p, then $(u, v)(u^{''}, p - 2)(u^{'''}, p - 1)(u, v^{'})$ is a path of length 3 between (u, v) and $(u^{'}, v^{'})$. If v and $v^{'}$ be non-multiple of p, then the path $(u, v)(u^{''}, p)(u^{'''}, 2v^{'})(u^{'}, v^{'})$ is connected. If v be multiple of p and $v^{'}$ be non-multiple of p, since $v - v^{'} \in \varphi_{2^{\alpha}p^{\beta}}$ then $(u, v)(u^{''}, v^{'})(u^{'''}, v)(u^{'}, v^{'})$ is a path of length 3 between (u, v) and $(u^{'}, v^{'})$.

ii) $u \neq u^{'}$ and $v = v^{'}$. In this case $(u^{''}, v - 1)$ where $u^{''} \neq u, u^{'}$ is common neighbor of (u, v) and $(u^{'}, v^{'})$. Hence $d((u, v), (u^{'}, v^{'})) = 2$.

iii) $u \neq u^{'}$ and $v \neq v^{'}$. If (u,v) and $(u^{'},v^{'})$ be adjacent then $d((u,v),(u^{'},v^{'})) = 1$. Now assume that (u,v) and $(u^{'},v^{'})$ are not adjacent. Let v and $v^{'}$ be both even or odd. Then by case *i)* of Lemma 3.1, we know that there is a vertex $(u^{''},v^{''})$, where $u^{''} \neq u, u^{'}$ and $v^{''}$ is adjacent to v and $v^{'}$, that is adjacent to both (u,v) and $(u^{'},v^{'})$. Thus $d((u,v),(u^{'},v^{'})) = 2$.

Now let one of either v or $v^{'}$ is even. Then by second paragraph of case *i)* and also by using of case *ii)* of Lemma 3.1, we see that $d((u,v),(u^{'},v^{'})) = 3$. Therefore $\mathrm{diam}(\Gamma) = 3$. □

Proposition 3.2. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$, *where* $\alpha, \beta \geq 1$. *Then*

i) $\gamma_t(\Gamma) = 6$.

ii) $\gamma_c(\Gamma)$ *is given by Table* 3.1.

Table 3.1: $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi))$ where $\alpha, \beta \geq 1$.

Γ	$\gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$	7	$p = 3$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$	6	$p \geq 5$

Proof. *i)* Let $(\alpha,\beta) = (1,1)$. By [8, Proposition 3.1], we see that $\gamma(\Gamma) = 4$ and $D = \{(0,0),(0,1),(1,p),(1,p+1)\}$ is a γ-set for Γ. Vertices of D are not adjacent to each other. Hence $\gamma_t(\Gamma) > 4$. Let a vertex say (u,v) dominates all vertices of D. Then (u,v) is adjacent to $(0,0)$ hence $(u,v) \in \Phi$. On the other hand (u,v) is adjacent to $(0,1)$ thus $(u,v) \notin \Phi$, which is impossible. We conclude that $\gamma_t(\Gamma) > 5$. Since vertex $(p-1,p-1)$ is adjacent to vertices $(0,1),(1,p)$ and also vertex $(p-1,2p-1)$ is adjacent to vertices $(0,0),(1,p+1)$. Hence $T = \{(0,0),(0,1),(1,p),(1,p+1),(p-1,p-1),(p-1,2p-1)\}$ is a γ_t-set for Γ.

Finally $(\alpha,\beta) \neq (1,1)$. In this case by [8, Proposition 3.3], $\gamma(\Gamma) = 6$ and $D' = \{(0,0),(0,1),(1,2),(1,3),(2,4),(2,5)\}$ is a γ-set for Γ. If $p = 3$, then we find that vertices $(0,0),(0,1),(1,3)$ are adjacent to vertices $(2,5),(1,2),(2,4)$, respectively and if $p \geq 5$ then vertices $(0,0),(1,3),(2,4),(0,1),(1,2)$ are adjacent to vertices $(1,3),(2,4),(0,1),(1,2),(2,5)$, respectively. Thus D' becomes a γ_t-set for Γ.

Note that both T and $D^{'}$ are two γ_t-sets for Γ, where $\alpha, \beta \geq 1$. Therefore $\gamma_t(\Gamma) = 6$.

ii) By using a similar argument given in the proof of case *i)*, we have $\gamma_c(\Gamma) \geq 6$.

Assume first that $p = 3$. Then the subgraphs generated by T and $D^{'}$ are disconnected. Since the subgraph generated by $D^{'}$ has exactly three connected components which are induced subgraphs generated by sets $\{(0,0),(2,5)\}$, $\{(0,1),(1,2)\}$ and $\{(1,3),(2,4)\}$, also the subgraph generated by T has exactly two connected components which are induced subgraphs generated by sets $\{(0,1),(1,p),(p-1,p-1)\}$ and $\{(0,0),(1,p+1),(p-1,2p-1)\}$. We conclude that $\gamma_c(\Gamma) \geq 7$.

Note that vertex $(0,p)$ is adjacent to vertices $(p-1,p-1)$ and $(1,p+1)$. Therefore $C=\{(0,0),(0,1),(1,p),(1,p+1),(p-1,p-1),(p-1,2p-1),(0,p)\}$ is a connected dominating set for Γ with minimum cardinality. Therefore $\gamma_c(\Gamma)=7$.

Now suppose that $p\geq 5$. According to the proof of final part of case $i)$, we see that $D^{'}$ becomes a connected dominating set for Γ with minimum cardinality. Therefore in this case $\gamma_c(\Gamma)=6$. $\square$

Proposition 3.3. *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{p^\alpha q^\beta},\Phi)$, *where* $p,q\geq 3$ *and* $\alpha,\beta\geq 1$. *Then* $\mathrm{diam}(\Gamma)=2$.

Proof. Let $(u,v),(u^{'},v^{'})\in V(\Gamma)$. Then we have the following three possibilities:

i) $u=u^{'}$ and $v\neq v^{'}$. Hence $d((u,v),\,(u^{'},v^{'}))\geq 2$. Let v and $v^{'}$ be multiple of pq, then $(u-1,pq-1)$ is common neighbor of (u,v) and $(u^{'},v^{'})$. Let $v,v^{'}\in\varphi_{p^\alpha q^\beta}$, then $(u-1,pq)$ is adjacent to both (u,v) and $(u^{'},v^{'})$. Let v and $v^{'}$ be multiple of p, then q is adjacent to both v and $v^{'}$. Also let v and $v^{'}$ be multiple of q, then p is adjacent to both v and $v^{'}$. So $d((u,v),\,(u^{'},v^{'}))=2$. Let v is multiple of p and $v^{'}$ is multiple of q. If v and $v^{'}$ be both even or odd, then we show that $(u-1,\frac{v+v^{'}}{2})$ is a common neighbor of (u,v) and $(u^{'},v^{'})$. Assume that $v=kp$ and $v^{'}=k^{'}q$; $k,k^{'}\in\mathbb{Z}$. Then $v-\frac{v+v^{'}}{2}=\frac{v-v^{'}}{2}=\frac{kp-k^{'}q}{2}$. Suppose that $\frac{kp-k^{'}q}{2}\notin\varphi_{p^\alpha q^\beta}$ and without loss of generality assume $\frac{kp-k^{'}q}{2}=k^{''}p$; $k^{''}\in\mathbb{Z}$. Then $kp-k^{'}q=2k^{''}p$ which implies $kp-2k^{''}p=k^{'}q$. Hence $(\frac{k-2k^{''}}{k^{'}})p=q$, which is impossible, since q is not a multiple of p. Hence $\frac{kp-k^{'}q}{2}\in\varphi_{p^\alpha q^\beta}$, and v is adjacent to $\frac{v+v^{'}}{2}$. Similarly $v^{'}$ is adjacent to $\frac{v+v^{'}}{2}$. If one of either v or $v^{'}$ be odd, then $2(v+v^{'})$ is adjacent to both v and $v^{'}$. Assume that $v=kp$ is even and $v^{'}=k^{'}q$ is odd. Without loss of generality let $2(v+v^{'})-v=v+2v^{'}=k^{''}p$. Then $kp+2k^{'}q=k^{''}p$. This implies $(\frac{k^{''}-k}{2k^{'}})p=q$, which is impossible. Thus v is adjacent to $2(v+v^{'})$. Similarly $v^{'}$ is adjacent to $2(v+v^{'})$. Hence $d((u,v),\,(u^{'},v^{'}))=2$. Let v be multiple of p or q and $v^{'}\in\varphi_{p^\alpha q^\beta}$. Assume that v and $v^{'}$ be both even or odd. If $v-v^{'}\in\varphi_{p^\alpha q^\beta}$ then it is easy to see that $\frac{v+v^{'}}{2}$ is adjacent to both v and $v^{'}$ and if $v-v^{'}\notin\varphi_{p^\alpha q^\beta}$ then $v-v^{'}$ is adjacent to both v and $v^{'}$. Now suppose that one of either v or $v^{'}$ is odd. If v be multiple of p then $v^{'}q$ is adjacent to both v and $v^{'}$. If v be multiple of q then $v^{'}p$ is adjacent to both v and $v^{'}$. Moreover if v be multiple of pq then $v^{'}(p+q)$ is adjacent to both v and $v^{'}$. Thus $d((u,v),\,(u^{'},v^{'}))=2$. Let one of either v or $v^{'}$ is multiple of p or q and other is multiple of pq. We know that $+2$ and -2 is adjacent to all of the multiple of pq. Since by proof of [8, Proposition 3.1], $\lambda=2$, hence v is adjacent to $+2$ or -2 or both of them. So we have a common neighbor between (u,v) and $(u^{'},v^{'})$. Therefore $d((u,v),\,(u^{'},v^{'}))=2$.

ii) $u\neq u^{'}$ and $v=v^{'}$. In this case the vertex $(u^{''},v-1)$ where $u^{''}\neq u,u^{'}$, is a common neighbor of (u,v) and $(u^{'},v^{'})$. Thus $d((u,v),\,(u^{'},v^{'}))=2$.

iii) $u\neq u^{'}$ and $v\neq v^{'}$. Hence by *i)* and *ii)*, $d((u,v),\,(u^{'},v^{'}))=2$.

Therefore $\mathrm{diam}(\Gamma)=2$. $\square$

Proposition 3.4. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$*, where* $p, q \geq 3$ *and* $\alpha, \beta \geq 1$*. Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table* 3.2.

Table 3.2: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)) = \gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$ where $p, q \geq 3$ and $\alpha, \beta \geq 1$.

Γ	$\gamma_t(\Gamma), \gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	5	one of the prime factors is 3
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	4	$p, q \geq 5$

Proof. Assume first that one of the prime factors is 3. Let $(\alpha, \beta) = (1, 1)$. Then by [8, Proposition 3.1], $\gamma(\Gamma) = 4$ and $D = \{(0,0), (0,1), (1,x'), (1,y')\}$ is a γ-set for Γ, where x, x' and y, y' are consecutive integers in $\mathbb{Z}_{pq}$, each of which shares a prime factor with pq where x' is a multiple of p and y' is a multiple of q. Note that vertices of D are not adjacent to each other. Hence $\gamma_t(\Gamma) > 4$. Also D is dominated by $\{(2,2)\}$. Thus $T = \{(0,0), (0,1), (1,x'), (1,y'), (2,2)\}$ is a γ_t-set and γ_c-set for Γ.

The next case is where $(\alpha, \beta) \neq (1, 1)$. By [8, Table 1], $\gamma(\Gamma) = 5$ and $D = \{(0,0), (0,1), (1,2), (2,3), (2,4)\}$ is a γ-set for Γ. Vertices $(0,0), (2,4), (1,2), (0,1)$ are adjacent to vertices $(2,4), (1,2), (0,1), (2,3)$, respectively. Hence D dominates all vertices of Γ and the subgraph generated by D is connected. Thus D becomes a γ_t-set and γ_c-set for Γ. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 5$.

Finally assume that $p, q \geq 5$. Then by [8, Proposition 3.1, Table 1], $\gamma(\Gamma) = 4$ and $D = \{(0,0), (1,1), (2,2), (3,3)\}$ is a γ-set for Γ. Since $p, q \geq 5$ then vertices of D dominate among themselves. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 4$. $\square$

As an immediate consequence of Lemma 3.2 and Propositions 3.2, 3.4, we have the following theorem.

Theorem 3.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$*, where* $p, q \geq 2$ *and* $\alpha, \beta \geq 1$*. Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table 3.3.*

Table 3.3: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$, $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi))$ where $\alpha, \beta \geq 1$.

Γ	$\gamma_t(\Gamma)$	$\gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta}, \Phi)$	8	does not exist	
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$	6	7	$p = 3$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta}, \Phi)$	6	6	$p \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	5	5	one of the prime factors is 3
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta}, \Phi)$	4	4	$p, q \geq 5$

Example 3.1. The graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2\times 3^2}, \Phi)$, which is shown in Figure 3.1, is a disconnected graph with two connected components, say Γ_1 and Γ_2. Thus γ_c-*set* does not exist for Γ. In this graph two sets $T_1 = \{(0,0),(0,4),(1,1),(1,3)\}$ and $T_2 = \{(0,1),(0,3),(1,0),(1,4)\}$ are γ_t-*sets* sets for Γ_1 and Γ_2, respectively. Hence $\gamma_t(\Gamma) = 8$.

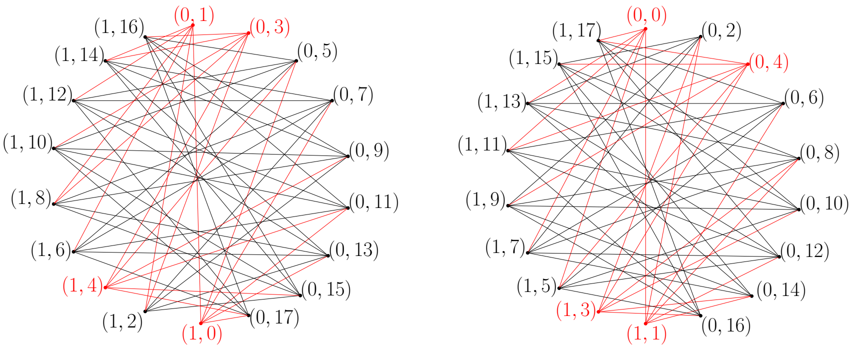

FIG. 3.1: Two connected components of $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2\times 3^2}, \Phi)$, left Γ_1, right Γ_2

Example 3.2. Let $p = 3$, $q = 5$. Then total and connected dominating set of $\Gamma = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_{15}, \Phi)$, which is shown in Figure 3.2, is $\{(0,0),(0,1),(1,6),(1,10),(2,2)\}$.

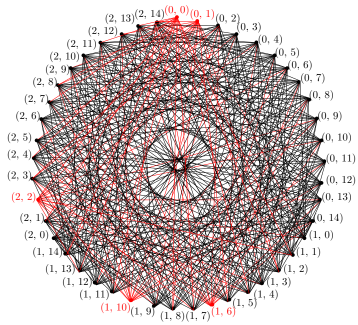

FIG. 3.2: The graph $\Gamma = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_{15}, \Phi)$ and its total dominating set.

4. Total and connected domination number and diameter of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$

Let p, q, r be three prime numbers, α, β, γ positive integers and $\Phi = \varphi_p \times \varphi_{p^\alpha q^\beta r^\gamma}$. In this section, we obtain the total and connected domination number of $\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$ and we extend the results in the previous section for diameter of this graph.

Lemma 4.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$*, where* p, q, r *are distinct prime numbers and* $\alpha, \beta, \gamma \geq 1$*. Then* $\mathrm{diam}(\Gamma) = 3$*.*

Proof. Γ is a disconnected graph with two connected components, say Γ_1 and Γ_2, where $V(\Gamma_1) = \{(1, v) | v \ is\ odd\} \cup \{(0, v) | v\ is\ even\}$ and $V(\Gamma_2) = \{(0, v) | v\ is\ odd\} \cup \{(1, v) | v\ is\ even\}$.

Let $(u, v), (u^{'}, v^{'}) \in V(\Gamma_1)$. Then we have the following two possibilities:

i) $u = u^{'}, v \neq v^{'}$. Since $u = u^{'}$ hence $d((u, v), (u^{'}, v^{'})) \geq 2$. Now by Table 4.1 we show that $d((u, v), (u^{'}, v^{'})) = 2$. In this table, when $v, v^{'}$ are odd we have $u = u^{'} = 1,\ u^{''} = 0$ and when $v, v^{'}$ are even we have $u = u^{'} = 0,\ u^{''} = 1$. We prove the rows $6, 8$ of the table and the rest is similarly proven.

Let $v, v^{'}$ are odd and $v = kq,\ v^{'} = k^{'}qr,\ k, k^{'} \in \mathbb{Z}$. If $\frac{v+v^{'}}{q}$ be non-multiple of q then we show that $\frac{v+v^{'}}{q}$ is adjacent to both v and $v^{'}$.

Let $k^{''} \in \mathbb{Z}$. If $v - \frac{v+v^{'}}{q} = 2k^{''}$, then $k(q-1) - k^{'}r = 2k^{''}$. This implies $k = \frac{2k^{''}+k^{'}r}{q-1}$. Since $2k^{''} + k^{'}r$ is odd and $q - 1$ is even hence k is non-integer, which is impossible. If $v - \frac{v+v^{'}}{q} = k^{''}q$, then $\frac{v+v^{'}}{q} = (k - k^{''})q$, which is inaccurate because $\frac{v+v^{'}}{q}$ is non-multiple of q. Moreover if $v - \frac{v+v^{'}}{q} = k^{''}r$, then $k = (\frac{k^{'}+k^{''}}{q-1})r$. But we know that k is non-multiple of r. So $v - \frac{v+v^{'}}{q} \in \varphi_{2^\alpha q^\beta r^\gamma}$ and similarly $v^{'}$ is adjacent to $\frac{v+v^{'}}{q}$. Since $u^{''}$ is adjacent to $u, u^{'}$ thus $(u^{''}, \frac{v+v^{'}}{q})$ is common neighbor between $(u, v), (u^{'}, v^{'})$. Similarly it is easy to see that if $\frac{v+v^{'}}{q}$ be multiple of q then $(u^{''}, \frac{v+v^{'}}{q} + 2r)$ is adjacent to both (u, v) and $(u^{'}, v^{'})$.

Let $v \in \varphi_{2^\alpha q^\beta r^\gamma}, v^{'} = kq$ is odd and $k, k^{''} \in \mathbb{Z}$. If $v - (v + v^{'})r = 2k^{''}$, then $v = 2k^{''} + (v+v^{'})r$. Hence v is even, which is inaccurate. Also if $v - (v+v^{'})r = k^{''}q$, then $v = (\frac{k^{''}+kr}{1-r})q$ and if $v - (v + v^{'})r = k^{''}r$, then $v = (k^{''} + v + v^{'})r$, which are impossible. Hence v is adjacent to $(v + v^{'})r$. Similarly it is easy to see that $v^{'}$ is adjacent to $(v+v^{'})r$. Therefore $(u^{''}, (v+v^{'})r)$ is adjacent to both (u, v) and $(u^{'}, v^{'})$.

ii) $u \neq u^{'}, v \neq v^{'}$. If v be adjacent to $v^{'}$, then $d((u, v), (u^{'}, v^{'})) = 1$. Suppose that v be non-adjacent to $v^{'}$, since $u \neq u^{'}$ and $u, u^{'} \in \mathbb{Z}_2$, hence we have no common neighbor between (u, v) and $(u^{'}, v^{'})$. This implies that $d((u, v), (u^{'}, v^{'})) \geq 3$. Without loss of generality assume that $u = 0$ and $u^{'} = 1$. Now by Table 4.2 we show that $d((u, v), (u^{'}, v^{'})) = 3$. In this table $u, u^{'''} = 0$ and also $u^{'}, u^{''} = 1$. Now we prove the fifth row and the rest is similarly proven. Let $v = 2kr$; $k \in \mathbb{Z}$ and $v^{'} \in \varphi_{2^\alpha q^\beta r^\gamma}$. Clearly $u, u^{'''}$ are adjacent to $u^{'}, u^{''}$.

First we show that v is adjacent to q. Let $k^{''} \in \mathbb{Z}$.

$$\text{If } v - q = 2k^{''}, \text{ then } q = 2(kr - k^{''}).$$

$$\text{If } v - q = k^{''}q, \text{ then } r = (\frac{k^{''}+1}{2k})q.$$

$$\text{If } v-q=k^{''}r, \text{ then } q=(2k-k^{''})r.$$

In all three cases, we came across a contradiction. So $v-q \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$.

Next we prove that q is adjacent to $(q+v^{'})r$.

$$\text{If } (q+v^{'})r-q=2k^{''}, \text{ then } k^{''}=\frac{(q+v^{'})r-q}{2}.$$

$$\text{If } (q+v^{'})r-q=k^{''}q, \text{ then } v^{'}=(\frac{k^{''}-r+1}{r})q.$$

$$\text{If } (q+v^{'})r-q=k^{''}r, \text{ then } q=(q+v^{'}-k^{''})r.$$

which is impossible, since $k^{''}$ is integer and $v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$ and also q is non-integer of r.

Finally we show that $(q+v^{'})r$ is adjacent to $v^{'}$.

$$\text{If } (q+v^{'})r-v^{'}=2k^{''}, \text{ then } k^{''}=\frac{(q+v^{'})r-v^{'}}{2}.$$

$$\text{If } (q+v^{'})r-v^{'}=k^{''}q, \text{ then } v^{'}=(\frac{k^{''}-r}{r-1})q.$$

$$\text{If } (q+v^{'})r-v^{'}=k^{''}r, \text{ then } v^{'}=(q+v^{'}-k^{''})r.$$

Again which are impossible. This implies that $(u,v)(u^{''},q)(u^{'''},(q+v^{'})r)(u^{'},v^{'})$ is shortest path between (u,v) and $(u^{'},v^{'})$. Thus $\text{diam}(\Gamma_1)=3$ and similarly $\text{diam}(\Gamma_2)=3$. Therefore $\text{diam}(\Gamma)=3$. $\square$

Lemma 4.2. *Let* $\Gamma = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha}q^{\beta}r^{\gamma}}, \Phi)$, *where* $\alpha,\beta,\gamma \geq 1$. *Then* $\gamma_c(\Gamma)$ *does not exist and* $\gamma_t(\Gamma)=12$.

Proof. Clearly Γ is a disconnected graph with two connected components say Γ_1 and Γ_2. Let $V_1=V(\Gamma_1)$ and $V_2=V(\Gamma_2)$. Then $V_1=\{(1,v)|v \textit{ is odd}\}\cup\{(0,v)|v \textit{ is even}\}$ and $V_2=\{(0,v)|v \textit{ is odd}\}\cup\{(1,v)|v \textit{ is even}\}$. Hence by the definition of connected dominating set, γ_c-*set* does not exist for Γ.

Let $(\alpha,\beta,\gamma)=(1,1,1)$. Then we find by [8, Lemma 4.1], that $\gamma(\Gamma)=8$ and $D_1=\{(0,0),(0,2),(1,x_4),(1,x_4')\}$ and $D_2=\{(0,1),(0,3),(1,x_5),(1,x_5')\}$ are minimal dominating sets for Γ_1 and Γ_2 respectively, where $X_i^5=\{x_1,x_2,x_3,x_4,x_5\}$ and $X_j^5=\{x_1',x_2',x_3',x_4',x_5'\}$ are consecutive integers in $\mathbb{Z}_{2qr}$, each of which shares a prime factor with $2qr$. Since vertices of D_1 are not adjacent to each other, we conclude that $\gamma_t(\Gamma_1)>4$. On the other hand it is clear that D_1 is not dominated by one vertex. Hence $\gamma_t(\Gamma_1)>5$. Vertex $(1,1)$ is adjacent to vertices $(0,0),(0,2)$ and vertex $(0,4)$ is adjacent to vertices $(1,x_4),(1,x_4')$. Thus $T_1=\{(0,0),(0,2),(0,4),(1,1),(1,x_4),(1,x_4')\}$ dominates all vertices of Γ_1. Similarly $T_2=\{(0,1),(0,3),(0,5),(1,2),(1,x_5),(1,x_5')\}$ dominates all vertices of Γ_2. Hence $\gamma_t(\Gamma)=12$.

Table 4.1: Common neighbor between $(u, v), (u', v')$ in $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$

$u = u', v \neq v'$	common neighbor	Comments
$v, v' \in \varphi_{2^\alpha q^\beta r^\gamma}$	$(u'', 2qr)$	
v, v' are odd and multiple of q	$(u'', 2r)$	
v, v' are odd and multiple of r	$(u'', 2q)$	
v, v' are odd and multiple of qr	$(u'', 2)$	
v is odd and multiple of q and v' is odd and multiple of r	$(u'', \frac{v+v'}{2})$ $(u'', \frac{v+v'}{2} + qr)$	if $\frac{v+v'}{2}$ be even if $\frac{v+v'}{2}$ be odd
v is odd and multiple of q and v' is odd and multiple of qr	$(u'', \frac{v+v'}{q})$ $(u'', \frac{v+v'}{q} + 2r)$	if $\frac{v+v'}{q} \neq kq,\ k \in \mathbb{Z}$ if $\frac{v+v'}{q} = k'q,\ k' \in \mathbb{Z}$
v is odd and multiple of r and v' is odd and multiple of qr	$(u'', \frac{v+v'}{r})$ $(u'', \frac{v+v'}{r} + 2q)$	if $\frac{v+v'}{r} \neq kr,\ k \in \mathbb{Z}$ if $\frac{v+v'}{r} = k'r,\ k' \in \mathbb{Z}$
$v \in \varphi_{2^\alpha q^\beta r^\gamma}$ and v' is odd and multiple of q	$(u'', (v+v')r)$	
$v \in \varphi_{2^\alpha q^\beta r^\gamma}$ and v' is odd and multiple of r	$(u'', (v+v')q)$	
$v \in \varphi_{2^\alpha q^\beta r^\gamma}$ and v' is odd multiple of qr	$(u'', (v+v')2)$	
v, v' are even and multiple of r	(u'', q)	
v, v' are even and multiple of q	(u'', r)	
v, v' are even and non-multiple of q and r	(u'', qr)	
v, v' are even and multiple of $2qr$	$(u'', 2qr - 1)$	
v is even and multiple of q and v' is even and multiple of r	$(u'', \frac{v+v'}{2} + qr)$ $(u'', \frac{v+v'}{2})$	if $\frac{v+v'}{2}$ be even if $\frac{v+v'}{2}$ be odd
v is even and multiple of qr and v' is even and non-multiple of q and r	$(u'', \frac{v'}{2})$ $(u'', \frac{v'}{2} + qr)$	if $\frac{v'}{2} \in \varphi_{2^\alpha q^\beta r^\gamma}$ if $\frac{v'}{2} \notin \varphi_{2^\alpha q^\beta r^\gamma}$
v is even and multiple of qr and v' is even and multiple of q	$(u'', \frac{v'}{q} + qr)$ $(u'', \frac{v'}{q} + r)$	if $\frac{v'}{q} \neq kq,\ k \in \mathbb{Z}$ if $\frac{v'}{q} = k'q,\ k' \in \mathbb{Z}$
v is even and multiple of qr and v' is even and multiple of r	$(u'', \frac{v'}{r} + qr)$ $(u'', \frac{v'}{r} + q)$	if $\frac{v'}{r} \neq kr,\ k \in \mathbb{Z}$ if $\frac{v'}{r} = k'r,\ k' \in \mathbb{Z}$

Let $(\alpha, \beta, \gamma) \neq (1, 1, 1)$. Then by [8, Lemma 4.3], $\gamma(\Gamma) = 12$. Indeed $D_1 = \{(0, 0), (0, 2), (0, 4), (1, 1), (1, 3), (1, 5)\}$ and $D_2 = \{(0, 1), (0, 3), (0, 5), (1, 0), (1, 2), (1, 4)\}$ are minimal dominating sets for Γ_1, Γ_2, respectively. Vertex $(1, 1)$ is adjacent to vertices $(0, 0), (0, 2)$ and vertex $(0, 4)$ is adjacent to vertices $(1, 3), (1, 5)$. Thus D_1 becomes a γ_t-set for Γ_1. Similarly D_2 becomes a γ_t-set for Γ_2. Therefore $\gamma_t(\Gamma) = 12$. □

Proposition 4.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$, *where* $\alpha, \beta, \gamma \geq 1$. *Then* $\mathrm{diam}(\Gamma) = 3$.

Table 4.2: Shortest path between $(u,v),(u^{'},v^{'})$ in $\Gamma = \mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha}q^{\beta}r^{\gamma}}, \Phi)$

$u \neq u^{'}, v \neq v^{'}$	shortest path between $(u,v),(u^{'},v^{'})$	Comments
$v = 2kqr, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$	$(u,v)(u^{'},v^{'})$	$d((u,v),(u^{'},v^{'})) = 1$
v is multiple of $2qr$ and $v^{'}$ is odd and multiple of q	$(u,v)(u^{''},1)(u^{'''},(1+v^{'})r)(u^{'},v^{'})$	
v is multiple of $2qr$ and $v^{'}$ is odd and multiple of r	$(u,v)(u^{''},1)(u^{'''},(1+v^{'})q)(u^{'},v^{'})$	
v is multiple of $2qr$ and $v^{'}$ is odd and multiple of qr	$(u,v)(u^{''},1)(u^{'''},(1+v^{'})2)(u^{'},v^{'})$	
$v = 2kr, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$	$(u,v)(u^{''},q)(u^{'''},(q+v^{'})r)(u^{'},v^{'})$	
v is multiple of $2r$ and $v^{'}$ is odd and multiple of r	$(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{2})(u^{'},v^{'})$ $(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{2}+qr)(u^{'},v^{'})$	if $\frac{q+v^{'}}{2}$ be even if $\frac{q+v^{'}}{2}$ be odd
v is multiple of $2r$ and $v^{'}$ is odd and multiple of q	$(u,v)(u^{''},q)(u^{'''},2r)(u^{'},v^{'})$	
v is multiple of $2r$ and $v^{'}$ is odd and multiple of qr	$(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{q})(u^{'},v^{'})$ $(u,v)(u^{''},q)(u^{'''},\frac{q+v^{'}}{q}+2r)(u^{'},v^{'})$	if $\frac{q+v^{'}}{q} \neq kq, k \in \mathbb{Z}$ if $\frac{q+v^{'}}{q} = k^{'}q, k^{'} \in \mathbb{Z}$
$v = 2kq, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$	$(u,v)(u^{''},r)(u^{'''},(r+v^{'})q)(u^{'},v^{'})$	
v is multiple of $2q$ and $v^{'}$ is odd and multiple of q	$(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{2})(u^{'},v^{'})$ $(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{2}+qr)(u^{'},v^{'})$	if $\frac{r+v^{'}}{2}$ be even if $\frac{r+v^{'}}{2}$ be odd
v is multiple of $2q$ and $v^{'}$ is odd and multiple of r	$(u,v)(u^{''},r)(u^{'''},2q)(u^{'},v^{'})$	
v is multiple of $2q$ and $v^{'}$ is odd and multiple of qr	$(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{r})(u^{'},v^{'})$ $(u,v)(u^{''},r)(u^{'''},\frac{r+v^{'}}{r}+2q)(u^{'},v^{'})$	if $\frac{r+v^{'}}{r} \neq kr, k \in \mathbb{Z}$ if $\frac{r+v^{'}}{r} = k^{'}r, k^{'} \in \mathbb{Z}$
$v = 2k, k \in \mathbb{Z}, v^{'} \in \varphi_{2^{\alpha}q^{\beta}r^{\gamma}}$	$(u,v)(u^{''},qr)(u^{'''},(qr+v^{'})2)(u^{'},v^{'})$	
v is multiple of 2 and $v^{'}$ is odd and multiple of q	$(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{q})(u^{'},v^{'})$ $(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{q}+2r)(u^{'},v^{'})$	if $\frac{qr+v^{'}}{q} \neq kq, k \in \mathbb{Z}$ if $\frac{qr+v^{'}}{q} = k^{'}q, k^{'} \in \mathbb{Z}$
v is multiple of 2 and $v^{'}$ is odd and multiple of r	$(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{r})(u^{'},v^{'})$ $(u,v)(u^{''},qr)(u^{'''},\frac{qr+v^{'}}{r}+2q)(u^{'},v^{'})$	if $\frac{qr+v^{'}}{r} \neq kr, k \in \mathbb{Z}$ if $\frac{qr+v^{'}}{r} = k^{'}r, k^{'} \in \mathbb{Z}$
v is multiple of 2 and $v^{'}$ is odd and multiple of qr	$(u,v)(u^{''},qr)(u^{'''},2)(u^{'},v^{'})$	

Proof. We proceed along the lines of Theorem 4.1, and $q := p$. Let $(u,v),(u^{'},v^{'})$ are arbitrary vertices of Γ. Then we have following three possibilities:

i) $u = u^{'}, v \neq v^{'}$. We know that $d((u,v),(u^{'},v^{'})) \geq 2$. Assume that v and $v^{'}$ are both even or odd. Thus by case *i)* of Theorem 4.1, we have $d((u,v),(u^{'},v^{'})) = 2$. Suppose that one of either v or $v^{'}$ is odd. Hence we have no path of length 2 between

$(u,v),(u^{'},v^{'})$. Now we show that $d((u,v),(u^{'},v^{'}))=3$. Without loss of generality assume that v is even and $v^{'}$ is odd. If v be multiple of $2pr$ and $v^{'}$ be multiple of pr, then $(u,v)(u^{''},pr-2)(u^{'''},pr-1)(u^{'},v^{'})$ is a path of length 3 between (u,v) and $(u^{'},v^{'})$, where $u=u^{'}\neq u^{''}\neq u^{'''}$. If v be multiple of $2pr$ and $v^{'}\in\varphi_{2^\alpha p^\beta r^\gamma}$, note that v and $v^{'}$ are adjacent, then $(u,v)(u^{''},v^{'})(u^{'''},v)(u^{'},v^{'})$ is a shortest path. For other cases of v and $v^{'}$ we are using of Table 4.2, where $u=u^{'}\neq u^{''}\neq u^{'''}$.

ii) $u\neq u^{'},v=v^{'}$. In this case vertex $(u^{''},v-1)$ is a common neighbor between (u,v) and $(u^{'},v^{'})$, where $u^{''}\neq u,u^{'}$. Thus $d((u,v),(u^{'},v^{'}))=2$.

iii) $u\neq u^{'},v\neq v^{'}$. Let v and $v^{'}$ be both even or odd. Then by Table 4.1, where $u^{''}\neq u,u^{'}$, we see that $d((u,v),(u^{'},v^{'}))=2$. Let one of either v or $v^{'}$ be even and other be odd. Then by Table 4.2, where $u=u^{'''}$ and $u^{'}=u^{''}$, we see that $d((u,v),(u^{'},v^{'}))=3$. Therefore $\mathrm{diam}(\Gamma)=3$. □

Proposition 4.2. *Let $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi)$, where $\alpha,\beta,\gamma\geq 1$. Then $\gamma_t(\Gamma)$ and $\gamma_c(\Gamma)$ is given by Table* 4.3.

Table 4.3: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi))$

Γ	$\gamma_t(\Gamma)$	$\gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi)$	10	12	one of the prime factors is 3 $p=3$
$\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi)$	10	10	one of the prime factors is 3 $p\geq 5$
$\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{2^\alpha p^\beta r^\gamma},\Phi)$	8	8	$p,r\geq 5$

Proof. Assume first that one of the prime factors is 3. In this case if $(\alpha,\beta,\gamma)=(1,1,1)$ then by [8, Lemma 4.2], $\gamma(\Gamma)=8$ and

$$D=\{(0,0),(0,1),(0,2),(0,3),(1,x_4),(1,x_4^{'}),(1,x_5),(1,x_5^{'})\}$$

is a γ-set for Γ. Vertices of D are not adjacent to each other. Hence $\gamma_t(\Gamma)>8$. Note that D is not dominated by one vertex, since every vertex $(u,v)\in V$, where v is an odd (even) integer, is not adjacent to the vertex $(u^{'},v^{'})$, where $v^{'}$ is an odd (even) integer. This implies that $\gamma_t(\Gamma)>9$. Now we take another dominating set with cardinality 10. By [8, Proposition 4.4], we have $D^{'}=\{(0,0),(0,1),(0,2),(0,3),(1,4),(1,5),(2,6),(2,7),(2,8),(2,9)\}$ is a dominating set of Γ. If the other prime factor is 5, then vertices $(0,0),(0,1),(0,2),(0,3),(1,5)$ are adjacent to vertices $(2,7),(2,8),(2,9),(1,4),(2,6)$, respectively. Also let other prime factor be ≥ 7 then vertices $(0,0),(0,1),(0,2),(0,3),(1,4)$ are adjacent to vertices $(1,5),(2,6),(2,7),(2,8),(2,9)$, respectively. Hence $D^{'}$ becomes a γ_t-set for Γ. Therefore $\gamma_t(\Gamma)=10$.

Let $(\alpha,\beta,\gamma)\neq(1,1,1)$. By [8, Proposition 4.4], $\gamma(\Gamma)=10$. By previous paragraph, $\gamma_t(\Gamma)=10$.

Now we find the connected domination number of Γ where one of the prime factors is 3. By above discussion $\gamma_c(\Gamma) > 9$. We use again from $D^{'}$.

Let $p = 3$. Without loss of generality assume that $r = 5$. Then the subgraph generated by $D^{'}$ has exactly five connected components which are induced the subgraphs generated by sets $\{(0,0),(2,7)\}$, $\{(0,1),(2,8)\}$, $\{(0,2),(2,9)\}$, $\{(0,3),(1,4)\}$ and $\{(1,5),(2,6)\}$. Hence $\gamma_c(\Gamma) > 10$. Let a vertex say $(u,v) \in V(\Gamma)$, where v is an odd integer, dominates all vertices $(0,0),(2,8),(0,2),(1,4),(2,6)$. Since $u \in \mathbb{Z}_3$, it is impossible. This implies that $\gamma_c(\Gamma) > 11$. Next consider another dominating with cardinality 12.

Let

$$A = \{(1,1),(2,2),(1,4),(2,5),(1,7),(2,8),(1,10),(2,11)\},$$

$$B = \{(0,0),(2,2),(0,3),(2,5),(0,6),(2,8),(0,9),(2,11)\}$$

and

$$C = \{(0,0),(1,1),(0,3),(1,4),(0,6),(1,7),(0,9),(1,10)\}.$$

Then A, B and C dominate $\{(0,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$, $\{(1,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$ and $\{(2,v)|v \in \mathbb{Z}_{2^\alpha 3^\beta r^\gamma}\}$ respectively. Thus

$$D^{''} = \{(0,0),(1,1),(2,2),(0,3),(1,4),(2,5),(0,6),(1,7),(2,8),(0,9),(1,10),(2,11)\}$$

is a dominating set for Γ. Both vertices next to each other in $D^{''}$ are adjacent. Hence the subgraph generated by $D^{''}$ is connected. Therefore $\gamma_c(\Gamma) = 12$.

Let $p \geq 5$. Then $D^{'''} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,5),(1,6),(2,7),(3,8),(4,9)\}$ is a dominating set for Γ. Both vertices next to each other in $D^{'''}$ are adjacent. Thus the subgraph generated by $D^{'''}$ is connected. Therefore $\gamma_c(\Gamma) = 10$.

Finally assume that $p, r \geq 5$. By [8, Lemma 4.2, Proposition 4.4], $\gamma(\Gamma) = 8$ and by using a proof of proposition 4.4, we know that $D^{''''} = \{(0,0),(1,1),(2,2),(3,3),(4,4),(2,5),(1,6),(0,7)\}$ is a γ-set for Γ, where $\alpha, \beta, \gamma \geq 1$. Both vertices next to each other in $D^{''''}$ are adjacent. Hence $D^{''''}$ is a γ_t-set and γ_c-set for Γ. Therefore $\gamma_t(\Gamma) = \gamma_c(\Gamma) = 8$. □

Proposition 4.3. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$*, where* $p, q, r \geq 3$ *and* $\alpha, \beta, \gamma \geq 1$*. Then* $\mathrm{diam}(\Gamma) = 2$*.*

Proof. Let $(u,v),(u^{'},v^{'})$ are arbitrary vertices of Γ. Then we have following three possibilities:

i) $u = u^{'}, v \neq v^{'}$. By Table 4.5, we show that $d((u,v),(u^{'},v^{'})) = 2$. In this table $u^{''} \neq u$.

ii) $u \neq u^{'}$ and $v = v^{'}$. In this case the vertex $(u^{''}, v-1)$ where $u^{''} \neq u, u^{'}$, is a common neighbor of (u,v) and $(u^{'},v^{'})$. Thus $d((u,v),\ (u^{'},v^{'})) = 2$.

iii) $u \neq u^{'}$ and $v \neq v^{'}$. Hence by (*i*) and (*ii*), $d((u,v),\ (u^{'},v^{'})) = 2$.

Therefore $\mathrm{diam}(\Gamma) = 2$. □

Table 4.4: Common neighbor between $(u,v),(u',v')$ in $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{p^\alpha q^\beta r^\gamma},\Phi)$

$u=u',v\neq v'$	common neighbor	Comments
$v,v'\in\varphi_{p^\alpha q^\beta r^\gamma}$	(u'',pqr)	
v,v' are multiples of pqr	$(u'',pqr-1)$	
v,v' are multiples of pq	(u'',r)	
v,v' are multiples of pr	(u'',q)	
v,v' are multiples of qr	(u'',p)	
v,v' are multiples of p	(u'',qr)	
$v\neq v'$ and each of them is multiple of one of the prime factor and both of them are even or odd	$(u'',\frac{v+v'}{2})$ $(u'',2(v+v'))$	if $v-v'\in\varphi_{p^\alpha q^\beta r^\gamma}$ if $v-v'\notin\varphi_{p^\alpha q^\beta r^\gamma}$
v is multiple of p and v' is multiple of pq	$(u'',(\frac{v}{p})r+pqr)$ $(u'',(\frac{v}{p})r+qr)$	if $\frac{v}{p}\in\varphi_{p^\alpha q^\beta r^\gamma}$ if $\frac{v}{p}$ be multiple of p
v is multiple of p and v' is multiple of pr	$(u'',(\frac{v}{p})q+pqr)$ $(u'',(\frac{v}{p})q+qr)$	if $\frac{v}{p}\in\varphi_{p^\alpha q^\beta r^\gamma}$ if $\frac{v}{p}$ be multiple of p
v is multiple of p and v' is multiple of qr	$(u'',\frac{v+v'}{2})$ $(u'',2(v+v'))$	if v,v' be both even or odd if one of them be odd and other be even
v is multiple of p and v' is multiple of pqr	$(u'',\frac{v}{p}+pqr)$ $(u'',\frac{v}{p}+qr)$	if $\frac{v}{p}$ be non-multiple of p if $\frac{v}{p}$ be multiple of p
v is multiple of p and $v'\in\varphi_{p^\alpha q^\beta r^\gamma}$	$(u'',(v+v')qr)$ $(u'',v'qr)$	if v,v' be both even or odd if one of them be odd and other be even
v,v' are multiples of q	(u'',pr)	
v is multiple of q and v' is multiple of pq	$(u'',(\frac{v}{q})r+pqr)$ $(u'',(\frac{v}{q})r+pr)$	if $\frac{v}{q}\in\varphi_{p^\alpha q^\beta r^\gamma}$ if $\frac{v}{q}$ be multiole of q
v is multiple of q and v' is multiple of qr	$(u'',(\frac{v}{q})p+pqr)$ $(u'',(\frac{v}{q})p+pr)$	if $\frac{v}{q}\in\varphi_{p^\alpha q^\beta r^\gamma}$ if $\frac{v}{q}$ be multiole of q
v is multiple of q and v' is multiple of pr	$(u'',\frac{v+v'}{2})$ $(u'',2(v+v'))$	if v,v' be both even or odd if one of them be odd and other be even
v is multiple of q and v' is multiple of pqr	$(u'',\frac{v}{q}+pqr)$ $(u'',\frac{v}{q}+pr)$	if $\frac{v}{q}$ be non-multiple of q if $\frac{v}{q}$ be multiple of q
v is multiple of q and $v'\in\varphi_{p^\alpha q^\beta r^\gamma}$	$(u'',(v+v')pr)$ $(u'',v'pr)$	if v,v' be both even or odd if one of them be odd and other be even

Table 4.5: Shortest path between $(u,v),(u^{'},v^{'})$ in $\Gamma=\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{2^{\alpha}q^{\beta}r^{\gamma}},\Phi)$

$u=u^{'},v\neq v^{'}$	common neighbor	Comments
$v,v^{'}$ are multiple of r	$(u^{''},pq)$	
v is multiple of r and $v^{'}$ is multiple of pr	$(u^{''},(\frac{v}{r})q+pqr)$ $(u^{''},(\frac{v}{r})q+pq)$	if $\frac{v}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $\frac{v}{r}$ be multiole of r
v is multiple of r and $v^{'}$ is multiple of qr	$(u^{''},(\frac{v}{r})p+pqr)$ $(u^{''},(\frac{v}{r})p+pq)$	if $\frac{v}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $\frac{v}{r}$ be multiole of r
v is multiple of r and $v^{'}$ is multiple of pq	$(u^{''},\frac{v+v^{'}}{2})$ $(u^{''},2(v+v^{'}))$	if $v,v^{'}$ be both even or odd if one of them be odd
v is multiple of r and $v^{'}$ is multiple of pqr	$(u^{''},\frac{v}{r}+pqr)$ $(u^{''},\frac{v}{r}+pq)$	if $\frac{v}{r}$ be non-multiple of r if $\frac{v}{r}$ be multiple of r
v is multiple of r and $v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$	$(u^{''},(v+v^{'})pq)$ $(u^{''},v^{'}pq)$	if $v,v^{'}$ be both even or odd if one of them be odd
v is multiple of pq and $v^{'}$ is multiple of pr	$(u^{''},\frac{v+v^{'}}{p})$ $(u^{''},\frac{v+v^{'}}{p}+qr)$	if $\frac{v+v^{'}}{p}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $\frac{v+v^{'}}{p}$ be multiple of p
v is multiple of pq and $v^{'}$ is multiple of rq	$(u^{''},\frac{v+v^{'}}{q})$ $(u^{''},\frac{v+v^{'}}{q}+pr)$	if $\frac{v+v^{'}}{q}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $\frac{v+v^{'}}{q}$ be multiple of q
v is multiple of pr and $v^{'}$ is multiple of rq	$(u^{''},\frac{v+v^{'}}{r})$ $(u^{''},\frac{v+v^{'}}{r}+pq)$	if $\frac{v+v^{'}}{r}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $\frac{v+v^{'}}{r}$ be multiple of r
$v=kpq,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ and $v,v^{'}$ are both even or odd one of the v or $v^{'}$ is odd	$(u^{''},\frac{v+v^{'}}{2})$ $(u^{''},v-v^{'})$ $(u^{''},v^{'}r)$	if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$
$v=kpr,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$and $v,v^{'}$ are both even or odd one of the v or $v^{'}$ is odd	$(u^{''},\frac{v+v^{'}}{2})$ $(u^{''},v-v^{'})$ $(u^{''},v^{'}q)$	if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$
$v=kqr,k\in\mathbb{Z},v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ and $v,v^{'}$ are both even or odd one of the v or $v^{'}$ is odd	$(u^{''},\frac{v+v^{'}}{2})$ $(u^{''},v-v^{'})$ $(u^{''},v^{'}p)$	if $v-v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$ if $v-v^{'}\notin\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$
$v=kpqr,k\in\mathbb{Z}$ and $v^{'}$ is multiple of pq or pr or qr	$(u^{''},+4)$ or $(u^{''},-4)$	by Proposition 4.5[8], $\lambda=4$ then pqr is adjacent by ±4 and pq,pr,qr are adjacent by $+4$ or -4
v is multiple of pqr and $v^{'}\in\varphi_{p^{\alpha}q^{\beta}r^{\gamma}}$	$(u^{''},\frac{v+v^{'}}{2})$ $(u^{''},2(v+v^{'}))$	if $v,v^{'}$ be both even or odd if one of them be odd

Proposition 4.4. *Let* $\Gamma=\mathrm{Cay}(\mathbb{Z}_p\times\mathbb{Z}_{p^{\alpha}q^{\beta}r^{\gamma}},\Phi)$, *where* $p,q,r\geq 3$ *and* $\alpha,\beta,\gamma\geq 1$. *Then* $\gamma_t(\Gamma)$ *and* $\gamma_c(\Gamma)$ *is given by Table 4.6.*

Table 4.6: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ where $p, q, r \geq 3$

Γ	$\gamma_t(\Gamma), \gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	$6 \leq \gamma_t(\Gamma), \gamma_c(\Gamma) \leq 8$	one of the prime factors is 3
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	5	$p, q, r \geq 5$

Proof. By using the [8, Proposition 4.5], $D, D^{'}, D^{''}, D^{'''}$ are minimal dominating sets for various cases in this graph. Clearly the subgraphs generated by D, $D^{'}$, $D^{''}$ and $D^{'''}$ are all connected. Therefore $\gamma(\Gamma) = \gamma_t(\Gamma) = \gamma_c(\Gamma)$. □

As an immediate consequence of Lemma 4.2 and Propositions 4.2, 4.4, we have the following theorem.

Theorem 4.1. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$, *where* $p, q, r \geq 2$ *and* $\alpha, \beta, \gamma \geq 1$. *Then* $\gamma_t(\Gamma)$*and* $\gamma_c(\Gamma)$ *is given by Table 4.7.*

Table 4.7: $\gamma_t(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ and $\gamma_c(\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi))$ where $\alpha, \beta, \gamma \geq 1$

Γ	$\gamma_t(\Gamma)$	$\gamma_c(\Gamma)$	Comments
$\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha q^\beta r^\gamma}, \Phi)$	12	does not exist	
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$	10	12	one of the prime factors is 3 $p = 3$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$	10	10	one of the prime factors is 3 $p \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{2^\alpha p^\beta r^\gamma}, \Phi)$	8	8	$p, r \geq 5$
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	$6 \leq \gamma_t(\Gamma) \leq 8$	$6 \leq \gamma_c(\Gamma) \leq 8$	one of the prime factors is 3
$\mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_{p^\alpha q^\beta r^\gamma}, \Phi)$	5	5	$p, q, r \geq 5$

As an immediate consequence of Lemmas 3.1, 4.1 and Propositions 3.1, 3.3, 4.1, 4.3, we have the following theorem.

Theorem 4.2. *Let* $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_m, \Phi)$, *where* $m \in \{p^\alpha q^\beta, p^\alpha q^\beta r^\gamma\}$. *Then*

1) $\mathrm{diam}(\Gamma) = 3$ *where one of the prime factors is* 2.

2) $\mathrm{diam}(\Gamma) = 2$ *where* $p, q, r \geq 3$.

Remark 4.1. Let $p_1, p_2, \ldots, p_k$ be consecutive prime numbers, $p_1 = 3$, $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, $\alpha \geq 2$, and $\Phi = \varphi_2 \times \varphi_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}$. Then by [8, Theorem 4.7], we have $\gamma(\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)) \geq 4k+4$. Therefore $\gamma_t(\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi)) \geq 4k+4$. Since Γ is a disconnected graph, the γ_c*-set* does not exist for Γ.

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A STUDY ON SCREEN TRANSVERSAL LIGHTLIKE SUBMANİFOLDS OF GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract. We have studied radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We have investigated several properties of such submanifolds and obtained necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. Moreover, we have studied totally umbilical radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and given the examples.
Key words: Geometric structures on manifolds, semi-Riemannian maifolds, Global submanifolds

1. Introduction

It is well known that in case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more different from the study of non-degenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact that in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle and also in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Lightlike submanifolds is developed by Duggal and Bejancu [5] and Duggal and Şahin [8]. The lightlike submanifolds have been studied by many authors in various spaces for example [1, 4, 13, 15, 17, 18, 19, 24, 27].

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Duggal and Bejancu [5] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds. Similar to CR-lightlike submanifolds, Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold [2]. Since CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases, Duggal and Şahin introduced Screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [7]. As a generalization of real null curves of indefinite Kaehler manifolds, Şahin introduced the notion of screen transversal lightlike submanifolds and obtained many interesting results [22]. In [25], Yıldırım and Şahin introduced screen transversal lightlike submanifolds of indefinite almost contact manifolds and show that such submanifolds contain lightlike real curves. Yıldırım and Erdoğan studied screen transversal lightlike submanifolds of semi-Riemannian product manifolds [26]. Khursheed Haider, Advin and Thakur studied totally umbilical screen transversal lightlike submanifolds of semi-Riemannian product manifolds [16].

Manifolds which are considered as differential geometric structures (such as almost complex manifolds, almost contact manifolds and almost product manifolds) are convenient when it comes to studying submanifold theory. One of the most studied manifold types are Riemannian manifolds with golden structures. Golden structures on Riemannian manifolds allow many geometric results. Hretcanu introduced golden structure on manifolds [14]. Crasmareanu and Hretcanu investigated the geometry of the golden structure on a manifold by using the corresponding almost product structure [3]. The integrability of golden structures has been investigated in [11]. In [23], Şahin and Akyol introduced golden maps between golden Riemannian manifolds, give an example and show that such map is harmonic. Erdoğan and Yıldırım studied totally umbilical semi-invariant submanifolds of golden Riemannian manifolds [10]. Gök, Keleş and Kılıç studied Schouten and Vrănceanu connections on golden manifolds [12]. Poyraz and Yaşar introduced lightlike submanifolds of a golden semi-Riemannian manifold [21]. Erdoğan studied the geometry of screen transversal lightlike submanifolds and radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds [9].

In this paper, we study radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We investigate several properties of such submanifolds and obtain necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. Moreover, we study totally umbilical radical screen transversal lightlike submanifolds and screen transversal anti-invariant lightlike submanifolds of golden semi-Riemannian manifolds and give examples. We also give different form of some theorems given in [9].

2. Preliminaries

Let $\tilde{M}$ be a $C^\infty-$differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I \tag{2.1}$$

then $\tilde{P}$ is named a golden structure on $\tilde{M}$, where I is the identity transformation [14].

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ holds the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y) \tag{2.2}$$

then $(\tilde{M}, \tilde{g}, \tilde{P})$ is named a golden semi-Riemannian manifold [20].

If $\tilde{P}$ is a golden structure, then the equation (2.2) is equivalent with

$$\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y) \tag{2.3}$$

for any $X, Y \in \Gamma(T\tilde{M})$.

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold with index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M, then M is named a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup \left\{ u \in T_x\tilde{M} : \tilde{g}(u, v) = 0, \forall v \in T_x\tilde{M}, x \in M \right\} \tag{2.4}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, called radical distribution on M of rank $r > 0$ then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S\,(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S\,(TM) \perp Rad\,(TM) \tag{2.5}$$

and $S\left(TM^{\perp}\right)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr\,(TM))$ and $ltr\,(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|M}$ and $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}$, respectively. Then we have

$$\begin{aligned} tr\,(TM) &= ltr\,(TM) \perp S\left(TM^{\perp}\right), & (2.6)\\ T\tilde{M}\,|_M &= TM \oplus tr\,(TM) & (2.7)\\ &= \{Rad(TM) \oplus ltr\,(TM)\} \perp S\,(TM) \perp S(TM^{\perp}). \end{aligned}$$

Theorem 2.1. *Let $(M, g, S(TM), S\left(TM^{\perp}\right))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose U is a coordinate neighbourhood of M and ξ_i, $i \in \{1, .., r\}$ is a basis of $\Gamma(Rad\,(TM)_{|U})$. Then, there exist a complementary vector subbundle $ltr\,(TM)$ of $Rad(TM)$ in $S\left(TM^{\perp}\right)^{\perp}_{|U}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr\,(TM)_{|U})$ such that*

$$\tilde{g}\,(N_i, \xi_j) = \delta_{ij}, \;\; \tilde{g}\,(N_i, N_j) = 0, \tag{2.8}$$

for any $i, j \in \{1, .., r\}$ [5].

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, using (2.7), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$
$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.10}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr\,(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr\,(TM)$, respectively. According to (2.7), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S(TM^{\perp})$, respectively, (2.9) and (2.10) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.11}$$
$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.12}$$
$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.13}$$

where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\{\nabla_X Y, A_N X, A_W X\} \in \Gamma(TM)$, $\{\nabla^l_X N, D^l(X, W)\} \in \Gamma(ltr\,(TM))$ and $\{\nabla^s_X W, D^s(X, N)\} \in \Gamma(S(TM^{\perp}))$. Thus taking account of (2.11)-(2.13) and the Levi-Civita connection $\tilde{\nabla}$ is a metric, we derive

$$g(h^s(X, Y), W) + g(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.14}$$
$$g(D^s(X, N), W) = g(A_W X, N). \tag{2.15}$$

Let J be a projection of TM on $S(TM)$. Thus using (2.5) we obtain

$$\nabla_X JY = \nabla^*_X JY + h^*(X, JY)\xi, \tag{2.16}$$
$$\nabla_X \xi = -A^*_\xi X - \nabla^{*t}_X \xi, \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\left\{\nabla^*_X JY, A^*_\xi X\right\}$ and $\{h^*(X, JY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.

Using the equations given above, we derive

$$g(h^l(X, JY), \xi) = g(A^*_\xi X, JY), \tag{2.18}$$
$$g(h^*(X, JY), N) = g(A_N X, JY), \tag{2.19}$$
$$g(h^l(X, \xi), \xi) = 0, A^*_\xi \xi = 0. \tag{2.20}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (2.11) we obtain

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y). \tag{2.21}$$

But, ∇^* is a metric connection on $S(TM)$.

Theorem 2.2. [5] *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$. Then the induced connection ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇.*

A lightlike submanifold M of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is named totally umbilical in $\tilde{M}$, if there is a smooth transversal vector field $H \in \Gamma(ltr\,(TM))$ of M which is named the transversal curvature vector of M, such that

$$h(X,Y) = Hg(X,Y), \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$.

It is known that M is totally umbilical if on each coordinate neighborhood U, there exists smooth vector fields $H^l \in \Gamma(ltr\,(TM))$ and $H^s \in \Gamma(S\left(TM^{\perp}\right))$ such that

$$h^l(X,Y) = g(X,Y)H^l, h^s(X,Y) = g(X,Y)H^s \text{ and } D^l(X,W) = 0, \tag{2.23}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$ [6].

3. Radical Screen Transversal Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Definition 3.1. Let M be a lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then we say that M is a screen transversal lightlike submanifold of $\tilde{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that

$$\tilde{P}(Rad(TM)) \subset S(TM^{\perp}). \tag{3.1}$$

Definition 3.2. Let M be a screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then M is said to be a radical screen transversal lightlike submanifold if $S(TM)$ is invariant with respect to $\tilde{P}$.

Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Thus, for any $X \in \Gamma(TM)$ we derive

$$\tilde{P}X = PX + wX, \tag{3.2}$$

where PX and wX are tangential and transversal parts of $\tilde{P}X$.

For any $V \in \Gamma(tr(TM))$ we write

$$\tilde{P}V = BV + CV, \tag{3.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$.

Throughout this paper, we assume that $\tilde{\nabla}\tilde{P} = 0$.

Lemma 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have*

$$P^2X = PX + X - BwX, \tag{3.4}$$

$$wPX = wX - CwX, \tag{3.5}$$

$$PBV = BV - BCV, \tag{3.6}$$

$$C^2 = CV + V - wBV, \tag{3.7}$$

$$g(PX, Y) - g(X, PY) = g(X, wY) - g(wX, Y), \tag{3.8}$$

$$\begin{aligned} g(PX, PY) &= g(PX, Y) + g(X, Y) + g(wX, Y) - g(PX, wY) \\ &\quad -g(wX, PY) - g(wX, wY). \end{aligned} \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Applying $\tilde{P}$ in (3.2) and using (2.1), we obtain

$$P^2X + wPX + BwX + CwX = PX + wX + X, \tag{3.10}$$

for any $X \in \Gamma(TM)$. From (3.10) we obtain (3.4) and (3.5). Using (2.1) and (3.3) we get

$$PBV + wBV + BCV + C^2V = BV + CV + V. \tag{3.11}$$

From (3.11) we get (3.6) and (3.7). From (2.2) and (3.2) we obtain

$$g(PX + wX, Y) = g(X, PY + wY), \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$ and from this we obtain (3.8). Also, from (2.3) and (3.2) we derive

$$g(PX + wX, PY + wY) = g(PX + wX, Y) + g(X, Y), \tag{3.13}$$

for any $X, Y \in \Gamma(TM)$ and we get (3.9). $\square$

Proposition 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then P is golden structure on $S(TM)$.*

Proof. By the definition of radical screen transversal lightlike submanifold we have $wX = 0$, for any $X \in \Gamma(S(TM))$. Then from (3.4) we have $P^2X = PX + X$. Thus P is golden structure on $S(TM)$. $\square$

Proposition 3.2. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on $ltr(TM)$.*

Proof. By the definition of radical screen transversal lightlike submanifold we have $BN = 0$, for any $N \in \Gamma(ltr(TM))$ From (3.7) we have $C^2N = CN + N$. Thus C is golden structure on $ltr(TM)$. $\square$

Example 3.1. Let $(\tilde{M} = \mathbb{R}^7_3, \tilde{g})$ be a 7−dimensional semi-Euclidean space with signature $(-,+,-,+,-,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ be the standard coordinate system of $\mathbb{R}^7_3$. If we set $\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = ((1-\phi)x_1, (1-\phi)x_2, \phi x_3, \phi x_4, (1-\phi)x_5, (1-\phi)x_6, (1-\phi)x_7)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{M}$. Suppose M is a submanifold of $\tilde{M}$ defined by

$$\begin{aligned} x_1 &= \phi u_1 + \cos u_2, x_2 = \phi u_1 - u_3, x_3 = \sqrt{2}u_1, x_4 = \sqrt{2}u_1, \\ x_5 &= \phi u_1 - \cos u_2, x_6 = \phi u_1 + \sqrt{2}u_3, x_7 = \cos u_2 - u_3, \end{aligned}$$

where u_i, $1 \leq i \leq 3$, are real parameters. Thus $TM = Span\{U_1, U_2, U_3\}$, where

$$\begin{aligned} U_1 &= \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial x_4} + \phi\frac{\partial}{\partial x_5} + \phi\frac{\partial}{\partial x_6}, \\ U_2 &= -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}, \\ U_3 &= -\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7}. \end{aligned}$$

Then M is a 1−lightlike submanifold. We have $Rad(TM) = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3\}$. Moreover, $\tilde{P}U_2 = \phi U_2$, $\tilde{P}U_3 = \phi U_3$ implies that $\tilde{P}(S(TM)) = S(TM)$. Lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = -\frac{1}{4(2+\phi)}(\phi\frac{\partial}{\partial x_1} - \phi\frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3} - \sqrt{2}\frac{\partial}{\partial x_4} + \phi\frac{\partial}{\partial x_5} - \phi\frac{\partial}{\partial x_6}).$$

Also, screen transversal bundle $S(TM^{\perp})$ is spanned by

$$\begin{aligned} W_1 &= \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}, \\ W_2 &= -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} + \sqrt{2}\phi\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}, \\ W_3 &= -\frac{1}{4(2+\phi)}(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\phi\frac{\partial}{\partial x_3} - \sqrt{2}\phi\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}). \end{aligned}$$

Then it is easy to see that $\tilde{P}\xi = W_2$, $\tilde{P}N = W_3$ and $\tilde{P}W_1 = \phi W_1$. Thus M is a radical screen transversal lightlike submanifold of $\tilde{M}$.

