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SELF-CONSISTENCY CONDITIONS IN STATIC THREE-BODY ELASTIC TANGENTIAL CONTACT

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Abstract. *The contact problem for an elastic third-body particle between two elastic half-spaces is considered. The contact is assumed to consist of three Hertzian contact spots. The normal and tangential contact problems are analyzed analytically considering partial slip in the contacts and the influence of third-body weight. Self-consistency conditions between global equilibrium and the contact solution are formulated to give criteria, under which circumstances static slip and stationary sliding are possible states for the third-body particle. The sliding case is solved in detail.*

Key words: *Three-body contact, Self-consistent sliding, Wear, Hertz-Mindlin theory*

1. INTRODUCTION

The tribological problem of the third body has recently attracted a lot of scientific interest, mostly in connection with the behavior and lifecycle of wear particles, whose understanding plays a critical role in a better description of both the wear process itself as well as the influence of wear and particle transport on other tribological phenomena in mechanical contacts [1]. Several aspects of the three-body problem have been analyzed, including the formation of wear debris particles – which was studied both experimentally [2] and numerically [3] – as well as their kinetics [4]. It was also shown that the third-body dynamics can have a massive influence on the frictional or other contact mechanical properties of a tribological system [5-7] and that *vice versa* contact properties like loading forces influence the mode of motion (sliding or rolling) of the wear debris particle [8]. While quite some research has been done on the wear and flow behavior of the debris particles, e.g., based on Monte Carlo methods [9] or Cellular Automata [10], there are hitherto very few works on the three-body system as a *contact mechanical* problem (which it obviously is). Li [11] analyzed the elastic three-body contact problem based on the Boundary Element method, using a starting configuration with only one

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contact spot on each of the first bodies, which at some threshold will result in rolling of the third body, because of the kinematic indeterminateness of a two-point fixation. An analytic, easy-to-use contact theory of the three-body problem, that might be very useful for practitioners or industrial applicants, to the best the author's knowledge, is lacking completely.

Whereas a (more or less) spherical particle between two surfaces, that are moved tangentially relative to one another, will usually simply roll, this may not be so easy for a third body of irregular shape. From experience we know that it will make a big difference for the apparent macroscopic friction between the first bodies, whether the third-body particle between them will roll or slide. In this context it is an interesting question whether static slip or stationary sliding are possible states for the particle. In a recent work it was shown analytically that that the same third-body particle can both slide and roll for a given coefficient of friction, depending on the particle's geometry and orientation [12].

Thus, in the present manuscript, the problem of self-consistency for static slip or stationary sliding of the third body will be investigated in analytic fashion, based on a Hertz-Mindlin formulation of the three-body contact problem. Note that the manuscript question is basically whether and how non-rolling configurations are possible for the third-body particle. So, effects of rolling are always excluded from the analysis.

2. Global Equilibrium Conditions

Let us consider the 2D-model of an elastic third-body particle of some irregular shape between two elastic half-spaces, as shown in Fig. 1. For static determinateness let there be three (axisymmetric) Hertzian contact spots, where the particle in the vicinity of the contact has radii of curvature R_i, i = 1,2,3. In the general 3D case, there should also be contact spots in the lateral direction (outside the plane shown in Fig. 1), to ensure lateral stability during tangential motion, but this will be neglected in the following analysis. As the lateral positions of the contact spots do not enter the following equations, the results can be applied directly to the 3D case, if the number of contact spots is adjusted appropriately. The gap height without any loads between the half-spaces is h_0.

