COSMOLOGICAL FLOWS ON HYPERBOLIC SURFACES[†]

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Abstract. *We outline the geometric formulation of cosmological flows for FLRW models with the scalar matter as well as certain aspects which arise in their study with methods originating from the geometric theory of dynamical systems. We briefly summarize certain results of numerical analysis which we carried out when the scalar manifold of the model is a hyperbolic surface of the finite or infinite area.*

Key words: *differential geometry, continuous dynamical systems, cosmology*

1. INTRODUCTION

The standard cosmological model of matter coupled to gravity involves an FLRW space-time with flat spatial section, with classical dynamics of matter fields reduced to spatially homogeneous configurations. The evolution of such models with respect to cosmological time is governed by generally covariant systems of ODEs which can be studied (Wainwright and Ellis, 1997; Coley, 2003) using the theory of dynamical systems, which in this context is most powerful when formulated geometrically (Palis Jr. and De Melo, 2012).

In most such applications up to date, the matter content of the cosmological model was taken to be relatively simple and was assumed not to carry any interesting 'internal' geometry. In particular, scalar inflationary models studied in the dynamical systems framework have largely been assumed to have kinetic terms described by scalar manifolds which are simply-connected and flat. However, arguments from string theory

and supergravity suggest that the matter content present in the universe immediately after the Big Bang could be quite complicated and, in particular, that it may involve multiple scalar fields whose kinetic terms are described by rather general scalar manifolds, which need not be flat or simply connected. In this context, there is currently substantial interest in the so-called multifield α-attractor models (Kallosh and Linde, 2015; Carrasco et al., 2015), whose most general incarnation involves an arbitrary hyperbolic scalar manifold (Lazaroiu and Shahbazi, 2017). When applied to such models, the dynamical systems approach leads to highly non-trivial problems which have deep connections with several branches of mathematics, while posing interesting challenges regarding the statistical interpretation of cosmological dynamics. These aspects are spectacularly illustrated already by the case of scalar two-field models whose scalar manifold is a hyperbolic surface. Indeed, the cosmological dynamics of such models relates not only to uniformization theory, Fuchsian groups and number theory (as already pointed out[1] in Lazaroiu and Shahbazi, 2017) but also, at least in certain limits, to the theory of hyperbolic and Morse-Smale flows, as well as to certain aspects of ergodic theory and of non-equilibrium classical statistical mechanics. In references (Babalic and Lazaroiu, 2018a) and (Babalic and Lazaroiu, 2017), we illustrated the complexity of cosmological dynamics for models whose scalar manifolds are certain hyperbolic surfaces of the finite and infinite area and discussed some of the first predictions made by such models for cosmological observables.

2. COSMOLOGICAL FLOWS

The cosmological equations of an FLRW model can be reduced to equations governing the time evolution of spatially homogeneous field configurations by eliminating the conformal scale factor $a(t)$ using the Friedmann constraint. The resulting differential equations (which involve only matter variables[2]) define a dynamical system on the relevant space of states, whose flow we call the *cosmological flow* of the model. For models involving only scalar fields, these differential equations form a geometric system of coupled second order ODEs governing the evolution of scalar variables, which is described by a point moving inside the scalar manifold $\mathcal{M}$. In such models, the cosmological flow is defined on the tangent bundle $T\mathcal{M}$, which is the space of states of the associated dynamical system.

2.1. Scalar triples and cosmological equations of motion

As explained in Lazaroiu and Shahbazi (2017), cosmological scalar field models of FLRW type with flat and simply-connected spatial section are parameterized by the choice of a *scalar triple* $(\mathcal{M}, \mathcal{G}, V)$. The *scalar manifold* $(\mathcal{M}, \mathcal{G})$ is a generally non-compact but complete Riemannian manifold which determines the kinetic terms of the scalar fields, while the *scalar potential* V is a smooth and everywhere non-negative real-valued function defined on $\mathcal{M}$. After eliminating the conformal scale factor $a(t)$ using the

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[1] See Schimmrigk (2015), Schimmrigk (2016), Schimmrigk (2017a), Schrimmrigk (2017b) for a different but related approach.
[2] The matter variables describe spatially homogeneous configurations of matter fields in the FLRW universe and hence depend only on the cosmological time.

