FACTA UNIVERSITATIS
Series: **Mechanical Engineering**
https://doi.org/10.22190/FUME231219010K

Original scientific paper

LONGITUDINAL-RADIAL VIBRATIONS OF A VISCOELASTIC CYLINDRICAL THREE-LAYER STRUCTURE

Khayrulla Khudoynazarov

Samarkand State University, Uzbekistan

Abstract. *The paper considers a cylindrical three-layer structure of arbitrary thickness made of viscoelastic material. It consists of two external bearing layers and a middle layer, the materials of which are generally different. The problem of nonstationary longitudinal-radial vibrations of such a structure is formulated. Based on the exact solutions in transformations of the three-dimensional problem of the linear theory of viscoelasticity for a circular cylindrical three-layer body, a mathematical model of its nonstationary longitudinal-radial vibrations is developed. Equations are derived that allow, based on the results of solving the vibration equations, to determine the stress-strain state of a cylindrical structure and its layers in arbitrary sections. The results obtained allow for special cases of transition into cylindrical viscoelastic and elastic two-layer structures, as well as into homogeneous single-layer cylindrical structures and round rods.*

Key words: *Three-layer structure, Vibration, Stress, Torsional displacement, Load-bearing layers, Non-stationary*

1. INTRODUCTION

Three-layer structural elements are widely used in aviation and shipbuilding, construction of buildings and structures, the space industry and other industries [1,2]. Therefore, the problem of developing effective methods for calculating the stress-strain state of three-layer structural elements, as well as generalizing classical theories using refined models reflecting the dynamic behavior of modern materials, is urgent [3,4]. In this regard, cylindrical structures and round rods are one of the main elements of various engineering structures [5] and studies of their dynamic behavior have important applied values [6,7]. Such elements are often under the influence of dynamic loads during operation, which lead to their vibrations [8,9].

Received: December 19, 2023 / Accepted March 03, 2024
Corresponding author: Khayrulla Khudoynazarov
Samarkand State University, 100104, 15 University Boulevard, Samarkand, Uzbekistan
E-mail: kh.khudoyn@gmail.com

Recently, special attention has been given to study vibrations of layered structures made of homogeneous and functionally graded materials (FGM) [10]. These include works where vibrations of three-layer plates are considered, considering imperfect [11], slipping [12] contacts between layers, cylindrical [13] and conical [14] FGM shells. In studies devoted to the dynamic behavior of elements of engineering structures, it is important to develop mathematical foundations for such structures [15,16] for design of new generations of improved lightweight structural materials [17].

One of the main problems in the study of the static and dynamic behavior of shells and rods is the choice of vibration equations, which should be implemented based on the specific physical and mechanical properties of their materials [18]. Various methods of derivation of vibration equations are used. One of these methods is the method of using general solutions in transformations of three-dimensional problems of elasticity theory [19,20]. The essence of the method is to study the constructed solutions for various types of external influences [21] and to clarify the conditions under which the displacements or their "main parts" satisfy simple vibration equations, and to find an algorithm that allows calculating approximate values of displacement and stress fields in any cross section for an arbitrary a moment in time. Similar studies, but considering more complex physical and mechanical properties, in particular viscoelastic ones, are considered in [22,23].

The analysis of the behavior of elements of engineering structures, taking into account the layering and heterogeneity of structures based on computational models, is relevant for applied problems, as evidenced by publications [24,25]. In addition, a fairly large number of studies of shell dynamics are carried out, which take into account the influence of hyperelastic [26], anisotropic, temperature and other physical and mechanical properties of the material.

Thus, it can be argued that at present there is a very limited number of works devoted to the practically important task of studying non-stationary longitudinal-radial vibrations of cylindrical three-layer structures of arbitrary thickness. Therefore, the problem of creating models for the dynamic calculation of such systems under the influence of dynamic loads, taking into account various physical and mechanical properties of their material, is urgent. In this article, a circular cylindrical three-layer viscoelastic structure of arbitrary thickness with sticking condition between the layers is considered. The task is to study its nonstationary longitudinal-radial vibrations based on the above-mentioned method of exact solutions in transformations. It is envisaged to build a mathematical model of it, including the derivation of general and refined vibration equations and the creation of an algorithm that allows determining the stress-strain state of an arbitrary section of the structure in coordinate and time using the field of desired functions.

2. Mathematical Model of the Problem

2.1 Formulation of the Problem

In the cylindrical coordinate system (r, θ, z), a three-layer circular cylindrical structure made of viscoelastic material is considered. It is assumed, that the structure consists of two layers, hereinafter called load-bearing layers, which are separated by a certain distance using the third layer. The intermediate layer holds the load-bearing layers at a distance. The axis Oz of the coordinate system is directed along the axis of symmetry of the structure perpendicular to the cross section and we schematic picture of layers is shown in Fig.1. Through a and b we denote the inner and outer radii of the cylindrical structure, and through r_1 and r_2 the inner and outer radii of the middle layer. When deriving the vibration equations, we assume that both the cylindrical structure as a whole and its layers separately strictly obey the mathematical theory of viscoelasticity and are described in an accurate formulation by its three-dimensional equations in a linear formulation.

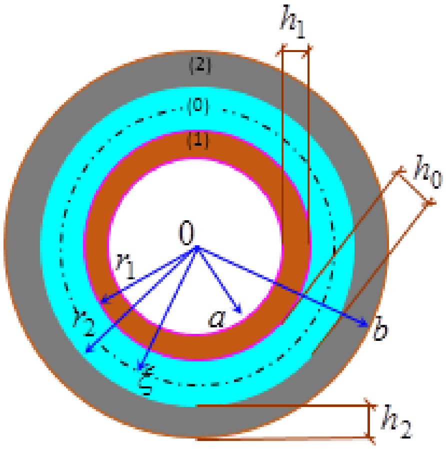

Fig. 1 Cross section of a three-layer structure

With longitudinal radial vibrations of the cylindrical structure, only the components of displacements w_m, u_m, and stresses $\boldsymbol{\sigma}_{rr}^{(m)}, \boldsymbol{\sigma}_{\theta\theta}^{(m)}, \boldsymbol{\sigma}_{zz}^{(m)}, \boldsymbol{\tau}_{zr}^{(m)}$ (m=0,1,2), will be different from zero [20]. Accordingly, the equations of motion of points of a viscoelastic structure are taken in the form of wave equations with respect to the potentials of longitudinal $\boldsymbol{\varphi}_m$ and transverse $\boldsymbol{\chi}_m$ waves in the layers of the structure:

$$\begin{cases} R_m\left(\Delta_0\boldsymbol{\varphi}_m\right) = \boldsymbol{\rho}_m \dfrac{\partial^2\boldsymbol{\varphi}_m}{\partial t^2},\ R_m = R_{\lambda_m} + 2R_{\mu_m}, \\ R_{\mu_m}\left(\Delta_0\boldsymbol{\chi}_m\right) = \boldsymbol{\rho}_m \dfrac{\partial^2\boldsymbol{\chi}_m}{\partial t^2},\ (m=0,1,2),\ a \le r \le r_1, \end{cases} \tag{1}$$

where $\Delta_0 = \dfrac{\partial^2}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial r^2}$, while R_{λ_m}, R_{μ_m} are the integral operators defined by the formula

$$R_{(\lambda,\mu)_m}(\varsigma) = (\boldsymbol{\lambda}_m, \boldsymbol{\mu}_m)\left[\varsigma(t) - \int_0^t K_{(\lambda,\mu)_m}(t-\tau)\varsigma(\tau)\,d\tau\right],$$

$\boldsymbol{\lambda}_m$, $\boldsymbol{\mu}_m$ are the (Lame)coefficients of layer materials, $K_{(\lambda,\mu)_m}(t-\tau)$ is the kernel of the integral operators. It is assumed that the viscoelastic operators $R_{(\lambda,\mu)_m}(\varsigma)$ are reversible, and their kernels are arbitrary. Here and everywhere else, the index m takes values 0, 1, 2. Therefore, in the following, this will not be emphasized every time, implying that this is always the case.

It is assumed that the cylindrical structure is at rest prior to loading, and at the moment t=0, external surface's stresses $F_r^{(i)}(z,t)$, $F_{rz}^{(i)}(z,t)$ (i=1,2) are applied, i.e. it is assumed that the boundary conditions have the form:

$$\begin{cases} \boldsymbol{\sigma}_{rr}^{(1)}(a,z,t)=F_r^{(1)}(z,t),\ \boldsymbol{\tau}_{rz}^{(1)}(a,z,t)=F_{rz}^{(1)}(z,t) \quad at \ \ r=a, \\ \boldsymbol{\sigma}_{rr}^{(2)}(b,z,t)=F_r^{(2)}(z,t),\ \boldsymbol{\tau}_{rz}^{(2)}(b,z,t)=F_{rz}^{(2)}(z,t) \quad at \ \ r=b. \end{cases} \tag{2}$$

In addition, the sticking conditions must be met on the contact surfaces between the layers of the structure, which require equal displacements and stresses, i.e. contact conditions have the following form at $r=r_i$, i=1,2:

$$\begin{cases} w_0(r_i,z,t)=w_i(r_i,z,t),\ u_0(r_i,z,t)=u_i(r_i,z,t), \\ \boldsymbol{\sigma}_{rr}^{(0)}(r_i,z,t)=\boldsymbol{\sigma}_{rr}^{(i)}(r_i,z,t), \quad \boldsymbol{\tau}_{rz}^{(0)}(r_i,z,t)=\boldsymbol{\tau}_{rz}^{(i)}(r_i,z,t), \ \ (i=1,2). \end{cases} \tag{3}$$

The initial conditions of the problem are considered zero, i.e. at t=0

$$\boldsymbol{\varphi}_m=0,\ \frac{\partial \boldsymbol{\varphi}_m}{\partial t}=0,\ \boldsymbol{\chi}_m=0,\ \frac{\partial \boldsymbol{\chi}_m}{\partial t}=0\,. \tag{4}$$

2.2 Derivation of Vibration Equations

To solve the formulated problem of torsional vibrations of a three-layer cylindrical viscoelastic shell, the functions $F_r^{(i)}(z,t)$, $F_{rz}^{(i)}(z,t)$ (i=1,2) of external influences under boundary conditions (3) are considered in the class of functions represented as [21]:

$$F_r^{(i)}=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\} dk \int_{(l)} f_r^{(i)}(k,p)e^{pt}dp,\ F_{rz}^{(i)}=\int_0^\infty \left.\begin{matrix}\cos kz\\ \sin kz\end{matrix}\right\} dk \int_{(l)} f_{rz}^{(i)}(k,p)e^{pt}dp\,. \tag{5}$$

Here (l) is an open contour in the plane p adjacent to the right section ($-i\boldsymbol{\omega}_0$, $i\boldsymbol{\omega}_0$) of the imaginary axis. FUrtgermore, the functions $F_r^{(i)}(z,t)$, $F_{rz}^{(i)}(z,t)$ are assumed to be such that the functions $f_r^{(i)}(z,t)$, $f_{rz}^{(i)}(z,t)$ are negligibly small outside the domain {0<k<k_0, Im|p|<$\boldsymbol{\omega}_0$}

Let's assume the potential functions $\boldsymbol{\varphi}_m$ and $\boldsymbol{\chi}_m$ also in the form of Eq. (5). Substituting them into the equations of motion (Eq. (1)), we obtain ordinary Bessel differential equations for the transformed potential functions $\tilde{\boldsymbol{\varphi}}_m$, $\tilde{\boldsymbol{\chi}}_m$ (m=0,1,2):

