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Original scientific paper

HAMILTONIAN-BASED FREQUENCY-AMPLITUDE FORMULATION FOR NONLINEAR OSCILLATORS

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Abstract. *Complex mechanical systems usually include nonlinear interactions between their components which can be modeled by nonlinear equations that describe the sophisticated motion of the system. In order to interpret the nonlinear dynamics of these systems, it is necessary to compute their nonlinear frequencies more precisely. The nonlinear vibration process of a conservative oscillator always follows the law of energy conservation. A variational formulation is constructed and its Hamiltonian invariant is obtained. This paper suggests a Hamiltonian-based formulation to quickly determine the frequency property of the nonlinear oscillator. An example is given to explicate the solution process.*

Key words: *He's frequency formulation, Ancient Chinese mathematics, Semi-inverse method, Periodic solution*

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1. INTRODUCTION

Small amplitude oscillation of a pendulum or vibration in a long slender beam with low amplitude represent examples of the systems that can be well described using linear vibration theories. However, as the system components shift toward more sophisticated interactions, both nonlinear oscillators and their nonlinear characteristic equations play a vital role in explaining the behavior of complex systems. The unique phenomenon that can be modeled only through nonlinear systems, such as jump phenomenon, chaos, multiple steady-state solutions, etc., are the main significance of using the nonlinear oscillators in the vast majority of fields, especially in engineering structures. Nonlinear stiffness and friction in dynamical systems [1], complex beam and piezoelectric plate-based self-sustainable electromechanical models [2,3], nonlinear reinforced nanofibers [4], vibration caused by the interaction between vehicle and bridge [5], large amplitude vibration of beams [6-12] and dynamics of micro/nanoelectromechanical systems [13-18] are a few examples of nonlinear systems in the field of mechanical engineering. From the mathematical point of view, the Duffing oscillator, Van der Pol and Mathieu are well-known nonlinear equations. Several nonlinear systems can be described by utilizing the Duffing equation, from a simple pendulum with harmonic motion to the vibration of arched structures [19]. The Duffing equation especially emerges in mechanical systems with the presence of nonlinear stiffness springs. In many cases, stiffness is a function of displacement, which leads to cubic terms in the governing equations. Ultimately, this forms a nonlinear relation between the applied force to the spring and the resulting displacement. For instance, Fig. 1 shows a truck's rear Leaf suspension. The chaotic vibration caused by road excitation in vehicles can be studied by modeling the leaf spring with magnets as a double-potential-well Duffing oscillator [20].

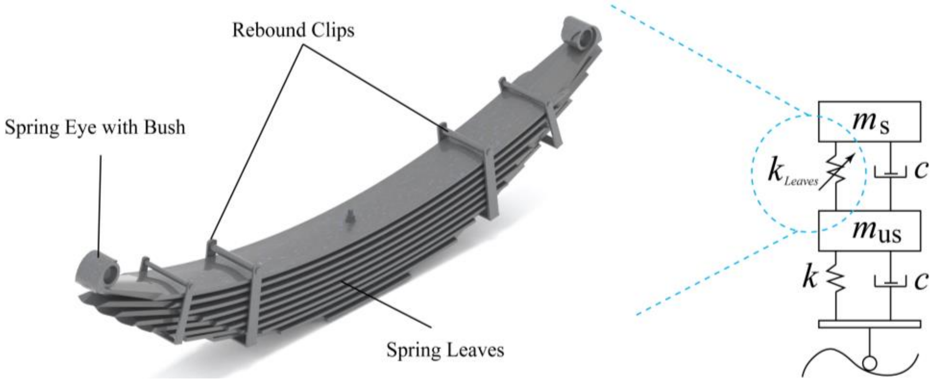

Fig. 1 Leaf Spring (Left) and quarter car diagram of a nonlinear suspension (Right)

Van der Pol is another example of nonlinear self-excited limit cycle oscillators that is widely used to describe various systems in electrical and mechanical engineering, seismology, economics, etc. A classical representation of the Van der Pol oscillator is in oscillator triode circuits [21]. This equation is also used to describe the Cardiac Pulse Modeling [22]. Another well-known nonlinear equation is Mathieu's equation. This equation was firstly encountered by Émile Léonard Mathieu when he was studying vibrating elliptical drumheads. Mathieu's equation tends to appear in the systems with harmonic motion and is a powerful tool for modeling systems with elliptic boundary conditions. For instance, a wind turbine blade under influence of wind shear force and gravitational cyclic force (Fig. 2) can be expressed using the forced Mathieu equation [23].