Friedmann constraint, the evolution of the model is described by smooth curves $\varphi \colon I \to \mathcal{M}$ (with $I \subset \mathbb{R}$ an interval) which satisfy the following geometric second order autonomous ODE, where t is the cosmological time:

$$\nabla_t \dot{\varphi}(t) + \left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}V)(\varphi(t)) = 0. \tag{1}$$

Here the norm, covariant derivative and gradient are taken with respect to the scalar manifold metric $\mathcal{G}$. The second term arising in the left hand side of (1) (known as the *Hubble friction term*) involves the *Hubble function H_φ* of the curve, which is defined through:

$$H_\varphi(t) \overset{\mathrm{def.}}{=} \mathcal{H}(\varphi(t), \dot{\varphi}(t)), \tag{2}$$

where the *absolute Hubble function $\mathcal{H}$* of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is defined on $T\mathcal{M}$ through the formula:

$$\mathcal{H}(u) \overset{\mathrm{def.}}{=} \frac{1}{3} \left[||u||^2_{\pi(u)} + 2V(\pi(u)) \right]^{\frac{1}{2}} = \frac{1}{3}\sqrt{2E(u)}, \ u \in T\mathcal{M}. \tag{3}$$

Here $\pi \colon T\mathcal{M} \to \mathcal{M}$ is the bundle projection and $E = \frac{9}{2}\mathcal{H}^2 \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is the *Hubble energy*:

$$E(u) \overset{\mathrm{def.}}{=} \frac{1}{2}||u||^2_{\pi(u)} + V(\pi(u)). \tag{4}$$

The Hubble energy and absolute Hubble function are continuous on $T\mathcal{M}$ and smooth on the slit tangent bundle $\dot{T}\mathcal{M} \overset{\mathrm{def.}}{=} T\mathcal{M} \backslash 0$ (where 0 denotes the image of the zero section of $T\mathcal{M}$). The following one-parameter deformation of (1):

$$\epsilon \nabla_t \dot{\varphi}(t) + \left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}V)(\varphi(t)) = 0 \tag{5}$$

interpolates between the geodesic equation $\nabla_t \dot{\varphi}(t) = 0$ of $(\mathcal{M}, \mathcal{G})$ (which is recovered for $\epsilon \to \infty$) and the equation $\left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}V)(\varphi(t)) = 0$ (which is obtained for $\epsilon \to 0$). As explained in (Lazaroiu and Shahbazi, 2017), the latter is equivalent with the gradient flow equation $\mathrm{d}\varphi(q)/\mathrm{d}q + (\mathrm{grad}V)(\varphi(q)) = 0$ of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ through the change of parameter $t \to q$ defined through:

$$\mathrm{d}t = 3H_\varphi(q)\mathrm{d}q. \tag{6}$$

Hence the cosmological equation of motion (which corresponds to $\epsilon = 1$) 'sits between' the geodesic and gradient flow equations. Unlike the geodesic equation (which is the Euler-Lagrange equation of the free particle Lagrangian $L(u) \overset{\mathrm{def.}}{=} \frac{1}{2}||u||^2_{\pi(u)}$), the cosmological equation of motion need *not* admit a Lagrangian description (see Remmen and Carroll, 2013 for a discussion of this point in the simple case of one-field models). Using the interpolation provided by (5), one can develop two perturbation expansions for the dynamics of the model, namely the *gradient flow expansion* (whose leading approximation was already discussed in Lazaroiu and Shahbazi, 2017) and the *geodesic flow expansion*. Besides these two, there are numerous other perturbation expansions which can be considered for such models.