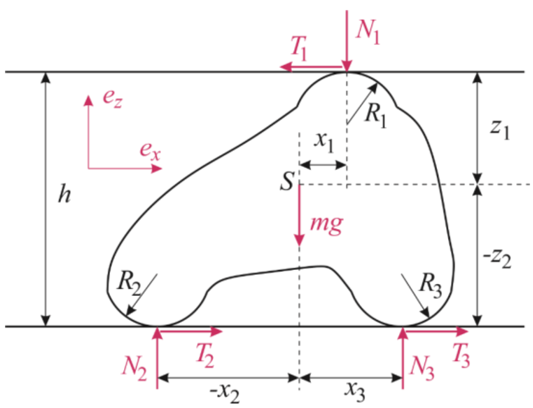


Fig. 1 Sketch and force diagram for the analyzed three-body contact problem

The global equilibrium conditions for the third body are

$$\begin{gathered}\sum F_z = 0 = N_2 + N_3 - N_1 - mg,\\ \sum F_x = 0 = T_2 + T_3 - T_1,\\ \sum M_y^{(S)} = 0 = N_1 x_1 - T_1 z_1 + N_2\left(-x_2\right) - T_2\left(-z_2\right) - N_3 x_3 - T_3\left(-z_2\right).\end{gathered} \tag{1}$$

Combining the last two equations we obtain

$$N_1 x_1 - N_2 x_2 - N_3 x_3 = T_1 h \approx T_1 h_0, \tag{2}$$

because the total indentation depth, $\Delta h = h_0 - h$, must be negligible for the Hertzian theory to be applicable.

3. Normal Contact Solution

For the contact solution in the following sections it shall be generally assumed that the characteristic length of the contact spots (e.g. their radius) is much smaller than the macroscopic dimensions of the third-body particle (so that one can neglect finite size effects for the elastic body) and that the contacting bodies are elastically similar to avoid elastic coupling of the normal and tangential contact problems, i.e. (introducing the shear moduli G_i and Poisson ratios ν_i, where the index "3" corresponds to the third-body particle, and "1" and "2" to the upper and lower half-spaces) [13]

$$\frac{1-2\nu_1}{G_1} = \frac{1-2\nu_2}{G_2} = \frac{1-2\nu_3}{G_3}. \tag{3}$$

With the effective elastic moduli on the upper and lower side [13],

$$E_1^* := \left(\frac{1-\nu_1}{2G_1} + \frac{1-\nu_3}{2G_3}\right)^{-1}, \quad E_2^* = E_3^* := \left(\frac{1-\nu_2}{2G_2} + \frac{1-\nu_3}{2G_3}\right)^{-1} \tag{4}$$

the Hertzian normal contact solution reads [13]

$$N_i = \frac{4}{3} E_1^* R_1^{1/2} d_i^{3/2}, \quad d_i > 0,\ i = 1,2,3. \tag{5}$$

The Hertzian solution can obviously only be used if the contact spots do not interact elastically. If the contact spots are very close to each other, one should consider interactions between them [14].

Without loss of generality let us assume that the lower half-space is fixed, and the upper half-space is macroscopically displaced by Δh. In general, the third-body particle will experience small elastic displacements in all its degrees of freedom (as a rigid body); if we denote the normal and tangential displacement of the center of gravity by w_S and u_S, and the small rotational angle by φ, the indentation depths for the three contact spots are given by

$$\begin{aligned} d_1 &= \Delta h - w_S - \varphi x_1, \\ d_2 &= w_S + \varphi x_2, \\ d_3 &= w_S + \varphi x_3. \end{aligned} \tag{6}$$

For the pure normal contact problem, the tangential forces are absent. The equilibrium conditions for F_z and M_y will then give a nonlinear equation system to determine the two unknown displacements, w_S and φ. This equation system will usually be unsolvable in closed analytical form and due to the plenty of influencing parameters a comprehensive solution cannot be shown here. For illustration purposes, however, let us give the solution, if the particle (e.g., due to symmetry) is not rotating.