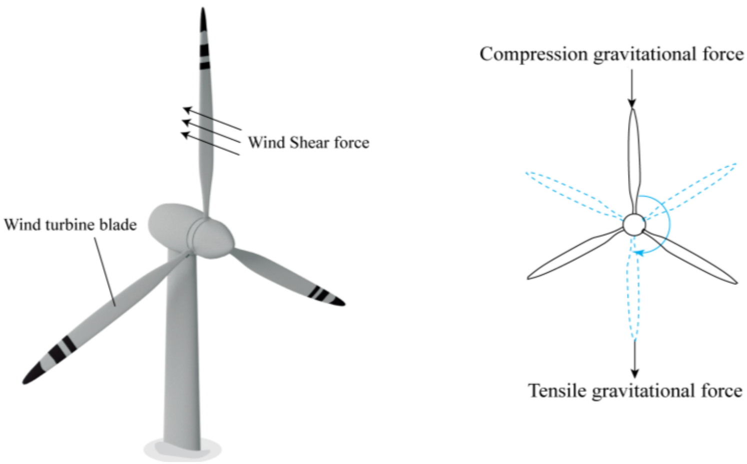

Fig. 2 Wind turbine (Left) and cyclic gravitational force on a blade (Right)

In this paper, based on the energy conservation, a modification of the frequency formulation is proposed in order to obtain the frequency-amplitude formulation of nonlinear systems. It is demonstrated that the proposed formulation is accurate enough for highly nonlinear differential equations containing large nonlinear terms. Several examples are also provided to exhibit the integrity of the introduced formulation.

2. Problem Statement

This paper focuses itself on the following conservative oscillator

$$w'' + p(w) = 0, w'(0) = 0 \quad w(0) = B \tag{1}$$

For a periodic solution, it requires $p(w)/w > 0$. There are many analytical methods available for solving Eq. (1), see some review articles in Refs. [24-26]. This paper will discuss the frequency-amplitude formulation, which was first proposed in 2006; it was obtained according to an ancient Chinese algorithm [27-29]. Due to its simplicity and accuracy, the formulation has been widely applied to solving various nonlinear oscillators; various modifications appeared in literature [30-38].

The formulation is to find a suitable solution in the form

$$w = B\cos\omega t \tag{2}$$

where ω is the frequency to be further determined. B residual equation is obtained by introducing Eq. (2) into Eq. (1), which results in

$$r(t) = -B\omega^2 \cos\omega t + p(B\cos\omega t) \tag{3}$$

The average residual can be calculated as

$$\tilde{r} = \frac{4}{T}\int_0^{T/4} r\cos\omega t dt \tag{4}$$

where $T = 2\pi/\omega$.

The formulation is to choose two trial frequencies, e.g., $\omega_1 = 1$ and $\omega_2 = 2$, and their residuals are respectively calculated as

$$\tilde{r}_1 = \frac{4}{T_1}\int_0^{T_1/4} r_1\cos\omega_1 t dt \tag{5}$$

$$\tilde{r}_2 = \frac{4}{T_2}\int_0^{T_2/4} r_2\cos\omega_2 t dt \tag{6}$$

The frequency-amplitude formulation is obtained as follows [27-29]

$$\omega^2 = \frac{\omega_2^2\tilde{r}_1 - \omega_1^2\tilde{r}_2}{\tilde{r}_1 - \tilde{r}_2} \tag{7}$$

There are many modifications of Eq. (7), see for examples, refs [30-38]. This paper will suggest an effective modification based on the Hamiltonian invariant.

3. HAMILTONIAN-BASED FREQUENCY-AMPLITUDE FORMULATION

The above frequency formulation is derived from a differential equation, here we suggests a modification from an energy form. The kinetic energy and the potential energy are changed during the oscillation process, but the total energy will keep unchanged for a conservative oscillator. In 2002, an energy approach to nonlinear oscillations was suggested [39].