2.2. The cosmological semispray and cosmological flow

Since the second-order ODE (1) is geometric, it is equivalent to the flow equation:

$$\dot{\gamma}(t) = S\big(\gamma(t)\big), \text{ where } \gamma: I \to T\mathcal{M}, \tag{7}$$

of a vector field S defined on $T\mathcal{M}$ which satisfies the *semispray condition* (see Antonelli and Bucataru, 2003):

$$\pi_*(S_u) = u, \forall u \in T\mathcal{M} \tag{8}$$

and which we call the *cosmological semispray* of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$. More precisely, a curve $\gamma: I \to T\mathcal{M}$ is a solution of (7) iff it coincides with the complete lift[3] $\tilde{\varphi}$ of a solution $\varphi: I \to \mathcal{M}$ of (1). One can show that the following relation holds:

$$S = S_0 + 3\mathcal{H}C + (\text{grad}V)^v, \tag{9}$$

where S_0 is the geodesic spray of $(\mathcal{M}, \mathcal{G})$, C is the Liouville vector field of $T\mathcal{M}$ and $X^v \in \mathcal{X}(T\mathcal{M})$ denotes the vertical lift of a vector field $X \in \mathcal{X}(\mathcal{M})$. Hence the classical dynamics of the model is described by the *cosmological dynamical system* defined by the vector field (9) on $T\mathcal{M}$, whose flow we call the *cosmological flow* of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$. The state space $T\mathcal{M}$ of this dynamical system is the space of positions and velocities of the spatially-homogeneous scalar field distributions described by the curve φ. When $\mathcal{H}$ and $\text{grad}V$ are small, the cosmological semispray (9) can be viewed as a vertical perturbation of the geodesic spray of $(\mathcal{M}, \mathcal{G})$.

The cosmological flow of a general scalar triple can be extremely complicated and its proper study requires deep ideas from the theory of dynamical systems, asymptotic analysis and singular perturbation theory. One useful perspective on this flow is provided by scale analysis, a.k.a. by 'renormalization group'[4] techniques. In the models discussed here, one can show that the IR limit corresponds to a reparameterized gradient flow, while the UV limit corresponds to the geodesic flow. In particular, the regime of most interest for inflation (which is the 'slow motion', i.e. the IR regime) can be understood by studying the gradient flow and gradient flow expansion. The UV and IR regimes of the model are markedly different, as illustrated quite dramatically by the case $\dim\mathcal{M} = 2$.

3. COSMOLOGICAL FLOWS ON HYPERBOLIC SURFACES

As mentioned above, the cosmological flow of a general scalar triple $(\mathcal{M}, \mathcal{G}, V)$ can be extremely complicated, so it is useful to consider the case of two-dimensional scalar manifolds. In this situation, $\mathcal{M}$ is a (generally non-compact and not simply connected) smooth surface which we shall denote by Σ.

The special case when $\mathcal{G} = 3\alpha G$ with α a positive parameter and G a complete hyperbolic metric on Σ of Gaussian curvature -1 produces a *generalized two-field α-attractor model* in the sense of reference (Lazaroiu and Shahbazi, 2017). Such models are

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[3] The *complete lift* of a curve $\varphi: I \to \mathcal{M}$ is the curve $\tilde{\varphi}: I \to T\mathcal{M}$ defined through $\tilde{\varphi}(t) \overset{def.}{=} (\varphi(t), \dot{\varphi}(t)) \in T\mathcal{M}$ for any $t \in I$.

[4] In our context, scale analysis is applied to a classical dynamical system, unlike Wilson's famous application of such methods to quantum field theories.

particularly interesting since, under mild assumptions on the scalar potential, they have universal behavior for certain special cosmological trajectories which are close to the Freudenthal ends of $\mathcal{M}$. More precisely, it was shown in (Lazaroiu and Shahbazi, 2017) that for scalar potentials having 'good' asymptotic behavior at the ends and in the slow-roll approximation for certain cosmological trajectories φ located close to the ends, the naive one-field truncation produces the same values of the spectral index n_s and tensor to scalar ratio r as ordinary one-field α-attractors, thus being in good agreement with current observations:

$$n_s \approx 1 - \frac{2}{N} \; , \; r \approx \frac{12\alpha}{N^2} , \tag{10}$$

where $N \stackrel{\text{def.}}{=} \int_{t_i}^{t_f} H_\varphi(t)\mathrm{d}t$ is the number of efolds. In fact, much stronger universality arguments can be made for such models using dynamical systems techniques.

Models based on hyperbolic surfaces are also interesting from a mathematical perspective, given their deep connection to Fuchsian groups and number theory (which stems from Poincaré's uniformization theorem) and the special behavior of their geodesic flow. For example, it is well-known that the geodesic flow of a hyperbolic surface of the finite area is ergodic and mixing.