Inserting Eqs. (6) (with $\varphi = 0$) into the first of Eqs. (1), we obtain

$$d_1^{3/2} + \frac{3mg}{4E_1^* R_1^{1/2}} = \frac{E_2^*}{E_1^*} \frac{R_2^{1/2} + R_3^{1/2}}{R_1^{1/2}} \left(\Delta h - d_1\right)^{3/2}. \tag{7}$$

Introducing dimensionless variables,

$$\delta := \frac{d_1}{\Delta h}, \quad \tilde{m} := \frac{3mg}{4E_1^* R_1^{1/2} \Delta h^{3/2}}, \quad \alpha^{-3/2} := \frac{E_2^*}{E_1^*} \frac{R_2^{1/2} + R_3^{1/2}}{R_1^{1/2}}, \tag{8}$$

Eq. (7) simplifies to

$$\delta^{3/2} + \tilde{m} = \alpha^{-3/2} \left(1 - \delta\right)^{3/2}. \tag{9}$$

From the derivation above it follows that this equation is independent of the number of contact spots on the upper and lower side – which only changes the value of α – if all contacts on one side have the same indentation depth, i.e., if all “asperities” have the same height. However, considering a height distribution as in classical asperity theories [15] would, of course, be possible without difficulties.

The nonlinear Eq. (9) cannot be solved in closed form. However, usually the particle weight will be small compared to the contact forces. In this case an asymptotic solution can be found easily. It is given by

$$\begin{aligned} \delta\left(\tilde{m}\right) &\approx \delta_0 + \psi \tilde{m}, \\ \delta_0 &= \delta\left(\tilde{m} = 0\right) = \left(1 + \alpha\right)^{-1}, \\ \psi &= \left.\frac{\mathrm{d}\delta}{\mathrm{d}\tilde{m}}\right|_{\tilde{m}=0} = \left.\frac{\mathrm{d}\delta}{\mathrm{d}\tilde{m}}\right|_{\delta=\delta_0} = -\frac{2}{3} \frac{1 - \delta_0}{\delta_0^{1/2}}. \end{aligned} \tag{10}$$

Note that δ itself has a weak dependence on the current total indentation Δh (via the normalized particle weight). Hence, the distribution of the total indentation into the upper and lower side will not be universal during the indentation process, if there are external forces acting on the debris particle.

4. TANGENTIAL CONTACT SOLUTION

Due to the equilibrium condition for the moments of forces, the normal and tangential contact problems are coupled macroscopically and therefore must be solved together. Suppose the upper half-space is displaced in the tangential direction by Δu, according to some loading history $\Delta u = \Delta u(\Delta h)$. Then, the relative tangential displacements in the contact spots are given by

$$\begin{aligned} u_1 &= \Delta u - u_S + \varphi z_1, \\ u_2 &= u_3 = u_S - \varphi z_2. \end{aligned} \tag{11}$$

All tangential forces are functions of these displacements,

$$T_i = T_i\left(u_i\right),\ i = 1, 2, 3. \tag{12}$$

However, the precise form of those functions depends on the loading history, and therefore, on all system parameters. Because of that it is not feasible – although theoretically possible based on the known solution procedures for tangential contact problems with arbitrary loading histories [16] – to give a comprehensive analytic solution for T_i. For example, it would be a gross simplification to use the classical Cattaneo-Mindlin solution, that is valid only for a specific loading history, namely a constant normal force and a subsequently applied increasing tangential force (which clearly contradicts the global equilibrium conditions). Nonetheless, a numerical determination for a concrete parameter set is easy, for example within the frameworks of the method of dimensionality reduction [17] or the method of memory diagrams [18]. Once the force laws for the tangential forces are known, the global equilibrium conditions (1) provide an equation system for the determination of the three displacements, w_S, u_S and φ.

It should be noted that the critical displacement, for which the contacts start to slide globally, is always [13]

$$u_{i,c} = \mu_i \frac{E_i^*}{G_i^*} d_i, \tag{13}$$

with the effective shear moduli

$$G_1^* := \left(\frac{2-\nu_1}{4G_1} + \frac{2-\nu_3}{4G_3}\right)^{-1},\ G_2^* = G_3^* := \left(\frac{2-\nu_2}{4G_2} + \frac{2-\nu_3}{4G_3}\right)^{-1}. \tag{14}$$

Hence, all contacts on one side will to start to slide at the same time if they have the same indentation depth, independent of their local radii of curvature.