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\alpha_m^2\right)\tilde{\varphi}=0,\ \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\beta_m^2\right)\tilde{\chi}_m=0 \tag{6}$$

where

$$\tilde{K}_{\mu_m}(p)=\int_0^\infty \tilde{K}_{\mu_m}(t)e^{-pt}dt, \tilde{R}_m=(\lambda_m+2\mu_m)\left[1-\tilde{K}_m(p)\right], \tilde{R}_{\mu_m}=\mu_m\left[1-\tilde{K}_{\mu_m}(p)\right],$$
$$\alpha_m^2=\rho_m p^2\tilde{R}_m^{-1}+k^2, \beta_m^2=\rho_m p^2\tilde{R}_{\mu_m}^{-1}+k^2,\ K_m(t)=\frac{\lambda_m K_{\lambda_m}(t)+2\mu_m K_{\mu_m}(t)}{\lambda_m+2\mu_m}. \quad (7)$$

The general solution of Eq. (6) is

$$\tilde{\varphi}_m(r)=A_m^{(1)}I_0(\alpha_m r)+A_m^{(2)}K_0(\alpha_m r),\ \tilde{\chi}_m(r)=C_m^{(1)}I_0(\beta_m r)+C_m^{(2)}K_0(\beta_m r), \quad (8)$$

where $A_m^{(i)}(k,p)$, $C_m^{(i)}(k,p)$, (m=0,1,2) (i=1,2) are arbitrary, regarding variable r, integration constants. The number of these constants is generally 12. It can be reduced based on the following considerations. The general solutions given by Eq. (8) for all three layers have the same structure. They should be finite in the limits $r\to 0$ and $r\to\infty$. At the same time, the boundaries of the inner layer are smooth a and r_1, that is $a \le r \le r_1$. It is bounded from below (from the inside) by the surface $r = a$, which may tend to zero in the limit, but exceed in no way the values of r_1, i.e. it cannot strive for infinity. Therefore, in general solutions for the potential functions of the inner layer $\tilde{\varphi}_1(r)$, $\tilde{\chi}_1(r)$, we can limit ourselves to taking into account their layer in the form of:

$$\tilde{\varphi}_1(r)=A_1I_0(\alpha_1 r),\ \tilde{\chi}_1(r)=D_1I_0(\beta_1 r). \quad (9)$$

Similarly, the boundaries of the second, outer layer are cylindrical surfaces with radii $r = r_2$ and $r = b$ that is $r_2 \le r \le b$. This layer is bounded externally by the surface of radius $r = b$, which may tend to infinity, i.e. $b\to\infty$. On the other hand, the inner surface of this layer cannot be tightened to a straight line, because this would lead to a homogeneous rod of circular cross-section of radius $r = b$. Therefore, in general solutions for the potential functions of the outer layer $\tilde{\varphi}_2(r)$, $\tilde{\chi}_2(r)$, we can limit ourselves to taking into account its limitations only at $r\to\infty$. Based on this, we will take the general solution, Eq. (8), for the outer layer as:

$$\tilde{\varphi}_2(r)=A_2K_0(\alpha_2 r),\ \tilde{\chi}_2(r)=D_2K_0(\beta_2 r),\ (r_2\le r\le b). \quad (10)$$

For the middle layer, we will take general solutions. Eq. (8), considering that these solutions, in the absence of the two outer layers, should transform into known solutions for a homogeneous cylindrical layer, limited at $r\to 0$ and $r\to\infty$ at the same time:

$$\tilde{\varphi}_0(r)=B_1I_0(\alpha_0 r)+B_2K_0(\alpha_0 r),\ \tilde{\chi}_0(r)=C_1I_0(\beta_0 r)+C_2K_0(\beta_0 r),\ r_1\le r\le r_2 \quad (11)$$

Thus, the number of integration constants to be determined from the contact conditions is reduced to eight.

Let us assume the stress components, both $\sigma_{rr}^{(m)}(r,z,t)$ and $\tau_{rz}^{(m)}(r,z,t)$ as

$$\sigma_{rr}^{(m)}=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\}dk\int_{(l)}\tilde{\sigma}_{rr}^{(m)}e^{pt}dp,\quad \tau_{rz}^{(m)}=\int_0^\infty \left.\begin{matrix}\cos kz\\ \sin kz\end{matrix}\right\}dk\int_{(l)}\tilde{\tau}_{rz}^{(m)}e^{pt}dp. \quad (12)$$

Substituting Eqs. (12) and (5) in the boundary conditions, Eq. (2), and expressing them in terms of solutions given by Eqs. (9) and (10), we obtain:

$$\begin{gathered}\left[\left(\beta_1^2+k^2\right)I_0\left(\alpha_1 a\right)-\frac{2\alpha_1}{a}I_1\left(\alpha_1 a\right)\right]A_1-\left[2k\beta_1^2 I_0\left(\beta_1 a\right)-\frac{2k\beta_1}{a}I_1\left(\beta_{1,}a\right)\right]D_1=\tilde{R}_{\mu_1}^{-1}\left[f_r^{(1)}\left(k,p\right)\right],\\ 2k\alpha_1 I_1\left(\alpha_1 a\right)A_1-\left(\beta_1^2+k^2\right)\beta_1 I_1\left(\beta_1 a\right)D_1=\tilde{R}_{\mu_1}^{-1}\left[f_{rz}^{(1)}\left(k,p\right)\right],\end{gathered} \tag{13}$$

$$\begin{gathered}\left[\left(\beta_2^2+k^2\right)K_0\left(\alpha_2 b\right)+\frac{2\alpha_2}{b}K_1\left(\alpha_2 b\right)\right]A_2-\left[2k\beta_2^2 K_0\left(\beta_2 b\right)+\frac{2k\beta_2}{b}K_1\left(\beta_2 b\right)\right]D_2=\tilde{R}_{\mu_2}^{-1}\left[f_r^{(2)}\left(k,p\right)\right],\\ 2k\alpha_2 K_1\left(\alpha_2 b\right)A_2-\left(\beta_2^2+k^2\right)\beta_2 K_1\left(\beta_2 b\right)D_2=-\tilde{R}_{\mu_2}^{-1}\left[f_{rz}^{(2)}\left(k,p\right)\right],\end{gathered} \tag{14}$$