Theorem 3.1. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is integrable iff*

$$h^s(X, \tilde{P}Y) = h^s(Y, \tilde{P}X), \tag{3.14}$$

for any $X, Y \in \Gamma(S(TM))$ [9].

Theorem 3.2. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is always integrable.*

Proof. Using the definition of a radical screen transversal lightlike submanifold, $S(TM)$ is integrable iff $\tilde{g}([X,Y],N)=0$, for any $X,Y\in\Gamma(S(TM))$ and $N\in\Gamma(ltr(TM))$. Using (2.3) and (2.11) and taking into account that M is a totally umbilicial, we obtain

$$\begin{aligned}\tilde{g}([X,Y],N) &= \tilde{g}(\tilde{\nabla}_X Y-\tilde{\nabla}_Y X,N)\\ &= \tilde{g}(\tilde{\nabla}_X\tilde{P}Y-\tilde{\nabla}_Y\tilde{P}X,\tilde{P}N)-\tilde{g}(\tilde{\nabla}_X Y-\tilde{\nabla}_Y X,\tilde{P}N)\\ &= \tilde{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),\tilde{P}N)-\tilde{g}(h^s(X,Y)-h^s(Y,X),\tilde{P}N)\\ &= \tilde{g}(h^s(X,\tilde{P}Y)-h^s(Y,\tilde{P}X),\tilde{P}N)\\ &= (\tilde{g}(X,\tilde{P}Y)-\tilde{g}(Y,\tilde{P}X))\tilde{g}(H^s,\tilde{P}N)\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the radical distribution is integrable iff*

$$A_{\tilde{P}\xi_1}\xi_2-A_{\tilde{P}\xi_2}\xi_1=A^*_{\xi_1}\xi_2-A^*_{\xi_2}\xi_1, \tag{3.15}$$

$\xi_1,\xi_2\in\Gamma(Rad(TM))$ [9].

Theorem 3.4. *Let M be a totally umbilical radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the radical distribution is always integrable.*

Proof. Using the definition of a radical screen transversal lightlike submanifold, $Rad(TM)$ is integrable iff $\tilde{g}([\xi_1,\xi_2],X)=0$, for any $X\in\Gamma(S(TM))$ and $\xi_1,\xi_2\in\Gamma(Rad(TM))$. Using (2.3), (2.11), and (2.23) and taking into account that $\tilde{\nabla}$ is a metric connection, we get

$$\begin{aligned}\tilde{g}([\xi_1,\xi_2],X) &= \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2-\tilde{\nabla}_{\xi_2}\xi_1,X)=\tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2,X)-\tilde{g}(\tilde{\nabla}_{\xi_2}\xi_1,X)\\ &= -\tilde{g}(\xi_2,\tilde{\nabla}_{\xi_1}X)+\tilde{g}(\xi_1,\tilde{\nabla}_{\xi_2}X)\\ &= -\tilde{g}(\tilde{P}\xi_2,\tilde{\nabla}_{\xi_1}\tilde{P}X)+\tilde{g}(\tilde{P}\xi_2,\tilde{\nabla}_{\xi_1}X)\\ &\quad +\tilde{g}(\tilde{P}\xi_1,\tilde{\nabla}_{\xi_2}\tilde{P}X)-\tilde{g}(\tilde{P}\xi_1,\tilde{\nabla}_{\xi_2}X)\\ &= -\tilde{g}(\tilde{P}\xi_2,h^s(\xi_1,\tilde{P}X))+\tilde{g}(\tilde{P}\xi_2,h^s(\xi_1,X))\\ &\quad +\tilde{g}(\tilde{P}\xi_1,h^s(\xi_2,\tilde{P}X))-\tilde{g}(\tilde{P}\xi_1,h^s(\xi_2,X))\\ &= -g(\xi_1,\tilde{P}X)\tilde{g}(\tilde{P}\xi_2,H^s)+g(\xi_1,X)\tilde{g}(\tilde{P}\xi_2,H^s)\\ &\quad +g(\xi_2,\tilde{P}X)\tilde{g}(\tilde{P}\xi_1,H^s)-g(\xi_2,X)\tilde{g}(\tilde{P}\xi_1,H^s).\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the screen distribution defines a totally geodesic foliation iff $h^s(X,\tilde{P}Y)-h^s(X,Y)$ has no compenents in $\tilde{P}(Rad(TM))$, for any $X,Y\in\Gamma(S(TM))$ [9].*

Now, we give different form of theorem given in [9].

Theorem 3.6. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution defines a totally geodesic foliation iff $h^s(\xi_1, X)$ has no compenents in $\tilde{P}(ltr(TM))$, for any $X \in \Gamma(S(TM))$ and $\xi_1 \in \Gamma(Rad(TM))$.*

Proof. Since $S(TM)$ is invariant, if $X \in \Gamma(S(TM))$ then $\tilde{P}X \in \Gamma(S(TM))$. Using the definition of radical screen transversal lightlike submanifold, $Rad(TM)$ defines a totally geodesic foliation iff $g(\nabla_{\xi_1}\xi_2, \tilde{P}X) = 0$, for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$. Since $\tilde{\nabla}$ is a metric connection, from (2.2) and (2.11), we derive

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, \tilde{P}X) &= \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}X) = \tilde{g}(\tilde{\nabla}_{\xi_1}\tilde{P}\xi_2, X) \\ &= -\tilde{g}(\tilde{P}\xi_2, \tilde{\nabla}_{\xi_1}X) = -\tilde{g}(\tilde{P}\xi_2, h^s(\xi_1, X)). \end{aligned}$$

Therefore we derive our theorem. $\square$

Taking into account that M is a totally umbilicial in Theorem 3.6 we get following theorem.

Theorem 3.7. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution always defines a totally geodesic foliation.*

Theorem 3.8. *Let M be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff there is no component of $h^s(X, Y)$ in $\tilde{P}(ltr(TM))$ or $A_{\tilde{P}\xi}X$ in $S(TM)$ for any $X, Y \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$* [9].

Theorem 3.9. *Let M be a totally umbilicial radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff H^s has no compenents in $\tilde{P}(ltr(TM))$.*

Proof. Considering Theorem 2.2, using (2.2), (2.11), (2.23) and taking into account that $\tilde{\nabla}$ is a metric connection, we obtain

$$\begin{aligned} g(\nabla_X\xi, \tilde{P}Y) &= \tilde{g}(\tilde{\nabla}_X\xi, \tilde{P}Y) = \tilde{g}(\tilde{\nabla}_X\tilde{P}\xi, Y) \; = -\tilde{g}(\tilde{P}\xi, \tilde{\nabla}_XY) \\ &= -\tilde{g}(\tilde{P}\xi, h^s(X, Y)) = -g(X, Y)\tilde{g}(H^s, \tilde{P}\xi), \end{aligned} \tag{3.16}$$

for any $X, Y \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$, which completes the proof. $\square$

4. Screen Transversal Anti-invariant Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Definition 4.1. Let M be a screen transversal lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then M is said to be a screen transversal anti-invariant lightlike submanifold if $S(TM)$ is screen transversal with respect to $\tilde{P}$, i.e.

$$\tilde{P}(S(TM)) \subset S(TM^{\perp}).$$

Let M be a screen transversal anti-invariant lightlike submanifold. Thus we have

$$S(TM^{\perp}) = \tilde{P}(Rad(TM)) \oplus \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM)) \perp D_0,$$

where D_0 is a non-degenerate orthogonal complementary distribution to

$$\tilde{P}(Rad(TM)) \oplus \tilde{P}(ltr(TM)) \perp \tilde{P}(S(TM)).$$

Proposition 4.1. *The distribution D_0 is an invariant distribution with respect to $\tilde{P}$ [9].*

Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have

$$\tilde{P}X = wX. \tag{4.1}$$

Let T_1, T_2, T_3 and T_4 be the projection morphisms on $\tilde{P}(Rad(TM))$, $\tilde{P}(S(TM))$, $\tilde{P}(ltr(TM))$ and D_0, respectively. Thus, for any $V \in \Gamma(S(TM^{\perp}))$ we obtain

$$V = T_1V + T_2V + T_3V + T_4V. \tag{4.2}$$

On the other hand, for any $V \in \Gamma(S(TM^{\perp}))$ we write

$$\tilde{P}V = BV + CV, \tag{4.3}$$

where BV and CV are tangential and transversal parts of $\tilde{P}V$. Then applying $\tilde{P}$ to (4.2), we derive

$$\tilde{P}V = \tilde{P}T_1V + \tilde{P}T_2V + \tilde{P}T_3V + \tilde{P}T_4V. \tag{4.4}$$

If we put $\tilde{P}T_1V = B_1V + C_1V$, $\tilde{P}T_2V = B_2V + C_2V$, $\tilde{P}T_3V = C_3^lV + C_3^sV$ and $\tilde{P}T_4V = C_4V$, we can rewrite (4.4) as follows:

$$\tilde{P}V = B_1V + B_2V + C_1V + C_2V + C_3^lV + C_3^sV + C_4V. \tag{4.5}$$

$B_1V \in \Gamma(S(TM))$, $B_2V \in \Gamma(Rad(TM))$, $C_1V \in \Gamma(\tilde{P}S(TM))$, $C_2V \in \Gamma(\tilde{P}Rad(TM))$, $C_3^lV \in \Gamma(ltr(TM))$, $C_3^sV \in \Gamma(\tilde{P}ltr(TM))$ and $C_4V \in \Gamma(D_0)$. From (4.3) and (4.5), we can write

$$BV = B_1V + B_2V, CV = C_1V + C_2V + C_3^lV + C_3^sV + C_4V. \tag{4.6}$$

Similar to the proof of Lemma 3.1, we have the following lemma.

Lemma 4.1. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then we have*

$$BwX = X, \tag{4.7}$$

$$CwX = wX, \tag{4.8}$$

$$BCV = BV - PBV, \tag{4.9}$$

$$C^2V = CV + V - wBV, \tag{4.10}$$

$$g(X, wY) = g(wX, Y), \tag{4.11}$$

$$g(wX, wY) = g(wX, Y) + g(X, Y), \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$.

Example 4.1. Let $(\tilde{M} = \mathbb{R}^9_3, \tilde{g})$ be a 7−dimensional semi-Euclidean space with signature $(-,-,+,+,-,+,+,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be the standard coordinate system of $\mathbb{R}^9_3$. If we set $\tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (\phi x_1, \phi x_2, \phi x_3, (1-\phi)x_4, (1-\phi)x_5, (1-\phi)x_6, (1-\phi)x_7, (1-\phi)x_8, \phi x_9)$, then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\tilde{M}$. Suppose M is a submanifold of $\tilde{M}$ defined by

$$\begin{aligned} x_1 &= u_1 + u_2, x_2 = u_1 - u_2, x_3 = u_1, x_4 = \phi u_1, \\ x_5 &= \sqrt{2}\phi u_1, x_6 = -\phi u_2, x_7 = \phi u_2, x_8 = \phi u_3, x_9 = u_3 \end{aligned}$$

where u_i, $1 \leq i \leq 3$, are real parameters. Thus $TM = Span\{U_1, U_2, U_3\}$, where

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \phi\frac{\partial}{\partial x_4} + \sqrt{2}\phi\frac{\partial}{\partial x_5}, \\ U_2 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \phi\frac{\partial}{\partial x_6} + \phi\frac{\partial}{\partial x_7}, \\ U_3 &= \phi\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}. \end{aligned}$$

Then M is a 1−lightlike submanifold. We have $Rad(TM) = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3\}$. Lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = -\frac{1}{3(\phi+2)}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \phi\frac{\partial}{\partial x_4} + \sqrt{2}\phi\frac{\partial}{\partial x_5}\right).$$

Also, screen transversal bundle $S(TM^\perp)$ is spanned by

$$\begin{aligned} W_1 &= \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}, W_2 = \phi\frac{\partial}{\partial x_1} - \phi\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7}, \\ W_3 &= -\frac{\partial}{\partial x_8} + \phi\frac{\partial}{\partial x_9}, W_4 = \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \phi\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5}, \\ W_5 &= -\frac{1}{3(2+\phi)}\left(\phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} - \phi\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5}\right). \end{aligned}$$

It is easy to see that $\tilde{P}U_1 = W_4$, $\tilde{P}U_2 = W_2$, $\tilde{P}U_3 = W_3$, $\tilde{P}N = W_5$ and $\tilde{P}W_1 = \phi W_1$. Thus we have $\tilde{P}(S(TM)) \subset S(TM^\perp)$, $\tilde{P}(Rad(TM)) \subset S(TM^\perp)$ and $\tilde{P}(ltr(TM)) \subset S(TM^\perp)$. Then M is a screen transversal anti-invariant lightlike submanifold of $\tilde{M}$.

Proposition 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then w is golden structure on TM.*

Proof. From (4.12), we have

$$g(wX, wY) = g(wX, Y) + g(X, Y),$$

for any $X, Y \in \Gamma(TM)$, which completes the proof. □

Proposition 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on $ltr(TM)$.*

Proof. By the definition of screen transversal anti-invariant lightlike submanifold we have $BN = 0$, for any $N \in \Gamma(ltr(TM))$. From (4.10) we have $C^2N = CN + N$. Thus C is golden structure on $ltr(TM)$. □

In the similar way, we have the following.

Proposition 4.4. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then C is golden structure on D_0.*

Theorem 4.1. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is integrable iff*

$$\nabla^s_X \tilde{P}Y = \nabla^s_Y \tilde{P}X, \tag{4.13}$$

for any $X, Y \in \Gamma(S(TM))$ [9].

Theorem 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is integrable iff*

$$\nabla^s_{\xi_1} \tilde{P}\xi_2 = \nabla^s_{\xi_2} \tilde{P}\xi_1, \tag{4.14}$$

for any $\xi_1, \xi_2 \in \Gamma(Rad(TM))$ [9].

Theorem 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the screen distribution is parallel iff*

$$\tilde{g}(\nabla^s_X \tilde{P}Y, \tilde{P}N) = \tilde{g}(h^s(X, Y), \tilde{P}N), \tag{4.15}$$

for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Using the definition of screen transversal anti-invariant lightlike submanifold, $S(TM)$ is parallel iff $g(\nabla_X Y, N) = 0$, for any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. From (2.3), (2.11) and (2.13), we obtain

$$\begin{aligned} g(\nabla_X Y, N) &= \tilde{g}(\tilde{\nabla}_X Y, N) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \tilde{P}N) - \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}N) \\ &= \tilde{g}(\nabla^s_X \tilde{P}Y, \tilde{P}N) - \tilde{g}(h^s(X, Y), \tilde{P}N), \end{aligned} \tag{4.16}$$

which completes the proof. □

Theorem 4.4. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is parallel iff*

$$\tilde{g}(\nabla^s_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) = \tilde{g}(h^s(\xi_1, \xi_2), \tilde{P}X), \tag{4.17}$$

for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$.

Proof. Using the definition of screen transversal anti-invariant lightlike submanifold $Rad(TM)$ is parallel iff $g(\nabla_{\xi_1}\xi_2, X) = 0$ for any $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad(TM))$. From (2.3), (2.11) and (2.13), we get

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, X) &= \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, X) = \tilde{g}(\tilde{\nabla}_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) - \tilde{g}(\tilde{\nabla}_{\xi_1}\xi_2, \tilde{P}X) \\ &= \tilde{g}(\nabla^s_{\xi_1} \tilde{P}\xi_2, \tilde{P}X) - \tilde{g}(h^s(\xi_1, \xi_2), \tilde{P}X), \end{aligned} \tag{4.18}$$

which completes the proof. □

Taking into account that M is a totally umbilicial in Theorem 4.4 we get following theorem.

Theorem 4.5. *Let M be a totally umbilical screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the radical distribution is parallel iff $\nabla^s_{\xi_1} \tilde{P}\xi_2$ has no compenents in $\tilde{P}(S(TM))$, for any $\xi_1, \xi_2 \in \Gamma(Rad(TM))$.*

Now, we give different form of theorem given in [9].

Theorem 4.6. *Let M be a screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the induced connection ∇ on M is a metric connection iff $B_1 \nabla^s_X \tilde{P}\xi = B_1 h^s(X, \tilde{P}\xi)$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.*

Proof. Since $\tilde{\nabla}\tilde{P} = 0$, we have

$$\tilde{\nabla}_X \tilde{P}\xi = \tilde{P}\tilde{\nabla}_X \xi, \tag{4.19}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Applying $\tilde{P}$ in this equation and using (2.1), we get

$$\tilde{P}\tilde{\nabla}_X \tilde{P}\xi = \tilde{P}\tilde{\nabla}_X \xi + \tilde{\nabla}_X \xi. \tag{4.20}$$

From (2.11), (2.13), (4.5) and (4.20), we have

$$\begin{aligned}(4.21)\quad & -\tilde{P}A_{\tilde{P}\xi}X + B_1\nabla^s_X\tilde{P}\xi + B_2\nabla^s_X\tilde{P}\xi + C_1\tilde{\nabla}^s_X\tilde{P}\xi + C_2\nabla^s_X\tilde{P}\xi \\ & +C^l_3\nabla^s_X\tilde{P}\xi + C^s_3\nabla^s_X\tilde{P}\xi + C_4\nabla^s_X\tilde{P}\xi + \tilde{P}h^l(X,\tilde{P}\xi) \\ = \; & \tilde{P}\nabla_X\xi + \tilde{P}h^l(X,\xi) + B_1h^s(X,\xi) + B_2h^s(X,\xi) + C_1h^s(X,\xi) + C_2h^s(X,\xi) \\ & +C^l_3h^s(X,\xi) + C^s_3h^s(X,\xi) + C_4h^s(X,\xi) + \nabla_X\xi + h^l(X,\xi) + h^s(X,\xi).\end{aligned}$$

Then, taking the tangential parts of (4.21), we derive

$$\nabla_X\xi = B_1\nabla^s_X\tilde{P}\xi + B_2\nabla^s_X\tilde{P}\xi - B_1h^s(X,\xi) - B_2h^s(X,\xi). \tag{4.22}$$

Considering Theorem 2.2, the equation (4.22) completes the proof. □

Taking into account that M is a totally umbilicial in Theorem 4.6 we get following theorem.

Theorem 4.7. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then the induced connection ∇ on M is a metric connection iff $\nabla^s_X\tilde{P}\xi$ has no component in $\tilde{P}(S(TM))$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.*

Theorem 4.8. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then $H^l = 0$ iff $\nabla^s_X\tilde{P}X$ has no component in $\tilde{P}(ltr(TM))$, for any $X \in \Gamma(S(TM))$.*

Proof. Using (2.3) and (2.11) and taking into account that M is a totally umbilicial screen transversal anti-invariant lightlike submanifold of $\tilde{M}$, we get

$$\begin{aligned} g(\nabla^s_X\tilde{P}X,\tilde{P}\xi) &= \tilde{g}(\tilde{\nabla}_X\tilde{P}X,\tilde{P}\xi) = \tilde{g}(\tilde{\nabla}_X\tilde{P}X,\xi) + \tilde{g}(\tilde{\nabla}_XX,\xi) \\ &= \tilde{g}(h^l(X,\tilde{P}X),\xi) + \tilde{g}(h^l(X,X),\xi) \\ &= \tilde{g}(X,\tilde{P}X)\tilde{g}(H^l,\xi) + \tilde{g}(X,X)\tilde{g}(H^l,\xi) \\ &= \tilde{g}(X,X)\tilde{g}(H^l,\xi), \end{aligned}$$

for any $X \in \Gamma(S(TM))$ and $\xi \in \Gamma(Rad(TM))$, which completes the proof. □

Theorem 4.9. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M},\tilde{g},\tilde{P})$. Then H^s has no component in $\tilde{P}(S(TM))$ or $dim(S(TM)) = 1$.*

Proof. Using (2.2) and (2.11) and taking into account that $\tilde{\nabla}$ is a metric connection, we derive

$$\tilde{g}(\tilde{\nabla}_X\tilde{P}X,Y) = \tilde{g}(\tilde{\nabla}_XX,\tilde{P}Y) = \tilde{g}(h^s(X,X),\tilde{P}Y), \tag{4.23}$$

$$\tilde{g}(\tilde{\nabla}_X\tilde{P}X,Y) = -\tilde{g}(\tilde{P}X,\tilde{\nabla}_XY) = -\tilde{g}(\tilde{P}X,h^s(X,Y)), \tag{4.24}$$

for any $X, Y \in \Gamma(S(TM))$. Combining (4.23) and (4.24), we obtain

$$\tilde{g}(h^s(X,X), \tilde{P}Y) = -\ \tilde{g}(\tilde{P}X, h^s(X,Y)). \tag{4.25}$$

Using (2.23) in equation (4.25), we get

$$g(X,X)\breve{g}(H^s, \tilde{P}Y) = -g(X,Y)\ \tilde{g}(H^s, \tilde{P}X). \tag{4.26}$$

Interchanging X and Y in (4.26) and rearranging the terms, we derive

$$\breve{g}(H^s, \tilde{P}X) = -\frac{g(X,Y)}{g(Y,Y)}\ \tilde{g}(H^s, \tilde{P}Y). \tag{4.27}$$

From (4.26) and (4.27), we conclude that

$$\breve{g}(H^s, \tilde{P}X) = \frac{g(X,Y)^2}{g(X,X)g(Y,Y)}\ \tilde{g}(H^s, \tilde{P}X). \tag{4.28}$$

This completes the proof. $\square$

Theorem 4.10. *Let M be a totally umbilicial screen transversal anti-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then H^s has no component in $\tilde{P}(ltr(TM))$.*

Proof. From (2.2), (2.11) and (2.23), we get

$$\begin{aligned}
\tilde{g}(D^l(X, \tilde{P}Y), \xi) &= \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \xi) = \tilde{g}(\tilde{\nabla}_X Y, \tilde{P}\xi) \\
&= \tilde{g}(h^s(X,Y), \tilde{P}\xi) = g(X,Y)\ \tilde{g}(H^s, \tilde{P}\xi),
\end{aligned}$$

for any $X, Y \in \Gamma(S(TM))$, which completes the proof. $\square$

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BOUNDARY VALUE PROBLEM FOR NONLINEAR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATION WITH HADAMARD FRACTIONAL INTEGRAL AND ANTI-PERIODIC CONDITIONS

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Abstract. The aim of this work is to study a class of boundary value problem including a fractional order differential equation involving the Caputo-Hadamard fractional derivative. Sufficient conditions will be presented to guarantee the existence and uniqueness of solution of this fractional boundary value problem. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.
Key words: fractional differential equation, fractional derivatives and integrals, boundary value problem.

1. Introduction

Fractional differential equations is a subject of the domain of mathematics, which are basically used to describe the comportment of several complex and nonlocal systems with memory. Due to the effective memory function of fractional derivative, they have been widely used to describe many physical phenomena such as flow in porous media and in fluid dynamic traffic model. Moreover, fractional differential equations been widely used in engineering, physics, chemistry, biology, and other

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fields; see the monographs of Kilbas et al. [24], F. Jarad et al. [22], Miller and Ross [26], Samko et al. [28] and the papers of Delbosco and Rodino [17], Hazarika et al. [16], Diethelm et al. [18], El-Sayed [19], Kilbas and Marzan [23], Mainardi [25], H.M. Srivastava [29] and Podlubny et al. [27]. Moreover, several papers have been devoted to the study of the existence, stability, existence and uniqueness of solutions for fractional differential equations, among others we refer to the papers [2, 3, 4, 5, 7, 9, 10, 15, 16, 30, 31].

In 2008, Benchohra et al. [10] studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations:

$$\left\{\begin{array}{ll} D^{\alpha}y(t)=f(t,y(t)), & \\ & t\in J,\\ ay(0)+by(T)=c, & \end{array}\right.$$

where $J:=[0,T]$, D^{α} is the caputo fractional derivative of order α, $(0<\alpha<1)$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, and a,b,c are real constants with $a+b\neq 0$.

In 2017, Asghar Ahmadkhanlu [6] studied the existence and uniqueness of solutions of the following boundary value problem of fractional differential equation is considered:

$$\left\{\begin{array}{ll} D^{\alpha}y(t)=f(t,y(t)), & \\ & t\in J,\\ y(0)=\eta I^{\beta}y(\tau), 0<\tau<1. & \end{array}\right.$$

Where $J:=[0,1]$, D^{α} is the caputo fractional derivative of order α, $(0<\alpha<1)$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, $\eta\in\mathbb{R}$, I^{β}, $0<\beta<1$, is the Riemman-Liouville fractional integral of order β.

In 2018, Benhamida et al. [12, 13], studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations:

$$\left\{\begin{array}{ll} D^{\alpha}y(t)=f(t,y(t)), & \\ & t\in J,\\ ay(1)+by(T)=c, & \end{array}\right.$$

where $J:=[1,T]$, D^{α} is the caputo-Hadamard fractional derivative of order α, $(0<\alpha<1)$, $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and a,b,c are real constants with $a+b\neq 0$.

In 2018, Benhamida et al. [11], studied the existence of solutions to the boundary value problem for fractional order differential equations

$$\left\{\begin{array}{ll} D^{\alpha}y(t)=f(t,y(t)), & \\ & t\in J,\\ y(0)+y(T)=b\int_0^T y(s)ds, bT\neq 2, & \end{array}\right.$$

where $J := [0,T], T > 0$, D^α is the Caputo fractional derivative of order α, $(0 < \alpha < 1)$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and b are real constants .

In 2018, Abdo et al. [1] discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} D^\alpha y(t) = f(t, y(t)), & \\ & t \in J, \\ y(0) = b\int_0^1 y(s)ds + d. & \end{cases}$$

Where $J := [0,1]$, $0 < \alpha \leq 1$, $\lambda \geq 0$, $d > 0$, D^α is the standard Caputo fractional operator and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function .

In 2019, A. Ardjouni et al. [8] discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions:

$$\left\{ \begin{array}{ll} D_1^\alpha y(t) = f(t, y(t)), & \\ & t \in J, \\ y(1) = b\int_1^e y(s)ds + d, & \end{array} \right)$$

where $J := [1,e]$, D_1^α is the Caputo-Hadamard fractional derivative of order $0 < \alpha \leq 1$, $\lambda \geq 0$, $d > 0$ and $f : J \times [0,\infty) \to [0,\infty)$ is a given continuous function.

Motivated by the studies above, among others, in this paper, we concentrate on the following boundary value problem, of nonlinear fractional differential equation with fractional integral as well as integer and fractional derivative:

$$ {}_H^C D_{1^+}^r x(t) = f(t, x(t)), \quad t \in J := [1,T], \quad 0 < r \leq 1, \tag{1.1}$$

with fractional boundary conditions:

$$\alpha x(1) + \beta x(T) = \lambda I^q x(\eta) + \delta, \qquad q \in (0,1] \tag{1.2}$$

where ${}_H^C D_{1^+}^r$ denote the Caputo-Hadamard fractional derivative and I^q denotes the standard Hadamard fractional integral. Throughout this paper, we always assume that $0 < r, q \leq 1$, $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. $\alpha, \beta, \lambda, \delta$ are real constants, and $\eta \in (1,T)$.

The rest of the paper is organized as follows. We recall some basic concepts of fractional calculus and introduce the integral operator associated to the given problem in Sect.2. Existence results, which rely on Schauder's fixed point theorem nonlinear alternative for single valued maps, and Scheafer's fixed point theorem are given. Also, In Sect.3, we obtain uniqueness results by means of Boyd and Wong's and Banach's fixed point theorems. Example illustrating the obtained results are presented in Sect.4, and the paper concludes with some interesting observations in Sect.5.

2. Preliminaries and lemmas

At first, we recall some concepts on fractional calculus and present some additional properties that will be used later. For more details, we refer to [20, 22, 24, 32]. We present some basic definitions and results from fractional calculus theory.
Let $E = C([1,T],\mathbb{R})$ be the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ with the norm

$$\|u\| = \max_{t\in[1,T]} |u(t)|$$

Let bet the space

$$AC_\delta^n([a,b],\mathbb{R}) = \{h : [a,b] \to \mathbb{R} : \delta^{n-1}h(x) \in AC([a,b],\mathbb{R})\}.$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC([a,b],\mathbb{R})$ is the space of absolutely continuous functions on $[a,b]$.

Definition 2.1. (Hadamard fractional integral [24]) *The Hadamard fractional integral of order $\alpha > 0$ for a function $h : [1,+\infty) \to \mathbb{R}$ is defined as*

$$I_{a^+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\log\frac{t}{s})^{\alpha-1} h(s)\frac{ds}{s} \tag{2.1}$$

where Γ is the Gamma function.

Definition 2.2. (Hadamard fractional derivative [24]) *For a function h given on the interval $[1,+\infty)$, and $n-1 < \alpha < n$, the Hadamard derivative of order α is defined by*

$$\begin{aligned} D_{a^+}^\alpha h(t) &= \tfrac{1}{\Gamma(n-\alpha)}(t\tfrac{d}{dt})^n \int_a^t (\log\tfrac{t}{s})^{n-\alpha-1}h(s)\tfrac{ds}{s} \\ &= \delta^n I_{a^+}^{n-\alpha}h(t). \end{aligned} \tag{2.2}$$

where $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = t\frac{d}{dt}$. provided the right integral converges.

There is a recent generalization introduced by Jarad and al in [22], where the authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 2.3. (Caputo-Hadamard fractional derivative [22]) *Let $\alpha = 0$, and $n = [\alpha]+1$. If $h(x) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$ and*

$$AC_\delta^n[a,b] = \{h : [a,b] \to C : \delta^{n-1}h(x) \in AC[a,b]\}.$$

The left-sided Caputo-type modification of left-Hadamard fractional derivatives of order α is given by

$${}_H^C D_{a^+}^\alpha h(t) = D_{a^+}^\alpha \left(h(t) - \sum_{k=0}^{n-1} \frac{\delta^k h(a)}{k!}(\log\frac{t}{s})^k \right) \tag{2.3}$$

Theorem 2.4. (See [22]) *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $y(t) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$. Then ${}^C_H D^\alpha_{a+} f(t)$ exist everywhere on $[a,b]$ and*
(i) if $\alpha \notin \mathbb{N} - \{0\}$, ${}^C_H D^\alpha_{a+} f(t)$ can be represented by

$$\begin{aligned} {}^C_H D^\alpha_{a+} h(t) &= \tfrac{1}{\Gamma(n-\alpha)} \int_a^t (\log \tfrac{t}{s})^{n-\alpha-1} \delta^n h(s) \tfrac{ds}{s} \\ &= I^{n-\alpha}_{a^+} \delta^n h(t). \end{aligned} \tag{2.4}$$

(ii) if $\alpha \in \mathbb{N} - \{0\}$, then

$$ {}^C_H D^\alpha_{a+} h(t) = \delta^n h(t) \tag{2.5}$$

In particular

$$ {}^C_H D^0_{a+} h(t) = h(t) \tag{2.6}$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (2.4) provided that $h(t) \in AC_\delta^n(\mathbb{R}^+)$. Thus one has

$$ {}^C_H D^\alpha_{a+} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\log \frac{t}{s})^{n-\alpha-1} \delta^n h(s) \frac{ds}{s} \tag{2.7}$$

Proposition 2.5. (See [24]) *Let $\alpha > 0, \beta > 0, n = [\alpha] + 1$, and $a > 0$, then*

$$\begin{aligned} I^\alpha_{a+} (\log \tfrac{t}{a})^{\beta-1}(x) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log \tfrac{x}{a})^{\beta+\alpha-1} \\ {}^C_H D^\alpha_{a+} (\log \tfrac{t}{a})^{\beta-1}(x) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log \tfrac{x}{a})^{\beta-\alpha-1}, \beta > n, \\ {}^C_H D^\alpha_{a+} (\log \tfrac{t}{a})^k &= 0, k = 0, 1, ..., n-1. \end{aligned} \tag{2.8}$$

Theorem 2.6. (See [20]) *Let $u(t) \in AC_\delta^n[a,b], 0 < a < b < \infty$ and $\alpha \geq 0, \beta \geq 0$, Then*

$$\begin{aligned} {}^C_H D^\alpha_{a+} \left(I^\alpha_{a+} u\right)(t) &= \left(I^{\beta-\alpha}_{a^+} u\right)(t), \\ {}^C_H D^\alpha_{a^+} \left({}^C_H D^\beta_{a^+} u\right)(t) &= \left({}^C_H D^{\alpha+\beta}_{a^+} u\right)(t). \end{aligned} \tag{2.9}$$

Lemma 2.7. (See [22]) *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $u(t) \in AC_\delta^n[a,b]$, then the Caputo-Hadamard fractional differential equation*

$$ {}^C_H D^\alpha_{a+} u(t) = 0, \tag{2.10}$$

has a solution:

$$ u(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k, \tag{2.11}$$

and the following formula holds:

$$ I^\alpha_{a+} \left({}^C_H D^\alpha_{a+} u\right)(t) = u(t) + \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k, \tag{2.12}$$

where $c_k \in \mathbb{R}, k = 1, 2, ..., n-1$.

3. Main Results

First, we prove a preparatory lemma for boundary value problem of linear fractional differential equations with Caputo-Hadamard derivative.

Definition 3.1. *A function $x(t) \in AC^1_\delta(J,\mathbb{R})$ is said to be a solution of* (1.1), (1.2) *if x satisfies the equation ${}^C_H D^r x(t) = f(t, x(t))$ on J , and the conditions* (1.2).

For the existence of solutions for the problem (1.1), (1.2), we need the following auxiliary lemma.

Lemma 3.2. *Let $h : [1,+\infty) \to \mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\} \tag{3.1}$$

if and only if x is a solution of the fractional BVP

$$ {}^C_H D^r x(t) = h(t), t \in J, r \in (0,1] \tag{3.2}$$

$$\alpha x(1) + \beta x(T) = \lambda I^q x(\eta) + \delta, q \in (0,1] \tag{3.3}$$

Proof. Assume x satisfies (3.2). Then Lemma 2.7 (2.12) implies that

$$x(t) = I^r h(t) + c_1. \tag{3.4}$$

By applying the boundary conditions (3.3) in (3.4), we obtain

$$\alpha c_1 + \beta I^r h(T) + \beta c_1 = \lambda I^{r+q} h(\eta)) + c_1 \frac{\lambda(\log\eta)^q}{\Gamma(q+1)} + \delta.$$

Thus,

$$c_1\left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right) = \lambda I^{r+q} h(\eta)) - \beta I^r h(T) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta)) - \beta I^r h(T) + \delta\right\},$$

where,

$$\Lambda = \left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right).$$

Finally, we obtain the solution (3.1)

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\}.$$

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1), (1.2) by using a variety of fixed point theorems.

3.1. Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.3. *Assume the following hypothesis:*
(H1) There exists a constant $L > 0$ *such that*

$$|f(t,x) - f(t,y)| \le L|x-y|.$$

If

$$LM < 1, \tag{3.5}$$

with

$$M := \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\},$$

then the problem (1.1) has a unique solution on J.

Proof. Transform the problem 1.1), (1.2) into a fixed point problem for the operator $\mathfrak{F}$ defined by

$$\mathfrak{F}x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q} h(\eta) - \beta I^r h(T) + \delta\right\}. \tag{3.6}$$

Applying the Banach contraction mapping principle, we shall show that $\mathfrak{F}$ is a contraction.

Now let $x, y \in C(J,\mathbb{R})$. Then, for $t \in J$, we have

(3.7)

Thus

$$\|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)\|_\infty \le LM\|x-y\|_\infty.$$

We deduce that $\mathfrak{F}$ is a contraction mapping. As a consequence of Banach contraction principle. the problem (1.1)-(1.2) has a unique solution on J. This completes the proof. □

3.2. Existence result via Schaefer's fixed point theorem

Theorem 3.4. *Assume the hypotheses:*
(H2): The function $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(H3) There exists a constant $K > 0$, *such that*

$$|f(t,0)| \le K, \ \textit{for a.e. } t \in J.$$

Then, the problem (1.1)-(1.2) has a least one solution in J.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. The proof will be given in several steps.
Step 1: $\mathfrak{F}$ is continuous Let x_n be a sequence such that $x_n \to x$ in $C(J,\mathbb{R})$. Then for each $t \in J$,

$$\begin{aligned}\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\| &\leq \tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}\|f(s,x_n(s))-f(s,x(s))\|\tfrac{ds}{s}\\ &\leq\left\{\tfrac{(\log T)^r}{\Gamma(r+1)}+\tfrac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\times\\ &\quad\|f(s,x_n(s))-f(s,x(s)).\|\end{aligned}$$

Since f is continuous, we have $\|(\mathfrak{F}x_n)(t)-(\mathfrak{F}x)(t)\|_\infty\to 0$ as $n\to\infty$.

Step 2: $\mathfrak{F}$ maps bounded sets into bounded sets in $C(J,\mathbb{R})$
Indeed, it is enough to show that for any $r>0$, we take

$$u\in B_r=\{x\in C(J,\mathbb{R}),\|x\|_\infty\leq r\}.$$

From (H1) and (H3), Then we have

$$|f(s,x(s))|\leq|f(s,x(s))-f(t,0)|+|f(t,0)|\leq Lr+K.$$

For $x\in B_r$ and for each $t\in[1,T]$, we have

$$\begin{aligned}|(\mathfrak{F}x)(t)|&\leq\tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s}+\tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}|f(s,x(s))|\tfrac{ds}{s}\\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq\tfrac{Lr+K}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}\tfrac{ds}{s}+\tfrac{|\lambda|(Lr+K)}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}\tfrac{ds}{s}\\ &+\tfrac{|\beta|(Lr+K)}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}\tfrac{ds}{s}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq(Lr+K)\left\{\tfrac{(\log T)^r}{\Gamma(r+1)}+\tfrac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}+\tfrac{|\delta|}{|\Lambda|}\\ &\leq(Lr+K)M+\tfrac{|\delta|}{|\Lambda|}.\end{aligned}$$

Thus,

$$\|(\mathfrak{F}x)(t)\|\leq(Lr+K)M+\frac{|\delta|}{|\Lambda|}.$$

Step 3: $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.
Let $t_1,t_2\in J,t_1<t_2$, B_r be a bounded set of $C(J,\mathbb{R})$ as in Step 2, and let $x\in B_r$. Then

$$\begin{aligned}\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|&\leq\tfrac{1}{\Gamma(r)}\int_1^{t_1}\left[(\log\tfrac{t_2}{s})^{r-1}-(\log\tfrac{t_1}{s})^{r-1}\right]\|f(s,x(s))\|\tfrac{ds}{s}\\ &+\tfrac{1}{\Gamma(r)}\int_{t_1}^{t_2}(\log\tfrac{t_2}{s})^{r-1}\|f(s,x(s))\|\tfrac{ds}{s}\\ &\leq\tfrac{Lr+K}{\Gamma(r)}\int_1^{t_1}\left[(\log\tfrac{t_2}{s})^{r-1}-(\log\tfrac{t_1}{s})^{r-1}\right]\tfrac{ds}{s}+\tfrac{K}{\Gamma(r)}\int_{t_1}^{t_2}(\log\tfrac{t_2}{s})^{r-1}\tfrac{ds}{s}\\ &\leq\tfrac{Lr+K}{\Gamma(r+1)}\left[(\log t_2)^r-(\log t_1)^r\right],\end{aligned}$$

which implies $\|\mathfrak{F}x(t_2)-\mathfrak{F}x(t_1)\|_\infty\to 0$ as $t_1\to t_2$, as consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that $\mathfrak{F}$ is continuous and completely continuous.

Step 4: A priori bounds.
Now it remains to show that the set

$$\Lambda = \{x \in C(J, \mathbb{R}) : x = \rho\mathfrak{F}(x) \text{ for some } 0 < \rho < 1\}$$

is bounded.

For such a $x \in \Lambda$. Thus, for each $t \in J$, we have

$$\begin{aligned} x(t) \quad &\leq \rho\Big\{\tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}f(s,x(s))\tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}f(s,x(s))\tfrac{ds}{s} \\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}f(s,x(s))\tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|}\Big\} \end{aligned}$$

For $\rho \in [0, 1]$, let x be such that for each $t \in J$

$$\begin{aligned} \|\mathfrak{F}x(t)\| \quad &\leq \tfrac{1}{\Gamma(r)}\int_1^t(\log\tfrac{t}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\tfrac{\eta}{s})^{r+q-1}|f(s,x(s))|\tfrac{ds}{s} \\ &+\tfrac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\tfrac{T}{s})^{r-1}|f(s,x(s))|\tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|} \\ &\leq (Lr + K)M + \tfrac{|\delta|}{|\Lambda|}. \end{aligned}$$

Thus

$$\|\mathfrak{F}x(t)\| \leq \infty$$

This implies that the set Λ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathfrak{F}$ has a fixed point which is a solution on J of the problem (1.1)-(1.2). □

3.3. Existence via the Leray-Schauder nonlinear alternative

Theorem 3.5. *Assume the following hypotheses:*
(H4) There exist $\omega \in L^1(J, \mathbb{R}^+)$ and $\psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|f(t, x)| \leq \omega(t)\psi(\|x\|), \text{ for a.e. } t \in J \text{ and each } x \in \mathbb{R}.$$

(H5) There exists a constant $\epsilon > 0$ such that

$$\frac{\epsilon}{\|\omega\|\psi(\epsilon)M + \frac{|\delta|}{|\Lambda|}} > 1.$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on J.

Proof. We shall use the Leray-Schauder theorem to prove that $\mathfrak{F}$ defined by (3.6) has a fixed point. As shown in Theorem 3.4, we see that the operator $\mathfrak{F}$ is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the

Arzela-Ascoli theorem $\mathfrak{F}$ is completely continuous.
Let x be such that for each $t \in J$, we take the equation $x = \lambda \operatorname{Im} x$ for $\lambda \in (0,1)$ and let x be a solution. After that, the following is obtained.

$$\begin{aligned}|x(t)| &\leq \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} + \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} \\ &+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} \omega(t)\psi(\|x\|)\tfrac{ds}{s} + \tfrac{|\delta|}{|\Lambda|} \\ &\leq \|\omega\|\psi(\|x\|)M + \tfrac{|\delta|}{|\Lambda|}.\end{aligned}$$

and consequently

$$\frac{\|x\|_\infty}{\|\omega\|\psi(\|x\|)M + \frac{|\delta|}{|\Lambda|}} \leq 1.$$

Then by condition (H5), there exists ϵ such that $\|x\|_\infty \neq \epsilon$. Let us set

$$\kappa = \{x \in C(J, \mathbb{R}) : \|x\| < \epsilon\}.$$

Obviously, the operator $\operatorname{Im} : \overline{\kappa} \to C(J, \mathbb{R})$ is completely continuous. From the choice of κ , there is no $x \in \partial\kappa$ such that $x = \lambda \operatorname{Im}(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, $\mathfrak{F}$ has a fixed point $x \in \kappa$ which is a solution of the (1.1)-(1.2).
The proof is completed. $\square$

Now we present another variant of existence-uniqueness result.

3.4. Existence and uniqueness result via Boyd-Wong nonlinear contraction

Definition 3.6. *Assume that E is a Banach space and $T : E \to E$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property: $\|Tx - Ty\| \leq \psi(\|x - y\|), \forall x, y \in E$. then, we say that T is a nonlinear contraction.*

Theorem 3.7. *(Boyd-Wong Contraction Principle)[14]*
Suppose that B is a Banach space and $T : B \to B$ is a nonlinear contraction. Then T has a unique fixed point in B.

Theorem 3.8. *Assume that $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $H > 0$ satisfying the condition*

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{H + |x-y|}, \text{ for } t \in J, x, y \in \mathbb{R}. \tag{3.8}$$

Then the fractional BVP (1.1)-(1.2) has a unique solution on J .

Proof. We define an operator $\mathfrak{F} : \chi \to \chi$ as in (3.6) and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \forall\varepsilon > 0,$$

where $M \leq H$. We notice that the function ψ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in \chi$, and for each $t \in J$, we obtain

$$\begin{aligned}
|(\mathfrak{F}x)(t) - (\mathfrak{F}y)(t)| &\leq \tfrac{1}{\Gamma(r)} \int_1^t (\log \tfrac{t}{s})^{r-1} \|f(s, x(s)) - f(s, y(s))\| \tfrac{ds}{s} \\
&+ \tfrac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \tfrac{\eta}{s})^{r+q-1} \|f(s, x(s)) - f(s, y(s))\| \tfrac{ds}{s} \\
&+ \tfrac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \tfrac{T}{s})^{r-1} \|f(s, x(s)) - f(s, y(s))\| \tfrac{ds}{s} \\
&\leq \tfrac{|x-y|}{H+|x-y|} \left\{ \tfrac{(\log T)^r}{\Gamma(r+1)} + \tfrac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \tfrac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \\
&:= M \tfrac{|x-y|}{H+|x-y|} \\
&\leq \psi(\|x - y\|).
\end{aligned}$$

Then, we get $\|\mathfrak{F}x - \mathfrak{F}y\| \leq \psi(\|x - y\|)$. Hence, $\mathfrak{F}$ is a nonlinear contraction. Thus, by Theorem 3.9 (Boyd-Wong Contraction Principle) the operator $\mathfrak{F}$ has a unique fixed point which is the unique solution of the fractional BVP (1.1)-(1.2). The proof is completed. □

4. Example

We consider the problem for Caputo-Hadamard fractional differential equations of the form:

$$\left\{ \begin{array}{l} {}^C_H D^{\frac{2}{3}} x(t) = f(t, x(t)), (t, x) \in ([1, e], \mathbb{R}^+), \\ \\ x(1) + x(e) = \frac{1}{2} \left(I^{\frac{1}{2}} x(2) \right) + \frac{3}{4}. \end{array} \right. \tag{4.1}$$

Here

$$\begin{array}{llll} r = \frac{2}{3}, & q = \frac{1}{2}, & \alpha = 1, & \beta = 1, \\ \delta = \frac{3}{4}, & \lambda = \frac{1}{2}, & \eta = 2, & T = e. \end{array}$$

With

$$f(t, y(t)) = \frac{1}{t^2 + 4} cosx, \qquad t \in [1, e]$$

Clearly, the function f is continuous.
For each $x \in \mathbb{R}^+$ and $t \in [1, e]$, we have

$$|f(t, x(t)) - f(t, y(t))| \leq \frac{1}{4}|x - y|$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{4}$.
Further,

$$M := \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \simeq 2.0286$$

and

$$LM \simeq 0.5071 < 1.$$

Therefore, by the conclusion of Theorem 3.3, It follows that the problem (4.1) has a unique solution defined on $[1, e]$.

5. Conclusion

In this paper, we have obtained some existence results for nonlinear Caputo-Hadamard type fractional differential equations with Hadamard integral boundary conditions by means of some standard fixed point theorems and nonlinear alternative of Leray-Schauder type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting some examples. Our results are new and generalize some available results on the topic. For instance,

✓ We remark that when $\alpha = \beta = 1$, $\lambda = 0$, problem (1.1)-(1.2) reduces to the case considered in [12, 13].

✓ If we take $\alpha = q = 1$, $\beta = 0$, in (1.2), then our results correspond to the case integral boundary conditions considered in [8].

✓ By fixing $\beta = \lambda = 0$, in (1.2), our results correspond to the ones for initial value problem take the form:$x(1) = \delta$.

✓ In case we choose $\alpha = \beta = 1$, $\lambda = \delta = 0$, in (1.2), our results correspond to anti-periodic type boundary conditions take the form: $x(1) = -x(T)$.

✓ When, $\alpha = \beta = 1$, $\delta = 0$, the (1.2), our results correspond to Fractional integral and anti-periodic type boundary conditions.

✓ If we take $\alpha = 1$, $\beta = \delta = 0$, in (1.2), then our results correspond to the case Fractional integral boundary conditions.

In the nutshell, the boundary value problem studied in this paper is of fairly general nature and covers a variety of special cases.

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THE STRUCTURE OF UNIT GRUOP OF $\mathbb{F}_{3^n}T_{39}$

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Abstract. Let RG be the group ring of a group G over ring R and let $\mathscr{U}(RG)$ be its unit group. In this paper, we study the structure of the unit group of $\mathbb{F}_{3^n}T_{39}$.
Key words: Group ring, unit group, group modules

1. Introduction

Let FG be the group ring of a group G over a field F and let $\mathscr{U}(FG)$ be its unit group, which is the multiplicative subgroup containing all invertible elements. The study of a unit group is one of the classical topics in ring theory that started in 1940 with a famous paper written by G. Higman [11]. In recent years many new results have been achived; however, only few group rings have been computed. Unit groups are useful, for instance, in the investigation of Lie properties of group rings (for example see [3]) and isomorphism problems (for example see [4]).

Up to now, the structure of unit groups of some group rings has been found. For instance, on an integral group ring [12], on a permutation group ring [18], on a commutative group ring [16], on a linear group ring [13], on a quaternion group ring [6], on a modular group ring [17] and on a pauli group ring [9]. In [7], the authors proved which groups can be unit groups as well as properties of unit elements themselves [2] and also we studied the structure of $\mathscr{U}(\mathbb{F}_{2^n}D_{14})$ in [1].

In this paper we will study the unit group of $\mathbb{F}_{3^n}T_{39}$. So far, some cases, in characteristic 3, have been studied. For instance, in [5], the authors obtained

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the structure of unit group of $\mathbb{F}_{3^k}D_6$, in [8], Gildea determined the structure of unit group of $\mathbb{F}_{3^k}(C_3 \times D_6)$ and in [10] Gildea and Monaghan studied groups of order 12 and recently in [15], Monaghan studied groups of order 24. In this paper we characterize the unit group structure of group T_{39} over any finite field with characteristic 3.

2. Preliminaries and Notations

In this section, we collect some notations and lemma which we need for the proofs of our main results. We denote the order of an element g in the group G by $Ord_G(g)$, the sum of all elements of subset X in ring R by $\widehat{X}$, which is $\sum_{r\in X} r$. Notice there is no need for X to be a subring or subgroup; it defines for any arbitrary subset. In group ring RG, when X is the subset of all different powers of g, an element of group G, we may simply write $\widehat{g}$ instead of $\widehat{X}$. Also when X is the right coset $\langle g\rangle h$, we may write $\widehat{g}h$ for $\widehat{X}$. In group, x^y denotes the conjugate of x by y, that is, $x^y = y^{-1}xy$. Let $f : X \to Y$ be an arbitrary function. Define $Supp_X(f) = \{x \in X \,|\, f(x) \neq 0\}$. Also, we use the following notations: $\mathrm{Ann}_R(a) = \{r \in R \,|\, ra = ar = 0\}$, we denote a finite field of characteristic p with order p^n by $\mathbb{F}_{p^n}$. If E is a vector space over F, then $Dim_F(E)$ is the dimension of E over F. Let $\mathscr{U}(R)$ be the unit group of ring R, which is $\mathscr{U}(R) = \{u \in R \,|\, u^{-1} \in R\}$ and let $J(R)$ be the Jacobson radical of ring R. Now we state a useful definition and recall a lemma.

Definition 2.1. Let RG be group ring of ring R over the group G, let p be a prime number and let S_p be subset of all *p–elements* including identity element of G, which is $S_p = \{g \in G \,|\, \exists n \in \mathbb{Z}^{\geqslant 0};\ \mathrm{Ord}_G(g) = p^n\}$. We define a binary map $T : G \to R$ as follows:

$$T(g) = \begin{cases} 1 & \textit{If} \quad g \in S_p \\ 0 & \textit{If} \quad g \notin S_p \end{cases}$$

As we know that T on G is the base of RG, so we can linearly extend it to whole RG, of course no more remains binary. Also if see elements of RG as functions from G to R, that map every group element (g) to its coefficient (r_g), then their supports will be feasible. Now we can define $\mathrm{Krn}(T) := \{\alpha \in RG \,|\, \forall g \in G;\ \alpha g \in \mathrm{Ker}_{RG}(T)\}$ and $\mathrm{Spr}(\alpha) := \mathrm{Supp}_G(\alpha)$. Also $\mathrm{Anh}(a) := \mathrm{Ann}_{RG}(a)$ and $\mathrm{Dmn}(S) := \mathrm{Dim}_F(S)$.

Lemma 2.1. *Let F be a finite field of characteristic p, let G be a finite group, let T be a function defined as above and $s = \widehat{S_p}$. Then:*

(1) $J(FG) \subseteq \mathrm{Krn}(T)$.
(2) $\mathrm{Krn}(T) = \mathrm{Anh}(s)$.
(3) $J(FG) \subseteq \mathrm{Anh}(s)$.

Proof. [19, Lemma 2.2 on p. 151]. □

In the next section we present our main results.

3. Unit Group of $\mathbb{F}_{3^n}T_{39}$

Let $T_{39} = \langle x, y \,|\, x^{13} = y^3 = 1,\, x^y = x^3 \rangle$, let C_n be the cyclic group of order n and let $GL_n(R)$ be the general linear group of degree n on ring R. Our main result is:

Theorem 3.1. *Let $G = T_{39}$ and $F = \mathbb{F}_{3^n}$. Then the structure of $\mathscr{U}(FG)$ can be obtained as follows:*

$$\mathscr{U}(FG) = C_3^{2n} \times C_{3^n-1} \times GL_3(F)^4.$$

Let $p = 3$, let s be defined as in Lemma 2.1, let $\langle x \rangle$ be the cyclic subgroup generated by x and let $\langle x \rangle y$ be a right coset of $\langle x \rangle$, that is, $\langle x \rangle y = \{x^i y \,|\, -6 \leqslant i \leqslant +6\}$, or equivalently, $\langle x \rangle y = \{x^{-6}y, x^{-5}y, x^{-4}y, x^{-3}y, x^{-2}y, x^{-1}y, y, xy, x^2y, x^3y, xy^4, x^5y, x^6y\}$. By definition, we have

$$\begin{aligned} \widehat{x} &= x^{-6} + x^{-5} + x^{-4} + x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\ \widehat{x}y &= x^{-6}y + x^{-5}y + x^{-4}y + x^{-3}y + x^{-2}y + x^{-1}y + y + xy + x^2y + x^3y \\ &\quad + x^4y + x^5y + x^6y. \end{aligned}$$

Now we show:

Proposition 3.1. *Let $p = 3$ and $G = T_{39}$. Then the structure of annihilator will be as follows:*

$$\mathrm{Anh}(s) = \{a^- \widehat{x}y^{-1} + a\widehat{x} + a^+ \widehat{x}y \mid a^- + a + a^+ = 0\}.$$

Proof. It is easy to find that the conjugacy classes of G are as below:

$$\begin{aligned} &\mathscr{C}_0 = \{1\} \\ &\mathscr{C}_{-1} = \{x^{-1}, x^{-3}, x^4\} \\ &\mathscr{C}_{+1} = \{x, x^3, x^{-4}\} \\ &\mathscr{C}_{-2} = \{x^{-2}, x^{-5}, x^{-6}\} \\ &\mathscr{C}_{+2} = \{x^2, x^5, x^6\} \\ &\mathscr{C}_{-3} = \langle x \rangle y^{-1} \\ &\mathscr{C}_{+3} = \langle x \rangle y \end{aligned} \tag{3.1}$$

It is clear that T_{39} has three types of elements: Identity, elements of the form $x^i y^{\pm 1}$ with order 3 and elements of the form $x^i \neq 1$ with order 13. Therefore, $S_3 = \mathscr{C}_{-3} \cup \mathscr{C}_0 \cup \mathscr{C}_{+3}$, so $\widehat{S}_3 = \widehat{\mathscr{C}}_{-3} + \widehat{\mathscr{C}_0} + \widehat{\mathscr{C}}_{+3} = \widehat{x}y^{-1} + 1 + \widehat{x}y$, sum of 3$-$*elements* including identity. Let $\alpha = \sum_{i=-3}^{+3} \alpha_i \in \mathrm{Anh}(s)$ where, $\mathrm{Spr}(\alpha_i) \subseteq \mathscr{C}_i$ and $s = \widehat{S}_3$.

Then we have

$$
\begin{aligned}
0 = \alpha.s &= (\sum_{i=-3}^{+3} \alpha_i).(\widehat{S}_3) \\
&= (\alpha_{-3} + \alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2} + \alpha_{+3})(\widehat{xy}^{-1} + 1 + \widehat{xy}) \\
&= (\alpha_{-3} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2}) + \alpha_{+3})(\widehat{xy}^{-1} + 1 + \widehat{xy}) \\
&= (\alpha_{-3} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2})\widehat{xy}^{-1} + \alpha_{+3}\widehat{xy}) \\
&\quad + (\alpha_{-3}\widehat{xy} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2}) + \alpha_{+3}\widehat{xy}^{-1}) \\
&\quad + (\alpha_{-3}\widehat{xy}^{-1} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2})\widehat{xy} + \alpha_{+3})
\end{aligned} \tag{3.2}
$$

Notice that for every j, we know:

$$
\begin{aligned}
x^j y \quad .\widehat{xy} &= x^j \ .\widehat{xy}^{-1} = \widehat{xy}^{-1} \\
x^j y^{-1}.\widehat{xy} &= x^j y.\widehat{xy}^{-1} = \widehat{x} \\
x^j y^{-1}.\widehat{xy}^{-1} &= x^j \ .\widehat{xy} \quad = \widehat{xy}
\end{aligned} \tag{3.3}
$$

So the conjugacy classes of three last parentheses of (3.2) are different and since the left hand side is zero, every parentheses should be zero separately. Hence,

$$
\begin{aligned}
&(\alpha_{-3} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2})\widehat{xy}^{-1} + \alpha_{+3}\widehat{xy}) = 0 \\
&(\alpha_{-3}\widehat{xy} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2}) + \alpha_{+3}\widehat{xy}^{-1}) = 0 \\
&(\alpha_{-3}\widehat{xy}^{-1} + (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2})\widehat{xy} + \alpha_{+3}) = 0.
\end{aligned}
$$

Similarly, using (3.3) we can conclude that:

$$
\begin{aligned}
\alpha_{-3} + \varepsilon((\alpha_{-2} + \cdots + \alpha_{+2}) + \alpha_{+3})\widehat{xy}^{-1} &= 0 \\
(\alpha_{-2} + \cdots + \alpha_{+2}) + \varepsilon(\alpha_{-3} + \alpha_{+3})\widehat{x} \quad &= 0 \\
\alpha_{+3} + \varepsilon(\alpha_{-3} + (\alpha_{-2} + \cdots + \alpha_{+2}))\widehat{xy} \quad &= 0.
\end{aligned} \tag{3.4}
$$

As mentioned above $\alpha = \sum_{i=-3}^{+3} \alpha_i = \alpha_{-3} + \alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_{+1} + \alpha_{+2} + \alpha_{+3}$ where $\mathrm{Spr}(\alpha_i) \subseteq \mathscr{C}_i$ and by definition of $\mathscr{C}_i$'s from (3.1), we can write:

$$
\begin{aligned}
\alpha_0 &= a_0 \\
\alpha_{-1} &= a_{-1}x^{-1} + a_{-3}x^{-3} + a_4x^4 \\
\alpha_{+1} &= a_1x + a_3x^3 + a_{-4}x^{-4} \\
\alpha_{-2} &= a_{-2}x^{-2} + a_{-5}x^{-5} + a_{-6}x^{-6} \\
\alpha_{+2} &= a_2x^2 + a_5x^5 + a_6x^6 \\
\alpha_{-3} &= \sum_{i=-6}^{6} a_i^- x^i y^{-1} \\
\alpha_{+3} &= \sum_{i=-6}^{6} a_i^+ x^i y
\end{aligned}
$$

By substitution of each α_i's in (3.4), we can calculate the coefficients of each element of the group in the left hand sides of equations and since the right hand sides are zero, so each coefficient must be zero too. Thus for every h, i and j we have

$$\begin{array}{lll} a_h^- = -\varepsilon(\sum_{r=-2}^{+2}\alpha_r + \alpha_{+3}) & a_i = -\varepsilon(\alpha_{+3} + \alpha_{-3}) & a_j^+ = -\varepsilon(\alpha_{-3} + \sum_{r=-2}^{+2}\alpha_r) \\ a_h^- = -\sum_{r=-6}^{6}(a_r + a_r^+) & a_i = -\sum_{r=-6}^{6}(a_r^+ + a_r^-) & a_j^+ = -\sum_{r=-6}^{6}(a_r^- + a_r) \\ a_{-6}^- = \cdots = a_6^- & a_{-6} = \cdots = a_6 & a_{-6}^+ = \cdots = a_6^+ \end{array}$$

So by knowing a_0^-, a_0 and a_0^+, all coefficients can be computed. Also since we deal with a field of characteristic 3, so $13 = 1$, therefore, we have $a_0^- + a_0 + a_0^+ = 0$, thus:

$$\mathrm{Anh}(s) = \{a_0^-\widehat{x}y^{-1} + a_0\widehat{x} + a_0^+\widehat{x}y \mid a_0^- + a_0 + a_0^+ = 0\}.$$

□

Let s be as in Proposition 3.1, that is $s = \widehat{S}_3$, then we have

Proposition 3.2. Anh(s) *is a nilpotent ideal.*

Proof. Let $\alpha, \beta, \gamma \in \mathrm{Anh}(s)$. According to Proposition 3.1, we have

$$\begin{aligned} \alpha &= a^-\widehat{x}y^{-1} + a\widehat{x} + a^+\widehat{x}y \\ \beta &= b^-\widehat{x}y^{-1} + b\widehat{x} + b^+\widehat{x}y \\ \gamma &= c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y. \end{aligned} \tag{3.5}$$

So their production is:

$$\begin{aligned} \alpha.\beta.\gamma &= (a^-\widehat{x}y^{-1} + a\widehat{x} + a^+\widehat{x}y).(b^-\widehat{x}y^{-1} + b\widehat{x} + b^+\widehat{x}y).(c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y) \\ &= (a^+ - a^-)(b^+ - b^-)\widehat{G}.(c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y) \\ &= (a^+ - a^-)(b^+ - b^-)(c^- + c + c^+)\widehat{G}|\langle x\rangle| \end{aligned} \tag{3.6}$$

By Proposition 3.1, $\alpha.\beta.\gamma = 0$, thus $\mathrm{Anh}^3(s) = 0$, therefore, Anh(s) is a nilpotent ideal. □

Let s be as in Proposition 3.2, that is $s = \widehat{S}_3$, then we have

Proposition 3.3. $\mathrm{Anh}(s) \subseteq J(FG)$.