In Babalic and Lazaroiu (2018a) and Babalic and Lazaroiu (2017), we performed a numerical study of cosmological flows for certain non-compact hyperbolic surfaces. Reference (Babalic and Lazaroiu, 2018a) considered cosmological flows on elementary hyperbolic surfaces (namely the Poincaré disk $\mathbb{D}$, the hyperbolic punctured disk $\mathbb{D}^*$ and the hyperbolic annuli $\mathbb{A}(R)$), while Babalic and Lazaroiu (2017) studied the case of the hyperbolic triply-punctured sphere $Y(2)$.

As explained in Lazaroiu and Shahbazi (2017), the equations of motion of any two-field generalized α-attractor model can be lifted from Σ to the Poincaré half-plane $\mathbb{H}$ by using the covering map $\pi_{\mathbb{H}}: \mathbb{H} \to \Sigma$ which uniformizes (Σ, G) to $\mathbb{H}$. This allows one to determine the cosmological trajectories $\varphi(t)$ by projecting to Σ the trajectories $\tilde{\varphi}(t)$ of a "lifted" model defined on $\mathbb{H}$, which is governed by the following system of second order non-linear ODEs:

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} + \frac{1}{M_0}\left[3\alpha\frac{\dot{x}^2+\dot{y}^2}{y^2} + 2\tilde{V}(x,y)\right]^{1/2}\dot{x} + \frac{1}{3\alpha}y^2\partial_x\tilde{V}(x,y) = 0 \tag{11}$$

$$\ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) + \frac{1}{M_0}\left[3\alpha\frac{\dot{x}^2+\dot{y}^2}{y^2} + 2\tilde{V}(x,y)\right]^{1/2}\dot{y} + \frac{1}{3\alpha}y^2\partial_y\tilde{V}(x,y) = 0.$$

Here $M_0 = \sqrt{3/2}M$ (where M is the reduced Planck mass), while $x = \mathrm{Re}\tau$, $y = \mathrm{Im}\tau$ are the Cartesian coordinates on the Poincaré half plane with complex coordinate τ and $\tilde{V} \stackrel{\text{def.}}{=} V \circ \pi_{\mathbb{H}}: \mathbb{H} \to \mathbb{R}$ is the *lifted potential*. Let $u = \pi_{\mathbb{H}}(\tau)$ be the complex coordinate on Σ. Let u_0 be any point of Σ and let $\tau_0 \in \mathbb{H}$ be chosen such that $\pi_{\mathbb{H}}(\tau_0) = u_0$. An initial velocity vector $v_0 = \dot{u}_0 \in T_{u_0}\Sigma$ and its unique lift $\tilde{v}_0 = \dot{\tau}_0 \in T_{\tau_0}\mathbb{H}$ through the differential of $\pi_{\mathbb{H}}$ at τ_0 are related through:

$$v_0 = (\mathrm{d}_{\tau_0}\pi_{\mathbb{H}})(\tilde{v}_0). \tag{12}$$

Writing $\tau = x + \mathbf{i}y$ and $\tau_0 = x_0 + \mathbf{i}y_0$, we have $\tilde{v}_0 = \tilde{v}_{0x} + \mathbf{i}\tilde{v}_{0y}$. A cosmological trajectory on Σ with initial condition (u_0, τ_0) can be written as $\varphi(t) = \pi_{\mathbb{H}}(\tilde{\varphi}(t))$, where $\tilde{\varphi}(t) = x(t) + \mathbf{i}y(t)$ is the solution of the lifted system (11) with initial conditions:

$$x(0) = x_0 \,, \ y(0) = y_0 \ \text{and} \ \dot{x}(0) = \tilde{v}_{0x} \,, \ \dot{y}(0) = \tilde{v}_{0y} \,. \tag{13}$$

In the next subsections, we limit ourselves to presenting examples of trajectories on $\mathbb{D}^*$ and $A(R)$ found in (Babalic and Lazaroiu, 2018a) for the corresponding globally well-behaved scalar potentials which lift to the following smooth function:

$$\hat{V}_+(\psi) = M_0 \cos^2 \frac{\psi}{2} \tag{14}$$

written in spherical coordinates (ψ, θ) on S^2. The trajectories were obtained by numerical computation of solutions (11) on the Poincaré half-plane, followed by projection to Σ through the explicitly-known uniformization maps.