Let us express the contact conditions also through general solutions. In this case, the number of unknown constants will be only 8. Therefore, due to the fact that we use four boundary conditions, Eq. (2), it is sufficient to take four of the eight contact conditions given by Eq. (3). In addition, the boundary conditions, Eq. (2), are dynamic, therefore, for contact conditions we will use only the kinematic part of the conditions given by Eq. (3). Let us also assume the displacements in form of Eq. (5) and apply together with Eq. (12) to the contact conditions, Eq. (3), in the displacements. Expressing the obtained contact conditions through general solutions, Eqs. (9-11), we find the constants A_i, D_i (i=1,2). Substituting the found expressions of the integration constants into Eqs. (13) and (14) gives the following system of four equations:

$$\begin{gathered}\bar{G}_{1i}(\alpha_i,\beta_i)\tilde{u}_0(r_i)+\frac{2}{r_i}\bar{G}_{2i}(\alpha_i,\beta_i)\tilde{w}_0(r_i)=\frac{r_i^2}{a^2\left(b^2\right)}\tilde{R}_{\mu_i}^{-1}\left[\left(\alpha_i^2+k^2\right)f_r^{(i)}\left(k,p\right)\right],\\ \bar{G}_{3i}(\alpha_i,\beta_i)\tilde{u}_0(r_i)+\frac{2}{r_i}\bar{G}_{4i}(\alpha_i,\beta_i)\tilde{w}_0(r_1)=\frac{4r_i^2}{a^3\left(b^3\right)}\tilde{R}_{\mu_i}^{-1}\left[\left(\alpha_i^2+k^2\right)f_{rz}^{(i)}\left(k,p\right)\right],\end{gathered} \tag{15}$$

where
$$\begin{cases}\bar{G}_{1i}\left(\alpha_i,\beta_i\right)=k\left(\beta_i^2+k^2+\frac{1}{4}\alpha_i^2\right), \ \bar{G}_{2i}\left(\alpha_i,\beta_i\right)=\pm\left[\beta_i^2\mp\frac{1}{4}(2\alpha_i^2-k^2)\right]\\ \bar{G}_{3i}\left(\alpha_i,\beta_i\right)=\alpha_i^2(\beta_i^2+3k^2), \quad \bar{G}_{4i}\left(\alpha_i,\beta_i\right)=k\left(2\alpha_1^2-\beta_1^2-k^2\right)\end{cases}$$

$$a^2\left(b^2\right)=\begin{cases}a^2, \ if \ \ i=1,\\ b^2, \ if \ \ i=2.\end{cases}$$

Let us highlight the main parts of the longitudinal radial displacement of the middle layer of the shell. For this, the displacements of the points of the median layer $\tilde{u}_0(r,k,p)$ and $\tilde{w}_0(r,k,p)$ transformed by Eq. (5) are expressed in terms of general solutions, Eq. (11). We will have

$$\begin{gathered}\tilde{u}_0(r,k,p)=k\left[B_1 I_0(\alpha_0 r)+B_2 K_0(\alpha_0 r)\right]-\beta_0^2\left[C_1 I_0(\beta_0 r)+C_2 K_0(\beta_0 r)\right],\\ \tilde{w}_0(r,k,p)=\alpha_0\left[B_1 I_1(\alpha_0 r)-B_2 K_1(\alpha_0 r)\right]-k\beta_0\left[C_1 I_1(\beta_0 r)-C_2 K_1(\beta_0 r)\right].\end{gathered} \tag{16}$$

We introduce the radius of some intermediate surface of the middle layer of the structure according to the following formula [27]:

$$\xi = \frac{r_1}{2}\left(\chi - \frac{r_1}{r_2}\right).$$

Writing Eqs. (16) through the standard decompositions of the Bessel functions I_i and K_i (i=0,1) into series according to the degrees of the radial coordinate r and, limiting them to the first terms at r=ξ, we obtain the indicated main parts, which we denote by $\tilde{u}_{0,0}$, $\tilde{u}_{0,1}$ and $\tilde{w}_{0,0}$, $\tilde{w}_{0,1}$:

$$\begin{aligned} \tilde{u}_{0,0} &= kB_{10} - \beta_0^2 C_{10}, \tilde{u}_{0,1} = \left(kB_2 - \beta_0^2 C_2\right)/\xi \\ \tilde{w}_{0,0} &= \alpha_0^2 B_{10} - k\beta_0^2 C_{10}, \tilde{w}_{0,1} = \left(B_2 - kC_2\right)/\xi \end{aligned} \tag{17}$$

where:

$$B_{10} = B_1 - B_2\left[\ln\frac{\alpha_0\xi}{2} - \psi(1) - \frac{1}{2}\right],\ C_{10} = C_1 - C_2\left[\ln\frac{\beta_0\xi}{2} - \psi(1) - \frac{1}{2}\right],$$

and $\psi(n)$, n=1,2, … is the logarithmic derivative of the Gamma function.

Writing Eqs.(16) for the transformed displacements $\tilde{u}_0(r,k,p)$ and $\tilde{w}_0(r,k,p)$ through the standard decompositions of the Bessel functions I_i and K_i (i=0,1) into series according to the degrees of the radial coordinate r and substituting in them the values of the constants B_{10}, C_{10}, B_2 and C_2, found from the system of Eqs. (17), we obtain

$$\begin{aligned} \tilde{u}_0(r,k,p) = \sum_{n=0}^{\infty}\Big\{&\alpha_0^{2n}\tilde{u}_{0,0} + \tilde{q}_1^{(0)}\tilde{Q}_n^{(0)}(k\tilde{w}_{0,0} - \alpha_0^2\tilde{u}_{0,0}) - \\ &-\xi\left[\alpha_0^{2n}\tilde{u}_{0,1} + \beta_0^2\tilde{q}_2^{(0)}\tilde{Q}_n^{(0)}(k\tilde{w}_{0,1} - \tilde{u}_{0,1})\right]\eta_{3,n}(r)\Big\}\frac{(r/2)^{2n}}{(n!)^2}, \end{aligned} \tag{18}$$