Proof. Since every nilpotent ideal is a nil ideal, so Proposition 3.2 shows $\mathrm{Anh}(s)$ is a nil ideal. On the other hand, by [14, Lemma 2.7.13 on p. 109], Jacobson radical contains all of the nil ideals, so,

$$\mathrm{Anh}(s) \subseteq J(FG).$$

□

In the next corollary, we will show that the equality hold:

Corollary 3.1. $J(FG) = \mathrm{Anh}(s)$.

Proof. By Proposition 3.3, $\mathrm{Anh}(s) \subseteq J(FG)$ and we know from Lemma 2.1 part (3) that $J(FG) \subseteq \mathrm{Anh}(s)$, so the equality is hold:

$$J(FG) = \mathrm{Anh}(s).$$

□

We will need the following proposition in the next steps:

Proposition 3.4. $\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 2$.

Proof. By Proposition 3.1 and Corollary 3.1 we have

$$J(FG) = \mathrm{Anh}(s) = \{a_0^- \widehat{x}y^{-1} + a_0\widehat{x} + a_0^+\widehat{x}y \mid a_0^- + a_0 + a_0^+ = 0\}. \tag{3.7}$$

That means, $J(FG)$ and $\mathrm{Anh}(s)$ are generated by three elements, with one restriction. Hence,

$$\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 3 - 1 = 2.$$

□

Let $H := \langle x \rangle = \{x^{-6}, x^{-5}, x^{-4}, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, x^4, x^5, x^6\} \trianglelefteq G$, a normal subgroup of G. Also we recall augmentation ideals $\Delta(G,H) := \langle h-1 |\ h \in H \rangle$, that in special case $H = G$, we denote $\Delta(G) := \Delta(G,G)$. Now it is obvious that, by using [14, Proposition 3.3.3 on p. 135], we have

$$\mathrm{Dmn}(\Delta(G,H)) = |G| - [G:H] = 39 - 3 = 36$$
$$\mathrm{Dmn}(\Delta(G,G)) = |G| - [G:G] = 39 - 1 = 38.$$

Therefore we obtain the following remark:

Remark 3.1. Dimensions of $\Delta(G,H)$ and $\Delta(G)$ can be computed as follows:

$$\mathrm{Dmn}(\Delta(G,H)) = 36$$
$$\mathrm{Dmn}(\Delta(G,G)) = 38.$$

We want to represent a decomposition for $\Delta(G)$ over $J(FG)$ and $\Delta(G,H)$. As both of them are included in $\Delta(G)$, first we show that they are disjoint:

Proposition 3.5. $J(FG)\cap\Delta(G,H)=0$.

Proof. Let $\alpha\in J(FG)\cap\Delta(G,H)$. By (3.7), $J(FG)=\langle\widehat{G}\rangle$. Now we compute $\alpha.\widehat{x}$ in two different ways, in order to see α as an element of $J(FG)$ or $\Delta(G,H)$ separately:

$$\begin{array}{cc} \alpha\in J(FG)=\langle\widehat{G}\rangle & \alpha\in\Delta(G,H)=\langle x-1\rangle \\ \alpha=a.\widehat{G} & \alpha=\beta(x-1) \\ \alpha\widehat{x}=a\widehat{G}\widehat{x}=a\widehat{G}|\langle x\rangle| & \alpha\widehat{x}=\beta(x-1)\widehat{x}=\beta(x\widehat{x}-1\widehat{x}) \\ =a.\widehat{G}.n=a.\widehat{G}=\alpha & =\beta.(\widehat{x}-\widehat{x})=\beta.0=0 \end{array}$$

So we conclude that:

$$\alpha=\alpha.\widehat{x}=0. \tag{3.8}$$

And therefore we have

$$J(FG)\cap\Delta(G,H)=0.$$

□

Now the decomposition can be achieved:

Proposition 3.6. $\Delta(G)=J(FG)\oplus\Delta(G,H)$.

Proof. By Proposition 3.4 and Remark 3.1, we have

$$\mathrm{Dmn}(J(FG))+\mathrm{Dmn}(\Delta(G,H))=2+36=38=\mathrm{Dmn}(\Delta(G))$$

Now Proposition 3.5 together with above equality shows that:

$$\Delta(G)=J(FG)\oplus\Delta(G,H).$$

□

In the next Proposition, we prove that $\Delta(G,H)$ is a semisimple ring:

Proposition 3.7. *$\Delta(G,H)$ is a semisimple ring.*

Proof. By Proposition 3.6, we have $\Delta(G,H)=\Delta(G)/J(FG)\subseteq FG/J(FG)$. From [14, Theorem 6.6.1 on p. 214], the group ring of a field over a finite group is Artinian, so FG is an Artinian ring, and [14, Lemma 2.4.9 on p. 87], implies its quotient ring, $FG/J(FG)$, is an Artinian ring too. Also from [14, Lemma 2.7.5 on p. 107] we know that $J(FG/J(FG))=0$. Now by using [14, Theorem 2.7.16 on p. 111] we can conclude that $FG/J(FG)$ is semisimple, and by [14, Proposition 2.5.2 on p. 91], all of its subrings are semisimple too. So $\Delta(G,H)$ is semisimple. □

By the Artin-Wedderburn Theorem, semisimple ring $\Delta(G,H)$, decomposes to its simple components that are division rings of matrices over extensions of F. Now we need to know their numbers and dimensions. First we show that the center of $\Delta(G,H)$ is included in the center of FG:

Proposition 3.8. $Z(\Delta(G,H)) \subseteq Z(FG)$.

Proof. For the proof of this proposition, we need show that each element of $Z(FG)$ must commute with all of elements of FG. Since F is commutative and G is generated by x and y, so it suffices to show they commute with x and y. Let $\alpha \in Z(\Delta(G,H))$, so it commutes with $x-1$ as it is in $\Delta(G,H)$:

$$\begin{gathered}\alpha.(x-1) = (x-1).\alpha \\ \alpha.x - \alpha = x.\alpha - \alpha \\ \alpha.x = x.\alpha\end{gathered}$$

So α commutes with x. Now we show that α also commutes with y. First we show that $\alpha y - y\alpha$ is in $\mathrm{Anh}(x-1)$. Notice we know that $(x-1)y = y(x^{-1}-1) \in \Delta(G,H)$, so,

$$\begin{array}{cc} (x-1)y \in \Delta(G,H) & y(x-1) \in \Delta(G,H) \\ \alpha.(x-1).y = (x-1).y.\alpha & \alpha.y.(x-1) = y.(x-1).\alpha \\ (x-1).\alpha y = (x-1).y\alpha & \alpha y.(x-1) = y\alpha.(x-1) \\ (x-1)(\alpha y - y\alpha) = 0 & (\alpha y - y\alpha)(x-1) = 0 \end{array}$$

So $(\alpha y - y\alpha) \in \mathrm{Anh}(x-1)$ and by [14, Lemma 3.4.3 on p. 139] we know that $\mathrm{Anh}(x-1) = \mathrm{Anh}(\Delta(G,H)) = FG\widehat{x}$. Now we compute $(\alpha y - y\alpha).\widehat{x}$ in two different ways, directly itself or consider $(\alpha y - y\alpha)$ as an element of $FG.\widehat{x}$ separately. Note that $\alpha \in Z(\Delta(G,H)) \subseteq \Delta(G,H)$, so by (3.8), $\alpha.\widehat{x} = 0$, and although x does not commute with y, but $\widehat{x}$ does, also $|\langle x\rangle| = Ord_G(x) = 7 = 1$. So we have

$$\begin{array}{cc} (\alpha y - y\alpha).\widehat{x} = \alpha.y.\widehat{x} - y.\alpha.\widehat{x} = & (\alpha y - y\alpha).\widehat{x} = \beta.\widehat{x}.\widehat{x} = \\ \alpha\widehat{x}.y - y.\alpha\widehat{x} = 0.y - y.0 = 0 & \beta.\widehat{x}.|\langle x\rangle| = \beta\widehat{x} = (\alpha y - y\alpha) \end{array}$$

Hence $\alpha y - y\alpha = (\alpha y - y\alpha).\widehat{x} = 0$. Thus $\alpha y = y\alpha$, which means α also commutes with y and therefore

$$Z(\Delta(G,H)) \subseteq Z(FG).$$

□

In the next proposition, we obtain the exact structure of $Z(\Delta(G,H))$:

Proposition 3.9. $Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3} \rangle$.

Proof. Let $\alpha \in Z(\Delta(G,H))$, from [14, Theorem 3.6.2 on p. 151] we know that $Z(FG) = \langle \widehat{\mathscr{C}_{-3}}, \widehat{\mathscr{C}_{-2}}, \widehat{\mathscr{C}_{-1}}, \widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_{+1}}, \widehat{\mathscr{C}_{+2}}, \widehat{\mathscr{C}_{+3}} \rangle$, so for center of augmentation ideal we have $Z(\Delta(G,H)) \subseteq \langle \widehat{\mathscr{C}_{-3}}, \widehat{\mathscr{C}_{-2}}, \widehat{\mathscr{C}_{-1}} \widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_{+1}}, \widehat{\mathscr{C}_{+2}}, \widehat{\mathscr{C}_{+3}} \rangle$, by using Proposition 3.8. So

$\alpha = \sum_{i=-3}^{+3} r_i\widehat{\mathscr{C}_i} = r_{-3}\widehat{\mathscr{C}}_{-3} + r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_0\widehat{\mathscr{C}_0} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2} + r_{+3}\widehat{\mathscr{C}}_{+3}$. By (3.8), $\alpha.\widehat{x} = 0$ and notice that $x^i\widehat{x} = \widehat{x}$, so for $i \in \{-2,-1,+1,+2\}$ we have $\widehat{\mathscr{C}_i}\widehat{x} = 3\widehat{x} = 0$. Hence,

(3.9)

$$\begin{aligned} 0 = \alpha\widehat{x} = \sum_{i=-3}^{+3} r_i\widehat{\mathscr{C}_i}\widehat{x} &= r_{-3}\widehat{\mathscr{C}}_{-3}\widehat{x} + (\sum_{i=-2}^{-1} r_i\widehat{\mathscr{C}_i}\widehat{x}) + r_0\widehat{\mathscr{C}_0}\widehat{x} + (\sum_{i=+1}^{+2} r_i\widehat{\mathscr{C}_i}\widehat{x}) + r_{+3}\widehat{\mathscr{C}}_{+3}\widehat{x} \\ &= r_{-3}y^{-1}\widehat{x} + 0 + r_0.1.\widehat{x} + 0 + r_{+3}y\widehat{x} = r_{-3}\widehat{x}y^{-1} + r_0.1.\widehat{x} + r_{+3}\widehat{x}y \end{aligned}$$

Since the left hand side of (3.9) is zero, so the right hand side coefficients must be zero too, hence we have $r_{-3} = r_0 = r_{+3} = 0$, terefore we conclude that $\alpha = r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}$. As α was an arbitrary element in center of $\Delta(G,H)$, thus $Z(\Delta(G,H)) \subseteq \langle\widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle$. Now it suffices to show that all of these types of elements are included in $\Delta(G,H)$. We must show that there is a β such that $\alpha = \beta(x-1)$. It is straightforward to find β's coefficients by solving a system of linear equations. So $\alpha \in \Delta(G,H)$, and therefore

$$Z(\Delta(G,H)) = \langle\widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle.$$

□

Now the dimension of the center of $\Delta(G,H)$ can be computed:

Corollary 3.2. $\mathrm{Dmn}(Z(\Delta(G,H))) = 4$.

Proof. By Proposition 3.9,we know that $Z(\Delta(G,H)) = \langle\widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle$. So,

$$\mathrm{Dmn}(Z(\Delta(G,H))) = 4.$$

□

Let $M_n(R)$ be the ring of the square matrices of order n on the ring R and let $GL_n(R)$ be its unit group. Also R^n be the direct sum of n copy of the ring R, which is $R^n = \oplus_{i=1}^{n} R$ and let F_n be the extension of the finite field F of the order n that is $[F_n : F] = n$. Now we are ready to prove Theorem 3.1:

Proof. [Proof of Theorem 3.1] Let $\alpha \in Z(\Delta(G,H))$. From Proposition 3.9, we know that α can be written as $\alpha = r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}$. Since $char(F) = 3$, we have

$$\begin{aligned} \alpha &= r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2} \\ \alpha^3 &= r_{-2}^3\widehat{\mathscr{C}}_{-2}^3 + r_{-1}^3\widehat{\mathscr{C}}_{-1}^3 + r_{+1}^3\widehat{\mathscr{C}}_{+1}^3 + r_{+2}^3\widehat{\mathscr{C}}_{+2}^3 \\ \alpha^3 &= r_{-2}^3\widehat{\mathscr{C}}_{-2} + r_{-1}^3\widehat{\mathscr{C}}_{-1} + r_{+1}^3\widehat{\mathscr{C}}_{+1} + r_{+2}^3\widehat{\mathscr{C}}_{+2} \\ \alpha^{3n} &= r_{-2}^{3n}\widehat{\mathscr{C}}_{-2} + r_{-1}^{3n}\widehat{\mathscr{C}}_{-1} + r_{+1}^{3n}\widehat{\mathscr{C}}_{+1} + r_{+2}^{3n}\widehat{\mathscr{C}}_{+2} \\ \alpha^{3n} &= r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}. \end{aligned}$$

Since $|F| = 3^n$, we know $r_i^{3^n} = r_i$, so $\alpha^{3^n} = \alpha$. Therefore we have

$$\Delta(G,H) \cong M_3(F)^4.$$

By [14, Proposition 3.6.7 on p. 153], $FG \cong F(G/H) \oplus \Delta(G,H)$, therefore, $\mathscr{U}(FG) \cong \mathscr{U}(F(C_3)) \times \mathscr{U}(\Delta(G,H))$. So we have

$$\mathscr{U}(FG) = C_3^n \times C_{3^n-1} \times GL_3(F)^4.$$

□

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NEW TYPE OF ALMOST CONVERGENCE

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Abstract. In [1] for a given sequence (λ_n) with $\lambda_n < \lambda_{n+1} \to \infty$ a new summability method C_λ was introduced which generalizes the classical Cesàro method. In this paper, we introduce some new almost convergence and almost statistical convergence definitions for sequences which generalize the classical almost convergence and almost statistical convergence.
Key words: sequence convergence, almost convergence, summability theory.

1. Introduction

Let (λ_n) be a given real valued sequence such that

$$0 \leq \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$$

and $[\lambda_n]$ denote the integer part of λ_n. The set of such sequences will be denoted Λ. Consider the mean

$$\sigma_n = \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} x_k, \quad n = 1, 2, 3, ...$$

of a given sequence (x_k) of real or complex numbers. If

$$\lim_{n\to\infty} \sigma_n = \ell,$$

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then we say that (x_k) is C_λ-summable to ℓ. In the particular case when $\lambda_n = n$ we see that σ_n is the $(C,1)$ mean of (x_k).Therefore, C_λ-method yields a submethod of the Cesàro method $(C,1)$, and hence it is regular for any λ. C_λ-matrix is obtained by deleting a set of rows from Cesàro matrix. $(C,1)$ and C_λ are equivalent for bounded sequences if and only if $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$. The basic properties of C_λ-method can be found in ([1],[24]).

Summability of matrix submethods was studied in [12] and [28]. The authors of [12] and [28] presented results showing when C_λ is equivalent to the Cesàro method C_1 for bounded sequences. Armitage and Maddox proved inclusion and Tauberian results for the C_λ method in [1]. In [24], inclusion properties of the C_λ method for bounded sequences and its relationship to statistical convergence are studied also a condensation test presented for statistical convergence.

In this study, firstly we will introduce $\hat{C}_\lambda$-almost convergent sequence and prove some inclusion relations. Later we will give definition of $\hat{C}_\lambda$- almost statistically convergent sequence and examine the relationship between $\hat{C}_\lambda$-almost convergence and $\hat{C}_\lambda$-almost statistically convergence. Finally, we will generalize the spaces $[C_\lambda]$ and $[\hat{C}_\lambda]$ to spaces $[C_\lambda(f)]$ and $[\hat{C}_\lambda(f)]$ by using a modulus function f. Thus, it will fill a gap in the literature.

2. Almost Convergence

Let ℓ_∞ be the Banach space of real valued bounded sequences (x_k) with the usual norm $||x|| := sup_k|x_k|$. There exists continuous linear functional $\phi : \ell_\infty \to \mathbb{R}$ called Banach limit if the following conditions hold:

$$\begin{array}{ll} (i) & \phi(ax_k + by_k) = a\phi(x_k) + b\phi(y_k), \quad a, b \in \mathbb{R} \\ (ii) & \phi(x_k) \geq 0 \quad \text{if} \quad x_k \geq 0, \quad k = 1, 2, 3.... \\ (iii) & \phi(Sx) = \phi(x), \quad Sx = (x_2, x_3, x_4, ...) \\ (iv) & \phi(e) = 1 \quad \text{where} \quad e = (1, 1, 1, ...). \end{array}$$

A sequence (x_k) in ℓ_∞ is said to be almost convergent if all of its Banach limits are equal. It is well known that any Banach limit of (x_k) lies between $\liminf x_k$ and $\limsup x_k$ [13].

Note that a convergent sequence is almost convergent, and its limit and its generalized limit are identical, but an almost convergent sequence need not be convergent. The sequence (x_k) defined as

$$x_k = \begin{cases} 1, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

is almost convergent to $1/2$ but not convergent.

Lorentz [13] gave the following characterization for almost convergence: A sequence (x_n) is said to be almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} x_{k+i} = \ell$$

uniformly in i.

Maddox[15] has defined strongly almost convergent sequence as follows:
A bounded sequence (x_k) is said to be strongly almost convergent to ℓ if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}|x_{k+i}-\ell|=0$$

uniformly in i.
Readers can refer to recently published articles ([2],[3],[14],[18],[19],[20],[22]) for more information.
Consider the mean

$$\hat{C}_\lambda x=\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}x_{k+i}$$

of a given sequence (x_k) of real numbers and $i=1,2,3,....$

Definition 2.1. A bounded sequence (x_k) is said to be $\hat{C}_\lambda$-almost convergent to ℓ if and only if .

$$\lim_{n\to\infty}\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}x_{k+i}=\ell$$

uniformly in i.

In this case we write $x_k\to\ell(\hat{C}_\lambda)$. In the particular case when $\lambda_n=n$ we get the definition of almost convergent sequence.

Theorem 2.1. *Let $\{\lambda_n\},\{\nu_n\}\in\Lambda$. If $\lim_{n\to\infty}\frac{\nu_n}{\lambda_n}=1$, then $\hat{C}_\lambda$- almost convergence is equivalent to $\hat{C}_\nu$- almost convergence on ℓ_∞.*

Proof. Let $x\in\ell_\infty$ and consider $M_n:=\max\{\lambda_n,\nu_n\}$ and $m_n:=\min\{\lambda_n,\nu_n\}$. Since $\lim_{n\to\infty}\frac{\nu_n}{\lambda_n}=1$, we can write $\lim_{n\to\infty}\frac{m_n}{M_n}=1$, then for each n and i

$$\begin{aligned}
|\hat{C}_\nu x-\hat{C}_\lambda x| &= |\frac{1}{\nu_n}\sum_{k=1}^{[\nu_n]}x_{k+i}-\frac{1}{\lambda_n}\sum_{k=1}^{[\lambda_n]}x_{k+i}| \\
&= |\frac{1}{M_n}\sum_{k=1}^{M_n}x_{k+i}-\frac{1}{m_n}\sum_{k=1}^{m_n}x_{k+i}| \\
&= |\sum_{k=1}^{m_n}(\frac{1}{M_n}-\frac{1}{m_n})x_{k+i}+\frac{1}{M_n}\sum_{k=m_n+1}^{M_n}x_{k+i}| \\
&\leq \sup_{k,i}|x_{k+i}|\sum_{k=1}^{m_n}\frac{M_n-m_n}{M_nm_n}+\sup_{k,i}|x_{k+i}|\frac{M_n-m_n}{M_n} \\
&= \sup_{k,i}|x_{k+i}|\frac{m_n(M_n-m_n)}{M_nm_n}+\sup_{k,i}|x_{k+i}|\frac{M_n-m_n}{M_n} \\
&= 2\sup_{k,i}|x_{k+i}|(1-\frac{m_n}{M_n})\to 0
\end{aligned}$$

as $n \to \infty$ uniformly in i. Hence, if $x \to L(\hat{C}_\lambda)$,

$$0 \leq |\frac{1}{\nu_n}\sum_{k=1}^{\nu_n} x_{k+i} - L| \leq |\hat{C}_\nu x - \hat{C}_\lambda x| + |\hat{C}_\lambda x - L| \to 0$$

as $n \to \infty$ uniformly in i. Similarly, if $x \to L(\hat{C}_\nu)$,

$$0 \leq |\frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n} x_{k+i} - L| \leq |\hat{C}_\lambda x - \hat{C}_{\nu\lambda} x| + |\hat{C}_\nu x - L| \to 0$$

as $n \to \infty$ uniformly in i. Thus, the proof is completed. $\square$

By using similar techniques to Theorem 1 of [1] we can prove following theorem:

Theorem 2.2. *Let $\{\lambda_n\}, \{\nu_n\} \in \Lambda$.*
(i) $\hat{C}_\lambda$ implies $\hat{C}_\mu$ if and only if $D(\mu) \setminus D(\lambda)$ is a finite set, where

$$D(\lambda) = \{[\lambda_n] : \quad n = 1, 2, ...\}.$$

(ii) $\hat{C}_\mu$ is equivalent $\hat{C}_\mu$ if and only if $D(\lambda) \triangle D(\mu)$ is a finite set.

Also by using similar techniques to Theorem 2.2 of [24] we can prove following theorem:

Theorem 2.3. *Let $\{\lambda_n\} \in \Lambda$. If $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, then $\hat{C}_\lambda$- almost convergence is equivalent to almost convergence on ℓ_∞.*

Definition 2.2. A bounded sequence (x_k) is said to be strongly $\hat{C}_\lambda$-almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]} |x_{k+i} - \ell| = 0$$

uniformly in i.

In this case we write $x_k \to \ell([\hat{C}_\lambda])$. In the particular case when $\lambda_n = n$ we get the definition of strongly almost convergent sequence.

Definition 2.3. A bounded sequence (x_k) is said to be p-strongly $\hat{C}_\lambda$-almost convergent to ℓ if and only if

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]} |x_{k+i} - \ell|^p = 0$$

uniformly in i where $0 < p < \infty$.

In this case we write $x_k \to \ell([\hat{C}_\lambda]_p)$. In the particular case when $\lambda_n = n$ we get the strongly p-almost convergent sequence definition.

3. Almost Statistical Convergence

The natural density of a set A of positive integers is defined if limit exists by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in A\}|,$$

where $|k \le n : k \in A|$ denotes the number of elements of A not exceeding n.

Statistical convergence, as it has recently been investigated, was defined by Fast [7]. Schoenberg [27] established some fundamental properties of the concept and studied as a summability method. The more recent times interest in statistical convergence arose after Fridy published his paper [8], and since then there have been many generalizations of the original concept(see [4]-[6],[9]-[11],[16],[21]).
A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |x_k - \ell| \ge \epsilon\}| = 0,$$

holds. In this case, we write $st - \lim x_k = \ell$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \ell$, then $st - \lim x_k = \ell$. The converse does not hold, in general. If a sequence $x = (x_k)$ is strongly Cesàro convergent to ℓ, then $x = (x_k)$ is statistically convergent to ℓ and the converse is also true when $x = (x_k)$ is a bounded sequence.

Definition 3.1. [24] A sequence $x = (x_k)$ is said to be C_λ-statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}|\{0 \le k \le [\lambda_n] : |x_k - \ell| \ge \epsilon\}| = 0,$$

holds.

In the particular case when $\lambda_n = n$, C_λ- statistically convergence coincide with statistically convergence.

Definition 3.2. A sequence $x = (x_k)$ is said to be C_λ- almost statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n}|\{0 \le k \le [\lambda_n] : |x_{k+i} - \ell| \ge \epsilon\}| = 0$$

holds uniformly in i.

In the particular case when $\lambda_n = n$ we get the definition of almost statistically convergent sequences was defined in [26].

Theorem 3.1. *If $x_k \to \ell([C_\lambda])$ then $x_k \to \ell(S_\lambda)$. The converse is true if (x_k) is bounded.*

Proof. Let $x_k \to \ell([C_\lambda])$. For an arbitrary $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| &= \left(\frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell| + \frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}|x_k-\ell|\right)\\
&\geq \frac{1}{1+\lambda_n}\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell|\\
&\geq \frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}|\epsilon.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}| = 0$$

that is, $x_k \to \ell(S_\lambda)$.
Now suppose that $x_k \to \ell(S_\lambda)$ and x_k is bounded, since x_k is bounded, say $|x_k-\ell| \leq M$ for all k. Given $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| &= \frac{1}{1+\lambda_n}\left(\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}|x_k-\ell| + \sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}|x_k-\ell|\right)\\
&\leq \frac{1}{1+\lambda_n}\left(M\sum_{\substack{k=0\\|x_k-\ell|\geq\epsilon}}^{[\lambda_n]}1 + \epsilon\sum_{\substack{k=0\\|x_k-\ell|<\epsilon}}^{[\lambda_n]}1\right)\\
&\leq M\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|\geq\epsilon\}|\\
&+ \epsilon\frac{1}{1+\lambda_n}|\{0\leq k\leq[\lambda_n] : |x_k-\ell|<\epsilon\}|
\end{aligned}$$

hence we have,

$$\lim_{x\to\infty}\frac{1}{1+\lambda_n}\sum_{k=0}^{[\lambda_n]}|x_k-\ell| = 0.$$

□

The proofs of the following theorems are similar to that of Theorem 3.1, so we state them without of proof.

Theorem 3.2. *Let $0 < p < \infty$. If $x_k \to \ell([C_\lambda]_p)$ then $x_k \to \ell(S_\lambda)$. The converse is true if (x_k) is bounded.*

Theorem 3.3. *Let $0 < p < \infty$. If $x_k \to \ell([\hat{C}_\lambda]_p)$ then $x_k \to \ell(\hat{S}_\lambda)$. The converse is true if (x_k) is bounded.*

Theorem 3.4. *Let $(\lambda_n) \in \Lambda$ with $\limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} < \infty$. If $x_k \to \ell(\hat{S}_\lambda)$ then $x_k \to \ell(\hat{S})$.*

Proof. Assume (x_k) is $x_k \to \ell(\hat{S}_\lambda)$ and $\limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} < \infty$. Consider $\Gamma = \mathbb{N}\backslash\Lambda := \{\nu_n\}$. If Γ is finite , then $\hat{S}$ is equivalent to $\hat{S}_\lambda$. Now assume that Γ is infinite. Then there exists an K such that $n \geq K, \nu_n > \lambda_1$. Since Γ and Λ are disjoint, for $n \geq K$, there exists an integer m such that $\lambda_m < \nu_n < \lambda_{m+1}$. We write $\nu_n = \lambda_{m+j}$ where $0 < j < \lambda_{m+1} - \lambda_m$. Then, for $n \geq K$,

$$\begin{aligned}
&\frac{1}{\nu_n}|\{k \leq \nu_n : |x_{k+i} - \ell| \geq \epsilon\}| \\
= \quad & \frac{1}{\lambda_{m+j}}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{1}{\lambda_{m+j}}|\{\lambda_{m+1} \leq k \leq \lambda_{m+j} : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
\leq \quad & \frac{1}{\lambda_m}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{1}{\lambda_{m+j}}|\{1 \leq k \leq \lambda_{m+1} : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
= \quad & \frac{1}{\lambda_m}|\{1 \leq k \leq \lambda_m : \quad |x_{k+i} - \ell| \geq \epsilon\}| \\
+ \quad & \frac{\lambda_{m+1}}{\lambda_{m+j}}\frac{1}{\lambda_{m+1}}|\{1 \leq k \leq \lambda_{m+1} : \quad |x_{k+i} - \ell| \geq \epsilon\}|.
\end{aligned}$$

Since, $0 < \frac{\lambda_{m+1}}{\lambda_{m+j}} < \frac{\lambda_{m+1}}{\lambda_m}$ and $\frac{\lambda_{m+1}}{\lambda_m}$ is bounded, then $\frac{\lambda_{m+1}}{\lambda_{m+j}}$ is bounded too. Thus, we see that $\frac{1}{n}|\{1 \leq k \leq n : \quad |x_{k+i} - \ell| \geq \epsilon\}|$ may be partitioned into two disjoint subsequences each having the common limit zero uniformly in i. Hence, we get $x_k \to \ell(\hat{S})$. $\square$

4. Convergence with respect to a modulus function

The notion of a modulus function was introduced by [23]. Ruckle [25] used the idea of a modulus function to construct the sequence space

$$L(f) = \{(x_k) : \quad \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK-space, and Ruckle proved that the intersection of all such $L(f)$ space is ϕ, the space of finite sequences, thereby answering negatively a question of A. Wilansky: "Is there a smallest FK-space in which the set $\{e_1, e_2, ...\}$ of unit vectors is bounded?" [17].

A real valued function f defined on $[0,\infty)$ is called a modulus function if it has following properties:

1. $f(x)\geq 0$ for each x,
2. $f(x)=0$ if and only if $x=0$,
3. $f(x+y)\leq f(x)+f(y)$ for all $x,y\geq 0$,
4. f is increasing,
5. $\lim_{x\to 0^+} f(x)=0$.

Since $|f(x)-f(y)|\leq f(x-y)$,(see [17]), it follows from conditions (3) and (5) that f is continuous on $[0,\infty)$.

Many new sequence spaces are defined by using the modulus function in the summability theory. Sequence spaces defined in this way generalize known sequence spaces. By using a modulus function f firstly Ruckle [25] defined the sequence space

$$L(f)=\{(x_k):\quad \sum_{k=1}^{\infty} f(|x_k|)<\infty\},$$

which generalization of the space

$$\ell_1=\{(x_k):\quad \sum_{k=1}^{\infty}|x_k|<\infty\}$$

and later Maddox [17] introduced following sequence spaces which are generalizations of the classical spaces of strongly summable sequences

$$w_0(f)=\{(x_k):\quad \frac{1}{n}\sum_{k=1}^{n} f(|x_k|)=0\},$$

$$w(f)=\{(x_k):\quad \frac{1}{n}\sum_{k=1}^{n} f(|x_k-\ell|)=0 \quad \text{for real number} \quad \ell\},$$

$$w_\infty(f)=\{(x_k):\quad \sup_n \frac{1}{n}\sum_{k=1}^{n} f(|x_k|)<\infty\}.$$

If we take $f(x)=x^p$ $(0<p<1)$ then the space $L(f)$ is the familiar space l_p.
It is known that,

$$\ell_1\subset L(f),\quad w_0\subset w_0(f),\quad w\subset w(f),\quad \text{and}\quad w_\infty\subset w_\infty(f).$$

Apart from these spaces, there are many sequence spaces defined using the modulus function in the literature. For example, Connor [5] introduced strongly A-summable sequences with respect to a modulus function.

In this section, by using a modulus function f, we will introduce the sequence spaces $[C_\lambda(f)]$ and $[\widehat{C}_\lambda(f)]$ which are generalization of the sequence spaces $[C_\lambda]$ and $[\widehat{C}_\lambda]$ and we are going to show that

$$[C_\lambda] \subset [C_\lambda(f)] \quad \text{and} \quad [\widehat{C}_\lambda] \subset [\widehat{C}_\lambda(f)]$$

holds.

Definition 4.1. Let (x_k) be a sequence of real or complex numbers, f be a modulus function and $(\lambda_n) \in \Lambda$ be a sequence. If

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_k - \ell|) = 0,$$

then we say that (x_k) is $[C_\lambda]$-summable to ℓ with respect to f and $\lambda = (\lambda_n)$.

The space of all sequences $[C_\lambda]$-summable to ℓ with respect to f and $\lambda = (\lambda_n)$ will be denoted by $[C_\lambda(f)]$.

Theorem 4.1. *For any modulus function f we have $[C_\lambda] \subset [C_\lambda(f)]$ holds for $\lambda = (\lambda_n) \in \Lambda$, that is, (x_k) is $[C_\lambda]$-summable to ℓ then (x_k) is $[C_\lambda]$-summable to ℓ with respect to the modulus function f.*

Proof. If (x_k) is $[C_\lambda]$-summable to ℓ, then we have

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} |x_k - \ell| = 0.$$

Let $\epsilon > 0$ and choose θ with $0 < \theta < 1$ such that $f(t) < \epsilon$ holds for $0 \le t \le \theta$. Now since for $|x_k - \ell| > \theta$,

$$|x_k - \ell| \le \frac{|x_k - \ell|}{\theta} < 1 + [\frac{|x_k - \ell|}{\theta}]$$

and

$$f(|x_k - \ell|) \le (1 + [\frac{|x_k - \ell|}{\theta}])f(1) < 2f(1)\frac{|x_k - \ell|}{\theta},$$

we can write

$$\begin{aligned} \sum_{k=0}^{[\lambda_n]} f(|x_k - \ell|) &= \sum_{k=0 |x_k-\ell|\le\theta}^{[\lambda_n]} f(|x_k - \ell|) + \sum_{k=0 |x_k-\ell|>\theta}^{[\lambda_n]} f(|x_k - \ell|) \\ &\le \epsilon[\lambda_n + 1] + \frac{2}{\theta} f(1)([\lambda_n] + 1)\frac{1}{[\lambda_n]+1} \sum_{k=0}^{[\lambda_n]} |x_k - \ell|. \end{aligned}$$

Hence, (x_k) is $[C_\lambda]$-summable to ℓ with respect to the modulus function f. $\square$

Definition 4.2. Let (x_k) be a sequence of real or complex numbers, f be a modulus function and $\lambda = (\lambda_n) \in \Lambda$. If

$$\lim_{n\to\infty} \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) = 0$$

holds uniformly in i, then we say that (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f.

The space of sequences $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f will be denoted by $[\widehat{C}_\lambda(f)]$.

Theorem 4.2. *If $(x_k) \to \ell[\widehat{C}_\lambda(f)]$ and $(x_k) \to \ell'[\widehat{C}_\lambda(f)]$ then $\ell = \ell'$.*

Proof. Let $(x_k) \to \ell[\widehat{C}_\lambda(f)]$ and $(x_k) \to \ell'[\widehat{C}_\lambda(f)]$. Then given $\epsilon > 0$, for all $i \in \mathbb{N}$ there exists $n > n_0$ such that

$$\frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) < \frac{\epsilon}{2}$$

and

$$\frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell'|) < \frac{\epsilon}{2}.$$

From these and the following inequality

$$f(|\ell - \ell'|) = f(|x_{k+i} + \ell - \ell' - x_{k+i}|) \leq f(|x_{k+i} - \ell|) + f(|x_{k+i} - \ell'|)$$

we can write

$$\begin{aligned} f(|\ell - \ell'|) &\leq \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell|) + \frac{1}{1+\lambda_n} \sum_{k=0}^{[\lambda_n]} f(|x_{k+i} - \ell'|) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $i \in \mathbb{N}$. Since $\epsilon > 0$ is arbitrary, we have $\ell = \ell'$ by the properties (2) and (5) of modulus function. $\square$

Theorem 4.3. *For any modulus function f we have $[\widehat{C}_\lambda] \subset [\widehat{C}_\lambda(f)]$, that is, (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ then (x_k) is $[\widehat{C}_\lambda]$-summable to ℓ with respect to the modulus function f.*

The proof of the theorem similar to the Theorem 4.1, so we omit it.

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SPACES OF FIBONACCI DIFFERENCE IDEAL CONVERGENT SEQUENCES IN RANDOM 2–NORMED SPACE

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Abstract. In this article, by using the same Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence of sequences in random 2–normed space in the same technique, we have introduced new spaces of Fibonacci difference ideal convergent sequences with respect to random 2 –norm and studied some inclusion relations, as well as topological and algebraic properties of these spaces.
Key words: ideals, statistical convergence, probabilistic metric spaces.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers respectively. By ω we denote the linear space of sequence of real numbers. c_0, c and ℓ_∞ represent sequence spaces of null convergent, convergent and bounded sequences respectively. The approach to statistical convergence was done by Fast [6] and Steinhaus [19] in 1951 independently. In 1999, Kostryko et al. [14] generalised the notion of statistical convergence to ideal convergence and some properties of this interesting generalization have been studied by Šalát et al. [17]. An ideal is a non–empty subset of the set of natural numbers $\mathbb{N}$ which satisfies hereditary and additivity property,

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i.e., $I \subseteq 2^{\mathbb{N}}$ such that $A \in I$ with $B \subset A$ implies $B \in I$ and $A \cup B \in I$ whenever $A, B \in I$. A non–empty family of sets $F \subseteq 2^{\mathbb{N}}$ is said to be a filter on $\mathbb{N}$ if only if $\phi \notin F, A \cap B \in F$ for $A, B \in F$ and any superset of an element of F is in F. An ideal I is non–trivial if $I \neq 2^{\mathbb{N}}$. A non–trivial ideal I is admissible if it contains all singletons. A sequence $x = (x_n) \in \omega$ is said to be $I-$ convergent to $L \in \mathbb{R}$ if the set $\{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in I$ for every $\epsilon > 0$. If $L = 0$, then we say the sequence is $I-$ null. The concept of ideal convergence was studied from the sequence point of view and linked with the summability theory by Hazarika and Savaş [11, 10]. The approach to construct sequence spaces by means of the domain of an infinite matrix and with the help of the notion of ideal convergence was firstly used by Śalát et. al [18] to introduce the sequence spaces $(c^I)_A$ and $(m^I)_A$. The theory of random 2–normed space was introduced by Gölet and studied some properties of convergence and Cauchy sequence with respect to random 2–norm as well. Recently, the notion of ideal convergence of sequences in the framework of random 2–normed spaces defined by Mursaleen and Alotaibi [15].

In 2013, Kara defined the double band matrix matrix $\hat{F} = (\hat{f}_{nk})$ by:

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & \text{if } k = n-1 \\ \frac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci sequence defined by the recurrence relation $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ satisfying some basic properties and addressed the approach to construct sequence spaces by means of an infinite matrix of particular limitation methods to introduced the Fibonacci difference sequence space

$$\ell_\infty(\hat{F}) = \Big\{x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \Big|\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}\Big| < \infty\Big\}.$$

The domains $c_0(\Delta^F), c(\Delta^F)$ and $l_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and l_∞ are introduced by Kizmaz [13]. Aftermore, the domain bv_p of the backward difference matrix Δ^B in the space l_p have recently been investigated for $0 < p < 1$ by Altay and Başar [1], and for $1 \leq p \leq \infty$ by Başar and Altay [2]. Quite recently, by combining the definitions of ideal convergence and the Fibonacci difference matrix $\hat{F}$, Khan et al. [12] have introduced some new Fibonacci difference sequence spaces

$$\lambda(\hat{F}) = \{x = (x_n) \in \omega : \hat{F}x = ((\hat{F}x)_n) \in \lambda\},$$

for $\lambda = c_0^I, c^I$ and ℓ_∞^I, the spaces of all I–null and I–convergent sequences,where the sequence $\hat{F}x = ((\hat{F}x)_n)$ is the $\hat{F}$–transform of the sequence $x = (x_n) \in \omega$ defined as follows:

$$\hat{F}(x) = ((\hat{F}x)_n) = \begin{cases} \frac{f_0}{f_1} x_0, & n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \geq 1. \end{cases}$$

For more work on difference sequence spaces and Fibonacci difference sequence space please see the references [16, 4, 5].

In this article, by using Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence in random 2–normed space, we introduce new sequence spaces and study their topological and algebraic properties.

We recall some definitions which will be used throughout this article.

Definition 1.1. [7] *A sequence $x = (x_n) \in \omega$ is said to be statistically convergent to $L \in \mathbb{R}$ if for every $\epsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ is zero. We write $st\text{–}\lim x_n = L$.*

Definition 1.2. [12] *An ideal is a subset of the set of natural numbers $\mathbb{N}$ which satisfies hereditary and additivity property, i.e., $I \subseteq 2^{\mathbb{N}}$ such that $A \in I$ with $B \subset A$ implies $B \in I$ and $A \cup B \in I$ whenever $A, B \in I$. A non–empty family of sets $F \subseteq 2^{\mathbb{N}}$ is said to be a filter on $\mathbb{N}$ if only if $\phi \notin F, A \cap B \in F$ for $A, B \in F$ and any superset of an element of F is in F. An ideal I is non–trivial if $I \neq 2^{\mathbb{N}}$. A non–trivial ideal I is admissible if it contains all singletons. A sequence $x = (x_n) \in \omega$ is said to be I–convergent to $L \in \mathbb{R}$ if $n \in \mathbb{N} : |x_n - L| \geq \epsilon \in I$ for every $\epsilon > 0$. If $L = 0$, then we say that the sequence is I–null.*

Definition 1.3. [15] *A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is said to be a distribution function if it is non–decreasing and left continuous such that $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions with $f(0) = 0$. For $a \in \mathbb{R}_0^+, H_a \in D^+$*

$$H_a(t) = \begin{cases} 1, & t > a \\ 0, & t \leq a \end{cases}$$

Definition 1.4. [15] *A triangular norm is a continuous map $* : [0, 1] \times [0, 1] \to [0, 1]$, $([0, 1], *)$ is an abelian monoid with unit one and $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$ for all $a, b, c, d \in [0, 1]$. A triangle τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.*

Definition 1.5. [8] *Let X be a vector space with dimension more than 1. A function $\|.,.\| : X \times X \to \mathbb{R}$ with the following properties:*

(1) $\|x_1, x_2\| = 0$ *if and only if* x_1, x_2 *are linearly dependent,*

(2) $\|x_1, x_2\| = \|x_2, x_1\|$,

(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|, \alpha \in \mathbb{R}$,

(4) $\|x_1 + x_2, x_3\| \leq \|x_1, x_3\| + \|x_2, x_3\|$.

Then $(X, \|.,.\|)$ is called a 2–normed space.

Definition 1.6. [9] *Let X be a linear space of dimension greater than 1, $*$ denote a t norm. $\mathcal{F} : X \times X \to D^+$ is said to be random 2–norm if the following conditions are satisfied:*

(1) $\mathcal{F}(x_1, x_2; t) = H_0(t)$ *if x_1, x_2 are linearly dependent,*

(2) $\mathcal{F}(x_1, x_2; t) \neq H_0(t)$ *if x_1, x_2 are linearly independent,*

(3) $\mathcal{F}(x_1, x_2; t) = \mathcal{F}(x_2, x_1; t)$ *for all $x_1, x_2 \in X$,*

(4) $\mathcal{F}(\alpha x_1, x_2; t) = \mathcal{F}(x_1, x_2; \frac{t}{|\alpha|})$ *for $t > 0, \alpha \neq 0$,*

(5) $\mathcal{F}(x_1, x_2, x_3; t_1 + t_2) \geq \mathcal{F}(x_1, x_3; t_1) * \mathcal{F}(x_2, x_3; t_2)$ *for all $x_1, x_2, x_3 \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$.*

Then $(X, \mathcal{F}, *)$ is called a random 2–normed space (R2NS).

Definition 1.7. [15] *A sequence $x = (x_n) \in X$ is $\mathcal{F}-$ convergent to L in $(X, \mathcal{F}, *)$ if there exists $n_0 > 0$ such that $\mathcal{F}(x_n - L, z; \epsilon) > 1 - \theta$ whenever $n \geq n_0$ for every $\epsilon > 0, \theta \in (0, 1)$ and non–zero $z \in X$. We denote it as $\mathcal{F}-\lim_n x_n = L$.*

Definition 1.8. [15] *Let $(X, \mathcal{F}, *)$ be a R2NS. A sequence $x = (x_n) \in X$ is $I-$convergent to L in $(X, \mathcal{F}, *)$ if for every $\epsilon > 0, \theta \in (0, 1)$ and non–zero $z \in X$ if the set $\{n \in \mathbb{N} : \mathcal{F}(x_n - L, z; \epsilon) \leq 1 - \theta\} \in I$. We write I^{R2N}–lim $x = L$.*

Definition 1.9. [17] *A sequence space E is said to be solid if $(\alpha_n x_n) \in E$ for $(x_n) \in E$ where (α_n) is a sequence of scalars such that $|\alpha_n| \leq 1$.*

Definition 1.10. [17] *Let $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and E be a sequence space. A K step space of E is a sequence space $\lambda_k^E = \{(x_{k_n} \in \omega : (x_n) \in E\}$. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_k^E$ is a sequence $(y_n) \in \omega$ defined as follows:*

$$y_n = \begin{cases} x_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_k^E is a set of canonical preimages of all elements in λ_k^E, i.e., y is in canonical preimage of λ_k^E if and only if y is canonical preimage of some $x \in \lambda_k^E$.

Definition 1.11. [17] *A sequence space E is said to be monotone if it contains the canonical preimage of all its step spaces i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_n) \in E$ the sequence $(\alpha_n x_n)$, where*

$$\alpha_n = \begin{cases} 1, & \text{if } n \in K \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.1. *Every solid sequence space is monotone.*

2. Main Results

2.1. Some New Fibonacci Difference Ideal Convergent Sequence Spaces

In the present section, we define Fibonacci difference spaces of I–convergent and I–null sequences in a random 2–normed space. Also, we discuss some inclusion relations topological and algebraic properties of these spaces. Throughout this paper, ideal I is admissible ideal. For $\epsilon > 0$, $0 < \theta < 1$ and non zero z in X, define

$$c_0^{I_{R2N}}(\hat{F}) := \{x = (x_n) \in X : \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n), z; \epsilon) \leq 1 - \theta\} \in I\},$$

$$c^{I_{R2N}}(\hat{F}) := \{x = (x_n) \in X : \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) \leq 1 - \theta\} \in I\}.$$

Remark 2.1. *We introduce an open ball with respect to R2N by means of the domain of the Fibonacci matrix, as follows:*

$$B(((\hat{F}x)_n), r, \epsilon) := \{y \in X : \mathcal{F}((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r \text{ for } \epsilon > 0, 0 < r < 1\}.$$

Theorem 2.1. *The spaces $c_0^{I_{R2N}}(\hat{F})$ and $c^{I_{R2N}}(\hat{F})$ are vector spaces over $\mathbb{R}$.*

Proof. We shall prove the result for $c^{I_{R2N}}(\hat{F})$. Let $x = (x_n)$ and $y = (y_n) \in c^{I_{R2N}}(\hat{F})$, then there exist $L_1, L_2 \in X$ such that for $\epsilon > 0, \theta \in (0,1)$ and non–zero $z \in X$, we have

$$A = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L_1, z; \frac{\epsilon}{2|\alpha|}) \leq 1 - \theta\} \in I,$$

$$B = \{n \in \mathbb{N}; \mathcal{F}(((\hat{F}y)_n) - L_2, z; \frac{\epsilon}{2|\beta|}) \leq 1 - \theta\} \in I,$$

where α and β are non–zero scalars in $\mathbb{R}$. Choose $\eta \in (0,1)$ such that $(1-\theta)*(1-\theta) > 1 - \eta$. Consider

$$C = \{n \in \mathbb{N} : \mathcal{F}((\alpha(\hat{F}x)_n) + (\beta(\hat{F}y)_n)) - (\alpha L_1 + \beta L_2)) \leq 1 - \eta\}.$$

We show $C \subseteq A \cup B$ or equivalently $A^c \cap B^c \subseteq C^c$. Since $A^c \cap B^c \in F(I)$ so is non–empty. Let $m \in A^c \cap B^c \in F(I)$, then

$$\begin{aligned}
& \mathcal{F}((\alpha(\hat{F}x)_n) + (\beta(\hat{F}y)_n) - (\alpha L_1 + \beta L_2), z; \epsilon) \\
\geq\ & \mathcal{F}((\alpha(\hat{F}x)_m) - L_1), z; \frac{\epsilon}{2}) * \mathcal{F}((\beta(\hat{F}y)_m) - L_2), z; \frac{\epsilon}{2}) \\
=\ & \mathcal{F}((\hat{F}x)_m) - L_1, z; \frac{\epsilon}{2|\alpha|}) * \mathcal{F}((\hat{F}y)_m) - L_2, z; \frac{\epsilon}{2|\beta|}) \\
>\ & (1-\theta) * (1-\theta) \\
>\ & 1 - \eta.
\end{aligned}$$

Thus $m \in C^c$ and therefore $A^c \cap B^c \subseteq C^c$. Hence $C \in I$. The proof for $c_0^{I_{R2N}}(\hat{F})$ can be given in the same manner. $\square$

Theorem 2.2. *Let $(X, \mathcal{F}, *)$ be a random 2–space. Every open ball $B((\hat{F}x)_n), r, \epsilon)$ is an open set.*

Proof.

$$B((\hat{F}x)_n, r, \epsilon) := \{y \in X : \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r, \epsilon > 0, 0 < r < 1\}$$

Let $y \in B((\hat{F}x)_n, r, \epsilon)$ then by definition $\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) > 1 - r$, there exists $\epsilon_0 \in (0, \epsilon)$ such that $\mathcal{F}(((\hat{F}x)_n - ((\hat{F}y)_n), z; \epsilon_0) > 1 - r$. Put $\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon_0) = r_0$, then for $r_0 > 1-r$ there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$. For $r_0 > 1-s$, there exists $r_1 \in (0,1)$ with $r_0 * r_1 > 1-s$. We show $B((\hat{F}y)_n), 1-r_1, \epsilon-\epsilon_0) \subset B((\hat{F}x)_n), r, \epsilon)$.
Let $w \in B(((\hat{F}y)_n), 1-r_1, \epsilon-\epsilon_0)$. Then $\mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z; \epsilon-\epsilon_0) > r_1$. Now,

$$\begin{aligned}
& \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \epsilon) \\
\geq\ & \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon_0) * \mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z, \epsilon - \epsilon_0) \\
\geq\ & r_0 * r_1 \\
>\ & 1 - s \\
>\ & 1 - r
\end{aligned}$$

Thus we have, $w \in B(((\hat{F}x)_n), r, \epsilon)$ so that
$B(((\hat{F}y)_n), 1-r_1, \epsilon-\epsilon_0) \subset B(((\hat{F}x)_n), r, \epsilon)$. □

Remark 2.2. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. Define $\tau^I_{\mathcal{F}}(\hat{F}) := \{A \subset c^{I_{R2N}}(\hat{F})$: for given $x \in A$, we can find $\epsilon > 0$ and $0 < r < 1$ such that $B((\hat{F}x)_n), r, \epsilon) \subset A\}$. Then $\tau^I_{\mathcal{F}}(\hat{F})$ is a topology on $c^{I_{R2N}}(\hat{F})$.*

Remark 2.3. *Since $\{B_x(\frac{1}{n}, \frac{1}{n})(\hat{F}) : n \in \mathbb{N}\}$ is a local base at x, the topology $\tau^I_{\mathcal{F}}(\hat{F})$ is first countable.*

Theorem 2.3. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. $c_0^{I_{R2N}}(\hat{F})$ and $c^{I_{R2N}}(\hat{F})$ are Hausdorff spaces.*

Proof. Let $x, y \in c^{I_{R2N}}(\hat{F})$ with $x \neq y$. For $\epsilon > 0$ and $z \neq 0 \in X, r = \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z, \epsilon) \in (0,1)$. Given $r_0 \in (r, 1)$ there exists r_1 such that $r_1 * r_1 \geq r_0$. We show the open balls $B(((\hat{F}x)_n), 1-r_1, \frac{\epsilon}{2})$ and $B(((\hat{F}y)_n), 1-r_1, \frac{\epsilon}{2})$ are disjoint. Suppose on contrary $w \in B(((\hat{F}x)_n), 1-r_1, \frac{\epsilon}{2}) \cap B(((\hat{F}y)_n), 1-r_1, \frac{\epsilon}{2})$, then

$$\mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) > r_1, \text{ and } \mathcal{F}(((\hat{F}y)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) > r_1$$

$$\begin{aligned}
r\ &=\ \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}y)_n), z; \epsilon) \\
&\geq\ \mathcal{F}(((\hat{F}x)_n) - ((\hat{F}w)_n), z; \frac{\epsilon}{2}) * \mathcal{F}(((\hat{F}w)_n) - ((\hat{F}y)_n), z\frac{\epsilon}{2}) \\
&>\ r_1 * r_1 \\
&>\ r_0 \\
&>\ r
\end{aligned}$$

which is a contradiction. Hence $c^{I_{R2N}}(\hat{F})$ is Hausdorff. Similarly we can prove for $c_0^{I_{R2N}}(\hat{F})$. $\square$

Theorem 2.4. *Let $(X, \mathcal{F}, *)$ be a random 2–normed space. Then $c^{R2N}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$, where by $c^{R2N}(\hat{F})$ we denote the space of all Fibonacci convergent difference sequences defined as*

$$\{x = (x_n) \in X : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) > 1 - \theta\}$$

where $\epsilon > 0, \theta \in (0, 1)$ and z is non–zero element in X.

Proof. Let $\mathcal{F}-\lim((\hat{F}x)_n) = L$. Then for every $\theta \in (0, 1)$, $\epsilon > 0$ and non–zero $z \in X$, there exists $N > 0$ such that for all $n \geq N$ $\mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) > 1 - \theta$. The set $K(\epsilon) = \{k \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_k) - L, z; \epsilon) \leq 1 - \theta\} \subseteq \{1, 2, 3 \cdots\}$ and since I is admissible, we have $K(\epsilon) \in I$. Hence $I^{R2N}-\lim \hat{F}_n(x) = L$.

To show the strictness of the inclusion let us consider $X = \mathbb{R}^2$ with 2–norm $\|x, y\| = |x_1y_2 - x_2y_1|, x = (x_1, x_2), y = (y_1, y_2)$ and $a * b = ab$ for all $a, b \in [0, 1]$. Define $\mathcal{F}(x, z; \epsilon) = \frac{\epsilon}{\epsilon + \|x,z\|}$, for all $x, z \in X$. Define a sequence $x = (x_n) \in X$ such that

$$((\hat{F}x)_n) = \begin{cases} (\sqrt{n}, 0) & \text{if } n \text{ is square,} \\ (0, 0) & \text{otherwise.} \end{cases}$$

For every $0 < \theta < 1$ and $\epsilon > 0$, write

$$A(\theta, \epsilon) = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) \leq 1 - \theta\}, L = (0, 0)$$

$$\mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) = \begin{cases} \frac{\epsilon}{\epsilon + \sqrt{n}z_2}, & \text{if } n \text{ is square,} \\ 1, & \text{otherwise.} \end{cases}$$

Hence

$$\lim_n \mathcal{F}(((\hat{F}x)_n) - L, z; \epsilon) = \begin{cases} 0, & \text{if } n \text{ is square,} \\ 1, & \text{otherwise.} \end{cases}$$

Therefore $x = (x_n)$ is not convergent in $(X, \mathcal{F}, *)$. If we take $I = I_\delta = \{M \subseteq \mathbb{N} : \delta(M) = 0\}$, then since $A(\theta, \epsilon) \subseteq \{1, 4, 9, 16, \cdots\}$, $\delta(A(\theta, \epsilon)) = 0$. Thus $I^{R2N}-\lim((\hat{F}x)_n) = L$. $\square$

Theorem 2.5. *The inclusion $c_0^{I_{R2N}}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$ is strict.*

Proof. The inclusion $c_0^{I_{R2N}}(\hat{F}) \subset c^{I_{R2N}}(\hat{F})$ is obvious. To show the strictness of the inclusion, consider $X = \mathbb{R}^2$ with 2– norm $\|x,z\| = |x_1z_2 - x_2z_1|$ and $a * b = ab$. Define $\mathcal{F}(x,z) = \frac{\epsilon}{\epsilon + \|x,z\|}$ for $\epsilon > 0$. Define $x = (x_n) \in X$ such that $((\hat{F}x)_n) = (1,1)$. Then I^{R2N}–$\lim((\hat{F}x)_n) = 1$, so $x = (x_n) \in c^{I_{R2N}}(\hat{F}) \setminus c_0^{I_{R2N}}(\hat{F})$. □

Theorem 2.6. *The space $c_0^{I_{R2N}}(\hat{F})$ is solid and monotone.*

Proof. Let $x \in c_0^{I_{R2N}}(\hat{F})$. For $\theta \in (0,1), \epsilon > 0$ and non–zero $z \in X$, we have

$$A = \{n \in \mathbb{N} : \mathcal{F}(\hat{F}x)_n), z; \frac{\epsilon}{|\alpha|}) \leq 1 - \theta\} \in I,$$

where $\alpha = (\alpha_n)$ is a sequence of scalars with $|\alpha| \leq 1$, then $A^c \in F(I)$. Consider

$$B = \{n \in \mathbb{N} : \mathcal{F}(((\hat{F}\alpha x)_n), z; \epsilon) \leq 1 - \theta\}.$$

If we show $A^c \subset B^c$, then we are done.

Let $m \in A^c$, then $\mathcal{F}(((\hat{F}x)_m), z; \epsilon) > 1 - \theta$. Now

$$\begin{aligned} \mathcal{F}(((\hat{F}\alpha\ x)_m), z; \epsilon) &= \mathcal{F}((\alpha(\hat{F}x)_m), z; \epsilon) = \mathcal{F}(((\hat{F}x)_m), z; \frac{\epsilon}{|\alpha|}) \\ &\geq \mathcal{F}(((\hat{F}x)_m), z; \epsilon) * \mathcal{F}(0, z; \frac{\epsilon}{|\alpha|} - \epsilon) \\ &> 1 - \theta * 1 = 1 - \theta \end{aligned}$$

Thus $B \in I$ so that $(\alpha\ x) \in c_0^{I_{R2N}}(\hat{F})$. Therefore $c_0^{I_{R2N}}(\hat{F})$ is solid. By Lemma 1.1, $c_0^{I_{R2N}}(\hat{F})$ is monotone. □

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NEW ASPECTS OF STRONGLY *Log*-PREINVEX FUNCTIONS

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Abstract. In this paper, we consider some new classes of log-preinvex functions. Several properties of the log-preinvex functions are studied. We also discuss their relations with convex functions. Several interesting results characterizing the log-convex functions are obtained. Optimality conditions of differentiable strongly log-preinvex are characterized by a class of variational-like inequalities. Results obtained in this paper can be viewed as significant improvement of previously known results.
Key words: Preinvex functions, variational inequalities, log-convex functions

1. Introduction

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. Hanson [5] introduced the notion of invex functions in mathematical programming, which inspired a great interest. Invex sets and preinvex functions were introduced by Ben-Israel and Mond [3]. They proved that the differentiable preinvex functions are invex functions and the converse is also true under certain conditions. Noor [15] proved that the minimum of the differentiable preinvex functions are characterized by variational-like inequalities. For the applications, numerical methods, variational-like inequalities and other aspects of preinvex functions,

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see [1, 2, 3, 5, 8, 14, 15, 18, 19, 20, 21, 23, 26, 24, 25, 30] and the references therein. It is known that more accurate and inequalities can be obtained using the log-convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave), the origin of exponentially convex functions can be traced back to Bernstein [4]. Noor and Noor [20, 21] introduced and discussed various aspects of exponentially preinvex functions and their variant forms. The exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See, for example, [1, 2, 3, 4, 5, 6, 13, 14, 20, 21, 22, 23, 26, 25] and the references therein.

Recently, Noor et al [23]considered the equivalent formulation of log-convex functions and proved that the log-convex functions have similar properties as the convex functions enjoy. For example. the function e^x is a log-convex function, but not convex. Hypergeometric functions including Gamma and Beta functions are log-convex functions, which have important applications in several branches of pure and applied sciences. Noor and Noor [22] introduced the concept of strongly *log*-biconvex functions and studied their characterization. It is shown that the optimality conditions of the biconvex functions can be characterized by the bivariational inequalities, which can be viewed as novel generalization of the variational inequalities.

Inspired and motivated by the ongoing research in this interesting, applicable and dynamic field, we reconsider the concept of strongly log-preinvex functions. We discuss the basic properties of the log-preinvex functions. It is has been shown that the log-preinvex(preincave) have nice properties. Several new concepts of strongly log-preinvex functions have been introduced and investigated. We show that the local minimum of the log-convex functions is the global minimum. The difference (sum) of the strongly log-preinvex function and affine strongly log-preinvex function is again a log-preinvex function. The optimal conditions of the differentiable strongly log-preinvex functions can be characterized by a class of variational-like inequalities, which is itself an interesting outcome of our main results. The ideas and techniques of this paper may be a starting point for further research in these different areas of mathematical programming, machine learning and related optimization problems.

2. Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K \to R$ be a continuous function.

Definition 2.1. [10] The set K in H is said to be convex set, if

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2.2. [7, 8, 9] A function F is said to be convex, if

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

Polyak [27] introduced the concept of strongly convex functions in optimization and mathematical programming.

Definition 2.3. A function F is said to be a strongly convex, if there exists a constant $\mu \geq 0$ such that

$$F((1-t)u+tv) \leq (1-t)F(u)+tF(v)-\mu t(1-t)\|v-u\|^2, \ \forall u,v \in K, \ t \in [0,1].$$

Clearly every strongly convex function is a convex function, but the converse is not true. For the applications of strongly convex functions in variational inequalities, differential equations and equilibrium problems, see [6, 7, 9, 10, 11, 17, 18, 19, 21, 27, 31] and the references therein.