3.1. Cosmological trajectories on the hyperbolic punctured disk $\mathbb{D}^*$

Figure 1 shows five lifted trajectories on the Poincaré half-plane $\mathbb{H}$ and their projections to $\mathbb{D}^* = \{u \in \mathbb{C} | 0 < |u| < 1\}$ (which is endowed with its unique complete hyperbolic metric) for the globally well behaved scalar potential:

$$V_+ = M_0 \frac{1}{1 + (\log|u|)^2}, \tag{15}$$

with the initial conditions listed in Table 1. Since V_+ has a minimum at the cusp end (which corresponds to the center of the disk), it produces an attractive force toward the cusp, which acts as a counterbalance to the repulsive effect of the hyperbolic metric. Out of the five trajectories, the one showed in yellow produces 55 efolds, which fits the observationally favored range of 50-60 efolds.
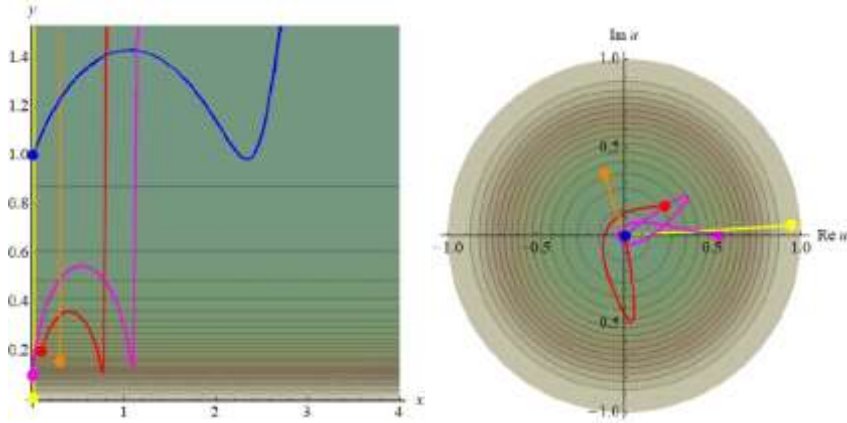
Fig. 1 Trajectories for $\tilde{V} = \tilde{V}_+$ on $\mathbb{H}$. Projection on $\mathbb{D}^*$ of the trajectories shown at the left. Numerical solutions for $V = V_+$ and $\alpha = \frac{M_0}{3}$.

In all figures, we show in the background the level sets of the potential $\tilde{V} = \tilde{V}_+$ on $\mathbb{H}$, respectively $V = V_+$ on Σ, where dark green indicates minima of the potential and light brown indicates maxima.

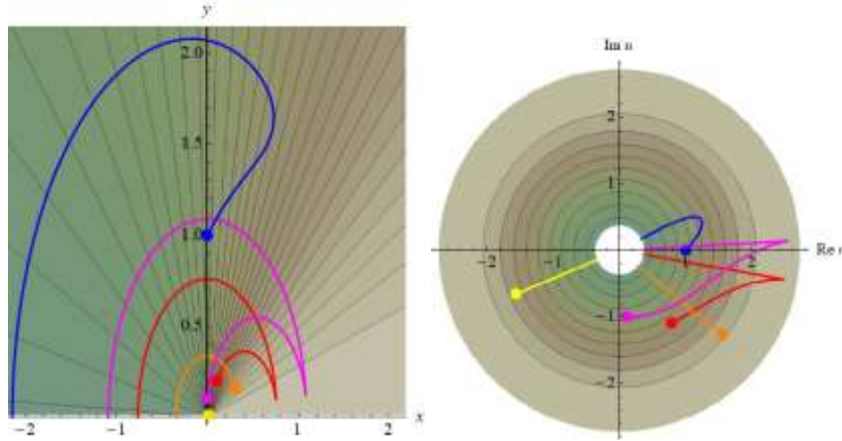**Table 1** Initial conditions for five trajectories on the Poincaré half-plane.