$$\begin{aligned} \tilde{w}_0(r,k,p) = -\frac{\xi}{r}\tilde{w}_{0,1} + \sum_{n=0}^{\infty}\Big\{&\beta_0^{2n}\cdot\tilde{w}_{0,0} + \alpha_0^2\tilde{q}_1^{(0)}\tilde{Q}_n^{(0)}(\tilde{w}_{0,0} - k\tilde{u}_{0,0}) - \\ &-\xi\left[\beta_0^{2n+2}\tilde{w}_{0,1} + \tilde{q}_2^{(0)}\tilde{Q}_{n+1}^{(0)}(\beta_0^2\tilde{w}_{0,1} - k\tilde{u}_{0,1})\right]\eta_{1,n}(r)\Big\}\frac{(r/2)^{2n+1}}{n!(n+1)!} \end{aligned} \tag{19}$$

where:

$$\tilde{q}_1^{(0)} = 1 - \tilde{R}_0\cdot\tilde{R}_{\mu_0}^{-1};\ \tilde{q}_2^{(0)} = \tilde{R}_0^{-1}\cdot\tilde{R}_{\mu_0} - 1;\ \eta_{3,n} = \ln\frac{r}{\xi} + \frac{1}{2} - \sum_{k=1}^{n}\frac{1}{k};$$

$$\tilde{Q}_n^{(0)} = \frac{\alpha_0^{2n} - \beta_0^{2n}}{\alpha_0^2 - \beta_0^2} = \sum_{k=1}^{n-1}\alpha_0^{2(n-1-k)}\beta_0^{2k};\ \tilde{Q}_0^{(0)} = 0;\ \tilde{Q}_1^{(0)} = 1;\ \tilde{Q}_2^{(0)} = \alpha_0^2 + \beta_0^2.$$

Let us introduce the following representations for functions $u_{0,i}(r,z,t)$ and $w_{0,i}(r,z,t)$:

$$u_{0,i}=\int_0^\infty \left.\begin{matrix}\cos kz\\ \sin kz\end{matrix}\right\} dk\int_{(l)}\tilde{u}_{0,i}e^{pt}dp,\quad w_{0,i}=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\} dk\int_{(l)}\tilde{w}_{0,i}e^{pt}dp \tag{20}$$

and operators $\lambda_m^n(\zeta)$, $\delta_m^n(\zeta)$, (m=0,1,2; n=0,1,2,…)

$$\left[\lambda_m^n(\zeta),\delta_m^n(\zeta)\right]=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\} dk\int_{(l)}\left(\beta_m^{2n},\alpha_m^{2n}\right)\tilde{\zeta}(r,k,p)e^{pt}dp, \tag{21}$$

Similarly to Eqs. (18) and (19), we express Eq. (15) in terms of the main parts of the transformed longitudinal $\tilde{u}_{0,0},\tilde{u}_{0,1}$ and radial $\tilde{w}_{0,0},\tilde{w}_{0,1}$ displacements. Limiting ourselves to the first approximations of (n=0) in infinite series in the obtained equations and applying operators given in Eqs. (20) and (21) to them, we obtain:

$$\begin{gathered}-G_{1i}\frac{\partial u_{0,0}}{\partial z}+G_{2i}w_{0,0}+\xi\left[G_{1i}\eta_{3,0}(r_i)-G_{2i}q_2^{(0)}\ln\frac{r_i}{\xi}\right]\frac{\partial u_{0,1}}{\partial z}-\\ -\xi G_{2i}\left[\left(1+q_2^{(0)}\right)\ln\frac{r_i}{\xi}\lambda_0+\frac{2}{r_i^2}\right]w_{0,1}=\frac{r_i^2}{a^2\left(b^2\right)}R_{\mu_1}^{-1}\left[\left(\delta_i-\frac{\partial^2}{\partial z^2}\right)F_r^{(i)}(z,t)\right]\end{gathered} \tag{22}$$

$$\begin{gathered}G_{3i}u_{0,0}-G_{4i}\frac{\partial w_{0,0}}{\partial z}-\xi\left[G_{3i}\eta_{3,0}(r_i)+G_{4i}q_2^{(0)}\ln\frac{r_i}{\xi}\frac{\partial^2}{\partial z^2}\right]u_{0,1}+\\ +\xi G_{4i}\left[\left(1+q_2^{(0)}\right)\ln\frac{r_i}{\xi}\lambda_0+\frac{2}{r_i^2}\right]\frac{\partial w_{0,1}}{\partial z}=\frac{4r_i^2}{a^3\left(b^3\right)}\tilde{R}_{\mu_2}^{-1}\left[\left(\delta_i-\frac{\partial^2}{\partial z^2}\right)F_{rz}^{(i)}(z,t)\right],\end{gathered} \tag{23}$$

Based on Eqs. (7) for the values α_m and β_m (m=0,1,2), it is easy to show that the operators δ_m^n and λ_m^n , m=0,1,2; n=0,1,2,… introduced by Eqs. (21) during the reverse transition along coordinate z and time t have the following form:

$$\begin{gathered}\delta_m^n(\zeta)=\left[\rho_m R_m^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n,\ \lambda_m^n(\zeta)=\left[\rho_m R_{\mu_m}^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n\\ m=0,1,2;\qquad n=1,2,3,...\end{gathered} \tag{24}$$

The system of Eqs. (22) and (23), in accordance with Eqs. (21) for operators λ_m^n , m=0,1,2; n=0,1,2,… are integro-differential equations. These equations contain the main parts $u_{0,0}$, $u_{0,1}$ and $w_{0,0}$, $w_{0,1}$, respectively, longitudinal – u_0 and radial – w_0 displacements of points of some “intermediate” surface of the middle layer of a three-layer cylindrical structure. The specified “intermediate” surface has a radius, the values of which are enclosed in the range $r_1\le\xi\le r_2$ In accordance with the numerical values of radius ξ, this “intermediate” surface can turn into a median surface at $\xi=(r_1+r_2)/2$ and contact surfaces between layers of the structure at $\xi=r_1$ and $\xi=r_2$. Therefore, the equations are a system of Eqs. (22) and (23) depending on the values of the radius ξ. The equations of vibration of a three-layer cylindrical structure relative to the main parts of the longitudinal and radial displacements of the points of the median or contact surfaces of the median layer can be.