In many problems, the underlying set may not a convex set. To overcome this deficiency, Ben-Israel and Mond [3] introduced the invex and preinvex functions with respect to an arbitrary bifunction, which can be viewed as important generalization of the convexity and inspired a great interest in nonlinear mathematical programming.

Definition 2.4. [3] The set K_η in H is said to be invex set with respect to an arbitrary bifunction $\eta(.,.)$, if

$$u+t\eta(v,u) \in K, \qquad \forall u,v \in K_\eta, t \in [0,1].$$

Note that, if $\eta(v,u) = v-u$, then the invex set becomes convex set. In particular, it follows that the set $K_\eta \subset K$.

Definition 2.5. A strictly positive function F is said to be preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(u+t\eta(v,u)) \leq (1-t)F(u)+tF(v), \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

It is known that the differentiable preinvex functions is an invex function, that is

Definition 2.6. A function F is said to be an invex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(v)-F(u) \geq \langle F'(u), \eta(v,u)\rangle, \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

The converse is also true under certain conditions, see [8].

Noor [15] has proved that $u \in K_\eta$ is a minimum of a differentiable preinvex functions F, if and only if, $u \in K_\eta$ satisfies the inequality

$$\langle F'(u), \eta(v,u)\rangle \geq 0, \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

which is known as the variational-like inequality. For the formulation, applications, numerical methods and other aspects of variational-like inequalities and related optimization problems, see [2, 3, 5, 8, 15, 16, 28, 29] and the references therein.

Noor [14] has also proved that a function F is a preinvex function, if and only if, F satisfies the inequality

$$F\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{2}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{F(a)+F(b)}{2},$$

which is known as the Hermite-Hadamard-Noor inequality. Such type of inequalities are used to find the upper and lower estimates of the integrals and have important applications in physical and material sciences.

Definition 2.7. A strictly positive function F is said to be log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$F(u+t\eta(v,u)) \leq (F(u))^{1-t}(F(v))^{-t}, \quad \forall u,v \in K_\eta, \quad t \in [0,1]. \tag{2.1}$$

We can rewrite the Definition 2.7 in the following equivalent form as

Definition 2.8. [14] A strictly positive function F is said to be log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\log F(u+t\eta(v-u)) \leq (1-t)\log F(u) + t\log F(v), \qquad \forall u,v \in K_\eta, t \in [0,1]. \tag{2.2}$$

We use this equivalent Definition 2.8 to discuss some new aspects of log-preinvex functions.

If $\log F = e^{f(u)}$, then we recover the concepts of the exponentially preinvex function, which are mainly due to Noor and Noor [19, 21] as:

Definition 2.9. [19, 21] A positive function f is said to be exponentially preinvex function, if

$$e^{f(u+t\eta(v,u))} \leq (1-t)e^{f(u)} + te^{f(v)}, \quad \forall u,v \in K_\eta, \quad t \in [0,1].$$

We remark that Definition 2.9 can be rewritten in the following equivalent way, which is mainly due to Antczak [2].

Definition 2.10. A function f is said to be exponentially preinvex function, if

$$f(u+t\eta(v,u)) \leq \log[(1-t)e^{f(u)} + te^{f(v)}], \quad \forall u,v \in K_\eta, t \in [0,1]. \tag{2.3}$$

A function is called the exponentially preincave function f, if $-f$ is exponentially preinvex function. For the applications and properties of exponentially preinvex functions, see [1, 2, 3, 17, 18].

We now introduce the concept of strongly log-preinvex functions and study their basic properties.

Definition 2.11. A strictly positive function F is said to be strongly log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if there exists a constant $\mu \geq 0$, such that

$$\begin{aligned} \log F(u+t\eta(v-u)) &\leq (1-t)\log F(u)+t\log F(v) \\ &\quad -\mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v\in K_\eta, \quad t\in[0,1]. \end{aligned} \tag{2.4}$$

Definition 2.12. A strictly positive function F on the invex set K_η is said to be strongly log-quasi preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u)) &\leq \max\{\log F(u),\log F(v)\}-\mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v\in K_\eta, t\in[0,1]. \end{aligned}$$

Definition 2.13. A strictly positive function F on the invex set K is said to be first kind of strongly log-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u))) &\leq (\log(F(u))^{1-t}(\log F(v))^t-\mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v\in K_\eta, t\in[0,1], \end{aligned}$$

where $F(\cdot)>0$.

From the above definitions, we have

$$\begin{aligned} \log F(u+t\eta(v,u)) &\leq (\log(F(u))^{1-t}(\log F(v))^t-\mu t(1-t)\|\eta(v,u)\|^2 \\ &\leq (1-t)\log F(u)+t\log F(v)-\mu t(1-t)\|\eta(v,u)\|^2 \\ &\leq \max\{\log F(u),\log F(v)\}-\mu t(1-t)\|\eta(v,u)\|^2. \end{aligned}$$

This shows that every fist kind of strongly log-preinvex function is a strongly log-preinvex function and strongly log-preinvex function is a strongly log-quasip reinvex function. However, the converse is not true.

If $t=1$, then Definitions 2.13 and 2.14, we have:

Condition A. $\log F(u+\eta(v,u)\leq F(v)), \quad \forall u,v\in K_\eta.$

Condition A plays an important part in the derivation of the main results.

Definition 2.14. A strictly positive function F is said to be strongly affine log-preinvex function with respect to an arbitrary bifunction $\eta(.,.)$, if

$$\begin{aligned} \log F(u+t\eta(v,u)) &= (1-t)\log F(u)+t\log F(v)-\mu t(1-t)\|\eta(v,u)\|^2, \\ &\quad \forall u,v\in K_\eta, t\in[0,1]. \end{aligned}$$

Let $K_\eta = I_\eta = [a, a+\eta(b,a)]$ be the interval. We now define the log-preinvex functions on the interval I_η.

Definition 2.15. Let $I_\eta = [a, a+\eta(b,a)]$. Then F is log-convex function, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\eta(b,a) \\ \log F(a) & \log F(x) & \log F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following are equivalent:

1. F is a log-preinvex function.
2. $\log F(x) \leq \log F(a) + \frac{\log F(b)-\log F(a)}{\eta(b,a)}(x-a)$.
3. $\frac{\log F(x)-\log F(a)}{x-a} \leq \frac{\log F(b)-\log F(a)}{\eta(b,a)}$.
4. $(a+\eta(b,a)-x)\log F(a) + \eta(a,b)\log F(x) + (x-a)\log F(b)) \geq 0$.
5. $\frac{\log F(a)}{\eta(b,a)(a-x)} + \frac{\log F(x)}{(x-a-\eta(b,a))(a-x)} + \frac{\log F(b)}{\eta(b,a)(x-b)} \leq 0$,

where $x = a + t\eta(b,a) \in [0,1]$.

We also need the following assumption regarding the bifunction $\eta(\cdot,\cdot)$, which played a crucial part in the field of variational and integral inequalities,

Condition C [8]. Let $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to H$ satisfy assumptions

$$\eta(u, u+\lambda\eta(v,u)) = -\lambda\eta(v,u)$$
$$\eta(v, u+\lambda\eta(v,u)) = (1-\lambda)\eta(v,u), \quad \forall u,v \in K_\eta, \lambda \in [0,1].$$

Clearly for $\lambda = 0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u,v \in K_\eta$. One can easily show that $\eta(u+\lambda\eta(v,u),u) = \lambda\eta(v,u), \forall u,v \in K_\eta$.

3. Properties of log-preinvex functions

In this section, we consider some basic properties of log-preinvex functions.

Theorem 3.1. *Let F be a strictly* log*-preinvex function. Then any local minimum of F is a global minimum.*

Proof. Let the log-preinvex function F have a local minimum at $u \in K_\eta$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K$. Since F is a log-preinvex function, so

$$\log F(u+t\eta(v,u)) < t\log F(v) + (1-t)\log F(u), \quad \text{for} \quad 0 < t < 1.$$

Thus

$$\log F(u+t\eta(v,u)) - \log F(u) < t[\log F(v) - \log F(u)] < 0,$$

from which it follows that

$$\log F(u+t\eta(v,u)) < \log F(u),$$

for arbitrary small $t > 0$, contradicting the local minimum. $\square$

Theorem 3.2. *If the function F on the invex set K_η is* log*-preinvex, then the level set*

$$L_\alpha = \{u \in K : \log F(u) \leq \alpha, \quad \alpha \in R\}$$

is an invex set.

Proof. Let $u, v \in L_\alpha$. Then $\log F(u) \leq \alpha$ and $\log F(v) \leq \alpha$.
Now, $\forall t \in (0,1), \quad w = v + t\eta(u,v) \in K_\eta$, since K_η is an invex set. Thus, by the log-preinvexity of F, we have

$$\begin{aligned} \log F(v + t\eta(u,v)) &\leq (1-t)\log F(v) + t\log F(u) \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which, it follows that $v + t\eta(u,v) \in L_\alpha$ Hence L_α is an invex set. $\Box$

Theorem 3.3. *A positive function F is a* log*-preinvex, if and only if*

$$epi(F) = \{(u,\alpha) : u \in K : \log F(u) \leq \alpha, \alpha \in R\}$$

is an invex set.

Proof. Assume that$F$ is log-preinvex function. Let $(u,\alpha),(v,\beta) \in epi(F)$. Then it follows that $\log F(u) \leq \alpha$ and $\log F(v) \leq \beta$. Thus, $\forall t \in [0,1], \quad u, v \in K_\eta$, we have

$$\begin{aligned} \log F(u + t\eta(v,u)) &\leq (1-t)\log F(u) + t\log F(v) \\ &\leq (1-t)\alpha + t\beta, \end{aligned}$$

which implies that

$$(u + t\eta(v,u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is an invex set. Conversely, let $epi(F)$ be an invex set. Let $u, v \in K_\eta$. Then $(u, \log F(u)) \in epi(F)$ and $(v, \log F(v)) \in epi(F)$. Since $epi(F)$ is an invex set, we must have

$$(u + t\eta(v,u), (1-t)\log F(u) + t\log F(v)) \in epi(F),$$

which implies that

$$\log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(u).$$

This shows that F is a log-preinvex function. $\Box$

Theorem 3.4. *A positive function F is quasi* log*-preinvex, if and only if, the level set*

$$L_\alpha = \{u \in K_\eta, \alpha \in R : \log F(u) \leq \alpha\}$$

is an invex set.

Proof. Let $u, v \in L_\alpha$. Then $u, v \in K_\eta$ and $\max(\log F(u), \log F(v)) \leq \alpha$.
Now for $t \in (0,1), w = u + t\eta(v-u) \in K_\eta$, We have to prove that $u + t\eta(v,u) \in L_\alpha$.
By the quasi log-preinvexity of F, we have

$$\log F(u + t(v-u)) \leq \max(\log F(u), \log F(v)) \leq \alpha,$$

which implies that $u + t\eta(v,u) \in L_\alpha$, showing that the level set L_α is indeed an invex set.
Conversely, assume that L_α is an invex set. Then $\forall u, v \in L_\alpha, t \in [0,1]$, $u + t(v-u) \in L_\alpha$. Let $u, v \in L_\alpha$ for

$$\alpha = max(\log F(u), \log F(v) \quad \text{and} \quad \log F(v) \leq \log F(u).$$

From the definition of the level set L_α, it follows that

$$\log F(u + t\ (v,u)) \leq \max(\log F(u), \log F(v)) \leq \alpha.$$

Thus F is a quasi log-preinvex function. This completes the proof. □

Theorem 3.5. *Let F be a* log*-preinvex function.. Let $\mu = \inf_{u \in K} F(u)$. Then the set $E = \{u \in K : \log F(u) = \mu\}$ is an invex set of K_η. If F is strictly* log*-preinvex, then E is a singleton.*

Proof. Let $u, v \ \in E$. For $0 < t < 1$, let $w = u + t\eta(v,u)$. Since F is a log-preinvex function,

$$F(w) = \log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t \log F(v) = t\mu + (1-t)\mu = \mu,$$

which implies that to $w \in E$. and hence E is an invex set. For the second part, assume to the contrary that $F(u) = F(v) = \mu$. Since K is an invex set, for $0 < t < 1, u + t\eta(v,u) \in K_\eta$. Further, since F is strictly log-preinvex,

$$\begin{aligned} \log F(u + t(v-u)) &< (1-t)\log F(u) + t \log F(v) \\ &= (1-t)\mu + t\mu = \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{u \in K} F(u)$ and hence the result follows. □

Theorem 3.6. *If F is a* log*-preinvex function such that*

$$\log F(v) < \log F(u), \forall u, v \in K,$$

then F is a strictly quasi log*-preinvex function.*

Proof. By the log-convexity of the function F, $\forall u, v \in K, t \in [0,1]$, we have

$$\log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t \log F(v) < \log F(u),$$

since $\log F(v) < \log F(u)$, which shows that the function F is strictly quasi log-preinvex. □

4. Strongly log-preinvex functions

In this section, we now discuss some properties of the strongly log-preinvex functions.

Theorem 4.1. *Let F be a differentiable function on the invex set K_η and Condition C hold. Then the function F is* log-*preinvex function, if and only if,*

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall v,u \in K_\eta. \tag{4.1}$$

Proof. Let F be a strongly log-preinvex function. Then, $\forall u, v \in K_\eta$,

$$\log F(u + t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2,$$

which can be written as

$$\begin{aligned}\log F(v) - \log F(u) \quad &\geq \quad \{\frac{\log F(u+t\eta(v,u)) - \log F(u)}{t}\} \\ &\quad +\mu(1-t)\|\eta(v,u)\|^2.\end{aligned}$$

Taking the limit in the above inequality as $t \to 0$, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

which is (4.1), the required result.
Conversely, let (4.1) hold. Then $\forall u, v \in K_\eta, t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$ and using Condition C, we have

$$\begin{aligned} logF(v) - \log F(v_t) \quad &\geq \quad \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,v_t))\rangle + \mu\|\eta(v,v_t)\|^2 \\ &= \quad (1-t)\langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle + (1-t)^2\mu\|\eta(v,u)\|^2. \end{aligned} \tag{4.2}$$

In a similar way, we have

$$\begin{aligned} \log F(u) - \log F(v_t) \quad &\geq \quad \langle \frac{F'(v_t)}{F(v_t)}, \eta(u,v_t)\rangle + \mu\|\eta(u.v_t)\|^2 \\ &= \quad -t\langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle + \mu t^2\|\eta(v,u)\|^2. \end{aligned} \tag{4.3}$$

Multiplying (4.2) by t and (4.3) by $(1-t)$ and adding the resultant, we have

$$\log F(u + t(v-u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2,$$
$$\forall u, v \in K_\eta, t \in [0.1],$$

showing that F is a strongly log-preinvex function. $\square$

Remark 4.1. From (4.1), we have

$$F(v) \geq F(u)exp\{\langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle + \mu\|\eta(v,u)\|^2\}, \quad \forall u,v \in K_\eta.$$

Changing the role of u and v in the above inequality, we also have

$$F(u) \geq F(v)exp\{\langle\frac{F'(v)}{F(v)},\eta(u,v)\rangle + \mu\|\eta(u,v)\|^2\}, \quad \forall u,v \in K_\eta.$$

Thus, we can obtain the following inequality

$$\begin{aligned} F(u)+F(v) \quad \geq \quad & F(v)exp\{\langle\frac{F'(v)}{F(v)},\eta(u,v)\rangle + \mu\|\eta(u,v)\|^2\}, \\ & +F(u)exp\{\langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle + \mu\|\eta(v,u)\|^2\}, \quad \forall u,v \in K. \end{aligned}$$

Theorem 4.1 enables us to introduce the concept of the log-monotone operators, which appears to be new ones.

Definition 4.1. The differential $F'(.)$ is said to be strongly log-monotone, if

$$\langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle + \langle\frac{F'(v)}{F(v)},\eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u,v \in H.$$

Definition 4.2. The differential $F'(.)$ is said to be log-monotone, if

$$\langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle + \langle\frac{F'(v)}{F(v)},\eta(u,v)\rangle \leq 0, \quad \forall u,v \in H.$$

Definition 4.3. The differential $F'(.)$ is said to be log-pseudo-monotone, if

$$\langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle \geq 0, \quad \Rightarrow -\langle\frac{F'(v)}{F(v)},\eta(u-v)\rangle \geq 0, \quad \forall u,v \in H.$$

From these definitions, it follows that strongly log-monotonicity implies log-monotonicity implies log-pseudo-monotonicity, but the converse is not true.

Theorem 4.2. *Let F be differentiable strongly* log*-preinvex function on the invex set K_η. Let Condition C and Condition A hold. Then (4.1) holds, if and only if, $F'(.)$ satisfies*

$$\begin{aligned} \langle\frac{F'(u)}{F(u)},\eta(v,u)\rangle \quad + \quad & \langle\frac{F'(v)}{F(v)},\eta(u,v)\rangle \\ & \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u,v \in K_\eta. \end{aligned} \tag{4.4}$$

Proof. Let F be a strongly log-preinvex function on the invex set K_η. Then, from Theorem 4.1, we have

$$\text{(4.5)}\quad \log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u, v \in K_\eta.$$

Changing the role of u and v in (4.5), we have

$$\text{(4.6)}\quad \log F(u) - \log F(v) \geq \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle + \|\eta(u,v)\|^2, \quad \forall u, v \in K_\eta.$$

Adding (4.5) and (4.6), we have

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\}, \quad \forall u, v \in K_\eta.$$

which shows that F' is a strongly log-monotone.
Conversely, from (4.4) and Condition C, we have

$$\text{(4.7)}\quad \langle \frac{F'(v)}{F(v)}, \eta(u,v)\rangle \leq -\mu\{\|\eta(v,u)\|^2 + \|\eta(u,v)\|^2\} - \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle, \quad \forall u, v \in K_\eta.$$

Since K is an invex set, $\forall u, v \in K_\eta, \quad t \in [0,1]$ $v_t = u + t\eta(v,u) \in K_\eta$.
Taking $v = v_t$ in (4.7), we have

$$\langle \frac{F'(v_t)}{F(v_t)}, \eta(u,v_t)\rangle \leq -\mu\{\|\eta(v_t,u)\|^2 + \|\eta(u,v_t)\|^2\} - \langle \frac{F'(u)}{F(u)}, \eta(v_t,u)\rangle, \quad \forall u, v \in K_\eta.$$

Using Condition C, we obtain

$$\text{(4.8)}\qquad \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2\rangle.$$

Consider the auxiliary function

$$\xi(t) = \log F(u + t\eta(v,u)),$$

from which, we have

$$\xi(1) = \log F(u + \eta(v,u)), \quad \xi(0) = \log F(u).$$

Then, from (4.8), we have

$$\text{(4.9)}\qquad \xi'(t) \quad = \quad \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + 2\mu t\|\eta(v,u)\|^2.$$

Integrating (4.9) between 0 and 1, we have

$$\xi(1)-\xi(0)=\int_0^1 \xi'(t)dt \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2.$$

Thus it follows, using Condition A, that

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

which is the required (4.1). $\square$

We now give a necessary condition for log-pseudoconvex function.

Theorem 4.3. *Let $F'(.)$ be a* log*-pseudomonotone and let Condition C and Condition A hold. Then F is a* log*-pseudo preinvex function.*

Proof. Let $F'(.)$ be a log-pseudomonotone. Then,

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle \geq 0, \quad \forall u,v \in K_\eta,$$

implies that

$$-\langle \frac{F'(v)}{F(v)}, \eta(v,u)\rangle \geq 0. \tag{4.10}$$

Since K_η is an invex set, $\forall u,v \in K_\eta$, $t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$. Taking $v = v_t$ in(4.10) and using Condition C, we have

$$\langle e^{F(v_t)}F'(v_t), \eta(v,u)\rangle \geq 0. \tag{4.11}$$

Consider the auxiliary function

$$\xi(t) = \log F(u + t\eta(v,u)) = \log F(v_t), \quad \forall u,v \in K_\eta, t \in [0,1],$$

which is differentiable. Then, using (4.11), we have

$$\xi'(t) = \langle \frac{F'(v_t)}{F(v_t)}, \eta(v,u)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1)-\xi(0) = \int_0^1 \xi'(t)dt \geq 0.$$

Using Condition A, we have

$$\log F(v) - \log F(u) \geq 0,$$

showing that F is a log-pseudo preinvex function. $\square$

Definition 4.4. The function F is said to be sharply log-pseudo preinvex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle &\geq 0 \\ &\Rightarrow \\ F(v) &\geq \log F(u + t\eta(v,u)), \quad \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

Theorem 4.4. *Let F be a sharply* log*-pseudo preinvex function on K_η. Then*

$$\langle \frac{F'(v)}{F(v)}, \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K_\eta.$$

Proof. Let F be a sharply log-pseudo preinvex function on K_η. Then

$$\log F(v) \geq \log F(v + t\eta(u,v)), \quad \forall u, v \in K_\eta, t \in [0,1].$$

from which we have

$$0 \leq \lim_{t\to 0}\{\frac{\log F(v + t\eta(u,v)) - \log F(v)}{t}\} = \langle \frac{F'(v)}{F(v)}, \eta(v,u)\rangle,$$

the required result. □

Definition 4.5. A function F is said to be a log-pseudo preinvex function with respect to a strictly positive bifunction $B(.,.)$, such that

$$\begin{aligned} \log F(v) &< \log F(u) \\ &\Rightarrow \\ \log F(u + t\eta(v,u)) &< \log F(u) + t(t-1)B(v,u), \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

Theorem 4.5. *If the function F is strongly* log*-preinvex function such that* $\log F(v) < \log F(u)$, *then the function F is strongly* log*-pseudo preinvex.*

Proof. Since $\log F(v) < \log F(u)$ and F is strongly log-preinvex function, then $\forall u, v \in K_\eta, t \in [0,1]$, we have

$$\begin{aligned} & \log F(u + t\eta(v,u)) \\ \leq\ & \log F(u) + t(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ <\ & \log F(u) + t(1-t)(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ =\ & \log F(u) + t(t-1)(\log F(u) - \log F(v)) - \mu t(1-t)\|\eta(v,u)\|^2 \\ <\ & \log F(u) + t(t-1)B(u,v) - \mu t(1-t)\|\eta(v,u)\|^2, \end{aligned}$$

where $B(u,v) = \log F(u) - \log F(v) > 0$. This shows that the function F is strongly log-preinvex function. □

We now show that the difference of strongly log-preinvex function and affine strongly log-preinvex function is again a log-preinvex function.

Theorem 4.6. *Let f be a affine strongly* log*-preinvex function. Then F is a strongly* log*-preinvex function, if and only if, $g = F - f$ is a* log*-preinvex function.*

Proof. Let f be an affine strongly log-preinvex function. Then

$$\log f((u+t\eta(v,u)) = (1-t)\log f(u) + t\log f(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t\in[0,1]. \tag{4.12}$$

From the strongly log-preinvexity of F, we have

$$\log F(u+t\eta(v,u)) \leq (1-t)\log F(u) + t\log F(v) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, t\in[0,1]. \tag{4.13}$$

From (4.12) and (4.13), we have

$$\begin{aligned} \log F((u+t\eta(v,u)) - \log f((u+t\eta(v,u)) &\leq (1-t)(\log F(u) - \log f(u)) \\ &\quad + t(\log F(v) - \log f(v)), \end{aligned} \tag{4.14}$$

from which it follows that

$$\begin{aligned} \log g((u+t\eta(v,u)) &= \log F((u+t\eta(v,u) - \log f((u+t\eta(v,u)) \\ &\leq (1-t)(\log F(u) - \log f(u)) + t(\log F(v) - \log f(v)), \end{aligned}$$

which shows that $g = F - f$ is a log-preinvex function.
The inverse implication is obvious. □

We now discuss the optimality condition for the differentiable strongly log-preinvex functions, which is the main motivation of our next result.

Theorem 4.7. *Let F be a differentiable strongly* log*-preinvex function. Then $u \in K_\eta$ is a minimum of the function F, if and only if, $u \in K_\eta$ satisfies the inequality*

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \quad \forall u,v \in K_\eta. \tag{4.15}$$

Proof. Let $u \in K_\eta$ be a minimum of the log-preinvex function F. Then

$$F(u) \leq F(v), \forall v \in K_\eta.$$

from which, we have

$$\log F(u) \leq \log F(v), \forall v \in K_\eta. \tag{4.16}$$

Since K is an invex set, so, $\forall u, v \in K_\eta, \quad t \in [0,1]$,

$$v_t = u + t\eta(v,u) \in K_\eta.$$

Taking $v = v_t$ in (4.16), we have

$$0 \leq \lim_{t\to 0}\{\frac{\log F(u+t\eta(v,u)) - \log F(u)}{t}\} = \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle. \tag{4.17}$$

Since F is differentiable strongly log-preinvex function, so

$$\log F(u+t\eta(v,u)) \leq \log F(u) + t(\log F(v) - \log F(u)) - \mu t(1-t)\|\eta(v,u)\|^2, \quad \forall u, v \in K_\eta, t \in [0,1].$$

Using (4.17), we have

$$\begin{aligned} \log F(v) - \log F(u) &\geq \lim_{t\to 0}\{\frac{\log F(u+t\eta(v,u)) - \log F(u)}{t}\} + \mu\|\eta(v,u)\|^2 \\ &= \langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0. \end{aligned}$$

Thus, it follows that

$$\log F(v) - \log F(u) \geq \mu\|\eta(v,u)\|^2,$$

which is the required result(4.15). □

Remark 4.2. We note that, if $u \in K_\eta$ satisfies the

$$\langle \frac{F'(u)}{F(u)}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \forall v \in K_\eta, \tag{4.18}$$

then $u \in K_\eta$ is a minimum of a strongly log-preinvex function F. The inequality of the type (4.18) is called the log-variational-like inequality and appears to be a new one. For the applications, formulations, numerical methods and other aspects of variational inequalities, see Noor [12, 13, 15, 16, 31].

We remark that, if a strictly positive function F is a strongly log-preinvex function, then, we have

$$\begin{aligned} \log F(u+t\eta(v,u)) &+ \log F(v+t\eta(u,v)) \leq \log F(u) \\ &+ \log F(v) - 2\mu t(1-t)\|\eta(v,u)\|^2, \forall u, v \in K_\eta, t \in [0,1], \end{aligned} \tag{4.19}$$

which is called the Wright strongly log-preinvex function.
From (4.19), we have

$$\begin{aligned} \log F(u+t\eta(v,u))F(v+t\eta(u,v)) &= \log F(u+t\eta(v,u)) + \log F(v+t\eta(u,v)) \\ &\leq \log F(u) + \log F(v) \\ &= \log F(u)F(v), \quad \forall u, v \in K_\eta, t \in [0,1]. \end{aligned}$$

This implies that

$$F((u+t\eta(v,u))F(tu+(1-t)v) \leq F(u)F(v), \quad \forall u,v \in K_\eta, t \in [0,1],$$

which shows that a strictly positive function F is a multiplicative Wright strongly log-preinvex function. It is an interesting problem to study the properties and applications of the Wright log-preinvex functions.

Conclusion

In this paper, we have studied some new aspects of log-preinvex functions. It has been shown that log-preinvex functions enjoy several properties which convex functions have. Several new classes of strongly log-preinvex functions have been introduced and their properties are investigated. We have shown that the minimum of the differentiable strongly log-preinvex functions can be characterized by a new class of variational inequalities, which is called the log-variational inequality. Using the technique of auxiliary principle technique [13, 15, 25, 31], one can discuss the existence of a solution and suggest iterative methods for solving the *log* variational-like inequalities. One can explore the applications of the log-variational inequalities in pure and applied sciences. This may stimulate further research.

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A NOTE FOR A GENERALIZATION OF THE DIFFERENTIAL EQUATION OF SPHERICAL CURVES

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Abstract. The differential equation characterizing a spherical curve in $\mathbb{R}^3$ expresses the radius of curvature of the curve in terms of its torsion. In this paper, we have given a generalization of this equation for a curve lying in an arbitrary surface in $\mathbb{R}^3$. Moreover, we have established the analogue of the Frenet equations for a curve lying in a surface of $\mathbb{R}^3$. We have also revisited some formulas for the geodesic torsion of a curve lying in a surface of $\mathbb{R}^3$.
Keywords: spherical curves, differential geometry, Frenet equations.

1. Introduction

The curves to be considered here are curves in the Euclidean space $\mathbb{R}^3$ of the form $\alpha = \alpha(s)$, $s \in [0, L]$, where s is the arc length which is of class C^3. For such a curve, the following facts are well known.

There exists two functions κ, τ defined on $[0, L]$ that determine completely the shape of the curve in $\mathbb{R}^3$. The functions κ and τ are respectively the curvature and the torsion of the curve. Such a curve $\alpha : [0, L] \longrightarrow \mathbb{R}^3$ have a Frenet frame (T, N, B) which is a map on $[0, L]$, $s \longmapsto (T(s), N(s), B(s))$ that satisfies the Frenet

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equations

$$\left\{\begin{array}{ccc} T' & = & \kappa N \\ N' & = & -\kappa T - \tau B \\ B' & = & \tau N \end{array}\right. , \tag{1.1}$$

where the prime ($'$) denotes the differentiation with respect to arc length. For more information see [1, 3].

The condition for a curve to be a spherical curve, (i.e) it lies on a sphere, is usually given in form

$$\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' + \frac{\tau}{\kappa} = 0. \tag{1.2}$$

One can ask what the analogous of the equation (1.2) is when the curve is assumed to be in an arbitrary surface in $\mathbb{R}^3$. One of the aims is to give an answer to this question.

When a curve such as the above mentioned is assumed to lie in a given surface $\Sigma \subset \mathbb{R}^3$, then there exists two other invariants κ_n and τ_g defined on $[0, L]$ which are unique except for the sign (depending on the orientation of Σ). The functions κ_n and τ_g defined on $[0, L]$ are the normal curvature and the geodesic curvature of the curve.

Let Σ be a surface on $\mathbb{R}^3$. We will assume that Σ is oriented by choice of a unit normal field

$$\xi : \Sigma \longrightarrow S^2. \tag{1.3}$$

For a curve $\alpha : [0, L] \longrightarrow \mathbb{R}^3$ given as above, and lying in Σ, there are two naturel frames along α (see [1]). The first is Frenet frame (T, N, B) given above. For the second, let denoted by $\xi = \xi(s)$ be the restriction of ξ on α; and we consider the second frame $(T, \xi \times T, \xi)$ where $\times$ is the vector product in $\mathbb{R}^3$. These two frames (T, N, B) and $(T, \xi \times T, \xi)$ are the positively oriented in $\mathbb{R}^3$ as we will see later.

In [2] it is shown that the differential equation characterizing a spherical curve can be solved explicitly to express the radius of curvature of the curve in terms of its torsion. The author of [6] gives a necessary condition for a curve to be a spherical curve. In Minkowski space the characterization of curve lying on pseudohyperbolical space and Lorentzian hypersphere are stated both depending on curvature functions and character of Serret-Frenet frame of the curve, respectively. For detail see [4, 5, 7]. The main results of this paper is to prove the following results.

Theorem 1.1. *Under the assumptions and notations above, we have the following*

i) the trihedron $(T, \xi, T\times\xi)$ and the functions κ, τ, κ_n and τ_g satisfy the following equation

$$\left\{\begin{array}{ccc} T' & = & \kappa_n \xi + \sqrt{\kappa^2 - \kappa_n^2}(\xi \times T) \\ \xi' & = & -\kappa_n T + \tau_g(\xi \times T) \\ (T \times \xi)' & = & -\sqrt{\kappa^2 - \kappa_n^2} T - \tau_g(\xi \times T) \end{array}\right. , \tag{1.4}$$

ii)

$$\left(\frac{\kappa_n}{\kappa}\right)' = -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2} \tag{1.5}$$

iii)

$$\tau_g^2 = -(K - 2H\kappa_n + \kappa_n^2) \tag{1.6}$$

where K and H are respectively the restriction of mean curvature and the Gauss curvature of Σ to α.

Corollary 1.1. *If the curve α lying in a sphere with τ and κ' are nowhere zero in $[0, L]$, then equation* (1.5) *implies* (1.2).

The paper is organized as follows: in Section 2, we recall some results and definitions which we use for the proof of our main results. In Section 3, we prove the main results of this paper.

2. Preliminaries

Let $\alpha = \alpha(s)$, $s \in [0, L]$ be a regular curve of classe C^3 lying on an oriented surface Σ in $\mathbb{R}^3$. An orientation of Σ is determined by a choice of a unit normal $\xi : \Sigma \longrightarrow S^2$.

If $p \in \Sigma$, a basis (u, v) of $T_p\Sigma$ is positively oriented if $(u, v, \xi(p))$ is a positive basis of $\mathbb{R}^3$. A basis of $\mathbb{R}^3$ of the form $(u, v, u \times v)$ is positively oriented. So the Frenet frame $(T(s), N(s), B(s))$ on α is positively oriented at every $s \in [0, L]$. The second frame $(T(s), \xi(s) \times T(s), \xi(s))$, $s \in [0, L]$ considered above have the same orientation that the basis $(\xi(s), T(s), \xi(s) \times T(s))$, $s \in [0, T]$. Therefore, on α the "trihedron" (T, N, B) and $(T, \xi \times T, \xi)$ are positively oriented.

For each $s \in [0, L]$, we define the angle $\theta = \theta(s)$ between $N(s)$ and $\xi(s)$ by

$$\langle N(s), \xi(s) \rangle = \cos\theta(s). \tag{2.1}$$

And we have the following relation

$$N(s) = \cos\theta(s)\xi(s) + \sin\theta(s)(\xi(s) \times T(s)), \quad s \in [0, T]. \tag{2.2}$$

Now let us recall some basic facts for a curve $\alpha = \alpha(s)$ given as above and lying on a surface $\Sigma \subset \mathbb{R}^3$.

If p is a point of Σ, the Gauss map $\xi : \Sigma \longrightarrow S^2$ is a differential map and its differential $d_p\xi$ at p is a self-adjoint endomorphism of $T_p\Sigma$. The fact that $d_p\xi : T_p\Sigma \longrightarrow T_p\Sigma$ is a self-adjoint map allows to associate a quadratic form Π_p in T_pS. The quadratic form Π_p is defined on $T_p\Sigma$ by

$$\Pi_p(v) = -\langle d_p\xi(v), v \rangle \tag{2.3}$$

is called the second fundamental form of Σ at p.

Definition 2.1. A curve α in Σ passing through p, κ the curvature of α at p and $\cos\theta = \langle N, \xi\rangle$, where N is the normal vector of α at p; the number

$$\kappa_n = \kappa \cos\theta \tag{2.4}$$

is called the normal curvature of $\alpha \in \Sigma$ at p.

If $p = p(s) \in \Sigma$, the following interpretation of Π_p is well known:

$$\begin{aligned}
\Pi_p\Big(\alpha'(s)\Big) &= -\langle d_p\xi\Big(\alpha'(s)\Big), \alpha'(s)\rangle \\
&= -\langle \xi'(s), \alpha'(s)\rangle \\
&= \langle N(s), \alpha''(s)\rangle \qquad (2.5)\\
&= \langle N(s), \kappa N\rangle(p) = \kappa_n(p) \qquad (2.6)
\end{aligned}$$

In the other words, the value of the second fundamental form Π_p at a unit vector $v \in T_p\Sigma$ is equal to the normal curvature of a regular curve passing through p and tangent to v.

Now let us come back to the linear map $d_p\xi$. It is known that for each $p \in \Sigma$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$ such that $d_p\xi(e_1) = -k_1e_1$, $d_p\xi(e_2) = -k_2e_2$. Moreover, k_1 and k_2 ($k_1 \geq k_2$) are the maximum and the minimum of the second fundamental form Π_p restricted to the unit circle of $T_p\Sigma$. That is, they are the extreme values of the normal curvature at p.

The point $p \in \Sigma$ is called an umbilic point if $k_1(p) = k_2(p)$.

Definition 2.2. In terms of the principal curvatures k_1, k_2, the Gauss curvature K and the mean curvature H are given by:

$$K = k_1k_2 \qquad H = \frac{k_1 + k_2}{2}. \tag{2.7}$$

3. Proof of the main results

3.1. Proof of the theorem

For three vectors $u, v, w \in \mathbb{R}^3$, the following formulas will be used:

$$u \times (v \times w) = \langle u, w\rangle v - \langle u, v\rangle w. \tag{3.1}$$

And for an orthonormal positive oriented basis (u, v, w) in $\mathbb{R}^3$, the following relations

$$u \times v = w, \qquad w \times u = v, \tag{3.2}$$

will be also used.

Now assume that for $s \in [0, L]$, $\alpha(s)$ lies in a surface Σ. For the geodesic torsion τ_g of α at $p = \alpha(s)$, $s \in]0, L[$ we have the well known two formulas:

$$\tau_g(s) = \tau - \frac{d\theta}{dt} = \cos\phi \sin\phi(k_1 - k_2), \tag{3.3}$$

where τ is the torsion of α, θ is the angle between $\xi(s)$ and $N(s)$, ϕ is the angle that T makes with the principal direction e_1 and k_1, k_2 are principal curvatures associated with the orthonormal basis $\{e_1, e_2\}$ (assumed to be positively oriented in $T_p\Sigma$).

Here we will use another formulas for τ_g with is given in the lemma below.

Lemma 3.1. *In the notations given above, we have*

$$\tau_g(s) = \langle \xi'(s), \xi \times T\rangle, \quad s \in]0, L[. \tag{3.4}$$

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of $T_p\Sigma$ such that

$$d_p\xi(e_1) = -k_1 e_1, \quad d_p\xi(e_2) = -k_2 e_2.$$

where $p = \alpha(s)$. We can assume that $e_1 \times e_2 = \xi(s)$; thus $(e_1, e_2, \xi(s))$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. We put $T = \cos\varphi e_1 + \sin\varphi e_2$ and we have

$$\begin{aligned}
\langle \xi'(s), \xi \times T\rangle &= \langle d_p\xi(T), \xi \times T\rangle \\
&= \langle -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2, \xi \times (\cos\varphi e_1 + \sin\varphi e_2)\rangle \\
&= \langle -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2, -\sin\varphi e_1 + \cos\varphi e_2)\rangle \\
&= \cos\varphi \sin\varphi(k_1 - k_2).
\end{aligned}$$

This show (3.4) by (3.3). $\square$

Let us show (i) in Theorem 1.1.
For convenience, we will drop the point $p = \alpha(s) \in \Sigma$ in the formulas.
- From θ defined by $\cos\theta = \langle \xi, N\rangle$ the normal N which is normal to T becomes

$$N = \cos\theta\xi + \sin\theta T \times \xi,$$

and

$$\begin{aligned}
T' &= \kappa N \\
&= \kappa\cos\theta\xi + \kappa\sin\theta\xi \times T \\
&= \kappa_n\xi + \kappa\sqrt{1 - \cos^2\theta}\xi \times T \\
&= \kappa_n\xi + \sqrt{\kappa^2 - \kappa_n^2}\xi \times T.
\end{aligned}$$

- Since $\langle \xi, \xi\rangle = 1$, then $\xi' = aT + bT \times \xi$ for some numbers a and b.
We have

$$\begin{aligned}
a &= \langle \xi', T\rangle \\
&= \langle \xi, T\rangle' - \langle \xi, T'\rangle \\
&= -\kappa\langle \xi, N\rangle \\
&= -\kappa\cos\theta \\
&= -\kappa_n
\end{aligned}$$

and by (3.4) we get

$$b = \langle \xi', \xi \times T \rangle = \tau_g.$$

Thus we get $\xi' = -\kappa_n T + \tau_g \xi \times T$.
- We have $(\xi \times T)' = cT + d\xi$ for some constants c and d. We get

$$\begin{aligned} c &= \langle (\xi \times T)', T \rangle \\ &= \langle \xi \times T, T \rangle' - \langle \xi \times T, T' \rangle \\ &= -\kappa \langle \xi \times T, N \rangle \\ &= -\kappa \langle \xi \times T, \cos\theta \xi + \sin\theta T \times \xi \rangle \\ &= -\kappa \sin\theta \\ &= -\sqrt{\kappa^2 - \kappa_n^2}. \end{aligned}$$

and by (3.4), we get

$$d = \langle (\xi \times T)', \xi \rangle = \langle (\xi \times T), \xi \rangle' - \langle \xi \times T, \xi' \rangle = -\tau_g.$$

Thus $(\xi \times T)' = -\sqrt{\kappa^2 - \kappa_n^2} T - \tau_g \xi$.
This show the (i) of the theorem.
Let us show (ii) in Theorem 1.1.
We have $\frac{\kappa_n}{\kappa} = \cos\theta$. Differentiating this relation, we get

$$\begin{aligned} \left(\frac{\kappa_n}{\kappa}\right)' &= -\frac{d\theta}{dt} \sin\theta \\ &= -(\tau - \tau_g)\sqrt{1 - \cos^2\theta} \\ &= -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2}. \end{aligned}$$

This show (ii).
Let us show (iii) in Theorem 1.1.
Let $\{e_1, e_2\}$ be the unit orthonormal basis of $T_p\Sigma$ such that $d_p\xi(e_1) = -k_1 e_1$ and $d_p\xi(e_2) = -k_2 e_2$ as the recalls in section 2. And let φ be defined by $\cos\varphi = \langle e_1, T \rangle$; and then we can write $T = \cos\varphi e_1 + \sin\varphi e_2$, under the assumption that $e_1 \times e_2 = \xi$, i.e (e_1, e_2, ξ) is a positive oriented basis of $\mathbb{R}^3 = T_p\mathbb{R}^3$.
We have $\xi' = -\kappa \cos\theta T + \tau_g \xi \times T$ by (i). Also we have

$$\begin{aligned} \xi' &= d_p\xi(T) \\ &= -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2. \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned} \xi' &= -\kappa \cos\theta T + \tau_g T \times \xi \\ &= -\kappa \cos\theta (\cos\varphi e_1 + \sin\varphi e_2) + \tau_g(-\cos\varphi e_2 + \sin\varphi e_1) \\ &= (-\kappa \cos\theta \cos\varphi + \sin\varphi \tau_g) e_1 + (\kappa \cos\theta \sin\varphi - \tau_g \cos\varphi) e_2. \end{aligned} \tag{3.6}$$

By the computation given in (3.5) and (3.6) above one gets easily that

$$\begin{cases} (k_1 - \kappa\cos\theta)\cos\varphi + \tau_g \sin\varphi = 0 \\ (k_2 - \kappa\cos\theta)\sin\varphi + \tau_g \cos\varphi = 0 \end{cases}.$$

By writing the last relation in matrix form:

$$\begin{pmatrix} k_1 - \kappa\cos\theta & -\tau_g \\ \tau_g & k_2 - \kappa\cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

one gets the determinant

$$\begin{vmatrix} k_1 - \kappa\cos\theta & -\tau_g \\ \tau_g & k_2 - \kappa\cos\theta \end{vmatrix} = 0$$

$\Rightarrow k_1k_2 - \kappa\cos\theta(k_1 + k_2) + \kappa^2\cos^2\theta + \tau_g^2 = 0$
$\Rightarrow K - 2\kappa_n H + \kappa_n^2 + \tau_g^2 = 0.$
Thus we have

$$\tau_g^2 = -(K - 2H\kappa_n + \kappa_n^2).$$

This shows (iii). So the theorem is proved.

3.2. Proof of the corollary

We assume that α lies in a sphere in $\mathbb{R}^3$ of radius R. We consider the equation (ii):

$$\left(\frac{\kappa_n}{\kappa}\right)' = -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2}.$$

It is well known that, on a sphere every point is an umbilic point. This fact is important in the proof that on the sphere the second fundamental form is a constant (see [8]). That is, for any unit tangent vector v at $p = \alpha(s)$ belong to this sphere we have $\Pi_p(v) = \pm\frac{1}{R}$ and the Gauss curvature K and mean curvature H are constants $(K = \frac{1}{R^2},\ H = \pm\frac{1}{R})$. This shows that the geodesic curvature τ_g of α is zero.
Thus the equation (ii) becomes

$$\pm\frac{1}{R}\left(\frac{1}{\kappa}\right)' = -\tau\sqrt{1 - \frac{1}{R^2\kappa^2}},$$

that implies

$$\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 + \left(\frac{1}{\kappa}\right)^2 = R^2.$$

By differentiating this equation and by using $\kappa' \neq 0$, one gets easily (ii). This shows the corollary.

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HYPERSPHERICAL AND HYPERCYLINDRICAL GENERALIZED HELICES IN THE SENSE OF HAYDEN IN $\mathbb{E}^{2n+1}$

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Abstract. In this paper, we investigate generalized helices in the sense of Hayden in $(2n+1)$-dimensional Euclidean space $\mathbb{E}^{2n+1}$. We obtain some results for such curves in $\mathbb{E}^{2n+1}$. Thereafter, we obtain two families of generalized helices which are hyperspherical and hypercylindrical generalized helices in the sense of Hayden. In addition, we give examples of hyperspherical and hypercylindrical generalized helices in the sense of Hayden in $\mathbb{E}^5$. Finally, we give examples of hyperspherical and hypercylindrical generalized helices in the sense of Hayden in $\mathbb{E}^3$ and plot the graphics of these curves with Mathematica 10.0.
Keywords: generalized helices, global submanifolds, Euclidean space

1. Introduction

Helical structures have many applications to the various branches of science such as biology, architecture, engineering, etc. [1]. One of the important research problem for differential geometry is helices. The notion of helix is stated in 3-dimensional Euclidean space by M. A. Lancret in 1802. Helix is a curve whose tangent vector field makes a constant angle with a fixed direction called the axis of

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the helix. The necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion should be constant, which is given by B. de Saint Venant in 1845 [2, 4]. If both curvature and torsion are non-zero constants, then the curve is called circular helix [2]. Also, in the n-dimensional Euclidean space, a general helix is defined similarly i.e., whose tangent vector field makes a constant angle with a fixed direction [9].

In [6], generalized helix notion is more restrictive in the n-dimensional Euclidean space for $n > 3$; a fixed direction makes a constant angle with all Frenet vector fields of the curve. This type of curves are called the generalized helix in the sense of Hayden [4]. In [6], the generalized helix in the sense of Hayden has the property that the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{n-4}}{\kappa_{n-3}}, \frac{\kappa_{n-2}}{\kappa_{n-1}}$ are constants if n is odd, where $\kappa_i\,(1 \leqslant i \leqslant n-1)$ denote ith curvature function of the curve. In this work, we study generalized helices in the sense of Hayden. For the sake of brevity, we call them generalized helices.

Notice that, a curve β is called a W-curve, if the curve has constant curvatures. Also, W-curves in $\mathbb{E}^{2n+1}$ are generalized helices [4].

This study is organized as follows: In section 2, we review differential geometry of regular curves in $\mathbb{E}^n$. In Section 3, we give a theorem for generalized helix. After that, we obtain some results for generalized helices based on angles which are between the Frenet vector fields of the curve and a fixed direction. In Section 4, we show that the family of curves in [2] are hyperspherical generalized helices. Thereafter, we obtain hypercylindrical generalized helices in $\mathbb{E}^{2n+1}$ by using a different method from [2]. Finally we give examples for such curves in $\mathbb{E}^5$ and $\mathbb{E}^3$.

2. Preliminary

In this section, we give the basic theory of local differential geometry of curves in the n-dimensional Euclidean space. For more detail and background about this space, see [3, 5].

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve in the n-dimensional Euclidean space denoted by $\mathbb{E}^n$. Recall that $\langle , \rangle$ denotes the standard inner product of $\mathbb{R}^n$ given by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

for each $x = (x_1, x_2, x_3, \ldots x_n)\,,\ y = (y_1, y_2, y_3, \ldots y_n) \in \mathbb{R}^n$. The norm of a vector $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Let $\{V_1, V_2, V_3, \ldots V_n\}$ be the moving Frenet frame along the arbitrary curve α, where $V_i\,(1 \leqslant i \leqslant n)$ is Frenet vector field. Then, the matrix form of Frenet formulas are given by

$$(2.2)\quad \begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-1}' \\ V_n' \end{pmatrix} = \begin{pmatrix} 0 & \nu\kappa_1 & 0 & \cdots & 0 & 0 \\ -\nu\kappa_1 & 0 & \nu\kappa_2 & \cdots & 0 & 0 \\ 0 & -\nu\kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\nu\kappa_{n-1} \\ 0 & 0 & 0 & \cdots & -\nu\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \\ V_n \end{pmatrix}$$

where $\nu = \langle \alpha', \alpha' \rangle$ and $\kappa_i \, (1 \leqslant i \leqslant n-1)$ denote the ith curvature function of the curve α [1]. To obtain $V_1, V_2, V_3, \ldots V_n$ it is sufficient to apply the Gramm-Schmidt orthogonalization process to $\alpha'(t), \alpha''(t), \ldots, \alpha^{(n)}(t)$. More precisely, $V_i \, (1 \leqslant i \leqslant n)$ and $\kappa_i \, (1 \leqslant i \leqslant n-1)$ are determined by the following formulas [8]:

$$\begin{aligned} F_1(t) &= \alpha'(t), \\ F_i(t) &= \alpha^i(t) - \sum_{j=1}^{i-1} \frac{\langle \alpha^i(t), F_j(t) \rangle}{\langle F_j(t), F_j(t) \rangle} F_j(t) \ \text{ for } 2 \leqslant i \leqslant n, \\ \kappa_i(t) &= \frac{\|F_{i+1}(t)\|}{\|F_1(t)\| \, \|F_i(t)\|} \ \text{ for } 1 \leqslant i \leqslant n, \\ V_i &= \frac{F_i}{\|F_i\|} \ \text{ for } 1 \leqslant i \leqslant n \end{aligned}$$

where $\alpha', \alpha'', \ldots, \alpha^{(n)}$ are linearly independent. Let $\beta : I \to S^n$ be a unit speed hyperspherical curve in $\mathbb{E}^{n+1}$ where I is an open interval in $\mathbb{R}$. In [10], Izumiya and Nagai defined generalized Sabban frame $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{n-1}\}$ of the unit speed curve β which is determined by the following formulas:

$$\begin{aligned} \mathbf{n}_1 &= \frac{\mathbf{t}' + \beta}{\|\mathbf{t}' + \beta\|}, \\ k_1 &= \|\mathbf{t}' + \beta\|, \\ \mathbf{n}_2 &= \frac{\mathbf{n}_1' + k_1\beta'}{\|\mathbf{n}_1' + k_1\beta'\|}, \\ k_2 &= \|\mathbf{n}_1' + k_1\beta'\|, \\ k_i &= \left\|\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}\right\|, \\ \mathbf{n}_i &= \frac{\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}}{\left\|\mathbf{n}_{i-1}' + k_{i-1}\mathbf{n}_{i-2}\right\|}, \end{aligned}$$

for $3 \leqslant i \leqslant n-2$ and $k_i \neq 0$ for all i and

$$\begin{aligned} \mathbf{n}_{n-1} &= \frac{\beta \times \mathbf{t}' \times \mathbf{n}_1 \times \cdots \times \mathbf{n}_{n-2}}{\|\beta \times \mathbf{t}' \times \mathbf{n}_1 \times \cdots \times \mathbf{n}_{n-2}\|}, \\ k_{n-1} &= \left\langle \mathbf{n}_{n-2}', \mathbf{n}_{n-1} \right\rangle \end{aligned}$$

where k_i $(1 \leqslant i \leqslant n-1)$ denote ith curvature function of the curve β. Also, in the same paper, Izumiya and Nagai gave the following Frenet-Serret type formula for the generalized Sabban frame of the spherical curve β.

$$\begin{pmatrix} \beta' \\ \mathbf{t}' \\ \mathbf{n}_1' \\ \vdots \\ \mathbf{n}_{n-2}' \\ \mathbf{n}_{n-1}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & k_1 & \cdots & 0 & 0 \\ 0 & -k_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \mathbf{t} \\ \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_{n-2} \\ \mathbf{n}_{n-1} \end{pmatrix}. \tag{2.3}$$

Definition 2.1. A Frenet curve of rank r for which $\kappa_1, \kappa_2, \ldots, \kappa_r$ are constants is called W-curve [7].

A unit speed W-curve of rank $2n$ has the parameterization of the form

$$\beta(s) = a_0 + \sum_{i=1}^{n} (a_i \cos \mu_i s + b_i \sin \mu_i s) \tag{2.4}$$

and a unit speed W-curve of rank $2n+1$ has the parameterization of the form

$$\beta(s) = a_0 + b_0 s + \sum_{i=1}^{n} (a_i \cos \mu_i s + b_i \sin \mu_i s) \tag{2.5}$$

where $a_0, b_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are constant vectors in $\mathbb{R}^n$ and $\mu_1 < \mu_2 < \ldots < \mu_n$ are positive real numbers. So, a W-curve of rank 1 is a straight line, a W-curve of rank 2 is a circle, a W-curve of rank 3 is a right circular helix [8].

3. Generalized Helix in $\mathbb{E}^{2n+1}$

Hayden gave the following theorems in [6].

Theorem 3.1. *Let α be a curve in a Riemannian $(2n+1)$-space, the Frenet vector fields $V_3, V_5, \ldots, V_{2n+1}$ of the curve make constant angle with a parallel vector-field along the curve, then the curve α is generalized helix; moreover, V_1 also make a constant angle with the given vector-field, and $V_2, V_4, \ldots, V_{2n}$ are each perpendicular to the given vector-field [6].*

Theorem 3.2. *Let α be a curve in a Riemannian $(2n+1)$-space, the Frenet vector fields $V_1, V_3, \ldots, V_{2n-1}$ of the curve make constant angle with a parallel vector-field along the curve, then the curve α is generalized helix; moreover, V_{2n+1} also make a constant angle with the given vector-field, and $V_2, V_4, \ldots, V_{2n}$ are each perpendicular to the given vector-field [6].*

In the light of the theorems mentioned above, we can give the following theorem.

Theorem 3.3. *Let α be a curve in $\mathbb{E}^{2n+1}$. If the Frenet vector fields $V_1, V_3, V_5, \dots, V_{2j-1}, V_{2j+3}, \dots, V_{2n+1}$, $(1 \leqslant j \leqslant n)$ of the curve α make constant angle with a unit vector U, then the curve α is generalized helix; moreover, the vector field V_{2j+1} makes a constant angle with the given vector U, and $V_2, V_4, \dots, V_{2n}$ are each perpendicular to the given vector U.*

Proof. Assume that the Frenet vector fields $V_1, V_3, V_5, \dots, V_{2j-1}, V_{2j+3}, \dots, V_{2n+1}$, $(1 \leqslant j \leqslant n)$ of the curve α make constant angle with a unit vector U. Then, we have

$$\langle V_i, U\rangle = \cos\theta_i, \quad i = 1, 3, 5, \dots, 2j-1, 2j+1, \dots, 2n+1. \tag{3.1}$$

If we take the derivative of 3.1 for $i = 1$ by using Frenet formulas in 2.2, we obtain that V_2 is perpendicular to U.

If we take the derivative of 3.1 for $i = 3$ by using Frenet formulas in 2.2 and the fact that $V_2 \perp U$, we obtain that V_4 is perpendicular to U.

Similarly, we take the derivative of 3.1 for $i = 5, 7, ..., 2j-1$ we obtain $V_6, V_8, \dots V_{2j}$ each are perpendicular to U.

If we take the derivative of 3.1 for $i = 2n+1$ by using Frenet formulas in 2.2, we get V_{2n} is perpendicular to U.

If we take the derivative of 3.1 for $i = 2n-1$ by using Frenet formulas in 2.2 and the fact that $V_{2n} \perp U$, we obtain that V_{2n-2} is perpendicular to U.

Similarly, we take the derivative of 3.1 for $i = 2n-3, 2n-5, ..., 2j+3$ we obtain $V_{2n-4}, V_{2n-6}, \dots V_{2j+2}$ each are perpendicular to U.

Finally, for $i = 2j+1$ from 2.2 we have

$$\langle V_{2j+1}, U\rangle' = \kappa_{2j+1}\langle V_{2j+2}, U\rangle - \kappa_{2j}\langle V_{2j}, U\rangle = 0 \tag{3.2}$$

since $\langle V_{2j+2}, U\rangle = 0$ and $\langle V_{2j}, U\rangle = 0$. So, $\langle V_{2j+1}, U\rangle$ is a constant. Therefore, V_{2j} makes a constant angle with U. □

The vector U is called the axes of generalized helix. It is obvious; if we take the derivative of 3.1 for $i = 2, 4, \dots 2n$ by using 2.2 we have

$$\frac{\kappa_2}{\kappa_1} = \frac{\cos\theta_1}{\cos\theta_3}, \qquad \frac{\kappa_4}{\kappa_3} = \frac{\cos\theta_3}{\cos\theta_5}, \qquad \dots, \qquad \frac{\kappa_{2n}}{\kappa_{2n-1}} = \frac{\cos\theta_{2n-1}}{\cos\theta_{2n+1}}. \tag{3.3}$$

From 3.3, we give the following corollary.

Corollary 3.1. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \dots, \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$\frac{\kappa_2\kappa_4\dots\kappa_{2n}}{\kappa_1\kappa_3\dots\kappa_{2n-1}} = \frac{\cos\theta_1}{\cos\theta_{2n+1}},$$

$$\cos\theta_j = \frac{\kappa_{j+1}}{\kappa_j}\cos\theta_{j+2} \text{ for } j = 1, 3, 5, \ldots, 2n-1$$

and the axis of a generalized helix has the form

$$U = \cos\theta_1 V_1 + \cos\theta_3 V_3 + \cdots + \cos\theta_{2n+1} V_{2n+1}.$$

Theorem 3.4. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \ldots \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$U = \cos\theta_1 \left(V_1 + \sum_{i=1}^{n} \frac{\kappa_1\kappa_3\ldots\kappa_{2i-1}}{\kappa_2\kappa_4\ldots\kappa_{2i}} V_{2i+1} \right)$$

and

$$\tan^2\theta_1 = \sum_{i=1}^{n} \left(\frac{\kappa_1\kappa_3\ldots\kappa_{2i-1}}{\kappa_2\kappa_4\ldots\kappa_{2i}} \right)^2$$

where θ_1 is the angle between V_1 and U.

Proof. It is clear from equation 3.3 and Corollary 3.1. □

Similarly, we have the following theorem.

Theorem 3.5. *Let α be a generalized helix with curvatures $\kappa_1, \kappa_2, \ldots \kappa_{2n}$ in $\mathbb{E}^{2n+1}$. Then,*

$$U = \cos\theta_{2n+1} \left(V_{2n+1} + \sum_{i=1}^{n} \frac{\kappa_2\kappa_4\ldots\kappa_{2i}}{\kappa_1\kappa_3\ldots\kappa_{2i-1}} V_{2i-1} \right) \tag{3.4}$$

and

$$\tan^2\theta_{2n+1} = \sum_{i=1}^{n} \left(\frac{\kappa_2\kappa_4\ldots\kappa_{2i}}{\kappa_1\kappa_3\ldots\kappa_{2i-1}} \right)^2 \tag{3.5}$$

where θ_{2n+1} is the angle between V_{2n+1} and U.

Proof. It is clear from equation 3.3 and Corollary 3.1. □

4. Families of Generalized Hypercylindrical and Hyperspherical Generalized Helices in $\mathbb{E}^{2n+1}$

In this section, we show that the curve in [2] is a hyperspherical generalized helix. Also, we used a W-curve to obtain a hypercylindrical generalized helix.

Lemma 4.1. $\beta : I \subset \mathbb{R} \to S^{2n}$,

$$\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_{2n+1}(t))$$

is given by

$$\begin{aligned}\beta_{2i-1}(t) &= \frac{\left(1-{c_i}^2\right)\sin\left(c_i\lambda t\right)}{\left(\sum\limits_{k=1}^{n} {c_k}^4-{c_k}^2\right)^{1/2}},\\ \beta_{2i}(t) &= \frac{\left(1-{c_i}^2\right)\cos\left(c_i\lambda t\right)}{\left(\sum\limits_{k=1}^{n} {c_k}^4-{c_k}^2\right)^{1/2}},\end{aligned}$$

for $i=1,2,\ldots n$ *and*

$$\beta_{2n+1}(t)=\left(\frac{\sum\limits_{k=1}^{n}{c_k}^2-n}{\sum\limits_{k=1}^{n}{c_k}^4-{c_k}^2}\right)^{\frac{1}{2}}$$

where $\lambda=\left(\frac{\sum\limits_{k=1}^{n}{c_k}^4-{c_k}^2}{\sum\limits_{k=1}^{n}{c_k}^2-2{c_k}^4+{c_k}^6}\right)^{\frac{1}{2}}$ *is a constant. Then,* β *is a* W*-curve of rank* $2n$*.*

Proof. It is clear from equation 2.4. □

Theorem 4.1. *Let* $\alpha : I\subset R\to E^{2n+1}$

$$\alpha(t)=(\alpha_1(t),\alpha_2(t),\ldots,\alpha_{2n+1}(t))$$

be a regular curve given by

$$\begin{aligned}\alpha_{2i-1}(t) &= \frac{1}{\left(\sum_{j=1}^{n}{c_j}^2\right)^{1/2}}\left(c_i\cos(t)\cos(c_it)+\sin(t)\sin(c_it)\right),\\ \alpha_{2i}(t) &= \frac{1}{\left(\sum_{j=1}^{n}{c_j}^2\right)^{1/2}}\left(\cos(c_it)\sin(t)-c_i\cos(t)\sin(c_it)\right),\end{aligned}$$

for $i=1,2,\ldots n$ *and*

$$\alpha_{2n+1}(t)=\left(1-\frac{n}{\sum\limits_{j=1}^{n}{c_j}^2}\right)^{1/2}\sin(t)$$

where $c_1,c_2,\ldots,c_n>1$ *with* $c_i\neq c_j$, $1\leqslant i<j\leqslant n$*. Then,* α *is a general helix which lies on* S^{2n} *[2].*

By means of the Teorem 4.1, we can give the following theorem.