trajectory	τ_0	$\tilde{v}_0$
orange	$0.3 + 0.159\,\mathbf{i}$	0
yellow	$0.01 + 0.009\,\mathbf{i}$	0
red	$0.1 + 0.2\,\mathbf{i}$	$2 + 3\,\mathbf{i}$
blue	$\mathbf{i}$	$1 + \mathbf{i}$
magenta	$0.1\,\mathbf{i}$	$1.3 + 7\,\mathbf{i}$

3.2 Cosmological trajectories on the annuli $A(R)$

Figure 2 shows five trajectories (orange, yellow, red, blue and magenta) on the Poincaré half-plane $\mathbb{H}$ and on the annulus $A(R) \stackrel{def.}{=} \{u \in \mathbb{C} \mid 1/R < |u| < R\}$ (which is endowed with its complete hyperbolic metric of modulus $\mu = 2\log R$) for $\alpha = \frac{M_0}{3}$, $R = e$ and scalar potential:

$$V = V_+ = M_0 \frac{1}{1 + [\log\frac{R - \frac{1}{R}}{|u| - \frac{1}{R}}]^2},$$ (16)

with the same initial conditions of Table 1. Here the potential induces an attractive force toward the inner funnel end, making some trajectories to turn at some point in $A(R)$ and evolve back toward the inner funnel end.

Fig. 2 Trajectories on $\mathbb{H}$ for $\tilde{V} = \tilde{V}_+$. Projection to $A(R)$ of the trajectories shown at the left. Five trajectories for $V = V_+$, $\alpha = \frac{M_0}{3}$ and $R = e$, with initial conditions of Table 1.

In this case, the orange and yellow trajectories give 76 and 74 efolds, respectively, while the remaining three trajectories do not start in the inflationary region of the chosen potential.

4. Conclusions

Cosmological models with multiple scalar fields described by general scalar triples $(\mathcal{M}, \mathcal{G}, V)$ have not been studied systematically from a global perspective. Such models lead to geometric dynamical systems defined by a certain semispray on the tangent bundle $T\mathcal{M}$, whose flow 'interpolates' in an appropriate sense between the geodesic flow of $(\mathcal{M}, \mathcal{G})$ and the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$. This flow becomes particularly interesting when the complete metric $\mathcal{G}$ has a negative sectional curvature. In particular, it relates to deep aspects of asymptotic analysis and ergodic theory.

When the scalar manifold is a surface Σ endowed with a metric of the form $\mathcal{G} = 3\alpha G$ with α a positive parameter and G a complete hyperbolic metric defined on Σ, the associated cosmological model is a two-field generalized α-attractor model in the sense of (Lazaroiu and Shahbazi, 2017). The cosmological flow of such models can already be very intricate, especially when (Σ, G) has a finite hyperbolic area. Numerical studies as well as arguments based on the gradient flow approximation indicate (Babalic and Lazaroiu, 2018a; Babalic and Lazaroiu, 2017; Babalic and Lazaroiu, 2018b) that such models can be compatible with current observational constraints.

The epistemological falsifiability of scalar cosmological models is limited by the largely arbitrary choice of the scalar potential, a problem which is only compounded in multi-field models. As such, it is natural to look for criteria which could constrain the choice of V. A natural way to achieve this is to require that the model admits a non-generic symmetry. In reference (Anguelova et al., 2018), we studied two-field cosmological α-attractors with (Σ, G) an elementary hyperbolic surface, determining those scalar potentials for which such models admit a 'separated' Noether symmetry. This approach can be extended to more general hyperbolic surfaces and could serve as one avenue for further constraining such models.

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KOSMOLOŠKI PROTOCI NA HIPERBOLIČNIM POVRŠIMA

Ukratko smo opisali geometrijsku formulaciju kosmoloskih protoka za FLRW modele sa skalarnom materijom, kao i određene aspekte nastale iz metoda razmatranja koje potiču od geometrijske teorije dinamičkih sistema. Sažeto smo izneli određene rezultate numeričke analize koju smo sproveli za slučaj kada je skalarna mnogostrukost modela konačna ili beskonačna hiperbolična površ.

Ključne reči: *diferencijalna geometrija, neprekidni dinamički sistemi, kosmologija*