These equations, in the absence of external layers, are general equations of longitudinal-radial vibrations [19] of a circular cylindrical viscoelastic structure, relative to the main parts of the longitudinal and radial displacements of points on the intermediate surface of the structure.

In addition, Eqs. (22) and (23) in their right parts correctly considering the forces acting on the outer and inner surfaces of the three-layer structure, reflect (approximately) the relationship and mutual influence of the layers-through the middle layer. It is not difficult to see in these equations the dependence on the viscoelastic operators of R_{μ_m} (m=0,1,2) layers.

If specific kernels K_m and K_{μ_m} are given for viscoelastic layer material operators, then it is not difficult to transform these equations for specific shell layer materials. Note that the operators given by Eqs. (22) and (23) are derived for the general case of arbitrary operator kernels R_{μ_m} (m=0,1,2).

Consider the following special case: suppose that the Poisson coefficients of the materials of all three layers of the structure are constant in time. Then the kernels of the viscoelastic operators R_m and R_{μ_m} will be the same, and taking this fact into account, the operators δ_m^n and λ_m^n, take the form:

$$\delta_m^n(\zeta)=\left[\frac{1}{a_m^2}M_m^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n,\ \lambda_m^n(\zeta)=\left[\frac{1}{b_m^2}M_m^{-1}\left(\frac{\partial^2\zeta}{\partial t^2}\right)-\frac{\partial^2\zeta}{\partial z^2}\right]^n \tag{25}$$
$$m=0,1,2;\qquad n=1,2,3,\ldots$$

Here the operator M_m is defined by the formula:

$$M_m(\zeta)=\zeta(t)-\int_0^t K_m(t-\tau)\zeta(\tau)d\tau; \tag{26}$$

a_m and b_m (m=0,1,2) are propagation velocities of longitudinal and transverse elastic waves in layer materials:

$$a_m=\sqrt{(\lambda_m+2\mu_m)/\rho_m};\quad b_m=\sqrt{\mu_m/\rho_m}. \tag{27}$$

In this case, the operators $q_1^{(0)}$ and $q_2^{(0)}$ are computed in then following way:

$$q_1^{(0)}=(\lambda_0+\mu_0)/\mu_0,\ q_2^{(0)}=-(\lambda_0+\mu_0)/(\lambda_0+2\mu_0) \tag{28}$$

Taking into account Eqs. (25-28), the equations of longitudinal-radial vibrations of a three-layer cylindrical viscoelastic structure are significantly simplified.

2.3 Determination of Displacements and Stresses

To determine the stress-strain state of the structure, it is necessary to determine the displacements and stresses. This procedure must be performed for all three layers. To do this, it will be necessary to express all displacements and stresses through the main parts $u_{0,0}$, $u_{0,1}$ and $w_{0,0}$, $w_{0,1}$ respectively, longitudinal – u_0 and radial – w_0 displacements of points of some “intermediate” surface of the middle layer of a three-layer cylindrical structure.

First, it is necessary to determine the displacements u_0, w_0 and the stresses $\sigma_{rr}^{(0)}, \sigma_{zz}^{(0)}$, $\sigma_{\theta\theta}^{(0)}, \tau_{zr}^{(0)}$ points of the middle layer. To find u_0, w_0 it is enough to reverse the expressions in Eqs. (18) and (19) for $\tilde{u}_0(r,k,p)$ and $\tilde{w}_0(r,k,p)$ by p and k. Applying transformations given by Eqs. (20) and (21) to Eqs. (18) and (19) and limiting ourselves to the zeroth order approximation, we obtain:

$$u_0(r,z,t) = u_{0,0} - \xi u_{0,1}\,;\; w_0(r,z,t) = -\frac{\xi}{r} w_{0,1} + \frac{r}{2} w_{0,0}. \tag{29}$$

Similarly, formulas for stresses are derived, for example

$$\begin{aligned}\sigma_{rr}^{(0)} = R_{\mu_0} \Bigg\{ \sum_{n=0}^{\infty} \Big[-q_1^{(0)} w_{0,0} - \left(1+q_1^{(0)}\right) \frac{\partial u_{0,0}}{\partial z} - \xi \Big[\eta_{3,n}(r) + \left(\delta_0 - q_2^{(0)} \frac{\partial^2}{\partial z^2} \right) \times \\ \times \eta_{1,n}(r) \Big] w_{0,1} - \xi \left[\left(-2q_2^{(0)} \frac{\partial^2}{\partial z^2} \right) \eta_{3,n}(r) - q_2^{(0)} \eta_{1,n}(r) \right] \frac{\partial u_{0,1}}{\partial z} \Big] - \frac{2}{r^2} \xi w_{0,1} \Bigg\}\end{aligned} \tag{30}$$

The displacements and stresses of the other layers are found in a similar way, for example:

$$\begin{aligned}\left(\delta_2 - \frac{\partial^2}{\partial z^2} \right) u_2(r,z,t) = \left(\delta_2 - \frac{\partial^2}{\partial z^2} + \frac{r^2 - r_2^2}{8} \delta_2 \left(\lambda_2 - \frac{\partial^2}{\partial z^2} \right) \right) \tilde{u}_0(r_2,z,t) + \\ + \frac{r^2 - r_2^2}{4r_2} \left(\lambda_2 - \delta_2 \right) \frac{\partial w_0(r_2,z,t)}{\partial z}\end{aligned} \tag{31}$$

Solutions of Eqs. (22) and (23) for $u_{0,0}$, $u_{0,1}$ and $w_{0,0}$, $w_{0,1}$, can be used for calculating the displacements u_0, w_0 of points of an arbitrary cross-section of the median layer for any moment in time using Eq. (29) with the desired accuracy along the radial coordinate r, for an arbitrary moment in time. Similarly, the stresses of the remaining displacements and stresses at the points of the layers are determined in the case when m=1 and m=2.

3. Results and Discussions

The obtained equations of longitudinal-radial vibrations of a three-layer circular cylindrical viscoelastic shell, Eqs. (22) and (23), are general. In the following, we consider several limiting cases and particular types of vibration equations.