Theorem 4.2. *Let $\alpha : I \subset R \to E^{2n+1}$*

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_{2n+1}(t))$$

be a regular curve given by

$$\begin{aligned}
\alpha_{2i-1}(t) &= \frac{1}{\left(\sum\limits_{j=1}^{n} {c_j}^2\right)^{1/2}} \left(c_i \cos(\lambda t) \cos(c_i \lambda t) + \sin(\lambda t) \sin(c_i \lambda t)\right), \\
\alpha_{2i}(t) &= \frac{1}{\left(\sum\limits_{j=1}^{n} {c_j}^2\right)^{1/2}} \left(\cos(c_i \lambda t) \sin(\lambda t) - c_i \cos(\lambda t) \sin(c_i \lambda t)\right),
\end{aligned}$$

for $i = 1, 2, \ldots n$ and

$$\alpha_{2n+1}(t) = \left(1 - \frac{n}{\sum\limits_{j=1}^{n} {c_j}^2}\right)^{1/2} \sin(\lambda t)$$

where $c_1, c_2, \ldots, c_n > 1$ with $c_i \neq c_j$, $1 \leqslant i < j \leqslant n$ and $\lambda = \left(\frac{\sum_{k=1}^{n} {c_k}^4 - {c_k}^2}{\sum_{k=1}^{n} {c_k}^2 - 2{c_k}^4 + {c_k}^6}\right)^{\frac{1}{2}}$.
Then, the curve $\alpha : I \subset R \to E^{2n+1}$ is a hyperspherical generalized helix on S^{2n}.

Proof. After straightforward calculations, we obtain

$$||\alpha(t)|| = 1, \qquad \alpha'(t) = \omega \cos t \, \beta(t),$$

where $\omega = \left(\frac{\sum_{k=1}^{n} {c_k}^4 - {c_k}^2}{\sum_{k=1}^{n} {c_k}^2}\right)^{\frac{1}{2}}$ and β is the W-curve in Lemma 4.1. Since $||\alpha(t)|| = 1$ the curve α lies on S^{2n}. If we apply the Gramm-Schmidt orthogonalization process to the curve α

$$\begin{aligned}
F_1(t) &= \omega \cos t \, \beta(t), \\
F_2(t) &= \omega \cos t \, \mathbf{t}(t), \\
F_i(t) &= \omega \cos t \, k_1(t) k_2(t) \ldots k_{i-2}(t) \mathbf{n}_{i-2}(t) \text{ for } 3 \leqslant i \leqslant n
\end{aligned}$$

where $k_i \, (1 \leqslant i \leqslant n-1)$ is the curvature functions of the curve β. Now, we can calculate the curvature functions $\kappa_i, (1 \leqslant i \leqslant n-1)$ of the curve α.

$$\begin{aligned}
\kappa_1(t) &= \frac{\|F_2(t)\|}{\|F_1(t)\|^2} = \omega^{-1} \sec t, \\
\kappa_i(t) &= \frac{\|F_{i+1}(t)\|}{\|F_1(t)\| \, \|F_i(t)\|} = \omega^{-1} k_{i-1}(t) \sec t
\end{aligned}$$

for $2 \leqslant i \leqslant 2n$. Since the curvature functions k_i are constants for $1 \leqslant i \leqslant 2n-1$, the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{2n-1}}{\kappa_{2n}}$ are constants. Therefore, α is a hyperspherical generalized helix on S^{2n}. □

Corollary 4.1. *From Theorem 4.2, the Frenet vector fields of the curve α are*

$$V_1 = \beta, \qquad V_2 = \mathbf{t}, \qquad V_3 = \mathbf{n}_1, \quad \ldots, \quad V_{2n+1} = \mathbf{n}_{2n-1} \tag{4.1}$$

where $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{2n-1}\}$ is the generalized Sabban frame of the unit speed curve β.

Example 4.1. If we choose $c_1 = 2$ and $c_2 = 4$ in Theorem 4.2, then

$$\alpha(t) = \left(\begin{array}{c} \frac{\cos(\lambda t)\cos(2\lambda t)}{\sqrt{5}} + \frac{\sin(2\lambda t)\sin(\lambda t)}{2\sqrt{5}}, \frac{\cos(2\lambda t)\sin(\lambda t)}{2\sqrt{5}} - \frac{\cos(\lambda t)\sin(2\lambda t)}{\sqrt{5}}, \\ \frac{2\cos(\lambda t)\cos(4\lambda t)}{\sqrt{5}} + \frac{\sin(4\lambda t)\sin(\lambda t)}{2\sqrt{5}}, \frac{\cos(4\lambda t)\sin(\lambda t)}{2\sqrt{5}} - \frac{2\cos(\lambda t)\sin(4\lambda t)}{\sqrt{5}}, \frac{3\sin(\lambda t)}{\sqrt{10}} \end{array} \right)$$

where $\lambda = \sqrt{\frac{7}{101}}$.

After straightforward calculations, we obtain the Frenet vector fields of the curve α

$$\begin{aligned}
V_1(t) &= \left(-\frac{\sin(2\lambda t)}{2\sqrt{7}}, -\frac{\cos(2\lambda t)}{2\sqrt{7}}, -\frac{5\sin(4\lambda t)}{2\sqrt{7}}, -\frac{5\cos(4\lambda t)}{2\sqrt{7}}, \frac{1}{\sqrt{14}}\right), \\
V_2(t) &= \left(-\frac{\cos(2\lambda t)}{\sqrt{101}}, \frac{\sin(2\lambda t)}{\sqrt{101}}, -\frac{10\cos(4\lambda t)}{\sqrt{101}}, \frac{10\sin(4\lambda t)}{\sqrt{101}}, 0\right), \\
V_3(t) &= \left(-\frac{73\sin(2\lambda t)}{2\sqrt{7189}}, -\frac{73\cos(2\lambda t)}{2\sqrt{7189}}, \frac{55\sin(4\lambda t)}{2\sqrt{7189}}, \frac{55\cos(4\lambda t)}{2\sqrt{7189}}, \frac{101}{\sqrt{14378}}\right), \\
V_4(t) &= \left(-\frac{10\cos(2\lambda t)}{\sqrt{101}}, \frac{10\sin(2\lambda t)}{\sqrt{101}}, \frac{\cos(4\lambda t)}{\sqrt{101}}, -\frac{\sin(4\lambda t)}{\sqrt{101}}, 0\right), \\
V_5(t) &= \sqrt{\frac{2}{1027}}\left(20\sin(2\lambda t), 20\cos(2\lambda t), -\sin(4\lambda t), -\cos(4\lambda t), \frac{15\sqrt{2}}{2}\right).
\end{aligned}$$

It is clear that the Frenet vector fields V_1, V_3 and V_5 of the curve α make constant angles $\theta_1 = \arccos\frac{1}{\sqrt{14}}$, $\theta_3 = \arccos\frac{101}{\sqrt{14378}}$ and $\theta_5 = \arccos\frac{15}{\sqrt{1027}}$ with vector $U = (0, 0, 0, 0, 1)$, respectively.

Also, after straightforward calculations, we have the curvatures of the curve α

$$\kappa_1(t) = \frac{1}{21}\sqrt{505}\sec(\lambda t), \quad \kappa_2(t) = \frac{1}{21}\sqrt{\frac{5135}{101}}\sec(\lambda t), \quad \kappa_3(t) = 40\sqrt{\frac{5}{103727}}\sec(\lambda t)$$

and

$$\kappa_4(t) = \frac{4}{3}\sqrt{\frac{1010}{7189}}\sec(\lambda t).$$

Since, α lies on hypersphere $S^4 = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}^5 \,\middle|\, \sum_{i=1}^{5} x_i^2 = 1\right\}$, then α is a hyperspherical generalized helix in $\mathbb{E}^5$.

Now, we have the following theorem for a curve γ which is integration of the curve β in Lemma 4.1.

Theorem 4.3. *Let* $\gamma : I \subset R \to E^{2n+1}$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_{2n+1}(t))$$

be a regular curve given by

$$\begin{aligned}
\gamma_{2i-1}(t) &= \frac{\left(c_i{}^2 - 1\right)\left(\sum\limits_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6\right)^{\frac{1}{2}}}{c_i\left(\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2\right)} \cos\left(c_i \lambda t\right), \\
\gamma_{2i}(t) &= \frac{\left(1 - c_i{}^2\right)\left(\sum\limits_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6\right)^{\frac{1}{2}}}{c_i\left(\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2\right)} \sin\left(c_i \lambda t\right),
\end{aligned}$$

for $i = 1, 2, \ldots n$ *and*

$$\gamma_{2n+1}(t) = \left(\frac{\sum\limits_{k=1}^{n} c_k{}^2 - n}{\sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2}\right)^{\frac{1}{2}} t$$

where $\lambda = \left(\frac{\sum_{k=1}^{n} c_k{}^4 - c_k{}^2}{\sum_{k=1}^{n} c_k{}^2 - 2c_k{}^4 + c_k{}^6}\right)^{\frac{1}{2}}$ *and* $c_1, c_2, \ldots, c_n > 1$ *with* $c_i \neq c_j$, $1 \leqslant i < j \leqslant n$. *Then,* γ *is a generalized helix which lies on hypercylinder*

$$\frac{1}{n\lambda^2 \sum\limits_{k=1}^{n} c_k{}^4 - c_k{}^2}\left(\frac{x_1^2 + x_2^2}{\left(\frac{c_1^2-1}{c_1}\right)^2} + \frac{x_3^2 + x_4^2}{\left(\frac{c_2^2-1}{c_2}\right)^2} + \cdots + \frac{x_{2n-1}^2 + x_{2n}^2}{\left(\frac{c_n^2-1}{c_n}\right)^2}\right) = 1.$$

Proof. After straightforward calculations, we have $\gamma'(t) = \beta(t)$ where β is a W-curve in Lemma 4.1. If we apply the Gramm-Schmidt orthogonalization process to the curve γ, we have

$$\begin{aligned}
F_1(t) &= \beta(t), \\
F_2(t) &= \mathbf{t}(t), \\
F_i(t) &= k_1(t)k_2(t)\ldots k_{i-2}(t)\mathbf{n}_{i-2}(t) \text{ for } 3 \leqslant i \leqslant 2n-1,
\end{aligned}$$

where $k_i\,(1 \leqslant i \leqslant n-1)$ is the curvature functions of the curve β. Now, we can calculate the curvature functions $\kappa_i, (1 \leqslant i \leqslant n-1)$ of the curve γ.

$$\begin{aligned}\kappa_1 &= \frac{\|F_2\|}{\|F_1\|^2} = 1,\\ \kappa_i &= \frac{\|F_{i+1}\|}{\|F_1\|\,\|F_i\|} = k_{i-1},\end{aligned}$$

for $2 \leqslant i \leqslant 2n$. Since the curvature functions k_i are constants for $1 \leqslant i \leqslant 2n-1$, the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, \ldots, \frac{\kappa_{2n-1}}{\kappa_{2n}}$ are constants. Therefore, γ is a hypercylindrical generalized helix. $\square$

Corollary 4.2. *From Theorem 4.3, the Frenet vector fields of the curve γ are*

$$V_1 = \beta, \qquad V_2 = \mathbf{t}, \qquad V_3 = \mathbf{n}_1, \quad \ldots, \quad V_{2n+1} = \mathbf{n}_{2n-1} \tag{4.2}$$

where $\{\beta, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{2n-1}\}$ is the generalized Sabban frame of the unit speed curve β.

Example 4.2. If we choose $c_1 = 3$ and $c_2 = 4$ in Theorem 4.3, then

$$\gamma(t) = \begin{pmatrix} \frac{4\sqrt{29}}{39}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), -\frac{4\sqrt{29}}{39}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{15\sqrt{29}}{104}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), -\frac{15\sqrt{29}}{104}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{\sqrt{23}}{2\sqrt{78}}t \end{pmatrix}$$

After straightforward calculations, we obtain the Frenet vector fields of the curve γ

$$\begin{aligned}
V_1(t) &= \begin{pmatrix} \frac{-2\sqrt{2}}{\sqrt{39}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{-2\sqrt{2}}{\sqrt{39}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-5\sqrt{3}}{2\sqrt{26}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{-5\sqrt{3}}{2\sqrt{26}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{\sqrt{23}}{2\sqrt{78}} \end{pmatrix},\\
V_2(t) &= \begin{pmatrix} \frac{-2}{\sqrt{29}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{2}{\sqrt{29}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-5}{\sqrt{29}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{5}{\sqrt{29}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), 0 \end{pmatrix},\\
V_3(t) &= \begin{pmatrix} \frac{-19\sqrt{2}}{\sqrt{4043}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{-19\sqrt{2}}{\sqrt{4043}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{85}{2\sqrt{8086}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{85}{2\sqrt{8086}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{29\sqrt{23}}{2\sqrt{8086}} \end{pmatrix},\\
V_4(t) &= \begin{pmatrix} -\frac{5}{\sqrt{29}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{5}{\sqrt{29}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{2}{\sqrt{29}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), -\frac{2}{\sqrt{29}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), 0 \end{pmatrix},\\
V_5(t) &= \begin{pmatrix} \frac{5\sqrt{23}}{\sqrt{933}}\sin\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \frac{5\sqrt{23}}{\sqrt{933}}\cos\left(\frac{\sqrt{39}}{\sqrt{58}}t\right), \\ \frac{-\sqrt{69}}{2\sqrt{311}}\sin\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{-\sqrt{69}}{2\sqrt{311}}\cos\left(\frac{2\sqrt{26}}{\sqrt{87}}t\right), \frac{35}{2\sqrt{933}} \end{pmatrix}.
\end{aligned}$$

It is clear that the Frenet vector fields V_1, V_3 and V_5 of the curve γ make constant angles $\theta_1 = \frac{\sqrt{23}}{2\sqrt{78}}$, $\theta_3 = \frac{29\sqrt{23}}{2\sqrt{8086}}$ and , $\theta_5 = \frac{35}{2\sqrt{933}}$ with vector $U = (0, 0, 0, 0, 1)$, respectively.
Also, after straightforward calculations, we have the curvatures of the curve γ

$$\kappa_1 = 1, \quad \kappa_2 = \frac{\sqrt{311}}{29\sqrt{3}}, \quad \kappa_3 = \frac{455}{29\sqrt{933}}, \quad \kappa_4 = \sqrt{\frac{299}{622}}.$$

Since, γ lies on the hypercylinder $\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}^5 \middle| \frac{x_1^2+x_2^2}{\frac{16}{351}} + \frac{x_3^2+x_4^2}{\frac{150}{1664}} = 1\right\}$, then γ is a hypercylindrical generalized helix in $\mathbb{E}^5$.

Remark 4.1. Even if the curve α and γ have different curvatures, they have same Frenet vectors.

Example 4.3. If we choose $c_1 = 2$ and in Theorem 4.2, then

$$\alpha(t) = \left(\frac{2\cos\frac{t}{\sqrt{3}}\cos\frac{2t}{\sqrt{3}} + \sin\frac{t}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}}{2}, \frac{\cos\frac{2t}{\sqrt{3}}\sin\frac{t}{\sqrt{3}} - 2\cos\frac{t}{\sqrt{3}}\sin\frac{2t}{\sqrt{3}}}{2}, \sqrt{\frac{3}{4}}\sin\frac{t}{\sqrt{3}} \right)$$

After straightforward calculations, we obtain the Frenet vector fields of the curve α

$$\begin{aligned} \mathrm{T}_\alpha(t) &= \left(-\frac{\sqrt{3}}{2}\sin\frac{2t}{\sqrt{3}}, -\frac{\sqrt{3}}{2}\cos\frac{2t}{\sqrt{3}}, \frac{1}{2}\right), \\ \mathrm{N}_\alpha(t) &= \left(-\cos\frac{2t}{\sqrt{3}}, \sin\frac{2t}{\sqrt{3}}, 0\right), \\ \mathrm{B}_\alpha(t) &= \left(\frac{1}{2}\sin\frac{2t}{\sqrt{3}}, \frac{1}{2}\cos\frac{2t}{\sqrt{3}}, \frac{\sqrt{3}}{2}\right). \end{aligned}$$

It is clear that the Frenet vector fields T_α and B_α of the curve α make constant angles $\theta_1 = \arccos\frac{1}{2}$ and $\theta_3 = \arccos\frac{\sqrt{3}}{2}$ with vector $U = (0, 0, 1)$, respectively. Also, after straightforward calculating, we have the curvatures of the curve α

$$\kappa_1 = \sec\frac{t}{\sqrt{3}}, \quad \kappa_2 = \frac{1}{\sqrt{3}}\sec\frac{t}{\sqrt{3}}.$$

Since, α lies on $S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{E}^3 \big| \sum\limits_{i=1}^{3} x_i^2 = 1 \right\}$, then α is a spherical generalized helix in $\mathbb{E}^3$.

Example 4.4. If we choose $c_1 = 2$ and in Theorem 4.3, then

$$\gamma(t) = \left(\frac{3}{4}\cos\frac{2t}{\sqrt{3}}, -\frac{3}{4}\sin\frac{2t}{\sqrt{3}}, \frac{t}{2}\right).$$

After straightforward calculations, we obtain the Frenet vector fields of the curve γ

$$\begin{aligned} \mathrm{T}_\gamma(t) &= \left(-\frac{\sqrt{3}}{2}\sin\frac{2t}{\sqrt{3}}, -\frac{\sqrt{3}}{2}\cos\frac{2t}{\sqrt{3}}, \frac{1}{2}\right), \\ \mathrm{N}_\gamma(t) &= \left(-\cos\frac{2t}{\sqrt{3}}, \sin\frac{2t}{\sqrt{3}}, 0\right), \\ \mathrm{B}_\gamma(t) &= \left(\frac{1}{2}\sin\frac{2t}{\sqrt{3}}, \frac{1}{2}\cos\frac{2t}{\sqrt{3}}, \frac{\sqrt{3}}{2}\right). \end{aligned}$$

It is clear that the Frenet vector fields T_γ and B_γ of the curve makes constant angles $\theta_1 = \arccos\frac{1}{2}$ and $\theta_3 = \arccos\frac{\sqrt{3}}{2}$ with vector $U = (0, 0, 1)$, respectively. Also, after straightforward calculating, we have the curvatures of γ

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{\sqrt{3}}.$$

Since, γ lies on $\frac{x_1^2 + x_2^2}{\left(\frac{3}{4}\right)^2} = 1$, then α is a circular helix in $\mathbb{E}^3$.

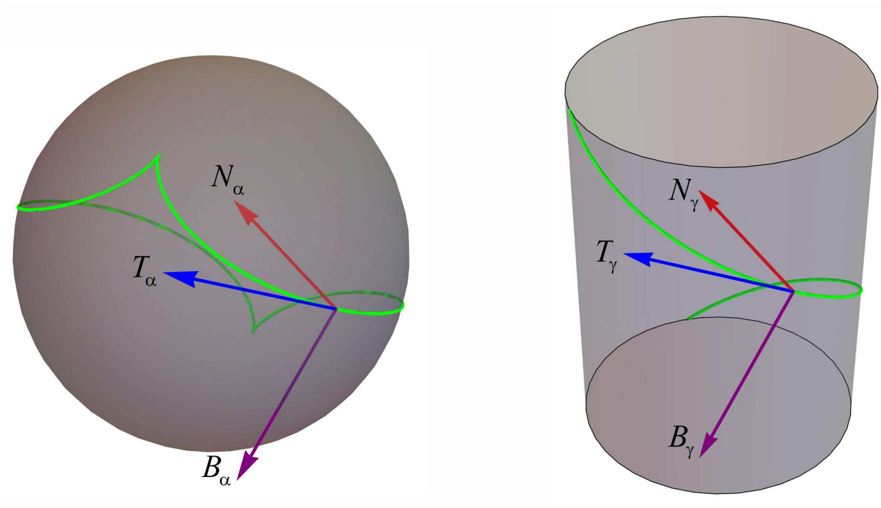

FIG. 4.1: Frenet vectors of the curves α and γ for $t = \frac{\pi}{6}$ in Example 4.3 and 4.4.

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ON CLAUSEN SERIES ${}_3F_2[-m, \alpha, \lambda + 3; \beta, \lambda; 1]$ WITH APPLICATIONS*

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Abstract. In this paper, a summation theorem for the Clausen series is derived. Further, a reduction formula is obtained for the Kampé de Fériet double hypergeometric function. Some special cases are given as applications. A generalization of the reduction and linear transformation formulas is also given in the form of the general double series identity.
Key words: Clausen series, hypergemoetric function, summation theorem

1. Introduction and preliminaries

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

(1.1)
$$ {}_pF_q\left[\begin{array}{ll} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] := {}_pF_q\left[\begin{array}{ll} \alpha_1, \alpha_2, \ldots, \alpha_p; & \\ & z \\ \beta_1, \beta_2, \ldots, \beta_q; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!} $$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume

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that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots \; ; \quad j = 1, 2, \ldots, q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Chu-Vandermonde theorem [5, p.69, Q.No. 4]:

$$ {}_2F_1\left[\begin{array}{ccc} -M,\ A & ; & \\ & & 1 \\ B & ; & \end{array}\right] = \frac{(B-A)_M}{(B)_M}; \quad M = 0, 1, 2, \cdots, \tag{1.2}$$

such that ratio of Pochhammer symbols in r.h.s. is well defined and $A,\ B \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions were unified and generalized by Kampé de Fériet [2, 1] who defined a general hypergeometric function of two variables.

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [6, p.423, Eq.(26)]:

(1.3)

$$F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{array}{lll} (a_p) & : (b_q)\ ;\ (c_k)\ ; & \\ & & x,\ y \\ (\alpha_\ell) & : (\beta_m)\ ;\ (\gamma_n)\ ; & \end{array}\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\frac{y^s}{s!},$$

where, for convergence,

$$(i) \quad p+q < \ell+m+1, \quad p+k < \ell+n+1, \quad |x| < \infty, \quad |y| < \infty, \text{ or} \tag{1.4}$$
$$(ii) \quad p+q = \ell+m+1, \quad p+k = \ell+n+1 \text{ and} \tag{1.5}$$

$$\begin{cases} |x|^{1/(p-\ell)} + |y|^{1/(p-\ell)} < 1, & \text{if } p > \ell \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq \ell. \end{cases} \tag{1.6}$$

An important development has been made by various authors in generalizations of the summation and transformation theorems, see [7, 4, 3]. In this work, our main motive is to find the summation theorem for the Clausen series ${}_3F_2[-m, \alpha, \lambda + 3; \beta, \lambda; 1]$ and to find its applications.

We shall use the following definition in proving our results in Sections 2 to 5:

Definition 1.1. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $r \in \mathbb{Z}^+ \cup \{0\}$, the following identity holds true:

$$\frac{(\lambda+3)_r}{(\lambda)_r} = 1 + \frac{3r}{\lambda} + \frac{3r(r-1)}{\lambda(\lambda+1)} + \frac{r(r-1)(r-2)}{\lambda(\lambda+1)(\lambda+2)}. \tag{1.7}$$

The proof of the above identity can be obtained smoothly.

2. Summation theorem

Theorem 2.1. *If γ, δ, σ are the roots of the cubic equation $Cm^3 + Dm^2 + Em + G = 0$ and α, β, λ, $-\gamma$, $-\delta$, $-\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $m \in \mathbb{N}_0$, then the following summation theorem holds true:*

$$ {}_3F_2\left[\begin{array}{ccc} -m, \ \alpha, \ \lambda+3 & ; & \\ & & 1 \\ \beta, \ \lambda & ; & \end{array}\right] = \frac{(-\gamma+1)_m \ (-\delta+1)_m \ (-\sigma+1)_m \ (\beta-\alpha-3)_m}{(-\gamma)_m \ (-\delta)_m \ (-\sigma)_m \ (\beta)_m}, \tag{2.1}$$

where the coefficients C, D, E and G are the polynomials in α, β, λ given as follows:

$$C = -2\alpha + 3\alpha^2 - \alpha^3 + 2\lambda - 6\alpha\lambda + 3\alpha^2\lambda + 3\lambda^2 - 3\alpha\lambda^2 + \lambda^3, \tag{2.2}$$

$$\begin{aligned} D = {} & 12\alpha - 9\alpha^2 - 3\alpha^3 - 6\alpha\beta + 6\alpha^2\beta - 12\lambda + 27\alpha\lambda + 3\alpha^2\lambda \\ & -3\alpha^3\lambda + 6\beta\lambda - 15\alpha\beta\lambda + 3\alpha^2\beta\lambda - 18\lambda^2 + 6\alpha\lambda^2 + 6\alpha^2\lambda^2 \\ & +9\beta\lambda^2 - 6\alpha\beta\lambda^2 - 6\lambda^3 - 3\alpha\lambda^3 + 3\beta\lambda^3, \end{aligned} \tag{2.3}$$

$$\begin{aligned} E = {} & -22\alpha - 12\alpha^2 - 2\alpha^3 + 24\alpha\beta + 6\alpha^2\beta - 6\alpha\beta^2 + 22\lambda - 21\alpha\lambda \\ & -27\alpha^2\lambda - 6\alpha^3\lambda - 24\beta\lambda + 30\alpha\beta\lambda + 15\alpha^2\beta\lambda + 6\beta^2\lambda - 9\alpha\beta^2\lambda \\ & +33\lambda^2 + 18\alpha\lambda^2 - 6\alpha^2\lambda^2 - 3\alpha^3\lambda^2 - 36\beta\lambda^2 - 3\alpha\beta\lambda^2 + 6\alpha^2\beta\lambda^2 \\ & +9\beta^2\lambda^2 - 3\alpha\beta^2\lambda^2 + 11\lambda^3 + 12\alpha\lambda^3 + 3\alpha^2\lambda^3 - 12\beta\lambda^3 \\ & -6\alpha\beta\lambda^3 + 3\beta^2\lambda^3 \end{aligned} \tag{2.4}$$

$$\begin{aligned} G = {} & -12\lambda - 22\alpha\lambda - 12\alpha^2\lambda - 2\alpha^3\lambda + 22\beta\lambda + 24\alpha\beta\lambda + 6\alpha^2\beta\lambda \\ & -12\beta^2\lambda - 6\alpha\beta^2\lambda + 2\beta^3\lambda - 18\lambda^2 - 33\alpha\lambda^2 - 18\alpha^2\lambda^2 - 3\alpha^3\lambda^2 \\ & +33\beta\lambda^2 + 36\alpha\beta\lambda^2 + 9\alpha^2\beta\lambda^2 - 18\beta^2\lambda^2 - 9\alpha\beta^2\lambda^2 + 3\beta^3\lambda^2 - 6\lambda^3 \\ & -11\alpha\lambda^3 - 6\alpha^2\lambda^3 - \alpha^3\lambda^3 + 11\beta\lambda^3 + 12\alpha\beta\lambda^3 + 3\alpha^2\beta\lambda^3 - 6\beta^2\lambda^3 \\ & -3\alpha\beta^2\lambda^3 + \beta^3\lambda^3 \end{aligned} \tag{2.5}$$

$$\begin{aligned} & = -C\gamma\delta\sigma \\ & = \lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha-1)\ (\beta-\alpha-2)\ (\beta-\alpha-3). \end{aligned}$$

Proof. Suppose the l.h.s. of equation (2.1) is denoted by Δ, then we have

$$\begin{aligned}
\Delta &= \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r\ (\lambda+3)_r}{(\beta)_r\ (\lambda)_r\ r!} \\
&= \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r}{(\beta)_r\ r!} \left[1+\frac{3r}{\lambda}+\frac{3r(r-1)}{\lambda(\lambda+1)}+\frac{r(r-1)(r-2)}{\lambda(\lambda+1)(\lambda+2)}\right] \\
&= {}_2F_1\left[\begin{array}{ll} -m,\ \alpha & ; \\ & 1 \\ \beta & ; \end{array}\right] + \frac{3}{\lambda}\sum_{r=0}^{m-1}\frac{(-m)_{r+1}\ (\alpha)_{r+1}}{(\beta)_{r+1}\ r!} \\
&\quad + \frac{3}{\lambda(\lambda+1)}\sum_{r=0}^{m-2}\frac{(-m)_{r+2}\ (\alpha)_{r+2}}{(\beta)_{r+2}\ r!} + \frac{1}{\lambda(\lambda+1)(\lambda+2)}\sum_{r=0}^{m-3}\frac{(-m)_{r+3}\ (\alpha)_{r+3}}{(\beta)_{r+3}\ r!} \\
&= {}_2F_1\left[\begin{array}{ll} -m,\ \alpha & ; \\ & 1 \\ \beta & ; \end{array}\right] + \frac{3}{\lambda}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}{}_2F_1\left[\begin{array}{ll} -(m-1),\ \alpha+1 & ; \\ & 1 \\ \beta+1 & ; \end{array}\right] + \\
&\quad + \frac{3}{\lambda(\lambda+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}{}_2F_1\left[\begin{array}{ll} -(m-2),\ \alpha+2 & ; \\ & 1 \\ \beta+2 & ; \end{array}\right] + \\
&\quad + \frac{1}{\lambda(\lambda+1)(\lambda+2)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}{}_2F_1\left[\begin{array}{ll} -(m-3),\ \alpha+3 & ; \\ & 1 \\ \beta+3 & ; \end{array}\right]. \qquad (2.6)
\end{aligned}$$

Using Chu-Vandermonde theorem (1.2) in r.h.s. of equation (2.6), we obtain

$$\begin{aligned}
\Delta &= \frac{(\beta-\alpha)_m}{(\beta)_m} + \frac{3}{\lambda}\frac{(-m)_1\ (\alpha)_1}{(\beta)_1}\frac{(\beta-\alpha)_{m-1}}{(\beta+1)_{m-1}} + \frac{3}{\lambda(\lambda+1)}\frac{(-m)_2\ (\alpha)_2}{(\beta)_2}\frac{(\beta-\alpha)_{m-2}}{(\beta+2)_{m-2}} + \\
&\quad + \frac{1}{\lambda(\lambda+1)(\lambda+2)}\frac{(-m)_3\ (\alpha)_3}{(\beta)_3}\frac{(\beta-\alpha)_{m-3}}{(\beta+3)_{m-3}} \\
&= \frac{(\beta-\alpha)_m}{(\beta)_m} + \frac{3(-m)_1\ (\alpha)_1}{\lambda}\frac{(\beta-\alpha)_{m-1}}{(\beta)_m} + \frac{3(-m)_2\ (\alpha)_2}{\lambda(\lambda+1)}\frac{(\beta-\alpha)_{m-2}}{(\beta)_m} + \\
&\quad + \frac{(-m)_3\ (\alpha)_3}{\lambda(\lambda+1)(\lambda+2)}\frac{(\beta-\alpha)_{m-3}}{(\beta)_m} \\
&= \frac{(\beta-\alpha)_m}{(\beta)_m}\left[1 - \frac{3m\ \alpha}{\lambda\ (\beta-\alpha+m-1)} + \frac{3(-m)_2\ (\alpha)_2}{\lambda(\lambda+1)\ (\beta-\alpha+m-2)_2} + \right. \\
&\quad \left. + \frac{(-m)_3\ (\alpha)_3}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-3)_3}\right] \\
&= \frac{(\beta-\alpha)_m}{(\beta)_m}\cdot \\
&\quad \cdot\left[\frac{\Omega(\alpha,\beta,\lambda,m)}{\lambda(\lambda+1)(\lambda+2)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right], \qquad (2.7)
\end{aligned}$$

where

$$\begin{aligned}\Omega(\alpha,\beta,\lambda,m) &= \lambda(\lambda+1)(\lambda+2)(\beta-\alpha+m-1)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\ &\quad -3m\alpha(\lambda+1)(\lambda+2)(\beta-\alpha+m-2)(\beta-\alpha+m-3)\\ &\quad +3(-m)(-m+1)(\alpha)(\alpha+1)(\lambda+2)(\beta-\alpha+m-3)\\ &\quad +(-m)(-m+1)(-m+2)(\alpha)(\alpha+1)(\alpha+2).\end{aligned}$$

Equation (2.7) can be written as

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{Cm^3+Dm^2+Em+G}{\lambda(\lambda+1)(\lambda+2)\;(\beta-\alpha+m-1)\;(\beta-\alpha+m-2)\;(\beta-\alpha+m-3)}\right], \tag{2.8}$$

Since γ, δ, σ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$, therefore equation (2.8) can be written as:

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{C(m-\gamma)(m-\delta)(m-\sigma)}{\lambda(\lambda+1)(\lambda+2)\;(\beta-\alpha+m-1)\;(\beta-\alpha+m-2)\;(\beta-\alpha+m-3)}\right]. \tag{2.9}$$

On simplification, we get assertion (2.1). □

3. Application in reducibility of the Kampé de Fériet function

The application of summation Theorem 2.1 is given by proving the following reduction formula:

Theorem 3.1. *For $b_1,\cdots,b_B$, α, β, λ, $-\gamma$, $-\delta$, $-\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$, the following reduction formula holds true:*

$$F^{A:0;2}_{B:0;2}\left[\begin{array}{ccccc}(a_A) & : & - & ; & \alpha,\ \lambda+3\ ; \\ & & & & & z,\ -z\\ (b_B) & : & - & ; & \beta,\ \lambda\ ; \end{array}\right] =$$

$$ {}_{A+4}F_{B+4}\left[\begin{array}{c}a_1,\cdots,a_A,\ -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3\ ;\\ \qquad\qquad\qquad\qquad\qquad\qquad z\\ b_1,\cdots,b_B,\ -\gamma,\ -\delta,\ -\sigma,\ \beta\ ;\end{array}\right], \tag{3.1}$$

subject to the convergence conditions:

$$\begin{cases}|z|<\frac{1}{2}, & \textit{if } A=B+1\\ |z|<\infty, & \textit{if } A\le B,\end{cases}$$

where γ, δ, σ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$ and C, D, E, G are given by equations (2.2)-(2.5).

Proof. Suppose l.h.s. of equation (3.1) is denoted by Φ, then we have

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_{m+n}\ (\alpha)_n\ (\lambda+3)_n(-1)^n z^{m+n}}{\prod_{i=1}^{B}(b_i)_{m+n}\ (\beta)_n\ (\lambda)_n\ m!\ n!}$$

$$= \sum_{m=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m}\frac{z^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n\ (\alpha)_n\ (\lambda+3)_n}{(\beta)_n\ (\lambda)_n\ n!}$$

$$= \sum_{m=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m}\frac{z^m}{m!}\ {}_3F_2\left[\begin{array}{c} -m,\ \alpha,\ \lambda+3\ \ ; \\ \qquad\qquad\qquad 1 \\ \beta,\ \lambda\ \ ; \end{array}\right]. \tag{3.2}$$

Using Theorem 2.1 in r.h.s. of above equation, it follows that

$$\Phi = \sum_{m=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m}\frac{(-\gamma+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\gamma)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}\ \frac{z^m}{m!}. \tag{3.3}$$

In view of equation (3.3), reduction formula (3.1) follows. □

4. Applications in linear transformations

If γ, δ, σ are the roots of the cubic equation $Cm^3 + Dm^2 + Em + G = 0$ and C, D, E, G are given by equations (2.2)-(2.5), we prove the following consequences of Theorem 3.1:

I. Taking $A = B = 0$ in equation (3.1), we get the following transformation formula:

$$ {}_4F_4\left[\begin{array}{c} -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3\ \ ; \\ \qquad\qquad\qquad z \\ -\gamma,\ -\delta,\ -\sigma,\ \beta\ \ ; \end{array}\right] = $$

$$\exp(z)\ {}_2F_2\left[\begin{array}{c} \alpha,\ \lambda+3\ \ ; \\ \qquad -z \\ \beta,\ \lambda\ \ ; \end{array}\right], \tag{4.1}$$

where $|z| < \infty$ and $\alpha, \beta, \lambda, -\gamma, -\delta, -\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

II. Taking $A = 1$, $a_1 = a$, $B = 0$ in equation (3.1) and using binomial theorem, we get the following transformation formula:

$$ {}_5F_4\left[\begin{array}{c} a,\ -\gamma+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3\ \ ; \\ \qquad\qquad\qquad z \\ -\gamma,\ -\delta,\ -\sigma,\ \beta\ \ ; \end{array}\right]$$

$$(4.2) \qquad = (1-z)^{-a}\ {}_3F_2\left[\begin{array}{lr} a,\ \alpha,\ \lambda+3 & ; \\ & \frac{-z}{1-z} \\ \beta,\ \lambda & ; \end{array}\right],$$

where $|z|<1$, $\left|\frac{-z}{1-z}\right|<1$ and $\alpha,\ \beta,\ \lambda,\ -\gamma,\ -\delta,\ -\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

5. General double series identity

Theorem 5.1. *Let* $\{\Theta(\ell)\}_{\ell=1}^{\infty}$ *is bounded sequence of arbitrary complex numbers,* $\Theta(0)\neq 0$ *and* $\alpha,\ \beta,\ \lambda,\ -\gamma,\ -\delta,\ -\sigma \in \mathbb{C}\setminus\mathbb{Z}_0^-$. *Then*

$$\sum_{m,n=0}^{\infty} \frac{\Theta(m+n)\ (\alpha)_n\ (\lambda+3)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n}\frac{z^{m+n}}{m!\ n!}$$

$$(5.1) \qquad = \sum_{m=0}^{\infty}\frac{\Theta(m)\ (-\gamma+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\gamma)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}\frac{z^m}{m!},$$

where $\gamma,\ \delta,\ \sigma$ *are the roots of cubic equation* $Cm^3+Dm^2+Em+G=0$ *and* C,D,E,G *are given by equations* (2.2)-(2.5) *with each of the multiple series involved is absolutely convergent.*

Remark 5.1. *For* $\Theta(\ell)=\frac{\prod_{i=1}^{A}(a_i)_\ell}{\prod_{i=1}^{B}(b_i)_\ell}$, *the above series identity reduces to the reduction formula* (3.1).

Appendix

The roots γ, δ, σ of the cubic equation $Cm^3+Dm^2+Em+G=0$ are calculated by using *Wolfram Mathematica 9.0* Software. The values of γ, δ and σ are given as follows:

$$\gamma = -\frac{D}{3C} - \frac{2^{1/3}\left(-D^2+3CE\right)}{3C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}} + \frac{\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{3\times 2^{1/3}C}$$

$$\delta = -\frac{D}{3C} + \frac{\left(1+i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}$$

$$-\frac{\left(1-i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}$$

$$\sigma=-\frac{D}{3C}$$

$$+\frac{\left(1-i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}$$

$$-\frac{\left(1+i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}$$

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SOLUTIONS FOR THE MIXED SYLVESTER OPERATOR EQUATIONS

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Abstract. This paper is devoted to investigating some system of mixed coupled generalized Sylvester operator equations. The block operator matrix decomposition is used to find the necessary and sufficient conditions for the solvability to these systems. The solutions of the system are expressed in terms of the Moore–Penrose inverses of the coefficient operators.
Keywords: Sylvester operator equations, Matrix equations, C^*-modules

1. Introduction and Preleminaries

The generalized Sylvester matrix equations have been attracting much attention from both practical and theoretical importance. The Sylvester matrix equation $AX - XB = C$ or generalized Sylvester matrix equation $AX - YB = C$ has massive applications in control theory [16, 15], singular system control [11], and widely used in many other fields such as signal and color image processing, orbital mechanics, robust control, neural network, computer graphics.

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Wimmer in [15] gave a necessary and sufficient condition for the existence of a simultaneous solution of

$$(1.1)\qquad \begin{cases} A_1X + YB_1 = C_1, \\ A_2X + YB_2 = C_2. \end{cases}$$

Kägström in [7] obtained a solution of (1.1) by using generalized Schur methods. Recently, some mixed Sylvester matrix equations have been investigated in some papers (see [12]). Lee and Vu [8] gave some solvability conditions to mixed Sylvester matrix equations

$$(1.2)\qquad \begin{cases} A_1X + YB_1 = C_1, \\ A_2Z + YB_2 = C_2. \end{cases}$$

The general solution of systems of coupled generalized Sylvester matrix equations to (1.2) was established by He and Wang in [1, 2, 3, 4, 5, 13, 14].

In this paper, by using the block operator matrix decomposition, we present a new approach to find the necessary and sufficient conditions for the solvability of mixed generalized coupled Sylvester operator equations. We obtain an arbitrary solutions of these systems that it is expressed in terms of the Moore–Penrose inverses of the coefficient operators.

Throughout this paper, we use $\mathcal{H}$ and $\mathcal{H}_i$ for denote Hilbert spaces. Also, $\mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$ instate the set of all bounded Linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$. For any $A \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$, the null and the range space of A are denoted by $\ker(A)$ and ran(A), respectively. In the case $\mathcal{H}_i = \mathcal{H}_j$, $\mathcal{L}(\mathcal{H}_i, \mathcal{H}_i)$ which is abbreviated to $\mathcal{L}(\mathcal{H}_i)$. The identity operator on $\mathcal{H}$ is denoted by $1_{\mathcal{H}}$ or 1 if there is no ambiguity.

Definition 1.1. Let $\mathcal{H}$ be Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. The Moore-Penrose inverse $A^\dagger$ of A is an element $X \in \mathcal{L}(\mathcal{H})$ which satisfies

$$(1) AXA = A, \quad (2) XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

From the definition of Moore-Penrose inverse, it can be proved that the Moore-Penrose inverse of an operator (if it exists) is unique and $A^\dagger A$ and $AA^\dagger$ are orthogonal projections, in the sense that they are self adjoint and idempotent operators. More precisely $A \in \mathcal{L}(\mathcal{H}_i, \mathcal{H}_j)$ have a closed range. Then $AA^\dagger$ is the orthogonal projection from $\mathcal{H}_j$ onto ran(A) and $A^\dagger A$ is the orthogonal projection from $\mathcal{H}_i$ onto ran(A*).

Clearly, A is Moore-Penrose invertible if and only if A^* is Moore-Penrose invertible, and in this case $(A^*)^\dagger = (A^\dagger)^*$. By Definition 1.1, it is concluded $\mathrm{ran(A)} = \mathrm{ran(AA^\dagger)}$, $\mathrm{ran(A^\dagger)} = \mathrm{ran(A^\dagger A)} = \mathrm{ran(A^*)}$, $\ker(A) = \ker(A^\dagger A)$ and $\ker(A^\dagger) = \ker(AA^\dagger) = \ker(A^*)$. For more related results, we refer the interested readers to [6] and [9] and references therein.

2. Solutions for the mixed Sylvester operator equations

In this section, by using some block matrix technique we find the conditions for solvability of the linear system equations (1.2) where A_i, B_i ($i \in \{1, 2\}$) are given matrices, X, Y and Z be arbiterary. First, we establish necessary and sufficient conditions for the solvability of (1.2) and the expression of the general solutions to the system when it is solvable.

When A_i, B_i ($i \in \{1,2\}$) are invertible operators. It can straightforward be seen that the proof of the following Theorem is valid in rings with involution.

So let A_i, B_i ($i \in \{1,2\}$) be Moore-Penrose invertible operators.

Theorem 2.1. *Suppose that $\{\mathcal{H}_i\}_{i=1}^4$ are Hilbert spaces and $B_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $A_i \in B(\mathcal{H}_4, \mathcal{H}_3)$; $i \in \{1,2\}$ are invertible operators and $C_1, C_2 \in B(\mathcal{H}_1, \mathcal{H}_3)$. Then the following statements are equivalent:*

(a) There exists solutions $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ of the system (1.2),

(b) $C_1 = C_2 B_2^{-1} B_1$.

In which case, the general solutions X, Y, Z to the system (1.2) are of the form

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1 B_1), \tag{2.1}$$

$$Y = \frac{1}{2}(C_2 B_2^{-1} + A_2 Z_2^*), \tag{2.2}$$

$$Z = \frac{1}{2}(A_2^{-1}C_2 - Z_2^* B_2), \tag{2.3}$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ satisfy $Z_2 = -Z_1^* A_1^* (A_2^*)^{-1}$.

Proof. $(a) \Rightarrow (b)$ It is clear.

$(b) \Rightarrow (a)$: By matrix representations, the system (1.2) become into the following form

$$\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} = \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}.$$

Let $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ be the general solutions to the system (1.2). Then

$$\begin{aligned}
\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \\
+ & \left(\frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \right. \\
- & \left. \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \right) \\
\times & \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
+ & \left(\frac{1}{2}\left(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right)\right. \\
- & \left.\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right)\begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
= & \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} \\
+ & \frac{1}{2}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} - \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right) \\
\times & \begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
= & \frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix}\begin{bmatrix} A_2^* & 0 \\ 0 & B_1 \end{bmatrix} \\
= & \frac{1}{2}\begin{bmatrix} 0 & A_1^{-1}C_1 + Z_1B_1 \\ (B_2^*)^{-1}C_2^* + Z_2A_2^* & 0 \end{bmatrix}.
\end{aligned}
$$

Where, Z_1, Z_2 take in the following matrix

$$
\begin{aligned}
\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} - \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & XB_1^{-1} - A_1^{-1}Y \\ Y^*(A_2^*)^{-1} - (B_2^*)^{-1}X^* & 0 \end{bmatrix}.
\end{aligned}
$$

Then,

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \tag{2.4}$$

$$Y = \frac{1}{2}(C_2B_2^{-1} + A_2Z_2^*). \tag{2.5}$$

Also,

$$
\begin{aligned}
\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} &= \begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
&- \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
& \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
= & \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
- & \begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix}.
\end{aligned}
$$

We have,

$$\begin{aligned}
&\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix} \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix} \\
+ \ & \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right) \\
- \ & \begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}+\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \\
& (\frac{1}{2}\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}+\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}\right) \\
- \ & \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}+\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix} \\
& \left(\frac{1}{2}\left(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_2^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & Y \\ Z^* & 0 \end{bmatrix}-\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}\right)\right) \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1 \\ C_2^* & 0 \end{bmatrix}\begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & B_1^{-1} \end{bmatrix}-\frac{1}{2}\begin{bmatrix} A_1 & 0 \\ 0 & B_2^* \end{bmatrix}\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix} \\
= \ & \frac{1}{2}\begin{bmatrix} 0 & C_1B_1^{-1}-A_1Z_1 \\ C_2^*(A_2^*)^{-1}-B_2^*Z_2 & 0 \end{bmatrix}.
\end{aligned}$$

Therefore,

$$Z = \frac{1}{2}(A_2^{-1}C_2 - Z_2^*B_2), \tag{2.6}$$

$$Y = \frac{1}{2}(C_1B_1^{-1} - A_1Z_1). \tag{2.7}$$

Since $C_1 = C_2B_2^{-1}B_1$ and $Z_2 = -Z_1^*A_1^*(A_2^*)^{-1}$ imply that Eqs. (1.2) and (2.7) coincide with other. This completes the proof. $\square$

Theorem 2.2. *Let $\{\mathcal{H}_i\}_{i=1}^4$ be Hilbert spaces and $B_i \in B(\mathcal{H}_1,\mathcal{H}_2)$ and $A_i \in B(\mathcal{H}_4,\mathcal{H}_3)$; $i \in \{1,2\}$ be invertible operators and $C_1, C_2 \in B(\mathcal{H}_1,\mathcal{H}_3)$. Then the following statements are equivalent:*

(a) *There exists solutions $X \in B(\mathcal{H}_1,\mathcal{H}_4)$ and $Y \in B(\mathcal{H}_2,\mathcal{H}_3)$ of the system (1.1),*

(b) $C_1 = C_2B_2^{-1}B_1$, *and* $C_2 = A_2A_1^{-1}C_1$.

If (a) or (b) is satisfied, then any solutions of the system (1.1) has the form

$$X = \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \tag{2.8}$$

$$Y = \frac{1}{2}(C_2B_2^{-1} + A_2Z_2^*), \tag{2.9}$$

where $Z_1 \in B(\mathcal{H}_2, H_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ *satisfy* $Z_2 = -Z_1^*A_1^*(A_2^*)^{-1}$ *and* $Z_1 = -Z_2^*B_2B_1^{-1}$.

Proof. The proof is quite similar to the proof of the previous theorem. □

Theorem 2.3. *Let* $\{\mathcal{H}_i\}_{i=1}^4$ *be Hilbert spaces and* $A_i \in B(\mathcal{H}_4, \mathcal{H}_3)$ *and* $B_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ *(* $i \in \{1, 2\}$*) have closed range operators such that* $\mathrm{ran}(B_1^*) = \mathrm{ran}(B_2^*)$, $\mathrm{ran}(B_1) = \mathrm{ran}(B_2)$ *and* $\mathrm{ran}(A_1) = \mathrm{ran}(A_2)$. *If* $C_1, C_2 \in B(\mathcal{H}_1, \mathcal{H}_3)$ *such that* $(1 - B_1^\dagger B_1)C_1B_1^\dagger = (1 - B_1^\dagger B_1)C_2B_2^\dagger$, *then the following statements are equivalent:*

(a) There exists solutions $X, Z \in B(\mathcal{H}_1, \mathcal{H}_4)$ *and* $Y \in B(\mathcal{H}_2, \mathcal{H}_3)$ *of the system (1.2),*

(b) $(1 - A_iA_i^\dagger)C_i(1 - B_i^\dagger B_i) = 0$ *(*$i \in \{1, 2\}$*) and* $B_1^\dagger B_1C_1A_1A_1^\dagger = B_1^\dagger B_1C_2B_2^\dagger B_1$

If (a) or (b) is satisfied, then the general solutions to the system (1.2) has the form

$$\begin{aligned}
X &= -\frac{1}{2}A_1^\dagger C_1B_1^\dagger B_1 + \frac{1}{2}A_1^\dagger A_1Z_1B_1 + A_1^\dagger C_1 + (1 - A_1^\dagger A_1)Z_3,\\
Y &= -\frac{1}{2}A_1A_1^\dagger C_2B_2^\dagger + \frac{1}{2}A_2Z_2^*B_1B_1^\dagger + C_2B_2^\dagger + Z_4(1 - B_1B_1^\dagger),\\
Z &= -\frac{1}{2}A_2^\dagger C_2B_2^\dagger B_2 - \frac{1}{2}A_2^\dagger A_2Z_2^*B_2 + A_2^\dagger C_2 + (1 - A_2^\dagger A_2)Z_5,
\end{aligned}$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$ *satisfy*

$$B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger,$$

and $Z_3, Z_5 \in B(\mathcal{H}_1, \mathcal{H}_4)$ *and* $Z_4 \in B(\mathcal{H}_2, \mathcal{H}_3)$ *are arbitrary.*

Proof. $(a) \Rightarrow (b)$ It is clear.

$(b) \Rightarrow (a)$ In view of [10, Corollary 1.2.] we can consider the matrix forms of the operators as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A_1^*) \\ \ker(A_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1) \\ \ker(A_1^*) \end{bmatrix},\\
A_2 &= \begin{bmatrix} A_{21} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A_2^*) \\ \ker(A_2) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1) \\ \ker(A_1^*) \end{bmatrix},\\
X &= \begin{bmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(B_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_1^*) \\ \ker(A_1) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{13} & Z_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{A}_2^*) \\ \ker(A_2) \end{bmatrix}, \\
B_1 &= \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1) \\ \ker(B_1^*) \end{bmatrix}, \\
B_2 &= \begin{bmatrix} B_{21} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1) \\ \ker(B_1^*) \end{bmatrix}, \\
Y &= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{13} & Y_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1) \\ \ker(B_1^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{A}_1) \\ \ker(A_1^*) \end{bmatrix},
\end{aligned}$$

where A_{11}, A_{21}, B_{11} and B_{21} are invertible. In addition, conditions $(1 - A_iA_i^\dagger)C_i(1 - B_i^\dagger B_i) = 0$, $(i \in \{1, 2\})$ in (b) implies that $C_{14} = C_{24} = 0$. Therefore,

$$\begin{aligned}
C_1 &= \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{A}_1) \\ \ker(A_1^*) \end{bmatrix}, \\
C_2 &= \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(\mathrm{B}_1^*) \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(\mathrm{A}_1) \\ \ker(A_1^*) \end{bmatrix}.
\end{aligned}$$

Hence, the mixed Sylvester operator equations (1.2) obtain as follow.

$$\begin{cases} \begin{bmatrix} A_{11}X_{11} & A_{11}X_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{11}B_{11} & 0 \\ Y_{13}B_{11} & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix}, \\ \\ \begin{bmatrix} A_{21}Z_{11} & A_{21}Z_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{11}B_{21} & 0 \\ Y_{13}B_{21} & 0 \end{bmatrix} = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix}. \end{cases}$$

Then, the following relations hold.

$$\begin{cases} A_{11}X_{11} + Y_{11}B_{11} = C_{11}, \\ A_{21}Z_{11} + Y_{11}B_{21} = C_{21}. \end{cases} \tag{2.10}$$

$$A_{11}X_{12} = C_{12}, \tag{2.11}$$

$$A_{21}Z_{12} = C_{22}, \tag{2.12}$$

$$Y_{13}B_{11} = C_{13}, \tag{2.13}$$

$$Y_{13}B_{21} = C_{23}. \tag{2.14}$$

[10, Corollary 1.2.] implies that A_{i1}, B_{i1} for $i \in \{1, 2\}$ are invertible and also condition $B_1^\dagger B_1 C_1 A_1 A_1^\dagger = B_1^\dagger B_1 C_2 B_2^\dagger B_1$ and their matrix representations on the following forms

$$B_1^\dagger B_1 C_1 A_1 A_1^\dagger = B_1^\dagger B_1 C_2 B_2^\dagger B_1.$$

Namely,

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{13} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & 0 \end{bmatrix} \\
&\times \begin{bmatrix} B_{21}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

which is implies that $C_{11} = C_{21}B_{21}^{-1}B_{11}$.

Now, by applying Theorem 2.1, general solutions of the system (2.10) can be stated as

$$\begin{aligned} X_{11} &= \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}), \\ Y_{11} &= \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}), \\ Z_{11} &= \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}), \end{aligned}$$

where, $(Z_1)_{11}$ and $(Z_2)_{11}$ satisfy $(Z_2)_{11} = -(Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1}$.

Condition $B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger$ is equal to

$$(Z_2)_{11} = -(Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1},$$

where $Z_1 \in B(\mathcal{H}_2, \mathcal{H}_4), Z_2 \in B(\mathcal{H}_4, \mathcal{H}_2)$.

Since with rewrite their matrix representations on the following forms

$$B_1B_1^\dagger Z_2A_1^\dagger A_1 = -B_1B_1^\dagger Z_1^*A_1^*(A_2^*)^\dagger.$$

In fact,

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (Z_2)_{11} & (Z_2)_{12} \\ (Z_2)_{21} & (Z_2)_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (Z_1^*)_{11} & (Z_1^*)_{21} \\ (Z_1^*)_{12} & (Z_1^*)_{22} \end{bmatrix} \\ &\times \begin{bmatrix} A_{11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (A_{21}^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

Thus,

$$\begin{bmatrix} (Z_2)_{11} & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} (Z_1^*)_{11}A_{11}^*(A_{21}^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Eqs. (2.11) and (2.12) imply that $X_{12} = A_{11}^{-1}C_{12}$ and $Z_{12} = A_{21}^{-1}C_{22}$.

Also, the condition $(1 - B_1^\dagger B_1)C_1B_1^\dagger = (1 - B_1^\dagger B_1)C_2B_2^\dagger$ ensures that $C_{13}B_{11}^{-1} = C_{23}B_{21}^{-1}$. Therefore, Eqs. (2.13) and (2.14) are solvable and $Y_{13} = C_{13}B_{11}^{-1} = C_{23}B_{21}^{-1}$.

Hence,

$$\begin{aligned} X &= \begin{bmatrix} \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}) & A_{11}^{-1}C_{12} \\ X_{13} & X_{14} \end{bmatrix}, \\ Y &= \begin{bmatrix} \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}) & Y_{12} \\ C_{23}B_{21}^{-1} & Y_{14} \end{bmatrix}, \end{aligned}$$

and

$$Z = \begin{bmatrix} \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}) & A_{21}^{-1}C_{22} \\ Z_{13} & Z_{14} \end{bmatrix},$$

X_{13}, X_{14}, Y_{12}, Y_{14}, Z_{13} and Z_{14} can be taken arbitrary.

By using the matrix forms, we get

$$\begin{aligned}
\frac{1}{2}(A_1^\dagger C_1 B_1^\dagger B_1 + A_1^\dagger A_1 Z_1 B_1) &= \begin{bmatrix} \frac{1}{2}(A_{11}^{-1}C_{11} + (Z_1)_{11}B_{11}) & 0 \\ 0 & 0 \end{bmatrix}, \\
A_1^\dagger C_1(1 - B_1^\dagger B_1) &= \begin{bmatrix} 0 & A_{11}^{-1}C_{12} \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

By taking $Z_3 = \begin{bmatrix} Z_{31} & Z_{32} \\ X_{13} & X_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1^*)} \\ \ker(A_1) \end{bmatrix}$ we conclude $(1 - A_1^\dagger A_1)Z_3 = \begin{bmatrix} 0 & 0 \\ X_{13} & X_{14} \end{bmatrix}$. Then

$$X = \frac{1}{2}(A_1^\dagger C_1 B_1^\dagger B_1 + A_1^\dagger A_1 Z_1 B_1) + A_1^\dagger C_1(1 - B_1^\dagger B_1) + (1 - A_1^\dagger A_1)Z_3.$$

Also,

$$\frac{1}{2}(A_1 A_1^\dagger C_2 B_2^\dagger + A_2 Z_2^* B_1 B_1^\dagger) = \begin{bmatrix} \frac{1}{2}(C_{21}B_{21}^{-1} + A_{21}(Z_2^*)_{11}) & 0 \\ 0 & 0 \end{bmatrix},$$

$$(1 - A_1 A_1^\dagger)C_2 B_2^\dagger = \begin{bmatrix} 0 & 0 \\ C_{23}B_{21}^{-1} & 0 \end{bmatrix}.$$

By taking $Z_4 = \begin{bmatrix} Z_{41} & Y_{12} \\ Z_{43} & Y_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1)} \\ \ker(B_1^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1)} \\ \ker(A_1^*) \end{bmatrix}$, we derive $Z_4(1 - B_1 B_1^\dagger) = \begin{bmatrix} 0 & Y_{12} \\ 0 & Y_{14} \end{bmatrix}$. Then

$$Y = \frac{1}{2}(A_1 A_1^\dagger C_2 B_2^\dagger + A_2 Z_2^* B_1 B_1^\dagger) + (1 - A_1 A_1^\dagger)C_2 B_2^\dagger + Z_4(1 - B_1 B_1^\dagger).$$

By using the matrix forms, we get

$$\begin{aligned}
\frac{1}{2}(A_2^\dagger C_2 B_2^\dagger B_2 - A_2^\dagger A_2 Z_2^* B_2) &= \begin{bmatrix} \frac{1}{2}(A_{21}^{-1}C_{21} - (Z_2^*)_{11}B_{21}) & 0 \\ 0 & 0 \end{bmatrix}, \\
A_2^\dagger C_2(1 - B_2^\dagger B_2) &= \begin{bmatrix} 0 & A_{21}^{-1}C_{22} \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

By taking $Z_5 = \begin{bmatrix} Z_{51} & Z_{52} \\ Z_{13} & Z_{14} \end{bmatrix} : \begin{bmatrix} \mathrm{ran(B_1^*)} \\ \ker(B_1) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(A_1^*)} \\ \ker(A_1) \end{bmatrix}$, we conclude $(1 - A_2^\dagger A_2)Z_5 = \begin{bmatrix} 0 & 0 \\ Z_{13} & Z_{14} \end{bmatrix}$. Then

$$Z = \frac{1}{2}(A_2^\dagger C_2 B_2^\dagger B_2 - A_2^\dagger A_2 Z_2^* B_2) + A_2^\dagger C_2(1 - B_2^\dagger B_2) + (1 - A_2^\dagger A_2)Z_5.$$

□

In the following theorem, consider the solvability and the expressions of the general solutions to the following systems of four coupled one sided Sylvester-type operator equations.

Theorem 2.4. *Suppose that $\mathcal{H}$ is Hilbert space and where $A_i, B_i, C_i \in B(\mathcal{H})$ ($i \in \{1,2,3,4\}$) are given operators such that $C_3 = A_2C_2B_3^{-1}$ and $X_1, ..., X_5 \in B(\mathcal{H})$ are unknowns operator A_i, B_i ($i \in \{1,2,3,4\}$) are invertible operators. Then the following statements are equivalent:*

(a) The system

$$\begin{cases} A_1X_1 + X_2B_1 = C_1, \\ A_2X_3 + X_2B_2 = C_2, \\ A_3X_4 + X_3B_3 = C_3, \\ A_4X_4 + X_5B_4 = C_4, \end{cases} \tag{2.15}$$

is solvable,

(b) $C_1 = C_3B_2^{-1}B_1$ and $C_4^* = C_2^*(A_3^*)^{-1}A_4^*$.

In which case, the general solution to the system (2.15) are of the form

$$\begin{aligned} X_1 &= \frac{1}{2}(A_1^{-1}C_1 + Z_1B_1), \\ X_2 &= \frac{1}{2}(C_3B_2^{-1} + A_2Z_4^*), \\ X_3 &= \frac{1}{2}(A_2^{-1}C_3 - Z_4^*B_2), \\ X_4 &= \frac{1}{2}(A_3^{-1}C_2 + Z_3B_3^*), \\ X_5 &= \frac{1}{2}(C_4B_4^{-1} + A_4Z_2^*), \end{aligned}$$

where $Z_1, Z_2, Z_3, Z_4 \in B(\mathcal{H})$ satisfy $Z_3 = -Z_2^*B_4B_3^{-1}$, $Z_4 = -Z_1^*A_1^*(A_2^*)^{-1}$ and $Z_3 = A_3^{-1}Z_4^*B_2$.