When all contacts are sliding, the tangential problem becomes trivial, and all tangential forces are given by the Amontons-Coulomb law

$$T_i = \mu_i N_i. \tag{15}$$

To illustrate the importance of the loading history once again for the tangential contact solution, let us consider a simple (but somewhat academic) non-sliding case, which can be easily solved in exact analytic form, namely the two-contact configuration without external forces.

If we set $N_3 = T_3 = mg = 0$, equilibrium of the forces demands that the contact forces on both sides are equal and opposite and the equilibrium condition for the moment of forces reduces to

$$T = N\frac{x_1 - x_2}{h} := N\tan\alpha. \tag{16}$$

Hence, during loading the tangential contact force always has to be proportional to the normal force and this, of course, also always has to be true for the force increments dN and dT. Consider a given equilibrium with the forces N and T and the contact radius a. Now, the normal force is increased by dN, which according to the Hertzian theory results in a new contact radius [13]

$$a + \mathrm{d}a = a\left(1 + \frac{\mathrm{d}N}{N}\right)^{1/3}. \tag{17}$$

Irrespective of the previous load history, the entire contact area will initially completely adhere (according to Amontons' law, the slip area is constantly at the limit of possible sticking, a slight increase in pressure leads to complete sticking). By applying an additional incremental force dT, however, local slip can again spread from the edge of the contact. The stick radius c is given by the Cattaneo-Mindlin theory and equals [13]

$$c = (a + \mathrm{d}a)\left(1 - \frac{dT}{\mu\left(N + \mathrm{d}N\right)}\right)^{1/3} = a\left(1 + \frac{\mathrm{d}N}{N} - \frac{\mathrm{d}T}{\mu N}\right)^{1/3}. \tag{18}$$

If $c > a$, the contact area increases faster than slip can propagate into the contact, i.e., the contact will always be completely sticking. This leads to the condition

$$\mu > \tan\alpha. \tag{19}$$

Hence, it was shown that – within the Hertz-Mindlin approximation – there is no partial slip for this loading history; the contact is always completely sticking if $\mu > \tan\alpha$ (and obviously completely sliding otherwise), because the contact area grows faster than slip can propagate from the contact edge. So, the self-consistency condition for elastic bodies resulting from Hertz-Mindlin contact mechanics is in this case the same as the one for rigid bodies (i.e., non-rolling configurations are only possible if $\tan\alpha \le \mu$)!

5. Self-Consistency Conditions for Gross Sliding

In the case of static slip (i.e., macroscopic "sticking") the tangential forces are bound by the friction law, which imposes a self-consistency condition. For stationary sliding there are two self-consistency conditions (because the tangential forces are given explicitly by the friction law), directly resulting from the global equilibrium,

$$\begin{gathered} \mu_1 N_1 h_0 = N_1 x_1 - N_2 x_2 - N_3 x_3, \\ \mu_1 N_1 = \mu_2\left(N_2 + N_3\right) = \mu_2\left(N_1 + mg\right) \Leftrightarrow mg = N_1\left(\frac{\mu_2}{\mu_1} - 1\right). \end{gathered} \tag{20}$$

So, if the upper and lower contacts have the same frictional properties, the weight of the particle will disturb the equilibrium and thus inhibit stationary sliding. That is not restrained to the particle weight or the number of contact spots on each side: in fact, if the coefficients of friction on both sides are the same and all contacts are sliding, obviously any force, which is not aligned with the friction angle, will violate the equilibrium condition.