3.1 Vibrations of a Two-Layer Viscoelastic Structure by a Layer, with Sticking between the Layers

At $a = r_1$ the three-layer cylindrical structure transforms into a two-layer structure. The intermediate surface of the structure of radius ξ passes into the intermediate surface of the inner layer of the structure. In this case, it should be assumed that the external action function $F_r^{(1)}(z,t), F_{rz}^{(1)}(z,t)$ acts on the surface $r = r_1$ and the operators $\lambda_1 = 0$, $\delta_1 = 0$.

Then, from the system of Eqs. (22) and (23), we obtain a system of equations of a two-layer cylindrical viscoelastic structure.

Similarly, it is possible to obtain a system of equations of longitudinal-radial vibrations of a two-layer cylindrical viscoelastic structure, where the main unknowns will be the main parts of the longitudinal-radial displacements of the intermediate surface of the outer layer. To do this, it is enough to assume that there is no outer layer and put $b = r_2$ assume that the functions of external action $F_r^{(2)}(z,t)$, $F_{rz}^{(2)}(z,t)$ act on the surface $r = r_2$, and the operators $\boldsymbol{\lambda}_2 = 0$, $\boldsymbol{\delta}_2 = 0$.

If there are no inner and outer layers of the cylindrical structure, i.e. it is homogeneous (single-layer), then $a = r_1$, $b = r_2$ and $\boldsymbol{\lambda}_i = 0$, $\boldsymbol{\delta}_i = (i=1,2)$ should be put into general Eqs. (22) and (23). In this case, we will have a system of equations for longitudinal-radial vibrations of a cylindrical homogeneous layer, which exactly coincides with the system of equations derived by Khudoynazarov et al. [19].

3.2 A Three-Layer Cylindrical Viscoelastic Structure with a Thin Middle Layer and Rigid Contacts between the Layers

If $r_2 = r_1(1+\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon} > 0$ is a small parameter, then the middle layer of the structure is thin (for example, a thin layer of glue, usually applied between layers). In this case, the values of $\ln(r_i/\boldsymbol{\varepsilon})$ can be assumed to be zero. Then Eq. (19) for $\boldsymbol{\eta}_{3,n}(r_i)$ is simplified and takes the form:

$$\boldsymbol{\eta}_{3,n}(r_i) = \frac{1}{2} - \sum_{k=1}^{n} \frac{1}{k}, \; n = 0,1,2,..., \; i = 1,2. \tag{32}$$

Consequently, Eqs. (22) and (23) are also equations of longitudinal-radial vibrations of a three-layer cylindrical structure with a thin middle layer, but with a different value for $\boldsymbol{\eta}_{3,n}(r_i)$, determined by Eq. (32).

3.3 Three-Layer Cylindrical Elastic Structure with Sticking between Layers

If the materials of the layers are elastic, then the expressions for viscoelastic operators will have the equality $K_{\boldsymbol{\mu}\mathrm{m}}(t)=0$, and therefore we will have $R_{\boldsymbol{\mu}_m} = \boldsymbol{\mu}_m$. Then we get a system of equations that coincides in structure with the system of Eqs. (22) and (23), where the integral operators R_{μ_i} ,i=1,2, are replaced by the Lame coefficients $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, respectively. In this case, the integral-differential operators $\boldsymbol{\lambda}_m^n(\boldsymbol{\zeta})$ and $\boldsymbol{\delta}_m^n(\boldsymbol{\zeta})$, defined by Eqs. (25), pass into the following differential operators:

$$\boldsymbol{\lambda}_m^n(\boldsymbol{\zeta}) = \left[\frac{1}{b_m^2}\left(\frac{\partial^2 \boldsymbol{\zeta}}{\partial t^2}\right) - \frac{\partial^2 \boldsymbol{\zeta}}{\partial z^2}\right]^n, \; \boldsymbol{\delta}_m^n(\boldsymbol{\zeta}) = \left[\frac{1}{a_m^2}\left(\frac{\partial^2 \boldsymbol{\zeta}}{\partial t^2}\right) - \frac{\partial^2 \boldsymbol{\zeta}}{\partial z^2}\right]^n, \; n = 1,2,3,...$$

The results obtained coincide with the results by Filippov and Filippov [20]. In particular cases, from the equations obtained in this way, it is easy to obtain equations of a two-layer elastic structure and a three-layer elastic structure with a thin middle layer, like the limiting cases discussed above.

4. CONCLUSIONS

A new mathematical model of longitudinal-radial vibrations of a circular cylindrical three-layer viscoelastic structure with sticking condition between the layers is proposed. The model includes new vibration equations and an algorithm for calculating the stress-strain state of an arbitrary point of the structure. The vibration equations of the considered structures, including the influence of the moments of inertia and transverse shear deformation, are derived for arbitrary external dynamic loads acting onto the structure surfaces. In the absence of external layers, the results obtained completely coincide with the results of Filippov and Filippov [20].

A new method has been developed for the dynamic calculation of circular cylindrical three-layer elastic and viscoelastic shells for the action of various external dynamic loads. The method consists in the derivation of vibration equations, both refined ones of the Timoshenko type and classical ones of the Kirchhoff-Love type, and in the development of an algorithm for calculating the stress-strain state system.

As special cases of the obtained results, new equations of longitudinal-radial unsteady vibrations of a circular cylindrical three-layer elastic structure are proposed. With restrictions in infinite series, their approximations of different order are followed by Kirchhoff-Love type vibration equations [21], Hermann-Mirsky and other refined equations of higher orders.

Equations have been developed that enable to determine the stress-strain state at any point of an arbitrary section of a circular cylindrical three-layer viscoelastic structure based on the results of solving longitudinal-radial vibrations.