Proof. By taking $T_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_4^* \end{bmatrix}$, $T_2 = \begin{bmatrix} A_2 & 0 \\ 0 & B_3^* \end{bmatrix}$, $S_1 = \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}$, $S_2 = \begin{bmatrix} A_3^* & 0 \\ 0 & B_2 \end{bmatrix}$, $U_1 = \begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix}$ and $U_2 = \begin{bmatrix} 0 & C_2 \\ C_3^* & 0 \end{bmatrix}$ that are given operators and $X = \begin{bmatrix} 0 & X_1 \\ X_5^* & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & X_2 \\ X_4^* & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 0 & X_3 \\ X_3^* & 0 \end{bmatrix}$ are unknowns operators. Hence system (2.15) get into

$$\begin{cases} T_1X + YS_1 = U_1, \\ T_2Z + YS_2 = U_2, \end{cases} \tag{2.16}$$

Condition (b) is equal to
$\begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_3^*)^{-1} & 0 \\ 0 & (B_2)^{-1} \end{bmatrix} \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}$. By applying Theorem 2.1, implies that system 2.15 are solvable, then any solutions have the following form

$$\begin{aligned}
\begin{bmatrix} 0 & X_1 \\ X_5^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} A_1^{-1} & 0 \\ 0 & (B_4^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_4^* & 0 \end{bmatrix} + W_1 \begin{bmatrix} A_4^* & 0 \\ 0 & B_1 \end{bmatrix}),\\
\begin{bmatrix} 0 & X_2 \\ X_4^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_3^*)^{-1} & 0 \\ 0 & (B_2)^{-1} \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & B_3^* \end{bmatrix} W_2^*),\\
\begin{bmatrix} 0 & X_3 \\ X_3^* & 0 \end{bmatrix} &= \frac{1}{2}(\begin{bmatrix} A_2^{-1} & 0 \\ 0 & (B_3^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C_3 \\ C_2^* & 0 \end{bmatrix} - W_2^* \begin{bmatrix} A_3^* & 0 \\ 0 & B_2 \end{bmatrix}),
\end{aligned}$$

where $W_1 = \begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0 & Z_3 \\ Z_4 & 0 \end{bmatrix}$.
Which is satisfy that $W_2 = -W_1^* T_1^* (T_2^*)^{-1}$ that is,
$\begin{bmatrix} 0 & Z_3 \\ Z_4 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & Z_2^* \\ Z_1^* & 0 \end{bmatrix} \begin{bmatrix} A_1^* & 0 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} (A_2^*)^{-1} & 0 \\ 0 & (B_3)^{-1} \end{bmatrix}$ that
$Z_3 = -Z_2^* B_4 B_3^{-1}$ and $Z_4 = -Z_1^* A_1^* (A_2^*)^{-1}$. Since, $C_3 = A_2 C_2 B_3^{-1}$ and Z_3, Z_4 satisfy $Z_3 = A_3^{-1} Z_4^* B_2$. Therefore,

$$\begin{aligned}
X_1 &= \frac{1}{2}(A_1^{-1} C_1 + Z_1 B_1),\\
X_2 &= \frac{1}{2}(C_3 B_2^{-1} + A_2 Z_4^*),\\
X_3 &= \frac{1}{2}(A_2^{-1} C_3 - Z_4^* B_2),\\
X_3^* &= \frac{1}{2}((B_3^*)^{-1} C_2^* - Z_3^* A_3^*),\\
X_4^* &= \frac{1}{2}(C_2^* (A_3^*)^{-1} + B_3 Z_3^*),\\
X_5^* &= \frac{1}{2}((B_4^*)^{-1} C_4^* + Z_2 A_4^*).
\end{aligned}$$

□

3. Conclusion

We have used the block operator matrix decomposition to find the general solutions of mixed Sylvester operator equations with three unknowns (1.2) and five unknowns (2.15) . We have provided some necessary and sufficient conditions for the existence of a solution to this system based on matrix representation. We have also derived the general solution to this system when it is solvable.

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TRANSLATION-FAVORABLE FLAT SURFACES IN 3-SPACES

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Abstract. In the paper, we obtain the complete classification of Translation-Factorable (TF-) surfaces with vanishing Gaussian curvature in Euclidean and Minkowski 3-spaces.
Keywords: flat surfaces, Gaussian curvatures, 3-spaces

1. Introduction

In the study of the differential geometries of surfaces in 3-spaces, it is the most popular to examine curvature properties or the relationships between the corresponding curvatures of them. Let M be a surface in 3-spaces and (x, y, z) rectangular coordinates. It is well known that M is called as translation or factorable (homothetical) surface if it is locally described as the graph of $z = f(x) + g(y)$ or $z = f(x)g(y)$, respectively. Translation surfaces having constant mean curvature (CMC) or constant Gaussian curvature (CGC) in 3-spaces have been studied in [1, 4, 15, 16, 22, 23]. Furthermore, translation surfaces in 3-spaces satisfying Weingarten condition have been studied by Dillen et. all in [10], by Sipus in [22] and also by Sipus and Dijvak in [23]. On the other hand, factorable (homothetical) surfaces whose curvatures satisfy certain conditions have been investigated in [2, 3, 17]. As an exception, surfaces with vanishing curvature have been also very much focused. It is well known that M is called as flat or minimal surface if the Gaussian curvature or the mean curvature vanishes, respectively. The study of flat or minimal surfaces have found many applications in differential geometry

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and also physics, (see in [5, 11, 24, 25]). Very recently, as a generalization of these surfaces, Difi, Ali and Zoubir described a new type surfaces called with translation-factorable (TF) surfaces in Euclidean 3-space in [9]. Moreover, author investigated these surfaces in Galilean 3-spaces, in [14]. In that paper, authors studied on the position vector of this new type surface in the 3-dimensional Euclidean space and Lorentzian-Minkowski space satisfying the special condition $\Delta r_i = \lambda_i r_i$, where Δ denotes the Laplace operator.

The main interest of this paper is to obtain the complete classification of Translation-Factorable (TF-) surfaces with vanishing Gaussian curvatures in 3-spaces, starting from this new type of surface, called as Translation-Factorable (TF-) surfaces, defined in [9]. In Sect. 2, we introduce the notations that we are going to use and give a brief summary of basic definitions in theory of surfaces in Euclidean and Minkowski 3-spaces. In Sect. 3 and 4, we give the complete classification of TF-flat surfaces in the Euclidean 3-space and Minkowski 3-space, respectively.

2. Preliminiaries

Let Euclidean and Minkowski 3-spaces denote with $\mathbb{E}^3$ and $\mathbb{E}^3_1$, respectively. One may introduce an euclidean and Lorentzian inner products between $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ as

$$\langle u, v\rangle = (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2 \quad \text{and} \quad \langle u, v\rangle_L = (d\xi_0)^2 + (d\xi_1)^2 - (d\xi_2)^2.$$

Here (ξ_0, ξ_1, ξ_2) is rectangular coordinate system of 3-spaces. These inner products induce in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ a norm in a natural way:

$$\|u\| = \sqrt{|\langle u, u\rangle|} \quad \text{and} \quad \|u\|_L = \sqrt{|\langle u, u\rangle|_L},$$

respectively. In addition, the corresponding cross products in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ shall be showed here by $\wedge$ and $\wedge_L$, respectively: notice that $\wedge_L$ should be computed as

$$u \wedge_L v = e_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - e_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} - e_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Let M^2 be a surface in $\mathbb{E}^3$ or $\mathbb{E}^3_1$. If M^2 is parameterized by an immersion

$$x(u^1, u^2) = \Big(x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\Big),$$

then M^2 is a regular surface if and only if the corresponding cross products of x_1 and x_2 don't vanish anywhere. Here, $x_k = \partial x/\partial u^k$, $k = 1, 2$. So, the normal vector field $\mathbf{N}$ of a regular surface M^2 in $\mathbb{E}^3$ or $\mathbb{E}^3_1$ is given by

$$\text{(2.1)} \qquad \mathbf{N} = \frac{x_1 \wedge x_2}{\|x_1 \wedge x_2\|} \quad \text{or} \quad \mathbf{N}_L = \frac{x_1 \wedge_L x_2}{\|x_1 \wedge_L x_2\|_L}.$$

The first fundamental form of $x : U \longrightarrow M^2 \subset \mathbb{E}^3$ (or $\mathbb{E}^3_1$) is defined as:

$$\text{(2.2)} \qquad I = g_{ij} du^i du^j, \quad g_{ij} = \langle x_i, x_j\rangle \quad \text{or} \quad g_{ij} = \langle x_i, x_j\rangle_L.$$

The second fundamental form II in simply and pseudo-isotropic spaces is with differentiable coefficients

$$II = h_{ij} du^i du^j, \quad h_{ij} = \langle \mathbf{N}, x_{ij} \rangle \quad \text{or} \quad h_{ij} = \langle \mathbf{N}, x_{ij} \rangle_L . \tag{2.3}$$

Therefore, the Gaussian curvature K and the mean curvature H of surface Σ are defined by, respectively,

$$K = \frac{h_{11}h_{22} - {h_{12}}^2}{W^2}, \tag{2.4}$$

$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2W^2}, \tag{2.5}$$

where $W = \sqrt{|g_{11}g_{22} - {g_{12}}^2|}$. Note that if $g_{11}g_{22} - {g_{12}}^2 < 0$ or $g_{11}g_{22} - {g_{12}}^2 > 0$, then the surface M^2 in $\mathbb{E}_1^3$ is called as time-like or space-like surface, respectively.

Now, first we would like to give the definition of the translation-factorable (TF-) surfaces in $\mathbb{E}^3$ defined in [9]. And then we would like to complete the definition of translation-factorable (TF-) surfaces in $\mathbb{E}_1^3$ given in same paper as follows:

Definition 2.1. Let M^2 be a surface in Euclidean 3-space. Then M is called a translation-factorable (TF-) surface if it can be locally written as following:

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))), \tag{2.6}$$

where f and g are some real functions and A, B are non-zero constants.

Definition 2.2. Let M^2 be a surface in Minkowski 3-space, $\mathbb{E}_1^3$. Then M is called a translation-factorable (TF-) surface if it can be locally written as one of the followings:

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))), \tag{2.7}$$

or

$$x(s,t) = (A(f(s) + g(t)) + B(f(s)g(t)), s, t), \tag{2.8}$$

which are called as first and second type and where f and g are some real functions and A, B are non-zero constants.

Remark 2.1. From Definition 2.2, one can be directly seen when taking $A = 0$ and $B \neq 0$, then surface becomes a factorable surface studied in [17]. On the other hand, if one can take $B = 0$ and $A \neq 0$, then surface is a translation surface studied in [15].

3. Classification of Translation-Factorable surfaces with vanishing Gaussian curvature in $\mathbb{E}^3$

As mentioned in the previous section, the TF-surfaces can be parametrized as in (2.6) in Euclidean 3-spaces. In this section, we calculate the Gaussian curvature for the TF-surfaces in $\mathbb{E}^3$. And then, we examine when it vanishes. Finally, we give the complete classification of of the TF-surfaces with vanishing Gaussian curvatures.

Let M^2 be a TF-surface in Euclidean 3-space, $\mathbb{E}^3$. Hence it can be parametrized as

$$x(s,t) = (s, t, B(f(s)g(t)) + A(f(s) + g(t))) \,. \tag{3.1}$$

Thus, the partial derivatives and $\mathbf{N}$, the unit normal vector field defined by (2.1) of this type surface are obtained by

$$\begin{aligned} x_s &= (1, 0, (Bg(t) + A)f'(s)), && (3.2)\\ x_t &= (0, 1, g'(t)(Bf(s) + A)), && (3.3)\\ \mathbf{N} &= \frac{1}{W}(-f'(s)(Bg(t) + A), -g'(t)(Bf(s) + A), 1). && (3.4)\end{aligned}$$

Here $W = \sqrt{1 + g'(t)^2(Bf(s) + A)^2 + f'(s)^2(Bg(t) + A)^2}$ and by $\prime$, we have denoted derivatives with respect to corresponding parameters. For readability, here and in the rest of the paper, we will lower the parameters of the $f(s)$ and $g(t)$ functions. Now, by considering the above into the second equalities in (2.2) and (2.3), respectively, we get

$$\begin{aligned} g_{11} &= 1 + f'^2(Bg + A)^2,\\ g_{12} &= g'f'(Bf + A)(Bg + A),\\ g_{22} &= 1 + g'^2(Bf + A)^2, \end{aligned} \tag{3.5}$$

and

$$h_{11} = \frac{f''(Bg + A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf + A)}{W}, \tag{3.6}$$

where $W^2 = 1 + g'^2(Bf + A)^2 + f'^2(Bg + A)^2$. Hence, by substituting of the last two statements into (2.4) gives

$$K = \frac{f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2}{1 + g'^2(Bf + A)^2 + f'^2(Bg + A)^2} \tag{3.7}$$

where f and g are some real functions and A, B are non-zero constants.

Now, we would like to investigate the vanishing Gaussian curvature problem for TF-surfaces in $\mathbb{E}^3$. As well known, the surfaces with vanishing Gaussian curvature are called flat. Now, we examine TF- flat surface in Euclidean 3-space, whose Gaussian curvature is identically zero. Then the following classification theorem is valid.

Theorem 3.1. *Let M^2 be a TF-surface defined by* (3.1) *in the Euclidean 3-space. Then, M^2 is a flat surface if and only if it can be parametrized as one of the followings:*

1. *M^2 is a part of a plane,*
2. *M^2 is a regular surface in $\mathbb{E}^3$ parametrized by*

$$x(s,t) = (s, t, g(t)(Bc + A) + Ac)\,, \tag{3.8}$$

where $f = c$ is a constant function or

$$x(s,t) = (s, t, f(s)(Bc + A) + Ac) \tag{3.9}$$

where $g = c$ is a constant function.

3. *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{3.10}$$

4. *f and g are given by*

$$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}}. \end{aligned} \tag{3.11}$$

Proof. Let M^2 be the TF- flat surface. Thus, from (3.7), it is clear that is sufficient that

$$f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2 = 0. \tag{3.12}$$

Let us consider on the following possibilities:

Case (1): $f' = 0$ and $g' = 0$. Then, the equation (3.12) is trivially satisfied. By considering these assumptions in (3.1), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1) of Theorem 3.1.

Case (2): $f' = 0$ or $g' = 0$. First, assume that $f' = 0$, i.e., f be constant. In case, the equation (3.12) is trivially satisfied. But, in case g is a arbitrary smooth function. Thus, we get (3.8). Similarly, by considering the assumption of g as $g' = 0$, we can get (3.9) in Theorem 3.1.

Case (3): Let $f'' = 0$ or $g'' = 0$, but not both. First, assume that $f'' = 0$, i.e., f be a linear function. In this case, one get $g' = 0$ to provide the equation (3.12). Second, let $g'' = 0$. Then by the similar way, $f' = 0$ must be. Note that one can easily see that these cases are covered by Case (2).

Case (4): Let f', g', f'' and g'' be non-zero. Then, the equation (3.12) can be rewritten as

$$\frac{f''(A + Bf)}{B(f')^2} = \frac{B(g')^2}{g''(A + Bg)} = C, \tag{3.13}$$

for non-zero constant C. We are going to consider the following cases seperately:

Case (4a): $C = 1$. In this case (3.13) implies that

$$f''(A + Bf) = B(f')^2 \quad \text{and} \quad B(g')^2 = g''(A + Bg), \tag{3.14}$$

from which, we get (3.10) in Case (3) in Theorem 3.1.

Case (4b): $C \neq 1$. In this case we solve (3.13) to obtain (3.11).

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 3.1 vanishes identically. □

4. Classification of Translation-Factorable surfaces with vanishing Gaussian curvature in $\mathbb{E}^3_1$

In this section, we study two types of TF-surfaces in the 3-dimensional Minkowski space. Let M^2 be a TF-surface parametrized in (2.7) or (2.8) in Minkowski 3-spaces. Namely, M^2 can be parametrized as

$$x(s,t) = (s, t, A(f(s) + g(t)) + Bf(s)g(t)), \tag{4.1}$$

or

$$x(s,t) = (A(f(s) + g(t)) + Bf(s)g(t), s, t), \tag{4.2}$$

which are called as first and second type TF-surfaces .

First, we would like to consider on the type I TF-surface parametrized as in (4.1). Thus, we have,

$$x_s = (1, 0, f'(A + Bg)), \tag{4.3}$$

$$x_t = (0, 1, g'(A + Bf)). \tag{4.4}$$

Also, $\mathbf{N}_L$ the unit normal vector field of M^2 defined by (2.1) is given by

$$\mathbf{N}_L = \frac{1}{W}(f'(A + Bg), -g'(A + Bf), 1). \tag{4.5}$$

Here with $'$, we have denoted derivatives with respect to corresponding parameters and

$$W = \sqrt{\left|1 - g'^2(A + Bf)^2 - f'^2(A + Bg)^2\right|}. \tag{4.6}$$

By considering (4.3), (4.4) and (4.5) into the third equalities in (2.2) and (2.3), respectively, we obtain

(4.7)
$$g_{11} = 1 - f'^2(A + Bg)^2, \quad g_{12} = -f'g'(A + Bf)(A + Bg), \quad g_{22} = 1 - g'^2(A + Bf)^2,$$

and

$$h_{11} = \frac{f''(Bg + A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf + A)}{W}. \tag{4.8}$$

Thus, by substituting of these above statements into (2.4) gives

$$K_L = \frac{f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2}{W^4} \tag{4.9}$$

where f and g are some real functions, A, B are non-zero constants and W is given as in (4.6).

Now, we would like to give the following theorem being the classification of type I TF-surfaces with vanishing Gaussian curvature in $\mathbb{E}^3_1$.

Theorem 4.1. *Let M^2 be a type I TF-surface defined by* (4.1) *in the Minkowski 3-space. Then,*

1. *M^2 is a type I space-like flat surface if and only if it can be parametrized as one of the followings:*
   (a) *M^2 is a part of a plane,*
   (b) *M^2 is a space-like surface in $\mathbb{E}^3_1$ parametrized by*

   $$x(s,t) = (s, t, g(t)(A + Bc) + Ac)\,, \tag{4.10}$$

   where $f = c$ is a constant function and $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$ or

   $$x(s,t) = (s, t, f(s)(A + Bc) + Ac) \tag{4.11}$$

   where $g = c$ is a constant function and $\frac{-1}{A+Bc} < f' < \frac{1}{A+Bc}$.
   (c) *f and g are given by*

   $$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}, \tag{4.12}$$

   such that satisfy the condition (4.18).
   (d) *f and g are given by*

   $$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \end{aligned} \tag{4.13}$$

   such that satisfy the condition (4.18).
2. *M^2 is a type I time-like flat surface if and only if it can be parametrized as one of the followings:*
   (a) *M^2 is a time-like surface in $\mathbb{E}^3_1$ parametrized by*

   $$x(s,t) = (s, t, g(t)(Bc + A) + Ac)\,, \tag{4.14}$$

   where $f = c$ is a constant function or

   $$x(s,t) = (s, t, f(s)(Bc + A) + Ac) \tag{4.15}$$

   where $g = c$ is a constant function.

(b) f and g are given by

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{4.16}$$

(c) f and g are given by

$$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}}. \end{aligned} \tag{4.17}$$

Proof. Let M^2 be a type I TF- flat surface. First, let M^2 be a type I space-like surface. Then from (4.6), we have

$$g'^2(A + Bf)^2 + f'^2(A + Bg)^2 < 1. \tag{4.18}$$

Since M^2 is a flat surface, then from (4.9), it is clear that is sufficient that

$$f''g''(A + Bf)(A + Bg) - B^2(f')^2(g')^2 = 0. \tag{4.19}$$

Let us consider on the following possibilities:

Case (1): $f' = 0$ and $g' = 0$. Then, the equation (4.18) and (4.19) are trivially satisfied. By considering these assumptions in (4.1), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1a) of Theorem 4.1.

Case (2): $f' = 0$ or $g' = 0$. First, assume that $f' = 0$, i.e., f be a constant. In case, the equation (4.19) is trivially satisfied and also from (4.18) yields g is satisfied $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$. Thus, we get (4.10). Similarly, by considering the assumption of g as $g' = 0$, we can get (4.11) in Theorem 4.1.

Case (3): Let $f'' = 0$ or $g'' = 0$, but not both. First, assume that $f'' = 0$, i.e., $f' = c_1$ and $f = c_1 s + c_2$ be a linear function. In this case, one get $g' = 0$, namely $g = C_1$, to provide the equation (4.19). Thus, from (4.18), we get the condition $1 < c_1^2 C_1^2$. Second, let $g'' = 0$. Then by the similar way, $f' = 0$ must be. Note that one can easily see that these cases are covered by Case (1b).

Case (4): Let f', g', f'' and g'' be non-zero. Then, the equation (4.19) can be rewritten as

$$\frac{f''(A + Bf)}{B(f')^2} = \frac{B(g')^2}{g''(A + Bg)} = C, \tag{4.20}$$

for non-zero constant C. We are going to consider the following cases seperately:

Case (4a): $C = 1$. In this case (4.20) implies that

$$f''(A + Bf) = B(f')^2 \quad \text{and} \quad B(g')^2 = g''(A + Bg), \tag{4.21}$$

from which, we get (4.12) in Case (1c) in Theorem 4.1.

Case (4b): $C \neq 1$. In this case we solve (4.20) to obtain (4.13).

Secondly, let M^2 be a type I time-like surface in $\mathbb{E}^3_1$. Then from (4.6), we have

$$g'^2(A+Bf)^2 + f'^2(A+Bg)^2 > 1. \tag{4.22}$$

In view of this condition, the proof of the second case can be made similar to the previous case.

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 4.1 vanishes identically. □

Now, secondly let M^2 be a type II TF-surfaces given as in (4.2). Thus, we have,

$$x_s = (f'(A+Bg), 1, 0), \tag{4.23}$$

$$x_t = (g'(A+Bf), 0, 1). \tag{4.24}$$

Also, $\mathbf{N}_L$ the unit normal vector field of M^2 defined by (2.1) is given by

$$\mathbf{N}_L = \frac{1}{W}(1, -f'(A+Bg), g'(A+Bf)). \tag{4.25}$$

Here with $'$, we have denoted derivatives with respect to corresponding parameters and

$$W = \sqrt{\left|1 + f'^2(A+Bg)^2 - g'^2(A+Bf)^2\right|}. \tag{4.26}$$

By considering (4.23), (4.24) and (4.25) into the third equalities in (2.2) and (2.3), respectively, we obtain

(4.27)

$$g_{11} = 1 + f'^2(A+Bg)^2, \quad g_{12} = f'g'(A+Bf)(A+Bg), \quad g_{22} = g'^2(A+Bf)^2 - 1,$$

and

$$h_{11} = \frac{f''(Bg+A)}{W}, \quad h_{12} = \frac{Bf'g'}{W}, \quad h_{22} = \frac{g''(Bf+A)}{W}. \tag{4.28}$$

Thus, by substituting of these above statements into (2.4) gives

$$K_L = \frac{f''g''(Bf+A)(Bg+A) - B^2(f')^2(g')^2}{W^4} \tag{4.29}$$

where f and g are some real functions, A, B are non-zero constants and W is given as in (4.26). As well knowing that if M^2 is a space-like surface then, from (4.26) yields

$$g'^2(A+Bf)^2 - f'^2(A+Bg)^2 < 1. \tag{4.30}$$

On the other hand, if M^2 is a time-like surface then, from (4.26) yields

$$g'^2(A+Bf)^2 - f'^2(A+Bg)^2 > 1. \tag{4.31}$$

Now we would like to give the following theorem being the classification of type II TF-flat surfaces in $\mathbb{E}^3_1$.

Theorem 4.2. *Let M^2 be a type II TF-surface defined by* (4.2) *in the Minkowski 3-space. Then,*

1. *M^2 is a type II space-like flat surface if and only if it can be parametrized as one of the followings:*
   (a) *M^2 is a part of a plane,*
   (b) *M^2 is a space-like surface in $\mathbb{E}^3_1$ parametrized by*

$$x(s,t) = (s, t, g(t)(A + Bc) + Ac)\,, \tag{4.32}$$

where $f = c$ is a constant function and $\frac{-1}{A+Bc} < g' < \frac{1}{A+Bc}$ or

$$x(s,t) = (s, t, f(s)(A + Bc) + Ac) \tag{4.33}$$

where $g = c$ is a constant function and $0 < f'^2(A + Bc)^2 + 1$.

   (c) *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s+c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t+c_2)} + \frac{A}{B}, \tag{4.34}$$

such that satisfy the condition (4.30).

   (d) *f and g are given by*

$$\begin{aligned} f(s) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \\ g(t) &= -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \end{aligned} \tag{4.35}$$

such that satisfy the condition (4.30).

2. *M^2 is a type I time-like flat surface if and only if it can be parametrized as one of the followings:*
   (a) *M^2 is a time-like surface in $\mathbb{E}^3_1$ parametrized by*

$$x(s,t) = (s, t, g(t)(Bc + A) + Ac)\,, \tag{4.36}$$

where $f = c$ is a constant function or

$$x(s,t) = (s, t, f(s)(Bc + A) + Ac) \tag{4.37}$$

where $g = c$ is a constant function.

   (b) *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s+c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t+c_2)} + \frac{A}{B}, \tag{4.38}$$

such that satisfy the condition (4.31).

(c) f and g are given by

$$f(s) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 s + c_2)\Big)^{\frac{1}{1-C}},$$
$$g(t) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((C-1)(c_1 t + c_2)\Big)^{\frac{1}{1-C}} \tag{4.39}$$

such that satisfy the condition (4.31)*.*

Proof. In view of the condition (4.6), the proof of this theorem can be made similar to the previous Theorem 4.1. □

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TANGENT BUNDLES ENDOWED WITH SEMI-SYMMETRIC NON-METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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Abstract. The differential geometry of the tangent bundle is an effective domain of differential geometry which reveals many new problems in the study of modern differential geometry. The generalization of connection on any manifold to its tangent bundle is an application of differential geometry. Recently a new type of semi-symmetric non-metric connection on a Riemannian manifold has been studied and a relationship between Levi-Civita connection and semi-symmetric non-metric connection has been established. The various properties of a Riemannian manifold with relation to such connection have also been discussed. The present paper aims to study the tangent bundle of a new type of semi-symmetric non-metric connection on a Riemannian manifold. The necessary and sufficient conditions for projectively invariant curvature tensors corresponding to such connection are proved and show many basic results on the Riemannian manifold in the tangent bundle. Furthermore, the properties of group manifolds of the Riemannian manifolds with respect to the semi-symmetric non-metric connection in the tangent bundle have been studied. Moreover, theorems on the symmetry property of Ricci tensor and Ricci soliton in the tangent bundle are established.
Keywords: Tangent bundle, Vertical and complete lifts, Riemannian manifold, semi-symmetric non-metric connection, Different curvature tensors.

1. Introduction

The concept of semi-symmetric linear connection on a differential manifold was introduced by Friedman and Schouten [8] in 1924. Hayden introduced the notion

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of metric connection on a Riemannian manifold in 1932 and known as Hayden connection [10].

Let M^n be a Riemannian manifold of n-dimensional with Riemannian metric g and ∇ be Levi-Civita connection on it. A linear connection $\tilde{\nabla}$ on M^n is said to be symmetric connection if its torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is of the form

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

is zero for all X and Y on M^n; otherwise it is non-symmetric . A linear connection $\tilde{\nabla}$ is said to be semi-symmetric connection if

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y \tag{1.2}$$

where

$$\pi(X) = g(P,X) \tag{1.3}$$

for all X and Y on M^n and π is 1-form and P is a vector field.

In 1969, Pak [18] studied the Hayden connection $\tilde{\nabla}$ and proved that it is a semi-symmetric metric and a linear connection $\tilde{\nabla}$ is said to be metric on M^n if $\tilde{\nabla}g = 0$ otherwise it is non-metric. In 1970, Yano [23] studied some curvature and derivational conditions for semi-symmetric connection in Riemannian manifolds. Agashe et al define a linear connection on a Riemannian manifold M^n which is semi-symmetric but non-metric in 1992 and studied some properties of the curvature tensor with respect to semi-symmetric non-metric connection [1]. In 1994, Liang [16] studied a type of semi-symmetric non-metric connection $\tilde{\nabla}$ which satiesfies $(\hat{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, u is 1-form and such connection called a semi-symmetric recurrent metric connection. In 2019, Chaubey at el [3] defined and studied a new type of semi-symmetric non-metric connection on a Riemannian manifold. Studies of various types of semi-symmetric non-metric connection and their properties include [2, 4, 5, 6, 9, 12, 15, 17, 19] and others.

In a Riemannian manifold of dimension n, the curvature tensor $\tilde{R}$ corresponding to $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y) = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \tag{1.4}$$

for all X, Y, Z om M_n.

The Ricci tensor $\tilde{S}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ is given by [3]

$$\begin{aligned} \tilde{S}(Y,Z) &= S(Y,Z) + \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i,Z)g(Y,e_i)) - \theta(Y,Z)g(e_i,e_i) \\ &- (g(Ae_i,Y)g(Z,e_i)) + (g(AY,e_i)g(Z,e_i))\} \end{aligned} \tag{1.5}$$

where S is a Ricci tensor with respect to ∇.

The projective curvature $\tilde{P}$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$ is defined as [7]

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{1.6}$$

for all vector fields X, Y and Z on M_n.

The conformal curvature tensor C [22] with respect to ∇ is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.7}$$

for arbitrary vector fields X, Y, Z on M_n.

The concircular curvature tensor $\breve{C}$ [3] on (M_n, g) with respect to ∇ is defined by

$$\begin{aligned} '\breve{C}(X,Y,Z,U) &= {}'R(X,Y,Z,U) \\ &- \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \end{aligned} \tag{1.8}$$

The conharmonic curvature tensor $'L$ of type (0,4) [3] is defined by

$$\begin{aligned} 'L(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{r}{n-2}\{S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)S(X,U) - g(X,Z)S(Y,U)\} \end{aligned} \tag{1.9}$$

On the other hand, the differential geometry of tangent bundles is an important domain of the differential geometry because the theory provides many new problems in the study of modern differential geometry. The theory of vertical, complete and horizontal lifts of geometrical structures and connections from a manifold to its tangent bundles was developed by Yano and Ishihara [24]. They defined and studied prolongations called vertical, complete and horizontal lifts and connections. Tani [21] developed the theory of surfaces prolonged to tangent bundle with respect to the metric tensor of the original manifold.

Most recently, the author [13, 14] studied tangent bundle endowed with respect to semy-symmetric non-metric connection on Kähler manifold and tangent bundle of an almost Hermitian manifold and an almost Kähler manifold with respect to quarter symmetric non-metric connection. Motivated by the previously mentioned studies, we study the tangent bundles of a new type of semi-symmetric non-metric connection on a Riemannian manifold.

The main contributions are summarized as follows:

- A new type of semi-symmetric non-metric connection is defined and studied on a Riemannian manifold to the tangent bundle.

- To prove the existence of such a connection on the tangent bundle and some theorems on it.
- Various curvature tensors such as projective, conformal and concircular curvature tensors corresponding to semi-symmetric non-metric connection on the tangent bundle are calculated.
- Symmetric property of Ricci tensor are established.
- To define Ricci soliton on the tangent bundle and discuss shrinking, steady and expanding properties of it.

The paper is organized as follows: Section 2 deals with a brief account of tangent bundle, vertical lift, complete lift and a new class of semi-symmetric non-metric connection. Section 3 presents semi-symmetric non-metric connection in the tangent bundle TM_n over a Riemannian manifold M_n and proves some basic results. Section 4 discusses the relation between curvature tensors of the Levi-Civita and semi-symmetric non-metric connections in the tangent bundle and some basic properties of the curvature tensor of $\tilde{\nabla}^C$. It is proved that such connection on a Riemannian manifold is projectively invariant curvature tensors under certain conditions and also proves some results on the curvature, concircular curvature, and conharmonic curvature tensors in the tangent bundle. Finally, Section 5 devotes the study of a group manifold with respect to a semi-symmetric non-metric connection in the tangent bundle. The symmetric property of Ricci tensor and Ricci soliton in the tangent bundle are established.

2. Preliminaries

Let M_n be an n-dimensional differentiable manifold and TM_n its tangent bundle. The projection bundle $\pi_{M_n} : TM_n \to M_n$ which denotes the natural bundle structure of TM_n over M_n. Let $\{U; x^i\}$ be coordinate neighborhood in M_n where $\{x^i\}$ is a system of local coordinates in neighborhood U. Let $\{x^i, y^i\}$ be a system of local coordinates in $\pi^{-1}_{M_n}(U) \subset TM_n$ i.e. $\{x^i, y^i\}$ the induced coordinate in $\pi^{-1}_{M_n}(U)$. Let $\wp^r_s(M_n)$ be the set of all tensor fields of type (r, s) in M_n, namely contravariant of degree r and covariant of degree s. If we denote by $\wp(M_n)$ the tensor algebra associated with M_n i.e. $\wp(M_n) = \wp^r_s(M_n)$. The set of tensor fields in tangent bundle represented by $\wp^r_s(TM_n)$ and tensor algebra in the tangent bundle by $\wp(TM_n)$. The set of functions, vector fields, 1-forms and tensor fields of type (1,1) are denoted by $\wp^0_0(TM_n), \wp^1_0(TM_n), \wp^0_1(TM_n)$ and $\wp^1_1(TM_n)$ respectively.

2.1. Vertical and complete lifts

The vertical and complete lifts of a function, a vector field, 1-form, tensor field of type (1,1) and affine connection ∇ are given by $f^V, X^V, \omega^V, F^V, \nabla^V$ and f^C, X^C, ω^C, F^C, ∇^C respectively [14, 24].

The following properties of complete and vertical lifts are given by

$$(fX)^V = f^V X^V, (fX)^C = f^C X^V + f^V X^C, \tag{2.1}$$

$$X^V f^V = 0, X^V f^C = X^C f^V = (Xf)^V, X^C f^C = (Xf)^C, \tag{2.2}$$

$$\omega^V(f^V) = 0, \omega^V(X^C) = \omega^C(X^V) = \omega(X)^V, \omega^C(X^C) = \omega(X)^C, \tag{2.3}$$

$$F^V X^C = (FX)^V, F^C X^C = (FX)^C, \tag{2.4}$$

$$[X, Y]^V = [X^C, Y^V] = [X^V, Y^C], [X, Y]^C = [X^C, Y^C]. \tag{2.5}$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \qquad \nabla^C_{X^C} Y^V = (\nabla_X Y)^V \tag{2.6}$$

We extend the vertical and complete lifts to a linear isomorphism of tensor algebra $\wp(M_n)$ into $\wp(TM_n)$ concerning constant coefficient. Let P^V and Q^V be vertical lift and P^C and Q^C be complete lift of arbitrary tensor fields P and Q of $\wp(M_n)$. Then by definition

$$(P \otimes Q)^V = P^V \otimes Q^V, (P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C$$

$$(P + Q)^V = P^V + Q^V, (P + Q)^C = P^C + Q^C.$$

2.2. Semi-symmetric non-metric connection

Let M_n be a Riemannian manifold of dimension n with Riemannian metric g. A linear connection $\tilde{\nabla}$ on M_n given by [3]

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\pi(Y)X - \pi(X)Y\} \tag{2.7}$$

where ∇ is a Levi-Civita connection, X, Y vector fields and π 1-form on M_n. The metric g have the relation

$$(\tilde{\nabla}_X g)(Y, Z) = \frac{1}{2}\{2\pi(X)g(Y, Z) - \pi(Y)g(X, Z) - \pi(Z)g(X, Y)\} \tag{2.8}$$

The connection $\tilde{\nabla}$ satisfying equations (1.2), (1.3), (2.7) and (2.8) is called a semi-symmetric non-metric connection.

3. Semi-symmetric non-metric connection of a Riemannian manifold in the tangent bundle

Let (M_n, g) be an n-dimensional Riemannian manifold with the Riemannian metric g and TM_n its tangent bundle. Then g^C is a Riemannian metric in TM_n. Taking complete lifts of equations (1.2), (1.3), (2.7) and (2.8), then obtained equations are [21]

$$\begin{aligned} \tilde{T}^C(X^C, Y^C) &= \pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \pi^C(X^C)Y^V - \pi^V(X^C)Y^C \end{aligned} \tag{3.1}$$

$$\pi^C(X^C) \quad = \quad g^C(X^C, P^C)$$

A linear connection $\tilde{\nabla}^C$ defined by

$$\begin{aligned} \tilde{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C \quad &+ \quad \frac{1}{2}\{\pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \quad \pi^C(X^C)Y^V - \pi^V(X^C)Y^C\} \end{aligned} \tag{3.2}$$

is said to be a semi-symmetric non-metric connection if the torsion tensor $\tilde{T}^C$ of TM_n with respect to $\tilde{\nabla}^C$ satisfies equations (3.1) and (3.2) and the Riemannian metric g^C holds the relation

$$\begin{aligned} (\tilde{\nabla}^C_{X^C} g^C)(Y^C, Z^C) \quad &= \quad \frac{1}{2}\{2\pi^C(X^C)g^C(Y^V, Z^C) + 2\pi^V(X^C)g^C(Y^C, Z^C) \\ &- \quad \pi^C(Y^C)g^C(X^V, Z^C) - \pi^V(Y^C)g^C(X^C, Z^C) \\ &- \quad \pi^C(Z^C)g^C(X^V, Y^C) - \pi^V(Z^C)g^C(X^C, Y^C)\} \end{aligned} \tag{3.3}$$

where ∇^C is Levi-Civita connection on TM_n.

In order to prove the existence of such connection on tangent bundle TM_n, it suffices to prove the following theorem:

Theorem 3.1. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with the Levi-Civita connection ∇^C. Then there exists a unique linear connection $\tilde{\nabla}^C$ on TM_n, called a semi-symmetric non-metric connection, given by (3.2), and it satisfies equations* (3.1) *and* (3.3).

Proof. Let M_n be a Riemannian manifold of dimension n equipped with a linear connection $\tilde{\nabla}$. Then the relation between the linear connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ are are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y) \tag{3.4}$$

Operating complete lifts of both sides of equation (3.4), we get

$$\tilde{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C + U^C(X^C, Y^C) \tag{3.5}$$

for arbitrary vector fields X^C and Y^C on TM_n, where U^C is complete lift of a tensor field U of type (1, 2). Using equations (1.1) and (3.5), the obtained equation is

$$\tilde{T}^C(X^C, Y^C) = U^C(X^C, Y^C) - U^C(Y^C, X^C) \tag{3.6}$$

which gives

$$g^C(\tilde{T}(X^C, Y^C), Z^C) = g^C(U^C(X^C, Y^C)Z^C) - g^C(U^C(Y^C, X^C)Z^C) \tag{3.7}$$

In the view of equations (3.1) and (3.7), then

$$\begin{aligned} g^C(U^C(X^C,Y^C),Z^C) &- g^C(U^C(Y^C,X^C),Z^C) \\ &= \pi^V(Y^C)g^C(X^C,Z^C)+\pi^C(Y^C)g^C(X^V,Z^C) \\ &- \pi^V(X^C)g^C(Y^C,Z^C)-\pi^C(X^C)g^C(Y^V,Z^C) \end{aligned} \tag{3.8}$$

Making use of (3.1), the obtained equation is

$$\begin{aligned} \tilde{\nabla}^C_{X^C} g^C(Y^C,Z^C) &= -g^C(\tilde{\nabla}^C_{X^C}Y^C-\nabla^C_{X^C}Y^C,Z^C) \\ &- g^C(Y^C,\tilde{\nabla}^C_{X^C}Z^C-\nabla^C_{X^C}Z^C) \\ &= -U'^C(X^C,Y^C,Z^C), \end{aligned} \tag{3.9}$$

where $U'^C(X^C,Y^C,Z^C)=g^C(U^C(X^C,Y^C),Z^C)+g^C(U^C(X^C,Z^C)Y^C)$.

Using equations (3.6), (3.7), and (3.9), the obtained equation is

$$\begin{aligned} g^C(\tilde{T}^C(X^C,Y^C),Z^C) &+ g^C(\tilde{T}^C(Z^C,X^C),Y^C)+g^C(\tilde{T}^C(Z^C,Y^C),X^C) \\ &= 2g^C(U^C(X^C,Y^C),Z^C)-U'^C(X^C,Y^C,Z^C) \\ &- U'^C(X^C,Y^C,Z^C)+U'^C(Z^C,X^C,Y^C) \\ &- U'^C(Y^C,X^C,Z^C) \end{aligned}$$

From equations (3.3) and (3.9), the above equation becomes

$$\begin{aligned} 2g^C(U^C(X^C,Y^C),Z^C) &= g^C(\tilde{T}^C(X^C,Y^C),Z^C)+g^C(\tilde{T'}^C(Z^C,X^C),Y^C) \\ &+ g^C(\tilde{T'}^C(Z^C,Y^C),X^C)-\pi^V(X^C)g^C(Y^C,Z^C) \\ &- \pi^C(X^C)g^C(Y^V,Z^C)-\pi^V(Y^C)g^C(X^C,Z^C) \\ &- \pi^C(Y^C)g^C(X^V,Z^C)+2\pi^V(Z^C)g^C(X^C,Y^C) \\ &+ \pi^C(Z^C)g^C(X^V,Y^C) \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} g^C(\tilde{T'}^C(X^C,Y^C),Z^C) &= g^C(\tilde{T}^C(Z^C,X^C),Y^C)=\pi^V(X^C)g^C(Z^C,Y^C) \\ &+ \pi^C(X^C)g^C(Z^V,Y^C)-\pi^V(Z^C)g^C(X^C,Y^C) \\ &- \pi^C(Z^C)g^C(X^V,Y^C) \end{aligned} \tag{3.11}$$

for all vector fields X^C, Y^C and Z^C on TM_n.

Making use of equation (3.11), then equation (3.10) becomes

$$\begin{aligned} 2U^C(X^C,Y^C) &= (\pi^C(Y^C))(X^V)+(\pi^V(Y^C))(X^C) \\ &- (\pi^C(X^C))(Y^V)+(\pi^V(X^C))(Y^C) \end{aligned} \tag{3.12}$$

and thus equations (3.5) and (3.12) give (3.1).

Conversely, it is easy to show that if the affine connection $\tilde{\nabla}^C$ satisfies (3.2) then it will also satisfy equations (3.1) and (3.3). Hence, the theorem is proved. □

The covariant derivative of equation (3.2) with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ on TM_n , then the obtained equation is

$$\begin{aligned}(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) &= (\nabla^C_{X^C}\pi^C)(Y^C) + \pi^C(P^C)g^C(X^V, Y^C)\\ &+ \pi^V(P^C)g^C(X^C, Y^C) - \pi^V(Y^C)\pi^C(X^C)\\ &- \pi^C(Y^C)\pi^V(X^C)\end{aligned} \tag{3.13}$$

for arbitrary vector fields X^C and Y^C on TM_n. Using equation (3.13), then

$$\begin{aligned}(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) &- (\tilde{\nabla}^C_{Y^C}\pi^C)(X^C) = (\nabla^C_{X^C}\eta^C)(Y^C)\\ &- (\nabla^C_{Y^C}\eta^C)(X^C).\end{aligned} \tag{3.14}$$

Hence, the following theorem is obtained:

Theorem 3.2. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, and then the necessary and sufficient condition for the 1-form π^C to be closed with respect to $\tilde{\nabla}^C$ is that it is also closed corresponding to the Levi-Civita connection ∇^C.*

Theorem 3.3. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then*

$$'\tilde{T}^C(X^C, Y^C, Z^C) +' \tilde{T}^C(Y^C, X^C, Z^C) = 0,$$
$$'\tilde{T}^C(X^C, Y^C, Z^C) +' \tilde{T}^C(Y^C, Z^C, X^C) +' \tilde{T}^C(Z^C, X^C, Y^C) = 0.$$

Proof: Let $\tilde{T}$ be the torsion tensor on TM_n and define $'\tilde{T}^C(X^C, Y^C, Z^C) = g^C(\tilde{T}^C(X^C, Y^C), Z^C)$ on TM_n.

In the view of equation (3.1), then obtained equation is

$$\begin{aligned}'\tilde{T}^C(X^C, Y^C, Z^C) &= \pi^V(Y^C)g^C(X^C, Z^C) + \pi^C(Y^C)g^C(X^V, Z^C)\\ &- \pi^V(X^C)g^C(Y^C, Z^C)\\ &- \pi^C(X^C)g^C(Y^V, Z^C)\end{aligned} \tag{3.15}$$

Making use of equation (3.15), it can easily prove theorem.

Theorem 3.4. *Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then $\tilde{T}^C$ is cyclic parallel if and only if the 1-form π^C is closed.*

Proof. Operating the covariant derivative of (3.1) with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$, the obtained equation is

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C, Z^C) = (\tilde{\nabla}_X\pi)^C(Z^C)Y^V + (\tilde{\nabla}_X\pi)^V(Z^C)Y^C$$

$$- \quad (\tilde{\nabla}_X\pi)^C(Y^C)Z^V - (\tilde{\nabla}_X\pi)^V(Y^C)Z^C \tag{3.16}$$

The cyclic sum of (3.16) for vector fields X^C, Y^C and Z^C gives

$$\begin{aligned}
(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) &+ (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C) \\
&= (\tilde{\nabla}_X\pi)^C(Z^C)Y^V + (\tilde{\nabla}_X\pi)^V(Z^C)Y^C \\
&- (\tilde{\nabla}_X\pi)^C(Y^C)Z^V - (\tilde{\nabla}_X\pi)^V(Y^C)Z^C \\
&+ (\tilde{\nabla}_Y\pi)^C(X^C)Z^V + (\tilde{\nabla}_Y\pi)^V(X^C)Z^C \\
&- (\tilde{\nabla}_Y\pi)^C(Z^C)X^V - (\tilde{\nabla}_Y\pi)^V(Z^C)X^C \\
&+ (\tilde{\nabla}_Z\pi)^C(Y^C)X^V + (\tilde{\nabla}_Z\pi)^V(Y^C)X^C \\
&- (\tilde{\nabla}_Z\pi)^C(X^C)Y^V - (\tilde{\nabla}_Z\pi)^V(X^C)Y^C
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) &+ (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C) \\
&= \{(\tilde{\nabla}_X\pi)^C(Z^C) - (\tilde{\nabla}_Z\pi)^C(X^C)\}Y^V \\
&+ \{(\tilde{\nabla}_X\pi)^V(Z^C) - (\tilde{\nabla}_Z\pi)^V(X^C)\}Y^C \\
&+ \{(\tilde{\nabla}_Y\pi)^C(X^C) - (\tilde{\nabla}_X\pi)^C(Y^C)\}Z^V \\
&+ \{(\tilde{\nabla}_Y\pi)^V(X^C) - (\tilde{\nabla}_X\pi)^V(Y^C)\}Z^C \\
&+ \{(\tilde{\nabla}_Z\pi)^C(Y^C) - (\tilde{\nabla}_Y\pi)^C(Z^C)\}X^V \\
&+ \{(\tilde{\nabla}_Z\pi)^V(Y^C) - (\tilde{\nabla}_Y\pi)^V(Z^C)\}X^C
\end{aligned} \tag{3.17}$$

From equation (3.17) and Theorem 3.3, it can easily show that

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) + (\tilde{\nabla}^C_{Y^C}\tilde{T}^C)(Z^C,X^C) + (\tilde{\nabla}^C_{Z^C}\tilde{T}^C)(X^C,Y^C) = 0$$

if and only if the 1-form π^C is closed. Hence, the theorem is proved. □

Theorem 3.5. *Let M_n, g be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then for any arbitrary vector fields X^C, Y^C and the vector field P^C defined as (3.2), the following relation holds:*

$$\begin{aligned}
(\tilde{\pounds}_P g)^C(X^C,Y^C) &= (\pounds_P g)^C(X^C,Y^C) + 2\{\pi^C(P^C)g^C(X^V,Y^C) \\
&+ \pi^V(P^C)g^C(X^C,Y^C) - \pi^V(Y^C)\pi^C(X^C) \\
&- \pi^C(Y^C)\pi^V(X^C)\}
\end{aligned} \tag{3.18}$$

where $\tilde{\pounds}^C_P$ and $\pounds^C_P$ denote the Lie derivatives along the vector field P^C corresponding to $\tilde{\nabla}^C$ and ∇^C, respectively.

Proof. The Lie derivative along P [3],

$$\pounds_P g(X,Y) = g(\nabla_X P, Y) + g(X, \nabla_Y P) \tag{3.19}$$

Taking complete lifts on both sides, then

$$(\pounds_P g)^C(X^C, Y^C) = g^C(\nabla^C_{X^C} P^C, Y^C) + g^C(X^C, \nabla^C_{Y^C} P^C) \tag{3.20}$$

holds for arbitrary vector fields X^C and Y^C on TM_n. From equations (2.7) and (3.18) and the definition of the Lie derivative, the obtained equation is

$$\begin{aligned}(\tilde{\pounds}_P g)^C(X^C, Y^C) &= (P^C)g^C(X^V, Y^C) + (P^V)g^C(X^C, Y^C) \\ &- g^C(\tilde{\nabla}^C_{P^C} X^C - g^C(\tilde{\nabla}^C_{X^C} P^C, Y^C) \\ &- g^C(Y^C, \tilde{\nabla}^C_{P^C} Y^C - g^C(\tilde{\nabla}^C_{Y^C} P^C) \\ &= (\pounds_P g)^C(X^C, Y^C) + 2\{\pi^C(P^C)g^C(X^V, Y^C) \\ &+ \pi^V(P^C)g^C(X^C, Y^C) - \pi^V(Y^C)\pi^C(X^C) \\ &- \pi^C(Y^C)\pi^V(X^C)\}\end{aligned} \tag{3.21}$$

Hence, the theorem is proved. $\square$

If the vector field P^C is Killing on (TM_n, g^C), then $(\pounds_P g)^C = 0$. From theorem 3.5, the following corollary is obtained:

Corollary 3.1. *If the vector field P^C defined as in (3.2) is Killing on TM_n equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then*

$$\begin{aligned}(\tilde{\pounds}_P g)^C(X^C, Y^C) &= 2\{\pi^C(P^C)g^C(X^V, Y^C) + \pi^V(P^C)g^C(X^C, Y^C) \\ &- \pi^V(Y^C)\pi^C(X^C) - \pi^C(Y^C)\pi^V(X^C)\}\end{aligned} \tag{3.22}$$

where X^C and Y^C are vector fields and π^C 1-form on TM_n.

4. Curvature tensor with respect to the semi-symmetric non-metric connection in the tangent bundle

Let M_n be an n-dimensional Riemannian manifold admitting a semi-symmetric non-metric connection $\tilde{\nabla}$. If the curvature tensor $\tilde{R}$ corresponding to $\tilde{\nabla}$ then there exists the curvature tensor $\tilde{R}^C$ corresponding to $\tilde{\nabla}^C$ in TM_n is defined by

$$\tilde{R}^C(X^C, Y^C)Z^C = \tilde{\nabla}^C_{X^C}\tilde{\nabla}^C_{Y^C} Z^C - \tilde{\nabla}^C_{Y^C}\tilde{\nabla}^C_{X^C} Z^C - \tilde{\nabla}^C_{[X^C, Y^C]} Z^C$$

for arbitrary vector fields X^C, Y^C and Z^C on (TM_n, g^C), then the Riemannian curvature tensor R^C of the Levi-Civita connection ∇^C is defined by

$$R^C(X^C, Y^C)Z^C = \nabla^C_{X^C}\nabla^C_{Y^C} Z^C - \nabla^C_{Y^C}\nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C$$

for arbitrary vector fields X^C, Y^C, and Z^C on (TM_n, g^C).

Making use of equation (3.2), we have

$$\tilde{R}^C(X^C, Y^C)Z^C = \tilde{\nabla}^C_{X^C}\{\nabla^C_{Y^C} Z^C + \frac{1}{2}(\pi^C(Z^C)(Y^V) + \pi^V(Z^C)(Y^C)$$

$$\begin{aligned}
&- \pi^C(Y^C)(Z^V) - \pi^V(Y^C)(Z^C)\})\\
&- \tilde{\nabla}^C_{Y^C}\{\nabla^C_{X^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(X^V) + \pi^V(Z^C)(X^C)\\
&- \pi^C(X^C)(Z^V) - \pi^V(X^C)(Z^C))\}\\
&- \{\nabla^C_{[X^C,Y^C]}Z^C + \frac{1}{2}(\pi^C(Z^C)([X,Y]^V) + \pi^V(Z^C)([X,Y]^C)\\
&- \pi^C([X,Y]^C)(Z^V) - \pi^V([X,Y]^C)(Z^C))\}\\
&= \nabla^C_{X^C}\{\nabla^C_{Y^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(Y^V) + \pi^V(Z^C)(Y^C)\\
&- \pi^C(Y^C)(Z^V) - \pi^V(Y^C)(Z^C)\})\\
&- \nabla^C_{Y^C}\{\nabla^C_{X^C}Z^C + \frac{1}{2}(\pi^C(Z^C)(X^V) + \pi^V(Z^C)(X^C)\\
&- \pi^C(X^C)(Z^V) - \pi^V(X^C)(Z^C))\}\\
&- \nabla^C_{[X^C,Y^C]}Z^C + \frac{1}{2}(\pi^C(\nabla^C_{Y^C}Z^C)(X^V) + \pi^V(\nabla^C_{Y^C}Z^C)(X^C)\\
&- \pi^C(X^C)(\nabla_Y Z)^V - \pi^V(X^C)(\nabla_Y Z)^C\}\\
&- \frac{1}{4}\{\pi^V(X^C)\pi^C(Z^C)Y^C) + \pi^C(X^C)\pi^V(Z^C)Y^C)\\
&+ \pi^C(X^C)\pi^C(Z^C)Y^V) - \pi^V(Y^C)\pi^C(Z^C)X^C)\\
&- \pi^C(Y^C)\pi^V(Z^C)X^C) - \pi^C(Y^C)\pi^C(Z^C)X^V)\\
&- \pi^C(Z^C)([X,Y]^V) - \pi^V(Z^C)([X,Y]^C)\\
&- \pi^C([X,Y]^C)(Z^V) - \pi^V([X,Y]^C)(Z^C)\\
&= R^C(X^C,Y^C)Z^C + \frac{1}{2}\{\theta^C(X^C,Z^C)Y^C - \theta^C(Y^C,Z^C)X^C\\
&- (\theta^C(X^C,Y^C) - \theta^C(Y^C,X^C))Z^C\} \qquad (4.1)
\end{aligned}$$

for arbitrary vector fields X^C, Y^C and Z^C on TM_n, where θ^C is a complete lift of a tensor field θ of type (0, 2) and is defined by

$$\begin{aligned}
\theta^C(X^C,Y^C) &= g^C(AX,Y)^C = (\nabla^C_{X^C}\pi^C)(Y^C)\\
&- \pi^V(X^C)\pi^C(Y^C) - \pi^C(X^C)\pi^V(Y^C) \qquad (4.2)
\end{aligned}$$

and

$$(AX)^C = (\nabla_X P)^C - \frac{1}{2}\{\pi^V(X^C)(P^C) - \pi^C(X^C)(P^V)\} \qquad (4.3)$$

for arbitrary vector fields X^C and Y^C on TM_n.

From equation (4.2), it is obvious that the tensor field θ^C is symmetric if and only if the 1-form π^C is closed. Taking the inner product of (4.1) with W^C and then setting $X^C = W^C = e_i^C, 1 \leq i \leq n$, where e_i^C is complete lift of $\{e_i, i = 1, 2, 3, ..., n\}$ which is an orthonormal basis of the tangent space at each point of the Riemannian manifold M_n, then obtained equation is

$$
\begin{aligned}
\tilde{S}^C(Y^C, Z^C) &= S^C(Y^C, Z^C) + \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i, Z)g(Y, e_i))^C - \theta^C(Y^C, Z^C)g^C(e_i, e_i) \\
&- (g(Ae_i, Y)g(Z, e_i))^C + (g(AY, e_i)g(Z, e_i))^C\} \\
&= S^C(Y^C, Z^C) + \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i, Z)^C g(Y, e_i))^V \\
&+ (g(Ae_i, Z)^V g(Y, e_i))^C - \theta^C(Y^C, Z^C)g^C(e_i, e_i) \\
&- (g(Ae_i, Y))^C(g(Z, e_i))^V - (g(Ae_i, Y))^V(g(Z, e_i))^C \\
&+ (g(AY, e_i))^C(g(Z, e_i))^V + (g(AY, e_i))^V(g(Z, e_i))^C\} \\
&= S^C(Y^C, Z^C) - \frac{n-1}{2}\theta^C(Y^C, Z^C) \\
&+ \frac{1}{2}\sum_{i=1}^{n}\{(g(Ae_i, e_i))^C g(Z, e_i))^C g(Y, e_i))^V \\
&+ (g(Ae_i, e_i))^C g(Z, e_i))^V g(Y, e_i))^C \\
&+ (g(Ae_i, e_i))^V g(Z, e_i))^C g(Y, e_i))^C \\
&- (g(Ae_i, e_i))^C g(Z, e_i))^C g(Y, e_i))^V \\
&- (g(Ae_i, e_i))^C g(Z, e_i))^V g(Y, e_i))^C \\
&- (g(Ae_i, e_i))^V g(Z, e_i))^C g(Y, e_i))^C\}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\tilde{S}^C(Y^C, Z^C) &= S^C(Y^C, Z^C) - \frac{n-1}{2}\theta^C(Y^C, Z^C) \\
&\Leftrightarrow \tilde{Q}^C(Y^C) = Q^C(Y^C) - \frac{n-1}{2}(AY)^C \qquad (4.4)
\end{aligned}
$$

for all vector fields Y^C and Z^C on TM_n. Here $\tilde{Q}^C$ and Q^C are the complete lift of Ricci operators corresponding to the Ricci tensors $\tilde{Q}^C$ and Q^C complete lifts $\tilde{S}^C$ and S^C Ricci tensors $\tilde{S}$ and S of the connections $\tilde{\nabla}^C$ and ∇^C, respectively; that is, $\tilde{S}^C(Y^C, Z^C) = g^C(\tilde{Q}^C Y^C, Z^C)$ and $S^C(Y^C, Z^C) = g^C(Q^C Y^C, Z^C)$

Again contracting eqution (4.4) along the vector field Y^C, then

$$\tilde{r} = r - (n-1)a, \qquad (4.5)$$

where $\tilde{r}$ and r denote the scalar curvatures corresponding to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ and the Levi-Civita connection ∇^C, respectively, and

$$a \overset{def}{=} \frac{1}{2} trA$$

Here trA represents the trace of A. From equation (4.5), the following Theorem is obtained:

Theorem 4.1. *Let (M_n, g) be n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. Then the necessary and sufficient condition for the scalar curvatures $\tilde{r}$ and r to coincide is that a be zero; that is, $trA = 0$.*

Interchanging Y^C and Z^C in equation (4.4), the obtained equation is

$$\tilde{S}^C(Z^C, Y^C) = S^C(Z^C, Y^C) - \frac{n-1}{2}\theta^C(Z^C, Y^C). \tag{4.6}$$

Subtracting equation (4.6) from equation (4.4) and then using equation (4.2) and the symmetric property of the Ricci tensor in it, the obtained equation is

$$\begin{aligned} \tilde{S}^C(Y^C, Z^C) - \tilde{S}^C(Z^C, Y^C) &= \frac{n-1}{2}\{\theta^C(Z^C, Y^C) - \theta^C(Y^C, Z^C)\} \\ &= -\frac{n-1}{2} d\pi^C(Y^C, Z^C), \end{aligned} \tag{4.7}$$

where d denotes the exterior derivative. In view of equation (4.7) and Theorem 3.2, the following theorem is obtained:

Theorem 4.2. *If an $n(>1)$-dimensional Riemannian manifold (M_n, g) and TM_n its tangent bundle admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then the Ricci tensor $\tilde{S}^C$ corresponding to the connection $\tilde{\nabla}^C$ is symmetric if and only if the 1-form π^C is closed.*

Theorem 4.3. *Let (M_n, g) be an $n(>1)$-dimensional Riemannian manifold and TM_n its tangent bundle equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ defined as in equation (3.1). Then the connection $\tilde{\nabla}^C$ is projectively invariant; that is, the projective curvature tensors with respect to $\tilde{\nabla}^C$ and ∇^C coincide if and only if the 1-form π^C is closed.*

Proof. If the 1-form π^C is closed and from equation (4.2) θ^C is symmetric. Using these in equation (4.1), then equation (4.1) becomes

$$\tilde{R}^C(X^C, Y^C)Z^C = R^C(X^C, Y^C)Z^C + \frac{1}{2}\{\theta^C(X^C, Z^C)Y^C - \theta^C(Y^C, Z^C)X^C \tag{4.8}$$

Contracting equation (4.8) along the vector field X^C, then

$$\tilde{S}^C(Y^C, Z^C) = S^C(Y^C, Z^C) - \frac{n-1}{2}\theta^C(Y^C, Z^C) \tag{4.9}$$

which gives

$$\tilde{Q}^C(Y^C) = Q^C(Y^C) - \frac{n-1}{2}(AY)^C \tag{4.10}$$

and

$$\tilde{r} = r - (n-1)a. \tag{4.11}$$

The projective curvature tensor $\tilde{P}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ is given in equation (1.6). Taking complete lift of equation (1.6), then

$$\begin{aligned} \tilde{P}^C(X^C, Y^C)Z^C &= \tilde{R}^C(X^C, Y^C)Z^C - \frac{1}{n-1}\{\tilde{S}^C(Y^C, Z^C)X^V \\ &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\ &- \tilde{S}^V(X^C, Z^C)Y^C\} \end{aligned} \tag{4.12}$$

for all vector fields X^C, Y^C and Z^C on TM_n, where $\tilde{P}^C$ is the complete lift the projective curvature $\tilde{P}$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$. In view of equations (4.8) and (4.9), equation (4.12) becomes

$$\tilde{P}^C(X^C, Y^C)Z^C = P^C(X^C, Y^C)Z^C, \tag{4.13}$$

where P^C denotes the complete lift of the projective curvature tensor P with respect to ∇^C and is defined by

$$\begin{aligned} P^C(X^C, Y^C)Z^C &= R^C(X^C, Y^C)Z^C - \frac{1}{n-1}\{\tilde{S}^C(Y^C, Z^C)X^V \\ &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\ &- \tilde{S}^V(X^C, Z^C)Y^C\} \end{aligned} \tag{4.14}$$

for arbitrary vector fields X^C, Y^C and Z^C on TM_n and P is given in (1.6). Conversely, suppose that (TM_n, g^C) equipped with $\tilde{\nabla}^C$ satisfies (4.13). Thus, use of equations (4.1), (4.4), (4.10), (4.12), and (4.14) in equation (4.13) gives

$$\{\theta^C(X^C, Y^C) - \theta^C(Y^C, X^C)\}Z^C = 0$$

Contracting the last equation along the vector field X^C, we find

$$\theta^C(X^C, Y^C) - \theta^C(Y^C, X^C) = 0$$

which shows that $\theta^C(Y^C, Z^C) = \theta^C(Z^C, Y^C)$.

