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DBI-TYPE TACHYONS FOR INVERSE *cosh* POTENTIAL *

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Abstract. We consider classical and quantum dynamics of a tachyonic system described by a DBI type Lagrangian and inverse *cosh* potential. This investigation is partially motivated by the string theory and D-brane dynamics, but mostly by their application in cosmological inflation. A formalism for describing dynamics of spatially homogenous tachyon scalar field with this kind of potentials is developed. Classical actions and corresponding quantum propagators in the Feynman path integral approach, both on real and nonarchimedean spaces, are calculated. Possibilities for a quantum adelic generalization of these models are noticed. Cosmological applications are pointed out and discussed.

1. Introduction

The main task of quantum cosmology [1] is to describe the evolution of the universe in a very early stage. Usually one takes that the universe is described by a complex-valued wave function. Since quantum cosmology is related to the Planck scale phenomena, it is logical to consider various geometries (in particular nonarchimedean [2] and noncommutative [3] ones) and parametrizations of the space-time coordinates: real, p-adic, or even adelic [4].

One of the most challenging period of the evolution of the Universe, despite its extremal shortness, is the inflation period, in particular, its very beginning. Some of the best candidates to give some physical background and understanding of the quantum origin of inflation are string theory, M-theory, string field theory, etc.

There have been a number of attempts to understand this description of the early Universe via (classical) nonlocal cosmological models, first of all via p-adic inflation models [5, 6], which are represented by a nonlocal p-adic string theory coupled to gravity. For these models, some rolling inflationary solutions were constructed

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and compared with CMB observations [5]. Another example is the investigation of the p-adic inflation near a maximum of the nonlocal potential when non-local derivative operators are included in the inflaton Lagrangian. It was found that higher-order derivative operators can support a (sufficiently) prolonged phase of slow-roll inflation [7].

The field theory of tachyon matter proposed by Sen [8], in a zero-dimensional version suggested by Kar [9] leads to a model of a particle moving in a constant external field with quadratic "damping"-like term. It leads to a dozen of interesting classical - toy models. Untill now, only a few of them are exactly solvable and quantized in the form of the path integrals (as an example see [10]).

The Lagrangian we study here is of a non-standard - DBI - type. It contains potential as a multiplicative factor, and a term with derivatives ("kinetic" term) inside the square root [8, 11]

$$\mathcal{L}_{tach} = \mathcal{L}(T, \partial_\mu T) = -V(T)\sqrt{1 + g_{\mu\nu}\partial^\mu T \partial^\nu T}, \tag{1.1}$$

where T is a tachyonic scalar field, $V(T)$ - potential of the theory, and $g_{\mu\nu}$ - components of the metric tensor with "mostly positive" signature. For the case of spatially homogenous tachyon field in flat spacetime, the Lagrangian and the Hamiltonian are

$$\mathcal{L}_{tach}(T, \dot{T}) = -V(T)\sqrt{1 - \dot{T}^2}, \tag{1.2}$$

$$\mathcal{H}_{tach}(T, P) = \sqrt{P^2 + V^2(T)}, \tag{1.3}$$

while the equation of motion is

$$\ddot{T}(t) - \frac{1}{V(T)}\frac{dV}{dT}\dot{T}^2(t) = -\frac{1}{V(T)}\frac{dV}{dT}. \tag{1.4}$$

In this paper we are focussed on one of the most interesting tachyonic potentials in the literature [12, 13, 14, 15]

$$V(T) = \frac{1}{\cosh(\beta T)}, \qquad \beta = const. > 0. \tag{1.5}$$

It is worth emphasizing that this system is not a dissipative, but rather a conservative one, i.e. Hamiltonian of the system is conserved quantity [16].

Quantum dynamics of tachyonic fields, in general, is purely investigated. We calculate the exact quantum propagator of the model, as well as the vacuum states and conditions necessary to construct a real and a p-adic model, and also the necessary step towards an adelic generalization.

This paper is organized as follows. In Section 2. we will introduce and study zero dimensional analogue of the theory, which is equivalent to the case of spatially homogenous theory in the flat space-time. Non-uniqueness of Lagrangian is studied in Section 3., while Section 4. and Section 5. deal with the propagator in the real and p-adic case, respectively. Vacuum sector of the theory is discussed in Section 6., while Section 7. is reserved for the discussion on cosmological implications and ideas for further investigations. Validity of the group property for the propagator in p-adic case is checked and confirmed in the Appendix.

2. "Classicalization" of the Field Model

To understand the theory we consider and investigate lower dimensional analogs of the tachyon field theory [9, 10, 17]. The corresponding zero dimensional analogue of a tachyon field can be obtained by the correspondence: $x^i \to t$, $T \to x$, $V(T) \to V(x)$. The action and the Lagrangian read

$$S = -\int dt V(x)\sqrt{1-\dot{x}^2}, \tag{2.1}$$

$$L_{tach} = -V(x)\sqrt{1-\dot{x}^2}. \tag{2.2}$$

In this article we will focus our attention on the potential of type

$$V(T) = \frac{1}{\cosh(\beta x)}, \qquad \beta = const. > 0. \tag{2.3}$$

Note that the potential (2.3) is never negative (bounded from below), symmetric under $x \to -x$, has a maximum at the origin $x = 0$, and goes to zero for a large x. This model has been considered in many articles in cosmology (see [11, 13, 14] and references therein).