Finally, if once again rotation of the particle is absent, using the normal contact solution from Eqs. (7) and (8), the first of Eqs. (20) can be written in the form

$$\mu_1 h_0 = x_1 - \frac{E_2^*}{E_1^*}\left(\frac{1-\delta(\tilde{m})}{\delta(\tilde{m})}\right)^{3/2}\left[x_2\left(\frac{R_2}{R_1}\right)^{1/2} + x_3\left(\frac{R_3}{R_1}\right)^{1/2}\right]. \quad (21)$$

If the particle weight is negligible compared to the contact forces, this simplifies to

$$\mu_1 h_0 = x_1 - \frac{R_2^{1/2} x_2 + R_3^{1/2} x_3}{R_2^{1/2} + R_3^{1/2}}, \quad (22)$$

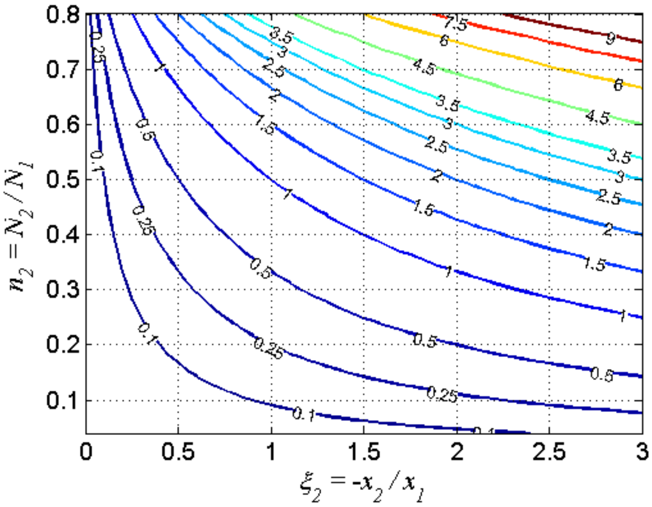
which, interestingly, does not depend on the elastic properties of the system.

One possibility for the absence of rotation would be complete symmetry, i.e., $x_1 = 0$, $R_2 = R_3$ and $x_2 = -x_3$. This results in the self-consistency condition $\mu h_0 = 0$, so stationary sliding in this case is impossible!

6. Example Case with Equal Coefficients of Friction

To illustrate the above findings let us consider a wear debris particle of some general shape, which in the contact plane forms three contact spots with the first bodies. We want to know whether stationary sliding is a possible state for the third body if the coefficients of friction on the upper and lower surfaces are equal and if we can neglect the particle weight. The contact enumeration shall be as in **Fig. 1**, i.e., spot number “one” is the singular one, “three” is the one on the other surface but on the same side (left/right with respect to the center of gravity) and “two” the one on the other surface and on the other side. All geometrical notations are the same as in **Fig. 1**. As we neglect the particle weight and the friction coefficients are equal on both surfaces, the second of Eqs. (20) is always fulfilled and the only remaining condition of self-consistent stationary sliding is the first of Eqs. (20).

Fig. 2 shows the contour line diagram of the normalized position of the third contact spot, $\xi_3 = x_3/x_1$, as a function of the normalized force distribution, $n_2 = N_2/N_1$ and the normalized position of the second contact spot, $\xi_2 = -x_2/x_1$, necessary to enable stationary sliding of the third-body particle. The normalized gap width between the first body surfaces was chosen to be $H = \mu h_0/x_1 = 1$. Note that the results for different values of H can be simply obtained by shifting the coordinate ξ_2 by $\Delta\xi_2 = \Delta H/n_2$.



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7. Discussion and Conclusions

In the considerations above several simplifying assumptions have been made to allow for analytical treatment of the problem, most prominently the Amontons-law, linear elasticity and the absence of surface roughness and elastic coupling. However, most findings are a consequence of the principal structure of the three-body contact problem and its resulting static indeterminateness. It was shown how global external forces on the third-body particle – like its weight – can influence the local contact problem and that static slip and stationary sliding are only possible (but they are possible, at least, if one neglects frictional instabilities) for specific system configurations. For example, for stationary sliding, if external forces are absent, the frictional properties of the upper and lower contacts must be the same. That condition is, however, not sufficient, as the equilibrium of the moments of force will impose another restriction for the geometrical "arrangement" of the contact spots. That restriction seems to be independent of the elastic properties but does depend on the local geometry in the vicinity of the contact spots (which strongly influences the respective normal contact solution).

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