The variational principle of Eq. (1) can be constructed by the semi-inverse method [40-43], which is

$$J(w) = \int\left\{\frac{1}{2}w'^2 - P(w)\right\}dt \tag{8}$$

where $p(w)$ is the potential, satisfying the following relation:

$$\frac{d}{dw}P(w) = p(w) \tag{9}$$

In the variational formulation given in Eq. (8), $\frac{1}{2}w'^2$ is the kinetic energy, and $P(w)$ is the potential energy. The total energy keeps unchanged during the oscillation:

$$\frac{1}{2}w'^2 + P(w) = H \tag{10}$$

where H is the Hamiltonian constant, which can be identified by the initial conditions given in Eq. (1). Finally we obtain the following first order differential equation,

$$\frac{1}{2}w'^2 + P(w) - P(B) = 0 \tag{11}$$

We use Eq. (11) instead of Eq. (1) to re-build the frequency-amplitude formulations. Substituting Eq. (2) into Eq. (11) results in the following residual equation,

$$R(t) = B^2\omega^2 \sin^2 \omega t + P(B\cos\omega t) - P(B) \tag{12}$$

Similarly we define two average residuals

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1 \cos\omega_1 t dt \tag{13}$$

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2 \cos\omega_2 t dt \tag{14}$$

A modification of the frequency- amplitude formulation is given as follows

$$\omega^2 = \frac{\omega_2^2\tilde{R}_1 - \omega_1^2\tilde{R}_2}{\tilde{R}_1 - \tilde{R}_2} \tag{15}$$

4. Example

Consider the following well-known Duffing equation,

$$w'' + w + \varepsilon w^3 = 0, w'(0) = 0 \quad w(0) = B \tag{16}$$

Eq. (16) can be reduced to the following first-order differential equation,

$$\frac{1}{2}w'^2 + \frac{1}{2}w^2 + \frac{1}{4}\varepsilon w^4 - \frac{1}{2}B^2 - \frac{1}{4}\varepsilon B^4 = 0 \tag{17}$$

We choose two arbitrary frequencies, e.g., $\omega_1 = 1$ and $\omega_2 = 2$, and obtain the following residual equations, respectively.

$$R_1 = \frac{1}{2}B^2 \sin^2 t + \frac{1}{2}B^2 \cos^2 t + \frac{1}{4}\varepsilon B^4 \cos^4 t - \frac{1}{2}B^2 - \frac{1}{4}\varepsilon B^4 \tag{18}$$

$$R_2 = 2B^2 \sin^2 2t + \frac{1}{2}B^2 \cos^2 2t + \frac{1}{4}\varepsilon B^4 \cos^4 2t - \frac{1}{2}B^2 - \frac{1}{4}\varepsilon B^4 \tag{19}$$

Their average residuals can be easily calculated:

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1 \cos t dt = \frac{-7B^4\varepsilon}{30\pi} \tag{20}$$

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2 \cos 2t dt = \frac{-7B^4\varepsilon + 30B^2}{30\pi} \tag{21}$$

According to the modified frequency-amplitude formulation, we obtain

$$\omega = \sqrt{\frac{\tilde{R}_1\omega_2^2 - \tilde{R}_2\omega_1^2}{\tilde{R}_1 - \tilde{R}_2}} = \sqrt{1 + \frac{7}{10}\varepsilon B^2} \tag{22}$$

To show its accuracy given in Eq. (22), we consider two extremes when $\varepsilon B^2 \to 0$ and $\varepsilon B^2 \to \infty$.

When $\varepsilon B^2 << 1$ Eq. (22) can be approximated as

$$\omega = 1 + \frac{7}{20}\varepsilon B^2 \tag{23}$$

While the perturbation solution is [32]

$$\omega = 1 + \frac{3}{8}\varepsilon B^2 \tag{24}$$

Table 1 shows that both Eq. (23) and Eq. (24) see good accuracy when $\varepsilon B^2 << 1$. Fig. 3 also shows the good agreement between the approximate and the exact solutions.