REFERENCES

1. Liang, W., Liu, T., Li, Ch., Wang, Q., 2023, *Three-Dimensional Vibration Model of Cylindrical Shells via Carrera Unified Formulation*, Materials, 16, 3345.
2. Naeem, M.N., Khan, A.G., Arshad, Sh.H., Abdul, G.Sh., Gamkhar, M., 2014, *Vibration of Three-Layered FGM Cylindrical Shells with Middle Layer of Isotropic Material for Various Boundary Conditions*, World Journal of Mechanics, 4(11), pp. 315–331.
3. Ye, T., Jin, G., Shi, S., Ma, X., 2014, *Three-dimensional free vibration analysis of thick cylindrical shells with general end conditions and resting on elastic foundations*, International Journal of Mechanical Sciences, 84, pp.120–137.
4. Banks, H.T., Hu, S., Kenz, Z.R., 2011, *A Brief Review of Elasticity and Viscoelasticity for Solids*, Advances in Applied Mathematics and Mechanics, 3(1), pp. 1–51.
5. Khudoynazarov, Kh.Kh., 2006, *Transversal vibrations of thick and thin cylindrical shells, interacting with deformable medium*, Proceedings of the 8th international conference on shell structures SSTA 2005, Jurata, Gdansk, Poland, Shell Structures: Theory and Applications, London: Taylor & Francis Group, pp. 343–347.
6. Mofakhami, M.R., Toudeshky, H.H., Hashemi, S.H., 2006, *Finite cylinder vibrations with different end boundary conditions*, Journal of Sound and Vibration, 297, pp. 293–314.
7. Sahoo, R., Grover, N., Singh, B.N., 2021, *Random vibration response of composite–sandwich laminates*, Archive of Applied Mechanics, 91, pp. 3755–3771.
8. Ghamkhar, M., Naeem, M.N., Imran, M., Soutis, C., 2019, *Vibration Analysis of a Three-Layered FGM Cylindrical Shell Including the Effect Of Ring Support*, Open Physics, 17(1), pp.587-600.
9. Shah., A.G., Mahmood, T., Naeem, M.N., Zafar, A.Sh., Arshad, Sh.H, 2010, *Vibrations of functionally graded cylindrical shells based on elastic foundations*, Acta Mechanica, 211(30), pp. 293–307.
10. Sayyad, A.S., Ghugal, Y.M., 2015, *On the free vibration analysis of laminated composite and sandwich plates: A review of recent literature with some numerical results*, Composite Structures, 129, pp. 177–201.
11. Altukhov, E.V., Fomenko, M.V., 2010, *Elastic equalization of a three-layer plate with imperfect layer contact*, Bulletin of Donetsk National University, Series A, Natural Sciences, 1, pp.78-87.

12. Altukhov, E.V., Fomenko, M.V., 2011, *Vibrations of three-layer plates in the case of boundary conditions of the flat end and slipping of layers*, Bulletin of Donetsk National University, Series.A, Natural Sciences, 2, pp. 34–41.
13. Iqbal, Z., Naeem, M.N., Sultana, N., 2009, *Vibration characteristics of FGM circular cylindrical shells*, Acta Mechanica, 208, pp. 237–248.
14. Sofiyev, A.H., Deniz, A., Akçay, I.H., Yusufoğlu, E.,2006, *The vibration and stability of a three-layered conical shell containing an FGM layer subjected to axial compressive load*, Acta Mechanica, 183, pp. 129–144.
15. Dimitrienko, Yu.I., Gubareva, E.A., Yakovleva, D.O., 2014, *Asymptotic theory of viscoelasticity of multilayer thin composite plates*, Science and Education of Bauman Moscow State Technical University, Electron Journal, 14, pp. 359–382.
16. Khudoynazarov, Kh., Abdurazakov, J., Kholikov, D., 2022, *Nonlinear torsional vibrations of a circular cylindrical elastic shell*, AIP Conference Proceedings, 2637, 020003.
17. Milić P., Marinković D., Klinge S., Ćojbašić Ž., 2023, *Reissner-Mindlin Based Isogeometric Finite Element Formulation for Piezoelectric Active Laminated Shells*, Tehnicki Vjesnik, 30(2), pp. 416 - 425.
18. Vyachkin E.S., Kaledin V.O., Reshetnikova E.V., Vyachkina E.A., Gileva A.E., 2018, *Mathematical modeling of static deformation of a layered construction with incompressible layers*, Tomsk State University Journal of Mathematics and Mechanics, 55, pp.72–83.
19. Khudoynazarov Kh.Kh., Khalmuradov R.I., Yalgashev B.F., 2021, *Longitudinal-radial vibrations of a elastic cylindrical shell filled with a viscous compressible liquid*, Tomsk State University, Journal of Mathematics and Mechanics, 69, pp. 139-154.
20. Filippov, I.G., Filippov, S.I., 2007, *Vibratory and wave processes in continuous compressible media*, Moskow: VINITI, 429 p.
21. Brekhovskikh, L., 2012, *Waves in Layered Media*, Elsevier, 574.
22. Khudayarov, B.A., Turaev, F.Z., 2019, *Mathematical simulation of nonlinear vibrations of viscoelastic pipelines conveying fluid*, Applied Mathematical Modelling, 66, pp. 662–679.
23. Netrebko, A.V., Pshenichnov, S.G., 2015, *Some problems of dynamics of linear-viscoelastic cylindrical shells of finite length*, Problems of strength and ductility, 77(1), pp. 67–74.
24. Rama G., Marinkovic D., Zehn M., 2018, *High performance 3-node shell element for linear and geometrically nonlinear analysis of composite laminates*, Composites Part B: Engineering, 151, pp. 118 - 126.
25. Safaei, B., Chukwueloka Onyibo, E., Goren, M., Kotrasova, K., Yang, Z., Arman, S., Asmael, M., 2023, *Free vibration investigation on rve of proposed honeycomb sandwich beam and material selection optimization*, Facta Universitatis-Series Mechanical Engineering, 21(1), pp. 31-50.
26. Zhao, Z., Yuan, X., Zhang, W., Niu, D., Zhang, H., 2021, *Dynamical modeling and analysis of hyperelastic spherical shells under dynamic loads and structural damping*, Applied Mathematical Modelling, 95, pp. 468-483.
27. Khudoynazarov, Kh., 2023, *A mathematical model of physically nonlinear torsional vibrations of a circular elastic rod*, Tomsk State University Journal of Mathematics and Mechanics, 84, pp. 152–166.