Hence, the proof is completed. □

Theorem 4.4. *Let (M_n, g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ whose curvature tensor $\tilde{R}^C$ vanishes identically, then (TM_n, g^C) is projectively flat if and only if θ^C is a symmetric tensor.*

Proof. Suppose that the curvature tensor with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ vanishes on (TM_n, g^C) i.e., $\tilde{R}^C = 0$, and the tensor field θ^C is symmetric. Then equation (4.8) takes the form

$$R^C(X^C,Y^C)Z^C = \frac{1}{2}\{\theta^C(Y^C,Z^C)X^C - \theta^C(X^C,Z^C)Y^C\} \tag{4.15}$$

which implies that

$$S^C(Y^C,Z^C) = \frac{n-1}{2}\theta^C(Y^C,Z^C), \qquad r = (n-1)a. \tag{4.16}$$

Using of equations (4.14), (4.15) and (4.16), then $P^C = 0$. Conversely, if the projective curvature tensor of ∇^C is zero and the curvature tensor $\tilde{R}^C$ is also zero, then equations (4.1) and (4.14) take the form

$$\begin{aligned} R^C(X^C,Y^C)Z^C &= \frac{1}{n-1}\{\tilde{S}^C(Y^C,Z^C)X^V + \tilde{S}^V(Y^C,Z^C)X^C \\ &- \tilde{S}^C(X^C,Z^C)Y^V - \tilde{S}^V(X^C,Z^C)Y^C\} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} R^C(X^C,Y^C)Z^C &= \frac{1}{2}\{\theta^C(Y^C,Z^C)X^C - \theta^C(X^C,Z^C)Y^C \\ &- (\theta^C(Y^C,X^C) - \theta^C(X^C,Y^C))Z^C\}. \end{aligned} \tag{4.18}$$

Equating equations (4.17) and (4.18) and then using equation (4.4), obtained equation is

$$\{\theta^C(X^C,Y^C) - \theta^C(Y^C,X^C)\}Z^C = 0$$

Contracting the above equation along the vector field Z^C, then

$$\theta^C(X^C,Y^C) = \theta^C(Y^C,X^C)$$

Hence, the proof is completed. □

Theorem 4.5. *Let (M_n,g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. If the curvature tensor with respect to $\tilde{\nabla}^C$ vanishes, then the tensor field θ^C is symmetric if and only if*

$$\begin{aligned} (n-2)\{'C^C(X^C,Y^C,Z^C,U^C) &+ {}'\breve{C}^C(X^C,Y^C,Z^C,U^C)g^C \\ &= -2'R^C(X^C,Y^C,Z^C,U^C) \end{aligned}$$

Proof. Let the curvature tensor $\tilde{R}^C$ with respect to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ vanish on TM_n. For necessary part, consider the tensor field θ^C is symmetric i.e., $\theta^C(X^C,Y^C) = \theta^C(Y^C,X^C)$. The conformal curvature tensor C

given in equation (1.7) with respect to ∇. Taking complete lift of equation (1.7), the obtained equation is

$$
\begin{aligned}
C^C(X^C,Y^C)Z^C &= R^C(X^C,Y^C)Z^C - \frac{1}{n-2}\{\tilde{S}^C(Y^C,Z^C)X^V \\
&+ \tilde{S}^V(Y^C,Z^C)X^C - \tilde{S}^C(X^C,Z^C)Y^V \\
&- \tilde{S}^V(X^C,Z^C)Y^C + g^C(Y^C,Z^C)(QX)^V \\
&+ g^V(Y^C,Z^C)(QX)^C - g^C(X^C,Z^C)(QY)^V \\
&- g^V(X^C,Z^C)(QY)^C\} \\
&+ \frac{r}{(n-1)(n-2)}\{g^C(Y^C,Z^C)X^V \\
&+ g^V(Y^C,Z^C)X^C - g^C(X^C,Z^C)Y^V \\
&- g^V(X^C,Z^C)Y^C\} \qquad (4.19)
\end{aligned}
$$

for arbitrary vector fields X^C, Y^C, Z^C on TM_n, where C^C is the complete lift of the conformal curvature tensor C with respect to ∇^C.

The inner product of equation (4.19) with U^C gives

$$
\begin{aligned}
{}'C^C(X^C,Y^C,Z^C,U^C) &= {}'R^C(X^C,Y^C,Z^C,U^C) \\
&- \frac{1}{n-1}\{\tilde{S}^C(Y^C,Z^C)g^V(X^V,U^C) \\
&+ \tilde{S}^V(Y^C,Z^C)g^V(X^C,U^C) \\
&- \tilde{S}^C(X^C,Z^C)g^V(Y^V,U^C) \\
&- \tilde{S}^V(X^C,Z^C)g^V(Y^C,U^C) \\
&+ g^C(Y^C,Z^C)\tilde{S}^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)\tilde{S}^C(X^C,U^C) \\
&- g^C(X^C,Z^C)\tilde{S}^V(Y^C,U^C) \\
&- g^V(X^C,Z^C)\tilde{S}^C(Y^C,U^C)\} \\
&+ \frac{r}{(n-1)(n-2)}\{g^C(Y^C,Z^C)g^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)g^C(X^C,U^C) \\
&- g^C(X^C,Z^C)g^V(Y^C,U^C) \\
&- g^V(X^C,Z^C)g^C(Y^C,U^C)\} \qquad (4.20)
\end{aligned}
$$

where ${}'C^C(X^C,Y^C),Z^C,U^C) = g^C(C^C(X^C,Y^C)Z^C,U^C)$. Using equations (4.15) and (4.16) in equation (4.20), the obtained equation is

$$
\begin{aligned}
{}'C^C((X^C,Y^C,Z^C,U^C) &= \frac{n}{(n-2)}{}'R^C(X^C,Y^C,Z^C,U^C) \\
&- \frac{a}{(n-2)}\{g^C(Y^C,Z^C)g^V(X^C,U^C) \\
&+ g^V(Y^C,Z^C)g^C(X^C,U^C)
\end{aligned}
$$

$$
\begin{aligned}
& - \quad g^C(X^C, Z^C) g^V(Y^C, U^C) \\
& - \quad g^V(X^C, Z^C) g^C(Y^C, U^C)\} \qquad (4.21)
\end{aligned}
$$

The concircular curvature tensor $\breve{C}$ is given in equation (1.8) with respect to ∇. Taking complete lift of equation (1.8), then

$$
\begin{aligned}
{}'\breve{C}^C((X^C, Y^C, Z^C, U^C) &= {}'R^C((X^C, Y^C, Z^C, U^C) \\
&- \frac{r}{(n-1)(n-2)}\{g^C(Y^C, Z^C) g^V(X^C, U^C) \\
&+ g^V(Y^C, Z^C) g^C(X^C, U^C) \\
&- g^C(X^C, Z^C) g^V(Y^C, U^C) \\
&- g^V(X^C, Z^C) g^C(Y^C, U^C)\} \qquad (4.22)
\end{aligned}
$$

for arbitrary vector fields X^C, Y^C, Z^C, U^C on TM_n, where $\breve{C}^C$ is complete lift of the concircular curvature tensor $\breve{C}$ and

$$
{}'\breve{C}^C(X^C, Y^C Z^C, U^C) = g^C(\breve{C}^C(X^C, Y^C) Z^C, U^C).
$$

Using equations (4.16) and (4.22) in equation (4.21), the obtained equation is

$$
\begin{aligned}
& (n-2)\{{}'C^C(X^C, Y^C Z^C, U^C){}'\breve{C}^C(X^C, Y^C Z^C, U^C)\} \\
= \; & -{}'R^C(X^C, Y^C Z^C, U^C). \qquad (4.23)
\end{aligned}
$$

For the sufficient part, Suppose that the Riemannian manifold (TM_n, g^C) equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ satisfies relation (4.23). Using equations (4.1), (4.20), (4.22), and (4.23), the obtained equation is:

$$
\begin{aligned}
\tilde{S}^C(Y^C, Z^C)X^V &+ \tilde{S}^V(Y^C, Z^C)X^C - \tilde{S}^C(X^C, Z^C)Y^V \\
&- \tilde{S}^V(X^C, Z^C)Y^C + g^C(Y^C, Z^C)(QX)^V \\
&+ g^V(Y^C, Z^C)(QX)^C - g^C(X^C, Z^C)(QY)^V \\
&- g^V(X^C, Z^C)(QY)^C\} \\
&= (n-10\{\theta^C(Y^C, Z^C)X^C - \theta^C(X^C, Z^C)Y^C \\
&- (\theta^C(Y^C, X^C) - \theta^C(X^C, Y^C))Z^C\}. \qquad (4.24)
\end{aligned}
$$

Contracting equation (4.24) along the vector field Z^C, implies that $\theta^C(X^C, Y^C) = \theta^C(Y^C, X^C)$. Hence, the proof is completed. $\square$

Corollary 4.1. *Let (M_n, g) be an $n > 2$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. The Riemannian manifold (TM_n, g^C) admits a semi-symmetric non-metric connection $\tilde{\nabla}^C$ whose curvature tensor vanishes and whose 1-form π^C is closed, then*

$$
(n-2)'L^C(X^C, Y^C, Z^C, U^C) + nR^C((X^C, Y^C, Z^C, U^C) = 0.
$$

Proof. The relation among $'C, '\breve{C}, ' L$ and $'R$ on a Riemannian manifold M_n is given by [3]

$$\begin{aligned} 'C(X,Y,Z,U) &+ {}'\breve{C}(X,Y,Z,U) \\ &= {}'L(X,Y,Z,U) \\ &+ {}'R((X,Y,Z,U). \end{aligned} \tag{4.25}$$

Taking complete lifts on both sides of above equation , the obtained equation is

$$\begin{aligned} 'C^C(X^C,Y^C,Z^C,U^C) &+ {}'\breve{C}^C(X^C,Y^C,Z^C,U^C) \\ &= {}'L^C(X^C,Y^C,Z^C,U^C) \\ &+ {}'R^C((X^C,Y^C,Z^C,U^C). \end{aligned} \tag{4.26}$$

where $'L^C$ is the complete lift of a conharmonic curvature tensor $'L$ of type (0, 4), which is obtain by taking complete lift of equation (1.9)

$$\begin{aligned} 'L^C(X^C,Y^C,Z^C,U^C) &= {}'R^C(X^C,Y^C,Z^C,U^C) \\ &- \frac{1}{n-1}\{\tilde{S}^C(Y^C,Z^C)g^V(X^V,U^C) \\ &+ \tilde{S}^V(Y^C,Z^C)g^V(X^C,U^C) \\ &- \tilde{S}^C(X^C,Z^C)g^V(Y^V,U^C) \\ &- \tilde{S}^V(X^C,Z^C)g^V(Y^C,U^C) \\ &+ g^C(Y^C,Z^C)\tilde{S}^V(X^C,U^C) \\ &+ g^V(Y^C,Z^C)\tilde{S}^C(X^C,U^C) \\ &- g^C(X^C,Z^C)\tilde{S}^V(Y^C,U^C) \\ &- g^V(X^C,Z^C)\tilde{S}^C(Y^C,U^C)\} \end{aligned} \tag{4.27}$$

From equations (4.23) and (4.26), the statement of Corollary 4.1 is obtained. □

5. Group manifolds with respect to the semi-symmetric non-metric connection in the tangent bundle

Let (M_n, g) be n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is said to be a group manifold [23] if

$$(\tilde{\nabla}^C_{X^C}\tilde{T}^C)(Y^C,Z^C) = 0 \quad \textit{and} \quad \tilde{R}^C(X^C,Y^C)Z^C = 0 \tag{5.1}$$

for arbitrary vector fields X^C, Y^C and Z^C on TM_n.

Making use of equations (3.17) and (5.1), the obtained equation is

$$\begin{aligned} (\tilde{\nabla}_X\pi)^C(Z^C)Y^V &+ (\tilde{\nabla}_X\pi)^V(Z^C)Y^C - (\tilde{\nabla}_X\pi)^C(Y^C)Z^V \\ &- (\tilde{\nabla}_X\pi)^V(Y^C)Z^C = 0 \end{aligned}$$

Using equation (3.13) and $n > 1$, then above equation gives

$$
\begin{aligned}
(\tilde{\nabla}^C_{X^C}\pi^C)(Y^C) = 0 \quad \Leftrightarrow \quad & (\nabla^C_{X^C}\pi^C)(Y^C) = \pi^V(Y^C)\pi^C(X^C) \\
+ \quad & \pi^C(Y^C)\pi^V(X^C) - \pi^C(P^C)g^C(X^V, Y^C) \\
- \quad & \pi^V(P^C)g^C(X^C, Y^C),
\end{aligned} \tag{5.2}
$$

Using equations (4.1) and (5.1), the curvature tensor R^C on TM_n is given by

$$
\begin{aligned}
R^C(X^C, Y^C)Z^C \quad = \quad & \frac{1}{4}\{\pi^V(Y^C)\pi^C(Z^C)X^C) + \pi^C(Y^C)\pi^V(Z^C)X^C) \\
+ \quad & \pi^C(Y^C)\pi^C(Z^C)X^V)\} - \pi^V(X^C)\pi^C(Z^C)Y^C) \\
- \quad & \pi^C(X^C)\pi^V(Z^C)Y^C) - \pi^C(X^C)\pi^C(Z^C)Y^V) \\
- \quad & \frac{\pi^C(P^C)}{2}\{g^C(Y^C, Z^C)X^V \\
+ \quad & g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
- \quad & g^V(X^C, Z^C)Y^C\} \\
- \quad & \frac{\pi^V(P^C)}{2}\{g^C(Y^C, Z^C)X^V \\
+ \quad & g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
- \quad & g^V(X^C, Z^C)Y^C\}
\end{aligned} \tag{5.3}
$$

Contracting equation (5.3) along the vector field X^C, then

$$
\begin{aligned}
S^C(Y^C, Z^C) \quad = \quad & \frac{n-1}{4}[\pi^V(Y^C)\pi^C(Z^C) + \pi^C(Y^C)\pi^V(Z^C) \\
- \quad & 2\pi^C(P^C)g^C(Y^V, Z^C) \\
- \quad & 2\pi^V(P^C)g^C(Y^C, Z^C)
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
Q^C(Y^C) \quad = \quad & \frac{n-1}{4}\{\pi^V(Y^C)(P^C) + \pi^C(Y^C)(P^V \\
- \quad & \pi^V(P^C)(Y^C) - \pi^C(P^C)(Y^V)\}
\end{aligned} \tag{5.5}
$$

Changing Z^C with P^C in equation (5.4) and using equation (3.2) in it, obtained equation is

$$
\begin{aligned}
S^C(Y^C, Z^C) \quad = \quad & -\frac{n-1}{4}[\pi^C(P^C)g^C(Y^V, P^C) \\
- \quad & 2\pi^V(P^C)g^C(Y^C, P^C).
\end{aligned}
$$

The following theorem is obtained:

Theorem 5.1. *Let (M_n, g) be $n(>1)$-dimensional group manifold and TM_n its tangent bundle with Riemannian metric g^C admit a semi-symmetric non-metric connection $\tilde{\nabla}^C$. Then $-\frac{n-1}{4}\pi^C(P^C)$ is an eigenvalue of S^C is the complete lift of the Ricci tensor S^C corresponding to the eigenvector P^C.*

Also contracting equation (5.5) along Y^C, then

$$r = -\frac{(n-1)(2n-1)\pi^C(P^C)}{4} \tag{5.6}$$

Using equations (5.3) and (5.4) in equation (4.14), then $P^C = 0$. Hence, the following theorem is obtained:

Theorem 5.2. *Let (M_n, g) be an $n > 1$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Every group manifold (M_n, g) in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is projectively flat.*

Theorem 5.3. *Let (M_n, g) be an $n > 2$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is P^C-conformally flat.*

Proof. From equations (5.3), (5.4), (5.5), and (5.6), then equation (4.19) takes the form

$$\begin{aligned}
C^C(X^C, Y^C)Z^C &= \frac{\pi^C(P^C)}{4(n-2)}\{g^C(Y^C, Z^C)X^V \\
&+ g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
&- g^V(X^C, Z^C)Y^C\} \\
&- \frac{\pi^V(P^C)}{4(n-2)}\{g^C(Y^C, Z^C)X^V \\
&+ g^V(Y^C, Z^C)X^C - g^C(X^C, Z^C)Y^V \\
&- g^V(X^C, Z^C)Y^C\} \\
&- \frac{1}{4(n-2)}\{\pi^V(X^C)\pi^C(Z^C)Y^C) \\
&+ \pi^C(X^C)\pi^V(Z^C)Y^C) + \pi^C(X^C)\pi^C(Z^C)Y^V) \\
&- \pi^V(Y^C)\pi^C(Z^C)X^C) - \pi^C(Y^C)\pi^V(Z^C)X^C) \\
&- \pi^C(Y^C)\pi^C(Z^C)X^V)\}
\end{aligned}$$

$$\begin{aligned}
&- \frac{n-1}{4(n-2)}\{\pi^C(Y^C)g^C(X^V, Z^C) \\
&+ \pi^V(Y^C)g^C(X^C, Z^C) - \pi^C(X^C)g^C(Y^V, Z^C) \\
&- \pi^V(X^C)g^C(Y^C, Z^C)\}P^C
\end{aligned}$$

$$
\begin{aligned}
& - \frac{n-1}{4(n-2)}\{\pi^C(Y^C)g^C(X^V, Z^C) \\
& + \pi^V(Y^C)g^C(X^C, Z^C) \\
& - \pi^C(X^C)g^C(Y^V, Z^C) \\
& - \pi^V(X^C)g^C(Y^C, Z^C)\}P^V.
\end{aligned} \tag{5.7}
$$

Let (M_n, g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Then (TM_n, g^C) is said to be P^C-conformally flat [3] if its nonvanishing conformal curvature tensor C^C satisfies $C^C(X^C, Y^C)P^C = 0$ for all vector fields X^C and Y^C on TM_n. Replacing Z^C by P^C in equation (5.7), it can easily show that $C^C(X^C, Y^C)P^C = 0$. Hence, Theorem 5.2 is verified. □

Theorem 5.4. *Let (M_n, g) be an $n(>2)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Every Ricci-symmetric group manifold (M_n, g) in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ satisfies $\pi^C(P^V) = 0$ and $\pi^V(P^C) = 0$.*

Proof. Let (M_n, g) be an n-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. The covariant derivative of equation (5.4) gives

$$
\begin{aligned}
(\nabla^C_{X^C} S^C)(Y^C, Z^C) &= \frac{n-1}{4}[(\nabla^C_{X^C}\pi^V)(Y^C)\pi^C(Z^C) \\
&+ \pi^V(Y^C)(\nabla^C_{X^C}\pi^C)(Z^C) \\
&+ (\nabla^C_{X^C}\pi^C)(Y^C)\pi^V(Z^C) \\
&+ \pi^C(Y^C)(\nabla^C_{X^C}\pi^V)(Z^C) \\
&- 2g^C(Y^V, Z^C)(\nabla^C_{\pi^C}(P^C) \\
&- 2g^C(Y^V, Z^C)\pi^C(\nabla^C_{X^C}P^C) \\
&- 2g^C(Y^C, Z^C)(\nabla^C_{\pi^V X^C}(P^C) \\
&- 2g^C(Y^C, Z^C)\pi^V(\nabla^C_{X^C}P^C)]
\end{aligned} \tag{5.8}
$$

which becomes

$$
\begin{aligned}
(\nabla^C_{X^C} S^C)(Y^C, Z^C) &= \frac{n-1}{4}\{2\pi^V(X^C)\pi^C(Y^C)\pi^C(Z^C) \\
&+ 2\pi^C(X^C)\pi^V(Y^C)\pi^C(Z^C) \\
&+ 2\pi^C(X^C)\pi^C(Y^C)\pi^V(Z^C) \\
&- [\pi^C(Y^C)g^C(X^V, Z^C) \\
&+ \pi^V(Y^C)g^C(X^C, Z^C) \\
&- \pi^C(X^C)g^C(Y^V, Z^C) \\
&- \pi^V(X^C)g^C(Y^C, Z^C)]\}
\end{aligned} \tag{5.9}
$$

where equation (5.2) is used.

A Riemannian manifold (M_n, g) of dimension n and TM_n its tangent bundle. Then tangent bundle TM_n is said to be Ricci symmetric if and only if $\nabla^C S^C = 0$. If possible, we suppose that the group manifold (M_n, g) in TM_n is Ricci-symmetric, and then the last equation gives $\pi^C(P^V) = 0$ and $\pi^V(P^C) = 0$. Hence, the Theorem 5.3 is proved. □

Theorem 5.5. *Let (M_n, g) be an $n(>1)$-dimensional Riemannian manifold and TM_n its tangent bundle with Riemannian metric g^C. Suppose (M_n, g) is a group manifold in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$. A Ricci soliton (g^C, P^C, λ) on (TM_n, g^C) to be shrinking, steady, and expanding according as $\pi^C(P^V)$ and $\pi^V(P^C)$ are $<, =,$ and > 0, respectively.*

Proof. If (M_n, g) is a group manifold in TM_n equipped with a semi-symmetric non-metric connection $\tilde{\nabla}^C$, then equation (5.2) and Theorem 3.5 give

$$\begin{aligned} (\pounds_P g)^C(X^C, Y^C) &= 2\{\pi^V(Y^C)\pi^C(X^C) \\ &+ \pi^C(Y^C)\pi^V(X^C) - \pi^C(P^C)g^C(X^V, Y^C) \\ &- \pi^V(P^C)g^C(X^C, Y^C)\} \end{aligned} \tag{5.10}$$

for arbitrary vector fields X^C and Y^C on TM_n. A triplet (g^C, P^C, λ) on an n-dimensional Riemannian manifold (M_n, g) in TM_n is said to be a Ricci soliton if it satisfies the relation

$$(\pounds_V g)^C + 2S^C + 2\lambda g^C = 0, \tag{5.11}$$

where $\pounds_V g + 2S + 2\lambda g = 0$ and V is a complete vector field on M_n and λ is a real constant [11]. A Ricci soliton (g^C, P^C, λ) on (TM_n, g^C) is said to be shrinking, steady, and expanding if λ is negative, zero, and positive, respectively. Changing V with P^C in equation (5.11) and then using equations (5.4) and (5.10), then the obtained equation is

$$\begin{aligned} (n-3)\{\pi^V(X^C)\pi^C(Y^C) &+ \pi^C(X^C)\pi^V(Y^C)\} \\ &- 2(n+1)\{\pi^C(P^C)g^C(X^V, Y^C) \\ &+ \pi^V(P^C)g^C(X^C, Y^C)\} \\ &+ 4\lambda g^C(X^C, Y^C) = 0 \end{aligned} \tag{5.12}$$

for arbitrary vector fields X^C and Y^C on TM_n.

Setting $Y^C = P^C$ in equation (5.12), then

$$\{\lambda - \frac{n-1}{4}\pi^C(P^C)\}\pi^C(X^V) + \{\lambda - \frac{n-1}{4}\pi^V(P^C)\}\pi^C(X^C) = 0,$$

$$\{\lambda - \frac{n-1}{4}(\pi(BP))^C\}\pi^C(X^V) + \{\lambda - \frac{n-1}{4}(\pi(BP))^V\}\pi^C(X^C) = 0,$$

which shows that $\lambda = \frac{n-1}{4}(\pi(BP))^C$ and $\lambda = \frac{n-1}{4}(\pi(BP))^V$, because $\pi^C(X^V) \neq 0$ and $\pi^C(X^C) \neq 0$ on TM_n (in general). In view of the last expression, it can easily observe that the Ricci soliton (g^C, P^C, λ) on TM_n is shrinking, steady, and expanding if $\pi(BP) <, =$ and > 0, respectively. Thus, Theorem 5.4 is satisfied. $\square$

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ON THE TRAJECTORIES OF STOCHASTIC FLOW GENERATED BY THE NATURAL MODEL IN MULTI-DIMENSIONAL CASE

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Abstract. Based on the same model stated in [3], we will study the differentiability of the stochastic flow generated by the natural model with respect to the initial data, based on an important idea of H-Kunita, R.M-Dudley and F-Ledrappier. This is the main motivation of our research.
Keywords: Sample path properties, stochastic flow, stochastic integrals

1. Introduction

The notion of the stochastic flow generated by a stochastic differential equation has been studied by several authors. For the differentiability of the stochastic flow, T-Fujiwara and H-Kunita [13] studied the differentiability of stochastic flows for stochastic differential equations with jumps then H-Kunita [6] demonstrated the differentiability of the stochastic flows with respect to the initial data for stochastic differential equations with smooth coefficients. Malliavin [14] demonstrated the differentiability of the solutions of stochastic differential equations according to the initial conditions for classical type equations on manifolds.

Recently, studies concerning the differentiability of the stochastic flow generated by the stochastic differential equations have been developed. A-Y-Pilipenko [15] demonstrated the differentiability of the solution of stochastic differential equations with reflection in the Sobolev space and he showed in [16] the same result

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but with Lipschitz continuous coefficients. In another work, he proved in collaboration with O-V-Aryasova [17] the differentiability of stochastic flow for stochastic differential equations with discontinuous drift in multidimensional case. In [18] K-Burdzy proved the differentiability of stochastic flow of reflected Brownian motions with respect to the initial data in a smooth multidimensional domain. A-Stefano [19] showed the differentiability of the solution for stochastic differential equations with discontinues drift in one-dimensional case. X-Zhang [20] obtained the differentiability of stochastic flow for stochastic differential equations without global Lipschitz coefficients. E-Fedrizzi and F-Flandoli [21] obtained weakly differentiable of solutions of stochastic differential equations with Non-regular drift. Qian Lin [22] studied the differentiability of the solutions of stochastic differential equations driven by G-Brownian motion with respect to the initial data and the parameter. S-Mohammed, T-Nilssen and F-Proske [23] demonstrated the differentiability of stochastic flow for stochastic differential equations with singular coefficients in the Sobolev sense.

In our paper, we consider a following stochastic differential equation:

$$(\natural_u) = \begin{cases} dX^x_{u,t} = X^x_{u,t}\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}N_t + f(X_t-(1-Z_t))dY_t\right), \, t\in[u,\infty[, \\ X^x_{u,u} = x, \end{cases}$$

where x is the initial condition.

This equation is called $\natural$-equation indicated in ([1],[3][5][24]),which is the priceless system in financial mathematics and it's one of the best ways to represent the evolution of a financial market after the default time, it's considered a prosperous system of parameters (Z, Y, f). the parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ.

Let's move to the multidimensional version of $\natural-$equation [3]. On a probability space $(\Omega, (\mathbb{F})_{t\geq 0}, \mathbb{P})$, we have:

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x)-(1-Z_t))dY_t\right), \, t\in[u,\infty[, \\ X_{u,u}(x) = x, \end{cases}$$

where $(\Lambda^1, ..., \Lambda^d)$ is d-dimensional is continuous increasing process null at the origin, $N_t = (N^1, ..., N^d)$ is a given $d-$dimensional continuous non-negative local martingale such that $0 < Z_t = N_t e^{-\Lambda_t} < 1, t > 0$ and $(Z(t,w) = (Z^1(t,w), ..., Z^d(t,w))$ presents the default intensity. $(Y(t,w) = (Y^1(t,w), ..., Y^n(t,w))$ is a given $n-$dimensional continuous local martingale and $F = (F_1, ..., F_n)$ on $\mathbb{R}^n$ is Lipschitz mapping null at the origin.

This equation has a unique solution $X_{u,t}(x)$ such as;

$$X^u_t = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s \sum_{i=1}^d \sum_{j=1}^n F^{ij}(X_s-(1-Z_s))dY^j_s, \, s\in[u,t]$$

where $X_u^u = x$ is the initial condition and F^{ij} is $i - th$ component of the vector function F^j.

The aim of this paper is to show the differentiability of the process X_t^u with respect to the initial value, this property was studied for several stochastic differential equations under different conditions like H-Kunita [6], Bismut [14], Malliavin [14], K.D.Elworthy and Z.Brzezniak [9]. Our paper is based mainly on an idea of R.M-Dudley, H-Kunita and F-Ledrappier [12], such that:

- We demonstrate the existence of the partial derivative for any s, t, x a.s if our stochastic flow generated by the $\natural$-equation in multidimensional case, has a continuous extension at $y = 0$ for any s, t, x a.s and this follows from the estimate given by the proposition of H.Kunita and also the Kolmogorov's theorem. Without forgetting the use of the usual estimation inequalities: Hölder Inequality, BDG inequality, and Gronwall's lemma. This means that the solution is continuously differentiable and the derivative is Hölder continuous.
- We assume the following hypothesis: the coefficients of $\natural-$equation are continuous and the processes represented in this equation take real values.

The rest of the paper is organized as follows: the second section contains generalities which we will need in what follows, the third section represents the obtained results about the differentiability of stochastic flow and the last section gives the main result of this paper.

2. Generalities

Theorem 2.1. *(BDG Inequality)*[11]. *Let $T > 0$ and ξ be a continuous local martingale such that $\xi_0 = 0$. For any $1 \leq p < \infty$ there exists positive constants c_p, C_p independent of T and $(\xi_t)_{0\leq t\leq T}$ such that,*

$$c_p\mathbb{E}[< \xi >_T^{p/2}] \leq \mathbb{E}[(\xi_t^*)^p] \leq C_p\mathbb{E}[< \xi >_T^{p/2}]$$

where $\xi_t^* = \sup_{0\leq t\leq T} |\xi_t|$.

Theorem 2.2. *(Hölder Inequality)*[11]. *Let $1 \leq p, q \leq \infty$ so that $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$ are Lebesgue measurable. Then*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

Proposition 2.1. [6] *Let $2 \leq p < \infty$. There exists a constant R such that, for any $(s, x), (s', x')$ belonging to $[0, T] \times \mathbb{R}^n$,*

$$\mathbb{E}\left[\sup_{s\leq t\leq T} |\xi_{s,t}^x - \xi_{s',t}^{x'}|^p\right] \leq R\left(|x - x'|^p + |s - s'|^{\frac{p}{2}}(1 + |x'|^p)\right) \tag{2.1}$$

Theorem 2.3. *(Kolmogorov's theorem)*[11]. *Let $\xi_\lambda(w)$ be a real valued random field with parameter $\lambda = (\lambda_1, ..., \lambda_d) \in [0,1]^d$. Suppose that there are constants $\gamma > 0$, $\alpha_i > d$, $i = 1, ..., d$ and $C > 0$ such that*

$$\mathbb{E}\left[|\xi_\lambda - \xi_\mu|^\gamma\right] \leq C \sum_{i=1}^{d} |\lambda_i - \mu_i|^{\alpha_i} \ \forall \lambda, \mu \in [0,1]^d. \tag{2.2}$$

Then ξ_λ has a continuous modification $\widetilde{\xi_\lambda}$.

We need also the following importan lemma.

Lemma 2.1. *(Gronwall's lemma)*[11]. *Let $(a,b) \in \mathbb{R}^2$ with $a < b$, φ, β and $\phi : [a,b] \to \mathbb{R}$ non-negative continuous functions, such that $\forall t \in [a,b]$,*

$$\varphi(t) \leq \beta(t) + \int_a^t \varphi(s)\phi(s)ds$$

Then,

$$\forall t \in [a,b], \varphi(t) \leq \beta(t) \exp\left(\int_a^t \phi(s)ds\right)$$

Lemma 2.2. [11] *Let $T > 0$ and p be any real number. Then there is a positive constant $C_{p,T}$ such that $\forall x, y \in \mathbb{R}^d$ and $\forall t \in [0,T]$,*

$$\mathbb{E}\,|J_t(x) - J_s(y)|^p \leq C_{p,T}\,|x - y|^p$$

3. The Found Results on the differentiability of the Solutions of SDE in multi-dimensional case

3.1. The case studied by Olga.V. Aryasova and Andrey.Yu. Pilipenko

This subsection is borrowed from [10]. We consider an SDE of the form:

$$\begin{cases} d\zeta_t(x) = a(\zeta_t(x))dt + dw_t, \\ \zeta_0(x) = x, \end{cases}$$

Where $x \in \mathbb{R}^d$,$d \geq 1$,$(w_t)_{t\geq 0}$ is a d-dimensional Wiener process, $a = (a^1, ..., a^d)$ is a bounded measurable mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$, this equation has a unique strong solution. The differentiability of this solution with respect to initial data is given in the following theorem.

Theorem 3.1. *Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be such that for all $1 \leq i \leq d$, a^i is a function of bounded variation on $\mathbb{R}^d$. Put $\mu^{ij} = \dfrac{\partial a^i}{\partial x_j}$, and assume that the measures $|\mu^{ij}|, 1 \leq$*

$i, j \le d$, belong to Kato's class. Let $\phi_t(x), t \ge 0$, be a solution to the integral equation

$$\phi_t(x) = E + \int_0^t dA_s(\zeta(x))\phi_s(x), \tag{3.1}$$

where E is $d \times d-$identity matrix, the integral on the right-hand side of (3.1) is the Lebesgue-Stieltjes integral with respect to the continuous function of bounded variation $t \to A_t(\zeta(x))$. Then $\phi_t(x)$ is the derivative of $\zeta_t(x)$ in L^p-sense, for all $p > 0$, $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d$, $t > 0$:

$$\mathbb{E}\left\|\frac{\zeta_t(x+h) - \zeta_t(x)}{\epsilon} - \phi_t(x)h\right\|^p \to 0,\, \epsilon \to 0, \tag{3.2}$$

where $\|.\|$ is a norm in the space $\mathbb{R}^d$. Moreover:

$$\mathbb{P}\{\forall t \ge 0 : \zeta_t(.) \in W^1_{p,loc}(\mathbb{R}^d, \mathbb{R}^d),\, \nabla\zeta_t(x) = \phi_t(x)\, for\, \lambda - a.a.x\} = 1,$$

where λ is the Lebesgue measure on $\mathbb{R}^d$.

3.2. The case studied by Philip E. Protter

This subsection is borrowed from [11]. Consider a following system:

$$D : \begin{cases} \varphi^i_t = x_i + \displaystyle\sum_{\alpha=1}^{m} \int_0^t f^i_\alpha(\varphi_{s^-})dZ^\alpha_s \\ D^i_{kt} = \delta^i_k + \displaystyle\sum_{\alpha=1}^{m}\sum_{j=1}^{n} \int_0^t \frac{\partial f^i_\alpha}{\partial x_j}(\varphi_{s^-})D^j_{ks}dZ^\alpha_s \end{cases}$$

$(1 \le i \le n)$ where D denotes an $n \times n$ matrix-valued process and $\delta^i_k = 1$ if $i = k$ and 0 otherwise (Kronecker's delta). A convenient convention, sometimes called the Einstein convention, is to leave the summations implicit. Thus, the system of equations (D) can be alternatively written as:

$$D : \begin{cases} \varphi^i_t = x_i + \displaystyle\int_0^t f^i_\alpha(\varphi_{s^-})dZ^\alpha_s \\ D^i_{kt} = \delta^i_k + \displaystyle\int_0^t \frac{\partial f^i_\alpha}{\partial x_j}(\varphi_{s^-})D^j_{ks}dZ^\alpha_s \end{cases}$$

Theorem 3.2. [11] *Let Z be as in (H_1) and let the functions (f^i_α) in (H_2) have locally Lipschitz first partial derivatives. Then for almost all w there exists a function $\varphi(t, w, x)$ which is continuously differentiable in the open set $\{x : \rho(x, w) > t\}$, where ρ is the explosion time. If (f^i_α) are globally Lipschitz then $\rho = \infty$. Let $D_k(t, w, x) \equiv \frac{\partial}{\partial x_k}\varphi(t, w, x)$. Then for each x the process $(\varphi(., w, x), D(., w, x))$ is identically càdlàg, and it is the solution of equations (D) on $[0, \rho(x, .)]$.*

3.3. The case studied by R.M.Dudley, H.Kunita and F.Ledrappier

This subsection is borrowed from [12]. We shall consider an Itô's stochastic differential equation:

$$d\chi_t = \overline{\xi}_0(t,\chi_t)dt + \sum_{k=1}^{m} \overline{\xi}_k(t,\chi_t)dB_t^k \tag{3.3}$$

has a solution χ_t, $t \in [s,T]$ such that for all $x \in \mathbb{R}^d$

$$\chi_t = x + \int_s^t \overline{\xi}_0(t,\chi_t)dt + \sum_{k=1}^{m} \int_s^t \overline{\xi}_k(t,\chi_t)dB_t^k$$

For the convenience of notations, we will often write dt as dB_t^0 and write the last equation as:

$$\chi_t = x + \sum_{k=1}^{m} \int_s^t \overline{\xi}_k(t,\chi_t)dB_t^k$$

where $x = \chi_s$ be initial condition.
The following theorem give the smoothness property of this solution.

Theorem 3.3. [12] *suppose that coefficients $\overline{\xi}^0, ..., \overline{\xi}^m$ of an Itô's stochastic differential equation, are globally Lipschitz continuous ($C_g^{1,\alpha}$) functions for some $\alpha > 0$ and their first derivatives are bounded. Then the solution $\chi_{s,t}(x)$ is a $C^{1,\beta}$ of x for any β less than α for each $s < t$ a.s.*

4. Main result

This section contains the main result which is concerning the differentiability of the solution of the natural equation with respect to the initial value. But before that we give a detailed description of the natural equation in multidimensional case, we have:

$$(\natural_u) = \begin{cases} dX_{u,t}^1(x) = X_{u,t}^1(x)\left(-\dfrac{e^{-\Lambda_t^1}}{1-Z_t^1}dN_t^1 + F_{11}dY_t^1 + ... + F_{1d}dY_t^n\right) \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ . \qquad\qquad\qquad . \\ dX_{u,t}^d(x) = X_{u,t}^d(x)\left(-\dfrac{e^{-\Lambda_t^d}}{1-Z_t^d}dN_t^d + F_{n1}dY_t^1 + ... + F_{nd}dY_t^n\right) \end{cases}$$

Then

$$(\natural_u) = \begin{cases} dX_{u,t}(x) = X_t(x)\left(-\dfrac{e^{-\Lambda_t}}{1-Z_t}dN_t + F(X_t(x) - (1-Z_t))dY_t\right), \ t \in [u,\infty[, \\ X_{u,u}(x) = x, \end{cases}$$

where $X_{u,t}(x) = (X^1_{u,t}(x), ..., X^d_{u,t}(x))^T$, $-\dfrac{e^{-\Lambda_t}}{1-Z_t} = \left(-\dfrac{e^{-\Lambda^1_t}}{1-Z^1_t}, ..., -\dfrac{e^{-\Lambda^d_t}}{1-Z^d_t}\right)^T$, $x = (x^1, ..., x^d)^T$ the initial condition and:

$$F = \begin{pmatrix} F_{11} & . & . & . & F_{1d} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ F_{n1} & . & . & . & F_{nd} \end{pmatrix}$$

Then we can write the solution X^u_t for $u \leq s \leq t$ in this form:

$$X^u_t = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)dN_s + \int_u^t X_s \sum_{i=1}^d \sum_{j=1}^n F^{ij}\left(X_s - (1-Z_s)\right)dY^j_s,\ s \in [u,t]$$

We introduce the stopping time $\tau_n = \inf\left\{t, 1 - Z_t < \dfrac{1}{n}\right\}$ on the quantity $\left(-\dfrac{e^{-\Lambda_s}}{1-Z_s}\right)$ (because we don't know if it's finite or not). Therefore, we assume the process $\widetilde{X}^x_{u,t}$ instead of $X^x_{u,t}$:

$$\widetilde{X^u_t} = x + \int_u^t \widetilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \widetilde{X}_s \sum_{i=1}^d \sum_{j=1}^n F^{ij}(\widetilde{X}_s - (1-Z_s))dY^j_s,\ s \in [u,t]$$

In order to prove the differentibility property, it's enough to apply the idea of R.M.Dudley, H.Kunita and F.Ledrappier [12]: For $y \in \mathbb{R} \setminus 0$, we define

$$\theta_{u,t}(x,y) = \frac{\partial \widetilde{X}^x_{u,t}}{\partial x_k} = \frac{1}{y}\left[\widetilde{X}^{x+ye_k}_{u,t} - \widetilde{X}^x_{u,t}\right]$$

where e_k is the unit vector $(0, ..., 0, 1, 0, ..., 0)$ for $k = 1...d$.
So we will demonstrate that $\theta_{u,t}(x,y)$ has a continuous extension at $y = 0$ for any (u,t,x). Depending on the following estimate and Kolmogorov's theorem, for any $p > 2$, there exits a positive constant C^p such that:

$$\begin{aligned} & \mathbb{E}|\theta_{u,t}(x,y) - \theta_{u',t'}(x',y')|^p \\ \leq\ & C^p\left[|x-x'|^{\alpha p} + |y-y'|^{\alpha p} + (1+|x|+|x'|)^{\alpha p}(|u-u'|^{\frac{\alpha p}{2}}\right. \qquad (4.1) \\ +\ & \left.|t-t'|^{\frac{\alpha p}{2}})\right] \qquad (4.2) \end{aligned}$$

Proof: Firstly we show the boundedness of $\mathbb{E}|\theta_{u,t}(x,y)|^p$, we have:

$$\theta_{u,t}(x,y) = \frac{1}{y}\left[\widetilde{X}^{x+ye_k}_{u,t} - \widetilde{X}^x_{u,t}\right]$$

we denote

$$M_t = -\frac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}}$$

$$\widetilde{F}^{ij}(\widetilde{X}_t^{x+ye_k}) = \widetilde{X}_t^{x+ye_k} F^{ij}\left(\widetilde{X}_t^{x+ye_k} - (1-Z_t)\right)$$

$$\widetilde{F}^{ij}(\widetilde{X}_t^{x}) = \widetilde{X}_t^{x} F^{ij}\left(\widetilde{X}_t^{x} - (1-Z_t)\right)$$

So

$$\begin{aligned}
\theta_{u,t}(x,y) &= e_k + \frac{1}{y}\left[\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s\right] \\
&+ \frac{1}{y}\left[\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right] \quad (4.3)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + \frac{1}{y}\mathbb{E}\left|\int_u^t \widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x} M_s dN_s\right|^p \\
&+ \frac{1}{y}\sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left|\int_u^t \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x}) dY_s^j\right|^p \quad (4.4)
\end{aligned}$$

Using BDG's inequality, we have:

$$\begin{aligned}
\mathbb{E}|\theta_{u,t}(x,y)|^p &\leq 1 + C_1^p\mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^2 |M_s|^2 ds\right]^{\frac{p}{2}} \\
&+ C_1^p\frac{1}{y}\sum_{i=1}^{d}\sum_{j=1}^{n}\mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})|^2 ds\right]^{\frac{p}{2}} \quad (4.5)
\end{aligned}$$

Now we apply the hölder inequality, noting q the conjugate of $\frac{p}{2}$:

$$\begin{aligned}
&\mathbb{E}|\theta_{u,t}(x,y)|^p \\
\leq\ & 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] + (t-u)^{\frac{p}{2q}} C_1^p \frac{1}{y} \\
&\times \sum_{i=1}^{d}\sum_{n}^{j=1}\mathbb{E}\left[\int_u^t |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^{x})|^p ds\right] \quad (4.6)
\end{aligned}$$

And as $\widetilde{F}^{ij}$ is Lipschitz, we have:

$$\left|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s)\right| \leq k_1 \left|\widetilde{X}_s^{x+ye_k} - \widetilde{X}_s^{x}\right|$$

Therefore

$$\begin{aligned}
\mathbb{E}\,|\theta_{u,t}(x,y)|^p &\leq 1 + (t-u)^{\frac{p}{2q}} C_1^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \\
&+ (t-u)^{\frac{p}{2q}} k_1 C_1^p \mathbb{E}\left[\int_u^t |\theta_{r,s}(x,y)|^p ds\right] \quad (4.7)
\end{aligned}$$

and by following, we have $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, so:

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^{2p}\right] + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \tag{4.8}$$

Then the proposition(2.1), yields for any $x \in \mathbb{R}^d$ and a constant c':

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq \frac{1}{2}c' + \frac{1}{2}\left[\int_u^t \mathbb{E}|M_s|^p ds\right]^2 \tag{4.9}$$

Furthermore, we have the quantity $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] < \infty$. Next, note that $\mathbb{E}\left[\int_u^t |M_s|^p ds\right] = \overline{R}$, then:

$$\mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y)|^p \int_u^t |M_s|^p ds\right] \leq C_2^p + C_3^p \overline{R}^2 \tag{4.10}$$

where $\frac{1}{2}c'(t-u)^{\frac{p}{2q}} C_1^p = C_2^p$ and $\frac{1}{2}(t-u)^{\frac{p}{2q}} C_1^p = C_3^p$. As a result:

$$\mathbb{E}\,|\theta_{u,t}(x,y)|^p \leq C_4^p + C_5^p \int_u^t \mathbb{E}|\theta_{r,s}(x,y)|^p ds \tag{4.11}$$

Where $C_4^p = C_2^p + C_3^p \overline{R}^2$ and $C_5^p = (t-u)^{\frac{p}{2q}} k_1 C_1^p$, therefore by Gronwall's lemma, we get:

$$\mathbb{E}\,|\theta_{u,t}(x,y)|^p \leq C_4^p \exp\left(C_5^p (t-u)\right) \tag{4.12}$$

Consequently $\mathbb{E}|\theta_{u,t}(x,y)|^p$ is bounded. Secondly we prove the estimate (4.1). In case $t = t'$, we suppose that $u < u' < t$. Other cases will be proven in the same way. Then we have:

$$\begin{aligned} &\theta_{u,t}(x,y) - \theta_{u',t}(x',y') \\ = \quad &\int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s + \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) \\ - \quad &\widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j \end{aligned} \tag{4.13}$$

Noting

$$\widetilde{I}_1 = \int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y') M_s dN_s$$

$$\widetilde{I}_2 = \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} \widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x) - \widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'}) dY_s^j$$

So

$$\mathbb{E}|\widetilde{I}_1|^p = \mathbb{E}\left|\int_u^{u'} \theta_{r,s}(x,y) - \theta_{r',s}(x',y')M_s dN_s\right|^p \tag{4.14}$$

The BDG's inequality leads to:

$$\mathbb{E}|\widetilde{I}_1|^p \leq C_6^p \mathbb{E}\left[\int_u^{u'} |\theta_{r,s}(x,y) - \theta_{r',s}(x',y')|^2 |M_s|^2 ds\right]^{\frac{p}{2}} \tag{4.15}$$

using Hölder's inequality, noting q^* the conjugate of $\frac{p}{2}$:

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_6^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^p \int_u^{u'} |M_s|^p ds\right] \tag{4.16}$$

and by following, we have $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$:

$$\begin{aligned}\mathbb{E}|\widetilde{I}_1|^p &\leq (u'-u)^{\frac{p}{2q^*}} C_7^p \mathbb{E}\left[\sup_{u<t<\infty} |\theta_{u,t}(x,y) - \theta_{u',t}(x',y')|^{2p}\right] \\ &+ (u'-u)^{\frac{p}{2q^*}} C_7^p \left[\int_u^{u'} \mathbb{E}|M_s|^p ds\right]^2 \end{aligned}\tag{4.17}$$

Then the proposition (2.1), gives:

$$\mathbb{E}|\widetilde{I}_1|^p \leq (u'-u)^{\frac{p}{2q^*}} C_7^p \left[R_1|y-y'|^{2p} + \overline{R}_1^2\right] \tag{4.18}$$

where $C_7^p = \frac{1}{2}C_6^p$.
it remains to study the term $\widetilde{I}_2$:

$$\begin{aligned}|\widetilde{I}_2| &\leq \frac{1}{y}\sum_{i=1}^d \sum_{j=1}^n \int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)| \\ &+ |-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})| dY_s^j \end{aligned}\tag{4.19}$$

Using again the BDG's inequality, we obtain:

$$\begin{aligned}&\mathbb{E}|\widetilde{I}_2|^p \\ &\leq \frac{1}{y}C_8^p \sum_{i=1}^d \sum_{j=1}^n \mathbb{E}\left[\int_u^{u'} |\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}_s^x)|^2\right. \\ &\left. +|-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}_s^{x'})|^2 ds\right]^{\frac{p}{2}} \end{aligned}\tag{4.20}$$

applying Hölder's inequality, noting q^* the conjugate of $\dfrac{p}{2}$, we have:

$$\begin{aligned}\mathbb{E}|\widetilde{I}_2|^p &\leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}\sum_{i=1}^{d}\sum_{j=1}^{n}\int_u^{u'}\mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}_s^{x})|^p \\ &+ \mathbb{E}|\widetilde{F}^{ij}(\widetilde{X}_s^{x'})-\widetilde{F}^{ij}(\widetilde{X}_s^{x'+y'e_k})|^p ds \end{aligned}\tag{4.21}$$

We have always $\widetilde{F}$ is Lipschitz:

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}C_8^p(u'-u)^{\frac{p}{2q^*}}k_1\int_u^{u'}\mathbb{E}|\widetilde{X}_s^{x+ye_k}-\widetilde{X}_s^{x}|^p+\mathbb{E}|\widetilde{X}_s^{x'}-\widetilde{X}_s^{x'+y'e_k}|^p ds \tag{4.22}$$

Thus, by lemma (2.2), we have:

$$\mathbb{E}|\widetilde{I}_2|^p \leq \frac{1}{y}K_{p,T}^1C_8^p\,(u'-u)^{\frac{p}{2q^*}+1}\,k_1\,(|y|^p+|y'|^p) \tag{4.23}$$

From (4.18)and (4.23), we obtain:

$$\mathbb{E}\left|\theta_{u,t}(x,y)-\theta_{u',t}(x',y')\right|^p \leq C_9^p\,(u'-u)^{\frac{p}{2q^*}} \tag{4.24}$$

Where $C_9^p = C_7^p(R_1|y-y'|^{2p}+\overline{R}_1^2)+\frac{1}{y}K_{p,T}^1C_8^p(u'-u)k_1(|y|^p+|y'|^p)$.
It remains Kolmogorov's theorem, we denote $G=\theta_{u,t}(x,y)-\theta_{u',t'}(x',y')$ and simply applying Itô's formula to the function $f(G)=|G|^p$ for $t=t'$, we obtain

$$|G|^p=\sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s+\frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

noting

$$\widehat{I}=\sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)dG_s$$

$$\overline{I}=\frac{1}{2}\sum_{i,j}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)d<G^i,G^j>_s$$

such that

$$\begin{aligned}\widehat{I} &= \sum_{i,j}\int_u^{u'}\frac{\partial f}{\partial G_i}(G)\Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}^{ij}(\widetilde{X}^{x+ye_k})-\widetilde{F}^{ij}(\widetilde{X}^{x}) \\ &\quad -\widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k})+\widetilde{F}^{ij}(\widetilde{X}^{x'})dY_s^j\Big]\end{aligned}$$

Then

$$\begin{aligned}\overline{I} &= \sum_{i,j,h,l}\int_u^{u'}\frac{\partial^2 f}{\partial G_iG_j}(G)\Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}_l^i(\widetilde{X}^{x+ye_k})-\widetilde{F}_l^i(\widetilde{X}^{x}) \\ &\quad -\widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k})+\widetilde{F}_l^i(\widetilde{X}^{x'})dY_s^l\Big] \\ &\times \Big[G_sM_sdN_s+\frac{1}{y}\widetilde{F}_h^j(\widetilde{X}^{x+ye_k})-\widetilde{F}_h^j(\widetilde{X}^{x})-\widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k})+\widetilde{F}_h^j(\widetilde{X}^{x'})dY_s^h\Big]\end{aligned}$$

For $\widehat{I}$, we denote:

$$\frac{\partial f}{\partial G_i}(G) = |p||G|^{P-1}$$

$$\widehat{I}_1 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G) G_s M_s dN_s$$

$$\widehat{I}_2 = \sum_i \int_u^{u'} \frac{\partial f}{\partial G_i}(G) \frac{1}{y} \widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'})$$

So, we have

$$\sum_i \left| \frac{\partial f}{\partial G_i}(G) G_s \right| \leq d|p||G|^{P-1}|G_s| \tag{4.25}$$

Then

$$|\widehat{I}_1| \leq d|p| \int_u^{u'} |G_s|^P ds \times \int_u^{u'} M_s dN_s \tag{4.26}$$

noting $\varphi_t = \int_u^{u'} M_s dN_s$, it's a local martingale (see [2]):

$$|\widehat{I}_1| \leq d|p|\varphi_t \int_u^{u'} |G_s|^P ds \tag{4.27}$$

And we have $\widetilde{F}^{ij}(\widetilde{X}^x)$ is Lipschitz function, therefore:

$$\sum_i \left| \frac{\partial f}{\partial G_i}(G) \frac{1}{y} \widetilde{F}^{ij}(\widetilde{X}^{x+ye_k}) - \widetilde{F}^{ij}(\widetilde{X}^{x}) - \widetilde{F}^{ij}(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^{ij}(\widetilde{X}^{x'}) \right| \leq d\, k_1\, |p|\, |G|^P \tag{4.28}$$

Then

$$|\widehat{I}_2| \leq d\, n\, k_1\, |p| \int_u^{u'} |G_s|^P ds \tag{4.29}$$

From (4.27) and (4.29), we get:

$$|\widehat{I}| \leq d\, |p|(\varphi_t + n\, k_1) \int_u^{u'} |G_s|^P ds \tag{4.30}$$

For $\overline{I}$, we denote

$$\overline{I}_1 = \sum_{i,j,h,l} \int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2 (M_s)^2 dN_s dN_s \tag{4.31}$$

$$\begin{aligned} \overline{I}_2 &= \frac{1}{y} \sum_{i,j,h,l} \int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G) G_s M_s \widetilde{F}^i_l(\widetilde{X}^{x+ye_k}) \\ & -\widetilde{F}^i_l(\widetilde{X}^{x}) - \widetilde{F}^i_l(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}^i_l(\widetilde{X}^{x'}) dN_s dY^l_s \end{aligned} \tag{4.32}$$

$$\overline{I}_3 = \frac{1}{y}\sum_{i,j,h,l}\int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G) G_s M_s \widetilde{F}_h^j(\widetilde{X}^{x+ye_k}) \tag{4.33}$$
$$-\widetilde{F}_h^j(\widetilde{X}^x) - \widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_h^j(\widetilde{X}^{x'}) dN_s dY_s^h$$

$$\overline{I}_4 = \frac{1}{y^2}\sum_{i,j,h,l}\int_u^{u'} \frac{\partial^2 f}{\partial G_i G_j}(G)\left[\widetilde{F}_l^i(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^i(\widetilde{X}^x) - \widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^i(\widetilde{X}^{x'})\right]$$
$$\times \left[\widetilde{F}_h^j(\widetilde{X}^{x+ye_k}) - \widetilde{F}_h^j(\widetilde{X}^x) - \widetilde{F}_h^j(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_h^j(\widetilde{X}^{x'})\right] dY_s^l dY_s^h$$

And note that

$$\frac{\partial^2 f}{\partial G_i G_j}(G) = p(p-1)|G|^{p-2}$$

Then for $\overline{I}_1$, we have

$$\sum_{i,j,h,l}\left|\frac{\partial^2 f}{\partial G_i G_j}(G)(G_s)^2\right| \leq d\,|p|\,|p-1|\,|G|^{p-2}|G|^2 \tag{4.34}$$

So

$$|\overline{I}_1| \leq d\,|p|\,|p-1|\int_u^{u'} |G_s|^p M_s^2 dN_s dN_s \tag{4.35}$$

$\int_u^{u'} M_s dN_s$ is always a local martingale, so

$$|\overline{I}_1| \leq d\,|p|\,|p-1|\,\varphi_t^2\int_u^{u'} |G_s|^p ds \tag{4.36}$$

For $\overline{I}_2$, we have

$$\sum_{i,j,h,l}\frac{1}{y}\left|\frac{\partial^2 f}{\partial G_i G_j}(G) G_s \widetilde{F}_l^i(\widetilde{X}^{x+ye_k}) - \widetilde{F}_l^i(\widetilde{X}^x) - \widetilde{F}_l^i(\widetilde{X}^{x'+y'e_k}) + \widetilde{F}_l^i(\widetilde{X}^{x'})\right|$$
$$\leq d\,n\,k_1|p|\,|p-1|\,|G|^{p-2}|G_s|^2 \tag{4.37}$$

Therefore we get

$$|\overline{I}_2| \leq d\,n\,k_1|p|\,|p-1|\,\varphi_t^2\int_u^{u'} |G_s|^p ds \tag{4.38}$$

For $\overline{I}_3$, we have

$$|\overline{I}_3| \leq d\,n\,k_1|p|\,|p-1|\,\varphi_t^2\int_u^{u'} |G_s|^p ds \tag{4.39}$$

For $\overline{I}_4$, we have

$$\overline{I}_4 \leq d\,n\,k_1^2|p|\,|p-1|\int_u^{u'} |G_s|^p ds \tag{4.40}$$

Then we have

$$\overline{I} = \frac{1}{2}\left[\overline{I}_1 + \overline{I}_2 + \overline{I}_3 + \overline{I}_4\right] \tag{4.41}$$

Such that

$$\overline{I} \leq \frac{1}{2}\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2)\,d\,|p|\,|p-1| \int_u^{u'} |G_s|^p ds \tag{4.42}$$

From these two inequalities (4.30) and (4.42), we get

$$|G|^p \leq d\,|p|\,(\frac{1}{2}\,|p-1|\,(2\,n\,k_1\,\varphi_t + \varphi_t^2 + n\,k_1^2) + \varphi_t + n\,k_1) \int_u^{u'} |G_s|^p ds \tag{4.43}$$

Therefore

$$\mathbb{E}|G|^p \leq C_{10}^p \int_u^{u'} \mathbb{E}|G_s|^p ds \tag{4.44}$$

By Grönwall's inequality we have

$$\mathbb{E}|G|^p \leq C_{11}^p \tag{4.45}$$

where C_{11}^p is $\exp\left(C_{10}^p(u'-u)\right)$.

The proof is completed.

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HORIZONTAL LIFT METRIC ON THE TANGENT BUNDLE OF A WEYL MANIFOLD

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Abstract. Let $(M,[g])$ be a Weyl manifold and TM its tangent bundle equipped with the horizontal lift of the base metric. The purpose of this paper is to study the tangent bundle TM endowed with a Weyl structure, and obtain the ide under which conditions such bundle is an Einstein-Weyl or a gradient Weyl-Ricci soliton.
Keywords: Riemannian metric, Weyl structure, tangent bundle

1. Introduction

Weyl geometry is, in a sense, midway between Riemannian geometry and affine geometry. A Weyl manifold is a conformal manifold equipped with an affine connection preserving the conformal structure, called a Weyl connection. It is said to be Einstein-Weyl if and only if the symmetric part of Ricci tensor is proportional to a Riemannian metric in the conformal class (see [5],[6] and [9]). As a generalization, in [4], the authors introduced a new notion, namely gradient Weyl-Ricci soliton, involving Hessian of a smooth function.

There exists a wide range of interesting studies on the geometry of tangent bundles with special types of metrics (Sasaki, Cheeger-Gromoll,...) or more generally g-natural metrics (see [1],[2] and [7]). A pseudo-Riemannian metric on the tangent bundle is defined by the horizontal lift of the base metric (see [8] and [10]).

Tangent bundle of a Weyl manifold is a very recent topic. In [3], Bejan and Gul constructed a Weyl structure on the tangent bundle and find conditions under which

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the tangent bundle is an Einstein-Weyl manifold. In [4], Bejan *et al.* obtained some conditions such that the Weyl structure on the tangent bundle is a gradient Weyl-Ricci soliton. In both studies, the tangent bundle is considered with the Sasaki metric.

In this paper, we introduce a Weyl structure on the tangent bundle of a Weyl manifold and prove that the tangent bundle cannot be an Einstein-Weyl manifold or a gradient Weyl-Ricci soliton unless the base manifold is locally flat. Here, the tangent bundle is endowed with horizontal lift metric.

Unless otherwise stated, throughout the paper, the Einstein summation convention is used and all geometric objects are considered as smooth.

2. Weyl manifolds

We recall the basic information about Weyl geometry from [3]. Let M be an $m-$dimensional manifold endowed with a conformal class of (pseudo) Riemannian metrics $[g]$. A torsion-free connection D is said to be a Weyl connection if it preserves the conformal class $[g]$. For a metric $g \in [g]$, there exists a 1-form ω determined by D as $Dg = -2\omega \otimes g$. If ∇ is the Levi-Civita connection of g, then D is expressed as follows:

$$D_X Y = \nabla_X Y + \omega(Y)X + \omega(X)Y - g(X,Y)\xi, \ \forall X, Y \in \Gamma(TM), \tag{2.1}$$

where ξ is the dual vector field of ω with respect to g. Conversely, if ω is given and if we use the equation (2.1) to define D, then D is a Weyl connection. Note that we have $g(\xi,\xi) = \|\xi\|^2 = \omega(\xi)$ and the relation (2.1) is invariant under the Weyl transformation $e \to e^{2f}g$, $\omega' = \omega - df$. The pair (g,ω) is called a Weyl structure on M.

Denote by $R_g = [\nabla,\nabla] - \nabla_{[,]}$ and $R_{[g]} = [D,D] - D_{[,]}$ the curvature tensors of the Levi-Civita connection ∇ and the Weyl connection D, respectively. Then the relation between them is given by

$$\begin{aligned} R_{[g]}(X,Y)Z \ = \ & R_g(X,Y)Z + d\omega(X,Y)Z - ((\nabla_Y\omega)(Z))X + ((\nabla_X\omega)(Z))Y \\ & +\omega(Y)\omega(Z)X - g(Y,Z)\nabla_X\xi - g(Y,Z)\omega(\xi)X \\ & +g(Y,Z)\omega(X)\xi - \omega(X)\omega(Z)Y + g(X,Z)\nabla_Y\xi \\ & +g(X,Z)\omega(\xi)Y - g(X,Z)\omega(Y)\xi, \forall X,Y,Z \in \Gamma(TM). \end{aligned} \tag{2.2}$$

From (2.2), the relation between the Ricci tensor field $Ric_{[g]}$ of the Weyl connection D and the Ricci tensor field Ric_g of the Levi-Civita connection ∇ is given by

$$\begin{aligned} Ric_{[g]}(X,Y) \ = \ & Ric_g(X,Y) + d\omega(X,Y) + (\delta\omega - (m-2)\,\|\xi\|^2)g(X,Y) \\ & -(m-2)(\nabla_X\omega)Y + (m-2)\omega(X)\omega(Y), \ \forall X,Y \in \Gamma(TM), \end{aligned}$$

where the co-differential $\delta\omega$ of ω is defined by $\delta\omega = -tr_g\{(U,V) \to (\nabla_U\omega)V\}$.