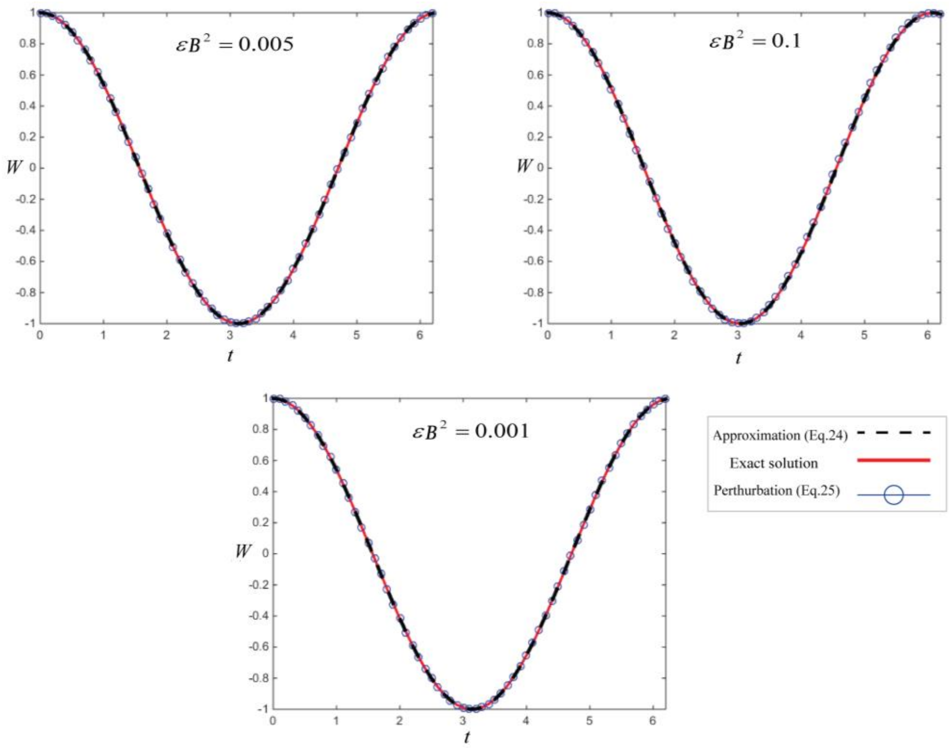

Fig. 3 Comparison of the approximate solution, the red continuous line is the exact solution, the black discontinuous line is the approximate solution, and the blue circles are perturbation solution

Table 1. Comparison of the approximate frequency of Eq. (23) with the exact one and the perturbation solution

εB^2	0	0.001	0.0025	0.003	0.005	0.007	0.009
Eq.(23)	1	1.00035	1.000875	1.00105	1.00175	1.00245	1.00315
Eq.(24)	1	1.000375	1.0009375	1.001125	1.001875	1.002625	1.003375
Exact frequency	1	1.000380	1.0009442	1.00113	1.0018726	1.002613	1.003369

When $\varepsilon B^2 \to \infty$, its approximate period becomes

$$\lim_{\varepsilon B^2 \to \infty} T_{app} = \frac{2\pi}{\omega} = \lim_{\varepsilon B^2 \to \infty} \frac{2\pi}{\sqrt{1+\frac{7}{10}\varepsilon B^2}} = \frac{7.5098}{\sqrt{\varepsilon B^2}} \tag{25}$$

The exact period, when $\varepsilon B^2 \to \infty$, is

$$T_{ex} = \frac{4}{\sqrt{\varepsilon B^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-0.5\sin^2 x}} = \frac{7.4164}{\sqrt{\varepsilon B^2}} \tag{26}$$

It is obvious that

$$\lim_{\varepsilon \to \infty} \frac{T_{ex}}{T_{app}} = 0.987 \tag{27}$$

The relative error is 1.317% when $\varepsilon B^2 \to \infty$.