The symmetric part $Ric^{sym}_{[g]}$ of $Ric_{[g]}$ is given by following formula:

$$\begin{aligned} Ric^{sym}_{[g]}(X,Y) &= Ric_g(X,Y) + (\delta\omega - (m-2)\,\|\xi\|^2)g(X,Y) \\ &\quad -\frac{1}{2}(m-2)[(\nabla_X\omega)Y + (\nabla_Y\omega)X] \\ &\quad +(m-2)\omega(X)\omega(Y), \forall X,Y \in \Gamma(TM). \end{aligned} \tag{2.3}$$

3. Tangent bundle

Let M be an $m-$dimensional manifold. Its tangent bundle is denoted by TM and $\pi : TM \to M$ is natural projection mapping. Recall that TM is a $2m-$dimensional differentiable manifold. Let (U, x^j) be a coordinate neighborhood of M, where (x^j) is a system of local coordinates defined in the neighborhood U. Let (u^j) be the system of cartesian coordinates in each tangent space of M with respect to the natural frame $\{\frac{\partial}{\partial x^j}\}$. Then, in $\pi^{-1}(U)$, we can introduce the local coordinates $(\pi^{-1}(U), x^j, u^j)$, which are called the induced coordinates. From now on, we denote the induced coordinates by $(x^J) = (x^j, x^{\bar{j}}) = (x^j, u^j)$, $j = 1, ..., m$, $\bar{j} = m+1, ..., 2m$. We also denote the natural frame in $\pi^{-1}(U)$ by $(\frac{\partial}{\partial x^J}) = (\frac{\partial}{\partial x^j}, \frac{\partial}{\partial u^j})$.

If $X = X^i\frac{\partial}{\partial x^i}$ is the local expression of a vector field X in U, then the vertical lift X^V and the horizontal lift X^H of X are given, with respect to the induced coordinates, by

$$X^V = X^i\frac{\partial}{\partial u^i}, \quad X^H = X^i\frac{\partial}{\partial x^i} - X^j\Gamma^i_{jk}u^k\frac{\partial}{\partial u^i},$$

where Γ^i_{jk} are the coefficients of a torsion-free affine connection ∇.

If f is a function on M, then the vertical lift f^V of f is defined by $f^V = f \circ \pi$. The horizontal lift f^H of f is $f^H = 0$.

Let ω be a 1-form on M. Then the horizontal lift ω^H of ω is given by the relations $\omega^H(X^H) = 0$, $\omega^H(X^V) = (\omega(X))^V$. The vertical lift ω^V of ω is given by the relations $\omega^V(X^V) = 0$, $\omega^V(X^H) = (\omega(X))^V$.

From [10], the horizontal lift metric G on the tangent bundle TM over the Riemannian manifold (M, g) is defined by the equations

$$\begin{aligned} G(X^H, Y^H) &= G(X^V, Y^V) = 0, \\ G(X^V, Y^H) &= G(X^H, Y^V) = g(X,Y), \forall X,Y \in \Gamma(TM). \end{aligned} \tag{3.1}$$

For the Levi-Civita connection $\bar{\nabla}$ of the metric G, we have

$$\begin{aligned} \overline{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H + (R_g(u,X)Y)^V, \\ \overline{\nabla}_{X^H}Y^V &= (\nabla_X Y)^V, \\ \overline{\nabla}_{X^V}Y^H &= 0, \\ \overline{\nabla}_{X^V}Y^V &= 0,\ \forall X,Y \in \Gamma(TM), \end{aligned} \tag{3.2}$$

where R_g is the curvature tensor field of the metric g. Non-zero components of the curvature tensor $\bar{R}_G$ and the Ricci tensor $\overline{Ric}_G$ are given by

$$\begin{aligned}
\bar{R}_G(X^H,Y^H)Z^H &= (R_g(X,Y)Z)^H + ((\nabla_u R_g)(X,Y)Z)^V, \\
\bar{R}_G(X^H,Y^H)Z^V &= \bar{R}_G(X^H,Y^V)Z^H = (R_g(X,Y)Z)^V, \\
\overline{Ric}_G(X^H,Y^H) &= 2Ric_g(X,Y), \forall X,Y \in \Gamma(TM),
\end{aligned}$$

where Ric_g is the Ricci tensor field of the metric g (see [8] and [10]).

4. A Weyl structure on tangent bundle

In this section, we construct a Weyl structure on (TM,G) using the vertical lift of a 1-form on M. Firstly, we write the following proposition from the definition of the metric G in (3.1).

Proposition 4.1. *Let (M,g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift metric G. Any conformal change $g \to e^{2f}g$ on M corresponds the change of the metric $G \to (e^{2f})^V G$ on TM.*

Now we can express the proposition below.

Proposition 4.2. *Let (M,g) be a Riemannian manifold and TM its tangent bundle with the horizontal lift metric G. If the pair (g,ω) is a Weyl structure on M, then the pair (G,ω^V) is a Weyl structure on TM and its Weyl connection is given by*

$$\begin{aligned}
\overline{D}_{X^H}Y^H &= (D_XY - g(X,Y)\xi)^H + (R_g(u,X)Y)^V, \\
\overline{D}_{X^H}Y^V &= (\nabla_XY + \omega(X)Y)^V - g(X,Y)\xi^H, \\
\overline{D}_{X^V}Y^H &= \omega(Y)X^V - g(X,Y)\xi^H, \\
\overline{D}_{X^V}Y^V &= 0,
\end{aligned} \tag{4.1}$$

where D is the Weyl connection on M, R_g is the curvature tensor field of g and ξ is the dual vector field of ω with respect to g.

Proof. Using the relations (3.2) in (2.1) give the result. □

Lemma 4.1. *Let M be an $m-$dimensional manifold $(m>2)$ endowed with the Weyl structure (g,ω) and TM its tangent bundle endowed with the Weyl structure (G,ω^V), where G is the horizontal lift metric. The symmetric part $\overline{Ric}^{sym}_{[G]}$ of the Ricci tensor field of the Weyl structure (G,ω^V) satisfies the following relations*

$$\begin{aligned}
\overline{Ric}^{sym}_{[G]}(X^H,Y^H) &= 2Ric_g(X,Y) - (m-1)[(\nabla_X\omega)Y + (\nabla_Y\omega)X] \\
&\quad +2(m-1)\omega(X)\omega(Y),
\end{aligned} \tag{4.2}$$

$$\overline{Ric}^{sym}_{[G]}(X^V,Y^H) = \delta\omega g(X,Y), \tag{4.3}$$

$$\overline{Ric}^{sym}_{[G]}(X^V,Y^V) = 0, \tag{4.4}$$

where ∇ is the Levi-Civita connection on M and Ric_g is the Ricci tensor field of g.

Proof. We use the formula (2.3). Since TM is a $2m-$dimensional manifold, we have

$$\begin{aligned}
\overline{Ric}^{sym}_{[G]}(X^H, Y^H) &= \overline{Ric}_G(X^H, Y^H) \\
&\quad +(\delta(\omega^V) - 2(m-1)G(\xi^H, \xi^H))G(X^H, Y^H) \\
&\quad -(m-1)[(\overline{\nabla}_{X^H}\omega^V)Y^H + (\overline{\nabla}_{Y^H}\omega^V)X^H] \\
&\quad +2(m-1)\omega^V(X^H)\omega^V(Y^H) \\
&= 2(Ric_g(X,Y))^V \\
&\quad -(m-1)[((\nabla_X\omega)Y)^V + ((\nabla_Y\omega)X)^V] \\
&\quad +2(m-1)[\omega(X)\omega(Y)]^V \\
&= 2Ric_g(X,Y) - (m-1)[(\nabla_X\omega)Y + (\nabla_Y\omega)X] \\
&\quad +2(m-1)\omega(X)\omega(Y).
\end{aligned}$$

By the same way, we obtain (4.3) and (4.4). □

Now we give the main results.

Theorem 4.1. *Let M be an $m-$dimensional manifold ($m > 2$) and TM be its tangent bundle such that M and TM are endowed with the Weyl structures (g, ω) and (G, ω^V), respectively. If the following conditions are satisfied, then TM is an Einstein-Weyl manifold:*

(i) (M, g) is flat.

$$(ii)(\nabla_X\omega)Y + (\nabla_Y\omega)X = 2\omega(X)\omega(Y), \forall X, Y \in \Gamma(TM).$$

Proof. It is known that TM is an Einstein-Weyl manifold if there exists a function $\bar{\alpha}$ such that $\overline{Ric}^{sym}_{[G]} = \overline{\alpha}G(\tilde{X}, \tilde{Y})$ for all vector fields $\tilde{X}, \tilde{Y}$ on TM.

Assume that $(\nabla_X\omega)Y + (\nabla_Y\omega)X = 2\omega(X)\omega(Y)$, then (2.3) becomes

$$Ric^{sym}_{[g]}(X, Y) = Ric_g(X, Y) + (\delta\omega - (m-2)\left\|\xi\right\|^2)g(X, Y), \tag{4.5}$$

$\forall X, Y \in \Gamma(TM)$. If we suppose M is flat, i.e. $R_g = 0$, then the formulas (4.2), (4.3) and (4.4) reduce to

$$\begin{aligned}
Ric^{sym}_{[G]}(X^H, Y^H) &= 0, \\
Ric^{sym}_{[G]}(X^V, Y^H) &= \delta\omega g(X, Y), \\
Ric^{sym}_{[G]}(X^V, Y^V) &= 0, \forall X, Y \in \Gamma(TM).
\end{aligned}$$

These equations show that if $\bar{\alpha} = (\delta\omega)^V$, then TM is an Einstein-Weyl manifold. This completes the proof. □

Theorem 4.2. *Let M be an $m-$dimensional manifold ($m > 2$) and TM be its tangent bundle such that M and TM are endowed with the Weyl structures (g, ω) and (G, ω^V), respectively. If the following conditions are satisfied, then the triple (G, ω^V, f^V) is a gradient Weyl-Ricci soliton:*

(i) (M,g) *is flat.*

(ii)

$$(4.6)\quad (\nabla_X\omega)Y+(\nabla_Y\omega)X-2\omega(X)\omega(Y)=Hess_g f(X,Y),\forall X,Y\in\Gamma(TM),$$

where $Hess_g f$ *denotes the Hessian of the function* f *on* M *with respect to the metric* g.

Proof. For (G,ω^V,f^V) to be a gradient Weyl Ricci soliton, it should satisfy

$$\overline{Ric}^{sym}_{[G]}+Hess_G f^V=\overline{\alpha}G, \tag{4.7}$$

where $\bar{\alpha}$ is a function on TM (see [4]).

For the Hessian of the function f^V with respect to G, we get the following relations by direct computations:

$$\begin{aligned}
Hess_G f^V(X^H,Y^H) &= (Hess_g f(X,Y))^V,\\
Hess_G f^V(X^H,Y^V) &= 0,\\
Hess_G f^V(X^V,Y^H) &= 0,\\
Hess_G f^V(X^V,Y^V) &= 0,\forall X,Y\in\Gamma(TM).
\end{aligned}$$

Suppose that (4.6) holds, then from (2.3) we have

$$\begin{aligned}
Ric^{sym}_{[g]}(X,Y) &= Ric_g(X,Y)+(\delta\omega-(m-2)\left\|\xi\right\|^2)g(X,Y)\\
&\quad -\frac{(m-2)}{2(m-1)}Hess_g f(X,Y),\forall X,Y\in\Gamma(TM).
\end{aligned}\tag{4.8}$$

If (M,g) flat, then the formulas (4.2), (4.3) and (4.4) turn into

$$\begin{aligned}
\overline{Ric}^{sym}_{[G]}(X^H,Y^H) &= -Hess_g f(X,Y)\\
\overline{Ric}^{sym}_{[G]}(X^V,Y^H) &= \delta\omega g(X,Y),\\
\overline{Ric}^{sym}_{[G]}(X^V,Y^V) &= 0,\forall X,Y\in\Gamma(TM).
\end{aligned}$$

So, for $\bar{\alpha}=(\delta\omega)^V$, TM is a gradient-Weyl Ricci soliton. This completes the proof. □

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NOTES ON LEFT IDEALS OF SEMIPRIME RINGS WITH MULTIPLICATIVE GENERALIZED $(\alpha, \alpha)-$ DERIVATIONS

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Abstract. Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative (generalized) $(\alpha, \alpha)-$derivation of R associated with a multiplicative $(\alpha, \alpha)-$derivation d. In this note, we will give the description of commutativity of semiprime rings with help of some identities involving a multiplicative generalized $(\alpha, \alpha)-$derivation and a nonzero left ideal of R.
Keywords: Derivations, ideals, semiprime rings.

1. Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $xoy = xy + yx$. Recall that a ring R is prime if for $x, y \in R$, $xRy = (0)$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = (0)$ implies $x = 0$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An immediate example of a derivation is the inner derivation (i.e., a mapping $x \to [a, x]$, where a is a fixed element). By the generalized inner derivation we mean an additive mapping $F : R \to R$ such that for fixed elements $a, b \in R, F(x) = ax + xb$ for all $x \in R$. It observed that F satisfies the relation $F(xy) = F(x)y + xI_{-b}(y)$ for all $x, y \in R$, where $I_{-b}(y) = [-b, y]$ is the inner derivation of R associated with the element $(-b)$. Motivated by these observations,

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M. Brešar [3] introduced the notion of generalized derivation. Accordingly, a generalized derivation $F : R \to R$ is an additive mapping which is uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Obviously, every derivation is a generalized derivation. Thus, generalized derivations cover both the concept of derivations and left multipliers (i.e., an additive mapping such that $F(xy) = F(x)y$, for all $x, y \in R$). Generalized derivations have been primarily studied on operator algebras.

In [4], the notion of multiplicative derivation was introduced by Daif motivated by Martindale in [13]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. In [10], Goldman and Šemrl gave the complete description of these maps. We have $R = C[0, 1]$, the ring of all continuous (real or complex valued) functions and define a mapping $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x)\log|f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that d is a multiplicative derivation, but d is not additive.

On the other hand, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [6]. $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking d is any mapping (not necessarily an additive mapping or a derivation) in [7]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, generalized derivations and the left multipliers.

Over the last several years, a number of authors studied commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations on appropriate subsets of R. Herstein proved that if R is a $2-$torsion free prime ring with a nonzero derivation d of R such that $[d(x), d(y)] = 0$, for all $x, y \in R$, then R is commutative ring. In [5], Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R contains a nonzero central ideal. Many authors extended these classical theorems to the class of derivations. (see [1], [2], [8], [9], [11], [12] for a partial bibliography).

In the present paper, we generalize the concept of multiplicative generalized derivations to multiplicative generalized $(\alpha, \alpha)-$derivations. A mapping $d : R \to R$ (not necessarily additive) is called a multiplicative $(\alpha, \alpha)-$derivation if there exists a map $\alpha : R \to R$ such that $d(xy) = d(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative generalized $(\alpha, \alpha)-$derivation if $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$, where d is a multiplicative $(\alpha, \alpha)-$derivation of R. Of course a multiplicative generalized $(1, 1)-$derivation where 1 is the identity map on R is a multiplicative generalized derivation. So, it would be interesting to extend some results concerning these notions to multiplicative generalized $(\alpha, \alpha)-$derivations. Our aim is to investigate some identities with multiplicative generalized (α, α) $-$derivations on a nonzero left ideal of semiprime ring R.

2. Results

Throughout the paper, R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and F a multiplicative (generalized) (α, α) $-$derivation of R associated with a multiplicative $(\alpha, \alpha)-$derivation d. Also, we will make some extensive use of the basic commutator identities:

i) $[x, yz] = y[x, z] + [x, y]z$

ii) $[xy, z] = [x, z]y + x[y, z]$

iii) $xyoz = (xoz)y + x[y, z] = x(yoz) - [x, z]y$

iv) $xoyz = y(xoz) + [x, y]z = (xoy)z - y[z, x]$

v) $[xy, z]_{\alpha,\alpha} = x\,[y, z]_{\alpha,\alpha} + [x, \alpha(z)]\,y = x[y, \alpha(z)] + [x, z]_{\alpha,\alpha}\,y$

vi) $[x, yz]_{\alpha,\alpha} = \alpha(y)\,[x, z]_{\alpha,\alpha} + [x, y]_{\alpha,\alpha}\,\alpha(z)$

vii) $(xz \circ y)_{\alpha,\alpha} = x(z \circ y)_{\alpha,\alpha} - [x, \alpha(y)]z.$

We remind some well known results which will be useful in our proofs:

Fact : Let R be a semiprime ring, then

i) The center of R contains no nonzero nilpotent elements.

ii) If P is a nonzero prime ideal of R and $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$.

iii) The center of a nonzero one sided ideal is contained in the center of R. In particular, any commutative one sided ideal is contained in the center of R.

Lemma 2.1. [12, Lemma 5] *Let R be a $2-$torsion-free semiprime ring and I a nonzero ideal of R. If $[I, I] \subseteq Z$, then R is a commutative ring.*

Theorem 2.1. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative generalized (α, α) $-$derivation of R associated with a multiplicative $(\alpha, \alpha)-$derivation d.*

If $[d(x), F(y)] = \pm\alpha\left([x, y]\right)$ for all $x, y \in I$, then $\alpha\,(I)\,[d(x), \alpha\,(x)] = (0)$ for all $x \in I$.

Proof. By the hypothesis, we have

$$[d(x), F(y)] = \pm\alpha\left([x, y]\right), \text{ for all } x, y \in I. \tag{2.1}$$

Replacing xz by x in (2.1) and using this, we get

$$d(x)[\alpha(z), F(y)] + [\alpha(x), F(y)]d(z) = 0, \text{ for all } x, y, z \in I. \tag{2.2}$$

Replacing zx by x in (2.2), we have

$$\begin{aligned} & d(z)\alpha(x)[\alpha(z), F(y)] + \alpha(z)d(x)[\alpha(z), F(y)] \\ & +\alpha(z)[\alpha(x), F(y)]d(z) + [\alpha(z), F(y)]\alpha(x)d(z) = 0. \end{aligned} \tag{2.3}$$

Left multiplying (2.2) by $\alpha(z)$, we arrive at

$$\alpha(z)d(x)[\alpha(z), F(y)] + \alpha(z)[\alpha(x), F(y)]d(z) = 0, \text{ for all } x, y, z \in I. \tag{2.4}$$

Subtracting (2.4) from (2.3), we find that

$$d(z)\alpha(x)[\alpha(z), F(y)] + [\alpha(z), F(y)]\alpha(x)d(z) = 0, \ \forall x, y, z \in I. \tag{2.5}$$

That is

$$d(z)\alpha(x)[\alpha(z), F(y)] = -[\alpha(z), F(y)]\alpha(x)d(z), \ \forall x, y, z \in I. \tag{2.6}$$

Replacing x with $x\alpha^{-1}(d(z))t$ in this equation, we have

$$\begin{aligned} & d(z)\alpha(x)d(z)\alpha(t)[\alpha(z), F(y)] = -[\alpha(z), F(y)]\alpha(x)d(z)\alpha(t)d(z), \\ & \forall x, y, z, t \in I. \end{aligned} \tag{2.7}$$

Right multiplying (2.6) by $\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]$, we get

$$\begin{aligned} & d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ & = -[\alpha(z), F(y)]\alpha(x)d(z)\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]. \end{aligned} \tag{2.8}$$

Using (2.7), it yields that

$$\begin{aligned} & d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ & = d(z)\alpha(x)d(z)\alpha(t)[\alpha(z), F(y)]\alpha(x)[\alpha(z), F(y)]. \end{aligned} \tag{2.9}$$

Using (2.5), (2.9) reduces to

$$\begin{aligned} & d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ & = -d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]. \end{aligned}$$

That is

$$2d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] = 0, \text{ for all } x, y, z, t \in I.$$

Since R is $2-$torsion free semiprime ring, we get

$$d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] = 0, \text{ for all } x, y, z, t \in I.$$

Replacing t with $rt, r \in R$ in this equation and left multiplying with $\alpha(t)$ gives that

$$\alpha(t)\, d(z)\alpha(x)\,[\alpha(z), F(y)]R\alpha(t)\, d(z)\alpha(x)\,[\alpha(z), F(y)] = (0),$$
$$\text{for all } x, y, z, t \in I, r \in R.$$

Since R is semiprime ring, we have

$$\alpha(t)\, d(z)\alpha(x)\,[\alpha(z), F(y)] = 0$$

and so

$$Vd(z)V[\alpha(z), F(y)] = (0), \text{ for all } y, z \in I.$$

where $\alpha(I) = V$ is a nonzero left ideal of R.

Let $\{P_\alpha | \alpha \in I\}$ be a family of prime ideals of R such that $\cap P_\alpha = (0)$. We can say

$$Vd(z) \subseteq P_\alpha \text{ or } V[\alpha(z), F(y)] \subseteq P_\alpha$$

and so

$$[\alpha(z), F(y)]Vd(z) \subseteq P_\alpha \text{ or } d(z)V[\alpha(z), F(y)] \subseteq P_\alpha.$$

By (2.6), $[\alpha(z), F(y)]Vd(z) \subseteq P_\alpha$ implies that $d(z)V[\alpha(z), F(y)] \subseteq P_\alpha$ and so,

$$d(z)V[\alpha(z), F(y)] \subseteq \cap P_\alpha.$$

That is

$$d(z)V[\alpha(z), F(y)] = (0), \text{ for all } y, z \in I.$$

Hence we have $d(z)\alpha(x)\,[\alpha(z), F(y)] = 0$ for all $x, y, z \in I$. Replacing y by yz in this equation and using this, we get

$$d(z)\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.10}$$

Left multiplying with $\alpha(zy)$ this equation, we have

$$\alpha(z)\alpha(y)\, d(z)\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.11}$$

Replacing x by zx in (2.10) and left multiplying with $\alpha(y)$, we obtain that

$$\alpha(y)d(z)\alpha(z)\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.12}$$

Subtracting (2.11) from (2.12), we find that

$$[\alpha(z), \alpha(y)\, d(z)]\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I$$

and so

$$\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)]\alpha(r)\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I, r \in R.$$

Since R is a semiprime ring, it follows that $\alpha(x)\,[\alpha(z), \alpha(y)\, d(z)] = 0$, for all $x, y, z \in I$. Replacing y with $\alpha^{-1}(d(z))\, y$, we have

$$\alpha(x)\,[\alpha(z), d(z)\alpha(y)\, d(z)] = 0. \tag{2.13}$$

Replacing y by $y\alpha^{-1}(d(z))\,u$ in (2.13) and using this, we obtain that

$$\alpha(x)\,d(z)\alpha(y)\,[d(z),\alpha(z)]\alpha(u)\,d(z)=0,\ \text{for all } x,y,z,u\in I.$$

This implies that

$$\alpha(x)\,[d(z),\alpha(z)]\alpha(y)\,[d(z),\alpha(z)]\alpha(u)\,[d(z),\alpha(z)]=0,\ \text{for all } x,y,z,u\in I.$$

That is $(V[d(z),\alpha(z)])^3=(0)$, for all $z\in I$ where $\alpha(I)=V$ is a nonzero left ideal of R. Since a semiprime ring contains no nonzero nilpotent left ideals, it follows that

$$V[d(z),\alpha(z)]=(0)$$

and so

$$\alpha(I)[d(z),\alpha(z)]=(0),\ \text{for all } z\in I.$$

The proof is completed. □

Theorem 2.2. *Let R be a 2−torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F:R\to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $[d(x),F(y)]=\pm\alpha(xoy)$ for all $x,y\in I$, then $\alpha(I)\,[d(x),\alpha(x)]=(0)$ for all $x\in I$.

Proof. We assume that

$$[d(x),F(y)]=\pm\alpha(xoy)\,,\ \text{for all } x,y\in I. \tag{2.14}$$

Replacing x by xz in (2.14) and using this equation, we get

$$d(x)[\alpha(z)\,,F(y)]+\alpha(x)\,[d(z),F(y)]+[\alpha(x)\,,F(y)]d(z)=\pm\alpha(x[z,y])\,. \tag{2.15}$$

Writing zx by x in (2.15), we find that

$$\begin{aligned}&d(z)\alpha(x)[\alpha(z)\,,F(y)]+\alpha(z)\,d(x)[\alpha(z)\,,F(y)]+\alpha(zx)\,[d(z),F(y)]\\&+\alpha(z)\,[\alpha(x)\,,F(y)]d(z)+[\alpha(z)\,,F(y)]\alpha(x)\,d(z)=\pm\alpha(zx[z,y])\,.\end{aligned} \tag{2.16}$$

Left multiplication of (2.15) by $\alpha(z)$ yields that

$$\begin{aligned}&\alpha(z)\,d(x)[\alpha(z)\,,F(y)]+\alpha(z)\,\alpha(x)\,[d(z),F(y)]\\&+\alpha(z)\,[\alpha(x)\,,F(y)]d(z)=\pm\alpha(z)\,\alpha(x[z,y])\,.\end{aligned} \tag{2.17}$$

Subtracting (2.17) from (2.16), we have

$$d(z)\alpha(x)[\alpha(z)\,,F(y)]+[\alpha(z)\,,F(y)]\alpha(x)\,d(z)=0,\ \text{for all } x,y,z\in I. \tag{2.18}$$

The last expression is the same as the relation (2.5). Using the similar arguments as used in the Theorem 2.1, we get the required result. □

Similarly, following theorem is straightforward.

Theorem 2.3. *Let R be a 2−torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative generalized (α,α) −derivation of R associated with a multiplicative (α,α)−derivation d.*

If $[d(x), F(y)] = 0$ for all $x, y \in I$, then $\alpha(I)[d(x), \alpha(x)] = (0)$ for all $x \in I$.

Theorem 2.4. *Let R be a 2−torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative generalized (α,α) −derivation of R associated with a multiplicative (α,α)−derivation d.*

If $g : R \to R$ is a multiplicative derivation of R such that $F([x, y]) \pm [g(x), g(y)] \pm \alpha([x, y]) = 0$ for all $x, y \in I$, then $\alpha(I)[g(x), \alpha(x)] = (0)$ and $\alpha(I)[d(x), \alpha(x)] = (0)$ for all $x \in I$.

Proof. By the hypothesis, we have

$$F([x, y]) \pm [g(x), g(y)] \pm \alpha([x, y]) = 0, \text{ for all } x, y \in I. \tag{2.19}$$

Replacing yx instead of y in (2.19), we get

$$\begin{aligned} &F([x, y])\alpha(x) + \alpha([x, y])\, d(x) \\ &+[g(x), g(y)\alpha(x)] + [g(x), \alpha(y)\, g(x)] + \alpha([x, y]x) = 0. \end{aligned} \tag{2.20}$$

Right multiplying (2.19) by $\alpha(x)$, we obtain

$$F([x, y])\alpha(x) \pm [g(x), g(y)]\alpha(x) \pm \alpha([x, y])\, \alpha(x) = 0, \text{ for all } x, y \in I. \tag{2.21}$$

Now subtracting (2.21) from (2.20), for all $x, y \in I$, we arrive at

$$\alpha([x, y])\, d(x) + g(y)[g(x), \alpha(x)] + [g(x), \alpha(y)\, g(x)] = 0. \tag{2.22}$$

Substituting xy instead of y in (2.22), we obtain

$$\begin{aligned} &\alpha(x)\, \alpha([x, y])\, d(x) + g(x)\alpha(y)\, [g(x), \alpha(x)] \\ &+\alpha(x)\, g(y)[g(x), \alpha(x)] + \alpha(x)\, [g(x), \alpha(y)\, g(x)] \\ &+[g(x), \alpha(x)]\alpha(y)\, g(x) = 0. \end{aligned} \tag{2.23}$$

Left multiplying (2.22) by $\alpha(x)$ and then subtracting from (2.23), we find that

$$g(x)\alpha(y)\, [g(x), \alpha(x)] + [g(x), \alpha(x)]\alpha(y)\, g(x) = 0$$

and so

$$g(x)\alpha(y)\, [g(x), \alpha(x)] = -[g(x), \alpha(x)]\alpha(y)\, g(x), \text{ for all } x, y \in I. \tag{2.24}$$

Replacing y with $y\alpha^{-1}(g(x))t$ in this equation, we have

$$g(x)\alpha(y)\, g(x)\, \alpha(t)\, [g(x), \alpha(x)] = -[g(x), \alpha(x)]\alpha(y)\, g(x)\, \alpha(t)\, g(x). \tag{2.25}$$

Now right multiplying (2.24) by $\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]$, for all $x,y,t\in I$, we get

$$\begin{aligned}&g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]\\&=-[g(x),\alpha(x)]\alpha(y)\,g(x)\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)].\end{aligned}\tag{2.26}$$

Using (2.25), this equation gives that

$$\begin{aligned}&g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]\\&=g(x)\alpha(y)\,g(x)\,\alpha(t)\,[g(x),\alpha(x)]\alpha(y)\,[g(x),\alpha(x)].\end{aligned}\tag{2.27}$$

Again using (2.24), it reduces to

$$\begin{aligned}&g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]\\&=-g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)].\end{aligned}\tag{2.28}$$

That is

$$2g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]=0,\ \text{for all } x,y,t\in I.$$

Since R is $2-$torsion free semiprime ring, we have

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]=0,\ \text{for all } x,y,t\in I.$$

Writing $tr, r\in R$ by t in this equation, we get

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,\alpha(r)g(x)\alpha(y)\,[g(x),\alpha(x)]=0.$$

This implies that

$$\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]R\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]=(0).$$

By the semiprimeness of R, we get

$$\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]=0$$

and so

$$\alpha(y)\,[g(x),\alpha(x)]R\alpha(y)\,[g(x),\alpha(x)]=(0)\,.$$

Since R is semiprime ring, we arrive at

$$\alpha(I)\,[g(x),\alpha(x)]=(0),\ \text{for all } x\in I.\tag{2.29}$$

Now, replacing y with $ry, r\in R$ in (2.22) and using (2.29), we obtain

$$\begin{aligned}&\alpha(r)\,\alpha([x,y])\,d(x)+\alpha([x,r]y)\,d(x)\\&+\alpha(r)\,g(y)[g(x),\alpha(x)]+\alpha(r)\,[g(x),\alpha(y)\,g(x)]\\&+[g(x),\alpha(r)]\alpha(y)\,g(x)=0.\end{aligned}\tag{2.30}$$

Left multiplying (2.22) by $\alpha(r)$, we get

$$\alpha(r)\,\alpha([x,y])\,d(x)+\alpha(r)\,g(y)[g(x),\alpha(x)]+\alpha(r)\,[g(x),\alpha(y)\,g(x)]=0.$$

Subtracting this equation from (2.30), we arrive at

$$\alpha\left([x,r]y\right)d(x)+[g(x),\alpha\left(r\right)]\alpha\left(y\right)g(x)=0,\ \text{for all}\ x,y,\in I,r\in R. \tag{2.31}$$

Replacing yx by y in (2.31), we get

$$\alpha\left([x,r]yx\right)d(x)+[g(x),\alpha\left(r\right)]\alpha\left(yx\right)g(x)=0,\ \text{for all}\ x,y,\in I,r\in R. \tag{2.32}$$

Right multiplying (2.31) by $\alpha\left(x\right)$ and subtracting from (2.32), we obtain

$$\alpha\left([x,r]y\right)[d(x),\alpha(x)]+[g(x),\alpha\left(r\right)]\alpha(y)[g(x),\alpha(x)]=0.$$

Using $\alpha\left(I\right)[g(x),\alpha\left(x\right)]=(0)$ in this equation, we find that

$$\alpha\left([x,r]y\right)[d(x),\alpha\left(x\right)]=0.$$

By (2.31), we get

$$[\alpha\left(x\right),r]\alpha(y)\left[d(x),\alpha\left(x\right)\right]=0,\ \text{for all}\ x,y\in I,r\in R.$$

In particular, $[d(x),\alpha\left(x\right)]\alpha\left(y\right)[d(x),\alpha\left(x\right)]=0$,
and so

$$\alpha\left(y\right)[d(x),\alpha\left(x\right)]R\alpha\left(y\right)[d(x),\alpha\left(x\right)]=(0),\ \text{for all}\ x,y\in I.$$

By the semiprimeness of R yields that $\alpha\left(I\right)[d(x),\alpha\left(x\right)]=(0)$ for all $x\in I$. This completes the proof. □

Theorem 2.5. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F:R\to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $g:R\to R$ is a multiplicative derivation of R such that $F(xoy)\pm g(x)og(y)\pm\alpha\left(xoy\right)=0$ for all $x,y\in I$, then $\alpha\left(I\right)[g(x),\alpha\left(x\right)]=(0)$ and $\alpha\left(I\right)[d(x),\alpha\left(x\right)]=(0)$ for all $x\in I$.

Proof. By our hypothesis, we have

$$F(xoy)\pm g(x)og(y)\pm\alpha\left(xoy\right)=0,\ \text{for all}\ x,y\in I. \tag{2.33}$$

Replacing yx by y in (2.33), we find that

$$F(xoy)\alpha\left(x\right)+\alpha\left(xoy\right)d\left(y\right)+g(x)o\left(g(y)\alpha\left(x\right)+\alpha\left(y\right)g\left(x\right)\right)\pm\alpha\left(xoy\right)\alpha\left(x\right)=0$$

and so

$$\begin{aligned}&F(xoy)\alpha\left(x\right)+\alpha\left(xoy\right)d\left(x\right)+\left(g(x)og(y)\right)\alpha\left(x\right)\\&-g\left(y\right)\left[g\left(x\right),\alpha\left(x\right)\right]+\left(g(x)o\alpha(y)\right)g\left(x\right)+\alpha\left(xoy\right)\alpha\left(x\right)=0.\end{aligned} \tag{2.34}$$

Right multiplying (2.33) by $\alpha\left(x\right)$ and subtracting from (2.34), for all $x,y\in I$, we get

$$\alpha\left(xoy\right)d\left(x\right)-g\left(y\right)\left[g\left(x\right),\alpha\left(x\right)\right]+\left(g(x)o\alpha(y)\right)g\left(x\right)=0. \tag{2.35}$$

Substituting xy instead of y in (2.35), we obtain

$$\begin{aligned}(2.36)\quad & \alpha(x)\alpha\left(xoy\right)d\left(x\right)-\alpha\left(x\right)g\left(y\right)\left[g\left(x\right),\alpha\left(x\right)\right]-g\left(x\right)\alpha(y)[g\left(x\right),\alpha\left(x\right)] \\ & +\alpha\left(x\right)\left(g(x)o\alpha(y)\right)g\left(x\right)-\left[g\left(x\right),\alpha\left(x\right)\right]\alpha\left(y\right)g\left(x\right)=0.\end{aligned}$$

Left multiplying (2.35) by y and subtracting from (2.36), we have

$$g(x)\alpha\left(y\right)\left[g(x),\alpha\left(x\right)\right]+\left[g(x),\alpha\left(x\right)\right]\alpha\left(y\right)g(x)=0$$

and so

$$g(x)\alpha\left(y\right)\left[g(x),\alpha\left(x\right)\right]=-[g(x),\alpha\left(x\right)]\alpha\left(y\right)g(x),\text{ for all }x,y\in I.$$

This equation is same as the relation (2.24). Using the similar arguments, we get the required result. □

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FUZZY IMPLICATIVE IDEALS OF SHEFFER STROKE BG-ALGEBRAS

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Abstract. In this paper, an (implicative) ideal and a fuzzy ideal of Sheffer stroke BG-algebra are defined and some properties are presented. Then a fuzzy implicative and a sub-implicative ideals of a Sheffer stroke BG-algebra are described. Morever, an implicative Sheffer stroke BG-algebra and a medial Sheffer stroke BG-algebra are defined, and it is expressed that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. Also, a fuzzy (completely) closed ideal and a fuzzy p-ideal are determined. Finally, the relationships between these structures are shown.
Keywords: Sheffer stroke BG-algebra, fuzzy ideal, fuzzy implicative ideal, fuzzy sub-implicative ideal.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6], [7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim introduced a new notion called a B-algebra [12]. C. B. Kim and H. S. Kim [8] introduced a BG-algebra as a generalization of B-algebra. Then a BG-algebra consists of a non-empty set X with a binary operation $*$ and a constant 0 satisfying some axioms.

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In 1965, Zadeh introduced the notion of a fuzzy set and fuzzy subset of a set [22]. As a generalization of this, intuitionistic fuzzy subset was defined by K. T. Atanassov ([2], [3], [4]) in 1986. In 1971, Rosenfield introduced the concept of fuzzy sub-group [20]. Ahn and Lee studied fuzzy subalgebra of BG-algebra in [1]. Muthuraj et al. presented fuzzy ideals in BG-algebras in [10]. Also, Muthuraj and Devi introduced a multi-fuzzy subalgebra of BG-algebras in [11].

Sheffer stroke (or Sheffer operation) was first introduced by H. M. Sheffer [21]. Because any Boolean function or axiom can be expressed by means of only this operation [9], the most important application is to have all diods on the chip forming processor in a computer, that is, it is enough to produce a single diod for Sheffer operation. Thus, it is simpler and cheaper than to produce different diods for other Boolean operations. In addition, it has many algebraic applications in algebraic structures such as Sheffer stroke BG-algebras [13], interval Sheffer stroke basic algebras [19], Sheffer stroke Hilbert algebras [14] and fuzzy filters [15], filters of strong Sheffer stroke non-associative MV-algebras [17], (fuzzy) filters of Sheffer stroke BL-algebras [18], Sheffer stroke UP-algebras [16] and Sheffer operation in ortholattices [5].

After giving basic definitions and notions about a Sheffer stroke BG-algebra, an (implicative) ideal of a Sheffer stroke BG-algebra is defined. It is proved that every implicative ideal of a Sheffer stroke BG-algebra is its ideal. By describing a fuzzy (implicative) ideal of this algebraic structure, the relationship between them is shown. After determining a fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra, it is proved that every fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra is the fuzzy ideal. An implicative Sheffer stroke BG-algebra is defined and it is indicated that every fuzzy ideal of a Sheffer stroke BG-algebra is its fuzzy sub-implicative ideal if the algebraic structure is implicative. Then a medial Sheffer stroke BG-algebra is described and it is expressed that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. Morever, a fuzzy (completely) closed ideal and a fuzzy p-ideal of a Sheffer stroke BG-algebra are determined and the relationships between them are indicated. It is shown that every fuzzy completely closed ideal of an implicative Sheffer stroke BG-algebra is the fuzzy implicative ideal under one condition. Finally, it is stated that every fuzzy p-ideal of a Sheffer stroke BG-algebra is the fuzzy implicative ideal if this algebra equals to the BCA-part.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BG-algebra.

Definition 2.1. [5] Let $\mathcal{A} = \langle A, |\rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:
$(S1)$ $a_1|a_2 = a_2|a_1$,
$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,

$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,
$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$.

Definition 2.2. [13] A Sheffer stroke BG-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:
$(sBG.1)$ $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0$,
$(sBG.2)$ $(0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = a_1|a_1$,
for all $a_1, a_2 \in A$.

Let A be a Sheffer stroke BG-algebra, unless otherwise is indicated.

Lemma 2.1. [13] *Let A be a Sheffer stroke BG-algebra. Then the following features hold:*

1. $(0|0)|(a_1|a_1) = a_1$,
2. $(a_1|(0|0))|(a_1|(0|0)) = a_1$,
3. $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_3|(a_2|a_2))|(a_3|(a_2|a_2))$ *implies* $a_1 = a_3$,
4. $(0|(0|(a_1|a_1))) = a_1|a_1$,
5. *If* $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ *then* $a_1 = a_2$,
6. *If* $(0|(a_1|a_1)) = (0|(a_2|a_2))$ *then* $a_1 = a_2$,
7. $(((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(a_1|a_1)) = a_1|a_1$,
8. $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$,

for all $a_1, a_2, a_3 \in A$.

3. Some Types Of Fuzzy Ideals

Definition 3.1. Let I be a nonempty subset of a Sheffer stroke BG-algebra. Then I is called an ideal of A if it satisfies:
$(sI1)$ $0 \in I$,
$(sI2)$ $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$ and $a_2 \in I$ imply $a_1 \in I$.

Definition 3.2. A nonempty subset I of a Sheffer stroke BG-algebra A is called an implicative ideal of A if
(i) $0 \in I$,
(ii) $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in I$ and $a_3 \in I$ imply $a_1 \in I$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.1. *Every implicative ideal of a Sheffer stroke BG-algebra A is an ideal of A.*

Proof. Let I be an implicative ideal of A. Then $0 \in I$ from Definition 3.2 (i). Assume that $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$ and $a_2 \in I$. Since
$(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2))|$
$(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2))$
$= (((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2))|(((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2))$
$= (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$,
from (S2), (sBG.1) and Lemma 2.1 (2), we obtain from Definition 3.2 (ii) that $a_1 \in I$. Therefore, I is an ideal of A. $\square$

Definition 3.3. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy ideal of A if it satisfies the following conditions:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}$,
for all $a_1, a_2 \in A$.

Lemma 3.1. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. If*

$$a_1 \leq a_2 \text{ if and only if } (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$$

holds for all $a_1, a_2 \in A$, then $\mu(a_1) \geq \mu(a_2)$ if $a_1 \leq a_2$.

Proof. Let $a_1 \leq a_2$. Then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. Thus,

$$\begin{aligned} \mu(a_1) &\geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} \\ &= \min\{\mu(a_2), \mu(0)\} \\ &= \mu(a_2) \end{aligned}$$

from Definition 3.3 (ii) and (i), respectively. $\square$

Lemma 3.2. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. If $\mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) = \mu(0)$, then $\mu(a_1) \geq \mu(a_2)$, for any $a_1, a_2 \in A$.*

Proof. It is obvious from Definition 3.3. $\square$

Definition 3.4. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy implicative ideal of A if it satisfies:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))$
$|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\}$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.2. *Every fuzzy implicative ideal of a Sheffer stroke BG-algebra A is a fuzzy ideal of A.*

Proof. Let μ be a fuzzy implicative ideal of a Sheffer stroke BG-algebra A. Then $\mu(0) \geq \mu(a_1)$ from Definition 3.4 (i). Also,

$$\begin{aligned}\mu(a_1) &\geq \min\{\mu(a_2), \mu((((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2\\ &\quad |a_2))|(((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_2|a_2)))\}\\ &= \min\{\mu(a_2), \mu((((a_1|(0|0))|(a_1|(0|0)))|(a_2\\ &\quad |a_2))|(((a_1|(0|0))|(a_1|(0|0)))|(a_2|a_2)))\}\\ &= \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}\end{aligned}$$

by Definition 3.4 (ii), (S2), (sBG.1) and Lemma 2.1 (2). Therefore, μ is a fuzzy ideal of A. $\square$

Theorem 3.1. *Let μ be a fuzzy subset of a Sheffer stroke BG-algebra A. Then μ is a fuzzy (implicative) ideal of A if and only if a level subset $\mu_x = \{a \in A : \mu(a) \geq x\} \neq \emptyset$ of A is an (implicative) ideal of A.*

Proof. Let μ be a fuzzy ideal of A and $\mu_x \neq \emptyset$. Since it follows from Definition 3.3 (i) that $\mu(0) \geq \mu(a) \geq x$, for $a \in \mu_x$, we get that $0 \in \mu_x$. Assume that $a_2, (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in \mu_x$. Since $\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \geq x$, it is obtained from Definition 3.3 (ii) that

$$\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} \geq x,$$

which implies that $a_1 \in \mu_x$. Thus, μ_x is an ideal of A. Also, let μ be a fuzzy implicative ideal of A and $\mu_x \neq \emptyset$. Suppose that a_3, $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in \mu_x$. Since

$$\begin{aligned}&\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))\\ &\quad |(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))) \geq x\end{aligned}$$

we have from Definition 3.4 (ii) that

$$\begin{aligned}&\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|\\ &(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\} \geq x,\end{aligned}$$

which means that $a_1 \in \mu_x$. Hence, μ_x is an implicative ideal of A.

Conversely, let $\mu_x \neq \emptyset$ be an ideal of A. Assume that $\mu(0) < \mu(a)$, for some $a \in A$. If $x = (\mu(0)+\mu(a))/2 \in (0, 1]$, then $\mu(0) < x < \mu(a)$. So, $0 \notin \mu_x$, which is contradiction with (sI1). Thereby, $\mu(0) \geq \mu(a)$, for all $a \in A$. Suppose that $x_1 = \mu(a_1) < \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\} = x_2$. If If $x_0 = (x_1 + x_2)/2 \in (0, 1]$, then $x_1 < x_0 < x_2$. Thus, $a_2, (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in \mu_{x_0}$ but $a_1 \notin \mu_{x_0}$, which contradicts with (sI2). Then $\mu(a_1) \geq \min\{\mu(a_2), \mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\}$, for all $a_1, a_2 \in A$. Hence, μ is a fuzzy ideal of A. Moreover, let $\mu_x \neq \emptyset$ be an implicative ideal of A. Assume that $y_1 = \mu(a_1) < \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\} = y_2$. If $x^* = (y_1 +$

$y_2)/2 \in (0,1]$, then $y_1 < x^* < y_2$. So, a_3, $(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)) \in \mu_{x^*}$ but $a_1 \notin \mu_{x^*}$, which contradicts with Definition 3.2 (ii). Hence,

$$\mu(a_1) \geq \min\{\mu(a_3), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))) |(a_3|a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\},$$

for all $a_1, a_2, a_3 \in A$. Therefore, μ is a fuzzy implicative ideal of A. $\square$

Definition 3.5. A fuzzy subset μ of a Sheffer stroke BG-algebra A is called a fuzzy sub-implicative ideal of A if it satisfies:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1 |(a_2|a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.3. *Let A be a Sheffer stroke BG-algebra. Then every fuzzy sub-implicative ideal of A is a fuzzy ideal of A.*

Proof. Let μ be a fuzzy sub-implicative ideal of A. Then $\mu(0) \geq \mu(a_1)$ from Definition 3.5 (i). We get from (sBG.1), Lemma 2.1 (2), Definition 3.5 (ii) that

$$\begin{aligned}
\mu(a_1) &= \mu((a_1|(0|0))|(a_1|(0|0))) \\
&= \mu((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_3|a_3))| \\
&\quad (((a_1|(a_1|(a_1|a_1)))|(a_1|(a_1|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\} \\
&= \min\{\mu((((a_1|(0|0))|(a_1|(0|0)))|(a_3|a_3))| \\
&\quad (((a_1|(0|0))|(a_1|(0|0)))|(a_3|a_3))), \mu(a_3)\} \\
&= \min\{\mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3))), \mu(a_3)\}.
\end{aligned}$$

Therefore, μ is a fuzzy ideal of A. $\square$

Theorem 3.2. *Let A be a Sheffer stroke BG-algebra and μ be a fuzzy ideal of A. Then μ is a fuzzy sub-implicative ideal of A if and only if*

$$\begin{aligned}
&\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \\
&\geq \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))).
\end{aligned} \tag{3.1}$$

Proof. Let μ be a fuzzy sub-implicative ideal of A. We have from Lemma 2.1 (2) and Definition 3.5 (ii) that

$$\begin{aligned}
\mu(a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))) &\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1| \\
&\quad (a_2|a_2))))|(0|0))|(((a_1|(a_1|(a_2|a_2))) \\
&\quad |(a_1|(a_1|(a_2|a_2))))|(0|0))), \mu(0)\} \\
&= \min\{\mu(a_1|(a_1|(a_2|a_2)))|, \\
&\quad (a_1|(a_1|(a_2|a_2)))\mu(0)\} \\
&= \mu(a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))).
\end{aligned}$$

Conversely, since μ is a fuzzy ideal, it follows that
(i) $\mu(0) \geq \mu(a_1)$,
(ii)

$$\begin{aligned}\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &\geq \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))\\ &\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2\\ &\quad |a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|\\ &\quad (a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.\end{aligned}$$

Therefore, μ is a fuzzy sub-implicative ideal of A. □

Definition 3.6. A Sheffer stroke BG-algebra is said to be implicative if it satisfies the condition

$$a_1|(a_1|(a_2|a_2)) = a_2|(a_2|(a_1|a_1)), \tag{3.2}$$

for all $a_1, a_2 \in A$.

Theorem 3.3. *Let A be an implicative Sheffer stroke BG-algebra. Then every fuzzy ideal of A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy ideal of A. Then
(i) $\mu(0) \geq \mu(a_1)$,
(ii)

$$\begin{aligned}&\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))\\ &\geq \min\{\mu((((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))|\\ &\quad (((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\}\\ &= \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|\\ &\quad (((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.\end{aligned}$$

Thereby, μ is a fuzzy sub-implicative ideal of A. □

Definition 3.7. A Sheffer stroke BG-algebra A is called medial if

$$a_1|(a_1|(a_2|a_2)) = a_2|a_2, \tag{3.3}$$

for all $a_1, a_2 \in A$.

Lemma 3.3. *In a Sheffer stroke BG-algebra A, the following property holds:*

$$((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)) = a_1|(a_1|(a_2|a_2)),$$

for all $a_1, a_2 \in A$.

Proof. It follows from (S1), (S2) and (S3) that

$$\begin{aligned}
&((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))\\
&= (((a_1|(a_2|a_2))|a_1)|((a_1|(a_2|a_2))|a_1))|(a_2|(a_1|a_1))\\
&= (a_1|(a_2|a_2))|((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))\\
&=(a_1|(a_2|a_2))|((((a_1|a_1)|(a_1|a_1))|(a_2|(a_1|a_1)))|(((a_1|a_1)|(a_1|a_1))|(a_2|(a_1|a_1))))\\
&= (a_1|(a_2|a_2))|((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2)))\\
&= (a_1|(a_2|a_2))|((a_1|a_1)|(a_1|a_1))\\
&= a_1|(a_1|(a_2|a_2)).
\end{aligned}$$

□

Theorem 3.4. *Every fuzzy ideal of a medial Sheffer stroke BG-algebra A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy ideal of a medial Sheffer stroke BG-algebra A. It is obtained from (S2), Definition 3.3, Definition 3.7 and Lemma 3.3 that
(i) $\mu(0) \geq \mu(a_1)$,
(ii)

$$\begin{aligned}
&\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))\\
&= \mu(a_1)\\
&\geq \min\{\mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3))), \mu(a_3)\}\\
&= \min\{\mu((((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))|\\
&\quad (((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\}\\
&= \min\{\mu(((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|\\
&\quad (((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))))|(a_3|a_3))|\\
&\quad ((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1)))|(((a_1|(a_1\\
&\quad |(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\}\\
&= \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|\\
&\quad (((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\}.
\end{aligned}$$

Hence, μ is a fuzzy sub-implicative ideal of A. □

Theorem 3.5. *Let A be a Sheffer stroke BG-algebra satisfying*

$$\mu((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) \geq \mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \tag{3.4}$$

for all $a_1, a_2, a_3 \in A$. Then every fuzzy ideal of A is a fuzzy sub-implicative ideal of A.

Proof. It is obtained from the inequality (3.3), (S2) and Lemma 3.3 that

$$\begin{aligned}\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &\geq \mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))\\ &\quad |(a_2|(a_1|a_1)))|(((a_1|(a_1|(a_2|a_2)))|\\ &\quad (a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))))\\ &= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))).\end{aligned}$$

Thus, μ is a fuzzy sub-implicative ideal of A by Theorem 3.2. □

Theorem 3.6. *Every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra.*

Proof. Let A be a medial Sheffer stroke BG-algebra. Then it follows from Lemma 3.3, Definition 3.7 and (S2) that

$$\begin{aligned}a_1|(a_1|(a_2|a_2)) &= ((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|(a_1|a_1))\\ &= ((a_2|a_2)|(a_2|a_2))|(a_2|(a_1|a_1))\\ &= a_2|(a_2|(a_1|a_1)).\end{aligned}$$

Therefore, A is an implicative Sheffer stroke BG-algebra. □

Theorem 3.7. *Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. Then μ is a fuzzy implicative ideal of A if and only if μ satisfies the following condition:*

$$\mu(a_1) \geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \tag{3.5}$$

for all $a_1, a_2 \in A$.

Proof. ($\Rightarrow$) Let μ be a fuzzy implicative ideal of A. Then we get from Lemma 2.1 (2) that

$$\begin{aligned}\mu(a_1) &\geq \min\{\mu(0), \mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0\\ &\quad |0))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0)))\}\\ &= \min\{\mu(0), \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))\}\\ &= \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))),\end{aligned}$$

for all $a_1, a_2 \in A$.

($\Leftarrow$) Let μ be a fuzzy ideal of A satisfying the inequality (3.4). Then it is clear that $\mu(0) \geq \mu(a_1)$, for all $a_1 \in A$. Since

$$\begin{aligned}\mu(a_1) &\geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))\\ &\geq \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))|\\ &\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3))), \mu(a_3)\},\end{aligned}$$

for all $a_1, a_2, a_3 \in A$, we have that μ is a fuzzy implicative ideal of A. □

Theorem 3.8. *Let A be a medial Sheffer stroke BG-algebra satisfying*

$$\mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \tag{3.6}$$

for all $a_1, a_2 \in A$. Then every fuzzy sub-implicative ideal of A is a fuzzy implicative ideal of A.

Proof. Let μ be a fuzzy sub-implicative ideal of a medial Sheffer stroke BG-algebra A satisfying the inequality (3.5). Then we obtain from Definition 3.7, (S2), Definition 3.5, Lemma 2.1 (2) and the inequality (3.5) that

$$\begin{aligned}
\mu(a_1) &= \mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(0|0)) \\
&\quad |(((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(0|0))), \mu(0)\} \\
&= \min\{\mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))), \mu(0)\} \\
&= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \\
&\geq \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))).
\end{aligned}$$

Thus, μ is a fuzzy implicative ideal of A by Theorem 3.7. □

Theorem 3.9. *Let A be an implicative Sheffer stroke BG-algebra. Then every fuzzy implicative ideal of A is a fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy implicative ideal of an implicative Sheffer stroke BG-algebra A. Then μ is a fuzzy ideal of A by Proposition 3.2. So, it is obvious that $\mu(0) \geq \mu(a_1)$, for all $a_1 \in A$. Thus, it follows from Definition 3.6 and Definition 3.3 (ii) that

$$\begin{aligned}
\mu((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))) &= \mu((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))) \\
&\geq \min\{\mu((((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2 \\
&\quad |a_2))))|(a_3|a_3))|(((a_1|(a_1|(a_2|a_2))) \\
&\quad |(a_1|(a_1|(a_2|a_2))))|(a_3|a_3))), \mu(a_3)\},
\end{aligned}$$

for all $a_1, a_2, a_3 \in A$. Hence, μ is a fuzzy sub-implicative ideal of A. □

Corollary 3.1. *Let A be a medial Sheffer stroke BG-algebra. Then every fuzzy implicative ideal of A is a fuzzy sub-implicative ideal of A.*

Definition 3.8. A fuzzy ideal μ of a Sheffer stroke BG-algebra A is said to be fuzzy closed if

$$\mu((0|(a_1|a_1))|(0|(a_1|a_1))) \geq \mu(a_1), \tag{3.7}$$

for all $a_1 \in A$.

Definition 3.9. Let μ be a fuzzy ideal of a Sheffer stroke BG-algebra A. Then μ is called a fuzzy completely closed ideal of A if

$$\mu((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \geq \min\{\mu(a_1), \mu(a_2)\},$$

for all $a_1, a_2 \in A$.

Theorem 3.10. *Let A be a Sheffer stroke BG-algebra satisfying*

$$((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3))))|(a_3|(a_2|a_2)) = 0|0, \tag{3.8}$$

for all $a_1, a_2, a_3 \in A$. Then A is implicative if and only if every fuzzy closed ideal of A is a fuzzy implicative ideal of A.

Proof. Let A be a Sheffer stroke BG-algebra satisfying the equation (3.8).

($\Rightarrow$) Assume that A is implicative and μ is a fuzzy closed ideal of A. Then μ is a fuzzy ideal of A. Thus,
(i) $\mu(0) \geq \mu(a_1)$.
(ii)

$$\begin{aligned}
\mu(a_1) &\geq \min\{\mu(a_3), \mu((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))\} \\
&= \min\{\mu(a_3), \mu(((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(a_3 \\
&\quad |a_3))|((((a_1|a_1)|(a_1|a_1))|((a_1|a_1)|a_2))|(a_3|a_3)))\} \\
&= \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3 \\
&\quad |a_3))|(((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(a_3|a_3)))\},
\end{aligned}$$

which means that μ is a fuzzy implicative ideal of A.

($\Leftarrow$) Suppose that every fuzzy closed ideal of A is a fuzzy implicative ideal of A. So, it follows from the equation (3.8), (S1)-(S2) and Lemma 2.1 (5) that $a_3|(a_2|a_2) = (a_1|(a_3|a_3))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))$. Since $a_3|(a_2|a_2) = (a_1|(a_3|a_3))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) = ((a_1|(a_1|(a_3|a_3)))|(a_1|(a_1|(a_3|a_3))))|(a_2|a_2)$ from (S1) and (S3), it is obtained from (S2) and Lemma 2.1 (3) that $a_3 = (a_1|(a_1|(a_3|a_3)))|(a_1|(a_1|(a_3|a_3)))$. Thus, we get from (S1)-(S3) and Lemma 2.1 (8) that

$$\begin{aligned}
a_1|(a_1|(a_2|a_2)) &= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(((a_2 \\
&\quad |(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|a_2)) \\
&= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|(((a_2 \\
&\quad |a_2)|(a_2|(a_1|a_1)))|((a_2|a_2)|(a_2|(a_1|a_1))))) \\
&= ((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_2|(a_2|a_2)) \\
&= (((a_2|(a_2|a_2))|((a_2|a_2)|(a_2|a_2)))|((a_2| \\
&\quad (a_2|a_2))|((a_2|a_2)|(a_2|a_2))))|(a_2|(a_1|a_1)) \\
&= a_2|(a_2|(a_1|a_1)),
\end{aligned}$$

for all $a_1, a_2 \in A$, which means that A is implicative. $\square$

Proposition 3.4. *Let A be an implicative Sheffer stroke BG-algebra satisfying the equation (3.8). Then every fuzzy completely closed ideal of A is a fuzzy implicative ideal of A.*

Proof. Let μ be a fuzzy completely closed ideal of an implicative Sheffer stroke BG-algebra A. Then μ is a fuzzy ideal of A. Since $\mu((0|(a_2|a_2))|(0|(a_2|a_2))) \geq \min\{\mu(0), \mu(a_2)\} = \mu(a_2)$, it is obtained that μ is a fuzzy closed ideal of A. Therefore, μ is a fuzzy implicative ideal of A from Theorem 3.10. $\square$

Corollary 3.2. *Let A be a medial Sheffer stroke BG-algebra satisfying the equation (3.8). Then every fuzzy completely closed ideal of A is a fuzzy implicative ideal of A.*

Definition 3.10. A fuzzy set μ of a Sheffer stroke BG-algebra A is called a fuzzy p-ideal of A if it satisfies:
(i) $\mu(0) \geq \mu(a_1)$,
(ii) $\mu(a_1) \geq \min\{\mu((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|$
$a_3)))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|a_3)))), \mu(a_2)\}$,
for all $a_1, a_2, a_3 \in A$.

Definition 3.11. Let A be a Sheffer stroke BG-algebra. Then the set $A_+ = \{a_1 \in A : (0|(a_1|a_1))|(0|(a_1|a_1)) = 0\}$ is called the BCA-part of A.

Theorem 3.11. *Let $A = A_+$ be a Sheffer stroke BG-algebra. Then every fuzzy p-ideal of A is a fuzzy implicative ideal of A.*

Proof. Let μ be a fuzzy p-ideal of A. Since

$$\begin{aligned}
\mu(a_1) &\geq \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|(a_2|(a_1|a_1))))| \\
&\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|(a_2|(a_1|a_1))))), \mu(0)\} \\
&= \min\{\mu((((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0))| \\
&\quad (((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1))))|(0|0))), \mu(0)\} \\
&= \min\{\mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))), \mu(0)\} \\
&= \mu((a_1|(a_2|(a_1|a_1)))|(a_1|(a_2|(a_1|a_1)))),
\end{aligned}$$

from Definition 3.10 (i)-(ii), (S2) and Lemma 2.1 (2), it follows from Theorem 3.7 that μ is a fuzzy implicative ideal of A. $\square$

4. Conclusion

In this study, we introduce a fuzzy ideal, a fuzzy implicative ideal, a fuzzy sub-implicative ideal, a fuzzy (completely) closed ideal and a fuzzy p-ideal of a Sheffer

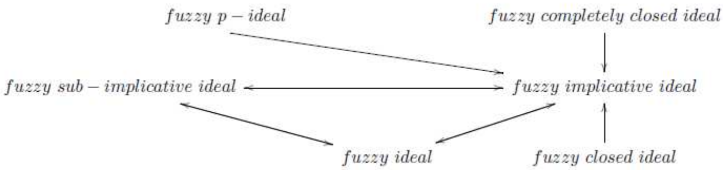

FIG. 3.1: Diagram of some types of fuzzy ideals

stroke BG-algebra and investigate some properties. After giving basic definitions and notions about a Sheffer stroke BG-algebra, we define an (implicative) ideal of a Sheffer stroke BG-algebra and prove that every implicative ideal of a Sheffer stroke BG-algebra is the ideal. Also, we determine a fuzzy ideal, a fuzzy implicative ideal and a fuzzy sub-implicative ideal on this algebraic structure. Besides, we construct an (implicative) ideal of a Sheffer stroke BG-algebra by means of its fuzzy (implicative) ideal and vice versa. It is shown that every fuzzy ((sub-)implicative) ideal of a Sheffer stroke BG-algebra is its fuzzy ideal. Besides, we examine the cases which the inverses hold. Morever, we describe an implicative Sheffer stroke BG-algebra and a medial Sheffer stroke BG-algebra and indicate that every medial Sheffer stroke BG-algebra is an implicative Sheffer stroke BG-algebra. It is demonstrated that every fuzzy ideal of an implicative (or medial) Sheffer stroke BG-algebra is the fuzzy sub-implicative ideal. It is indicated that every fuzzy sub-implicative ideal of a Sheffer stroke BG-algebra is the fuzzy implicative ideal when the algebra is a medial Sheffer stroke BG-algebra with a special condition, and every fuzzy implicative ideal of an implicative (or medial) Sheffer stroke BG-algebra is its fuzzy sub-implicative ideal. Finally, a fuzzy (completely) closed ideal and a fuzzy p-ideal of this algebraic structure are determined and the relationship between them are examined. By BCA-part of a Sheffer stroke BG-algebra, we prove that every fuzzy p-ideal of a Sheffer stroke BG-algebra is its fuzzy implicative ideal when the algebraic structure equals to the BCA-part.

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