The approximate period by the homotopy perturbation method is

$$\lim_{\varepsilon B^2 \to \infty} T_{\text{homotopy}} = \frac{2\pi}{\omega} = \lim_{\varepsilon B^2 \to \infty} \frac{2\pi}{\sqrt{1+\frac{3}{4}\varepsilon B^2}} = \frac{7.2552}{\sqrt{\varepsilon B^2}} \tag{28}$$

$$\lim_{\varepsilon \to \infty} \frac{T_{ex}}{T_{\text{homotopy}}} = 1.022 \tag{29}$$

The relative error is 2.153% even when $\varepsilon B^2 \to \infty$, see Fig. 4 and Table 2.

Table 2. Comparison of the approximate period of Eq. (25) with the exact one

εB^2	100	500	1000	1500	2000	$\varepsilon B^2 \to \infty$
Exact period	0.73629	0.33118	0.23435	0.19140	0.16577	$7.4164/\sqrt{\varepsilon B^2}$
Eq.(25)	0.74568	0.33537	0.23731	0.19381	0.16787	$7.5098/\sqrt{\varepsilon B^2}$
Relative error	1.275%	1.265%	1.263%	1.259%	1.267%	1.317%
Eq.(28)	0.72073	0.32403	0.22928	0.18725	0.16218	$7.2552/\sqrt{\varepsilon B^2}$
Relative error	2.113%	2.159%	2.163%	2.168%	2.166%	2.153%

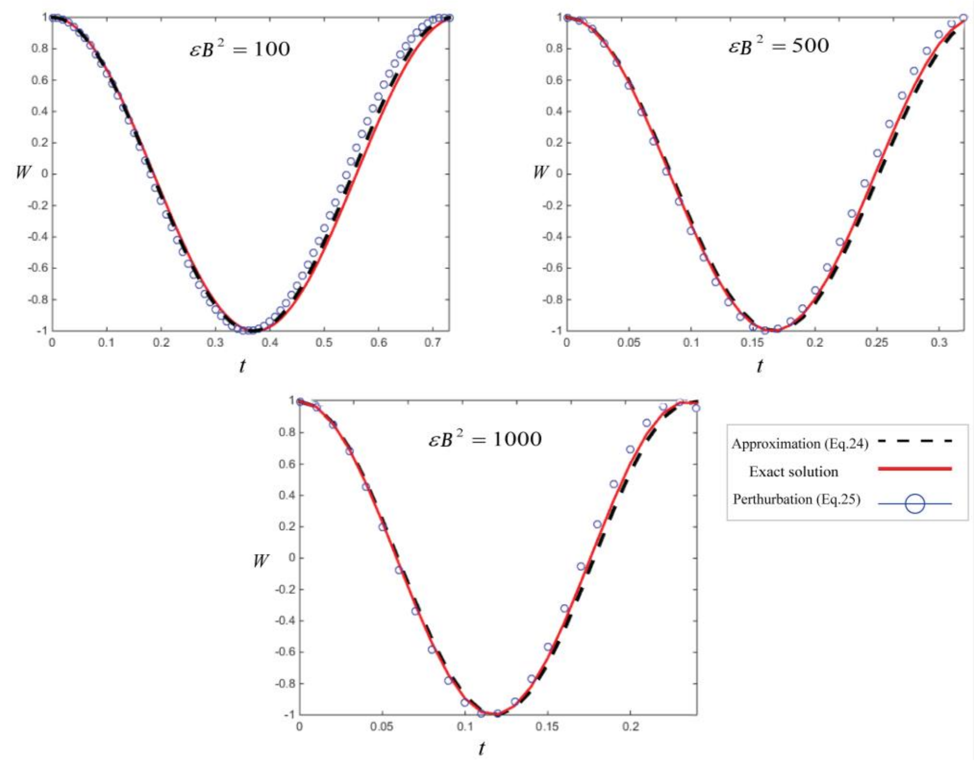

Fig. 4 Comparison of the approximate solution, the red continuous line is the exact solution, the black discontinuous line is the approximate solution, and the blue circles are perturbation solution

4. CONCLUSION

This paper suggests a modification of the frequency formulation based on the energy conservation, the obtained result is globally valid for $0 \leq \varepsilon B^2 < \infty$. The example shows that our result sees a good agreement with the perturbation solution for the weak nonlinearity. Even when $\varepsilon B^2 \to \infty$, our approximate frequency has also an extremely high accuracy, better than those obtained by the variational iteration method and the homotopy perturbation method.

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