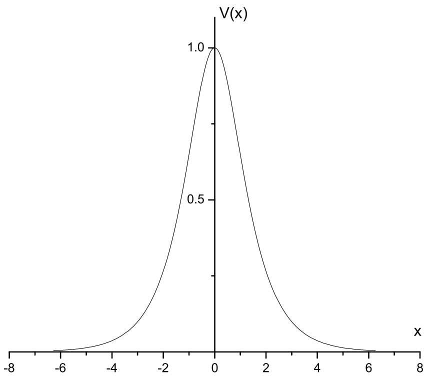

FIG. 2.1: Potential $V(x) = \frac{1}{\cosh(\beta x)}$

The equation of motion for action given by equation (2.1) has the form

$$\ddot{x} - \frac{1}{V(x)}\frac{dV}{dx}\dot{x}^2 = -\frac{1}{V(x)}\frac{dV}{dx}. \tag{2.4}$$

Substituting potential given by equation (2.3) in equation (2.4) we get a differential equation

$$\ddot{x}(t) + \beta \tanh(\beta x)\dot{x}(t)^2 = \beta \tanh(\beta x), \tag{2.5}$$

Equation (2.5) can be transformed to

$$\frac{\dot{x}d\dot{x}}{1-\dot{x}^2} = \beta \frac{\sinh(\beta x)}{\cosh(\beta x)} dx. \tag{2.6}$$

By solving the equation (2.6) for $\dot{x}$ we get

$$\dot{x}^2 = 1 - \frac{1}{C_1^2 \cosh^2(\beta x)}. \tag{2.7}$$

After integration of (2.7) we get the general solution of the equation of motion (2.4)

$$x(t) = \frac{1}{\beta}\text{arcsinh}\left[\pm\sqrt{1-\frac{1}{C_1^2}}\sinh(\beta C_2 \pm \beta t)\right]. \tag{2.8}$$

For the initial and final conditions $x(0) = x_1$ and $x(\tau) = x_2$ respectively, we get constants C_1 and C_2

$$C_1^2 = \left(1 - \frac{\sinh^2(\beta x_2) - \sinh^2(\beta x_1)}{\sinh^2(\beta C_2 \pm \beta\tau) - \sinh^2(\beta C_2)}\right)^{-1}, \tag{2.9}$$

$$C_2 = \frac{1}{2\beta}\ln\left[\frac{\sinh(\beta x_2) - e^{-\beta(\pm\tau)}\sinh(\beta x_1)}{\sinh(\beta x_2) - e^{\beta(\pm\tau)}\sinh(\beta x_1)}\right]. \tag{2.10}$$

Now, the solution (2.8) takes the form

$$x(t) = \frac{1}{\beta}\text{arcsinh}\left(\frac{\sinh(\beta t)\sinh(\beta x_2) - \sinh(\beta x_1)\sinh(\beta(t-\tau))}{\sinh(\beta\tau)}\right). \tag{2.11}$$

We will keep $\dot{x}^2 < 1$ through all calculations. Of course, case $\dot{x}^2 \geq 1$ is quite interesting (see [18]). However, will not be considered here.

3. Non-Uniqueness of Lagrangian

The task to quantize the system with the Lagrangian (2.2) is a non-trivial one. Looking at the classical level, we can choose another Lagrangian which will lead to the same equation of motion and will be more convenient [10, 17, 19]. Of course, one should be concerned about the equivalence of the Lagrangians at the quantum level, which is a very old problem in general. However, we are going to apply our model for a very short period of time - beginning of inflation, where a "local equivalence" of Lagrangians should be a reasonable assumption.

Once the convenient Lagrangian is taken, we can proceed with the quantization using the Feynman path integral approach for calculating the propagator for the system.

The equation of motion (2.5) can also be obtained from the (standard type) Lagrangian

$$L = \frac{1}{2}\dot{x}(t)^2 \cosh^2(\beta x) + \frac{1}{2}\cosh^2(\beta x). \tag{3.1}$$

Rescaling the coordinate $x(t) \to Y(t) = \frac{1}{\beta}\sinh(\beta x(t))$ Lagrangian (3.1) takes a very suitable quadratic form

$$L = \frac{1}{2}\dot{Y}(t)^2 + \frac{1}{2}\beta^2 Y(t)^2 + \frac{1}{2}, \tag{3.2}$$

where the last term can be omitted in the rest of the paper. The procedure is as follows [19]. We start with the equation of motion of the form

$$\ddot{x}(t) + b(x)\dot{x}^2(t) + g(x) = 0, \tag{3.3}$$

which is of the same form as (1.4) and (2.5). Let us stress that (3.3) can be obtained from the standard-type Lagrangian L_{st}, defined by the formula

$$L_{st}(x, \dot{x}) = \frac{1}{2}\dot{x}^2 e^{2I(x)} - \int^x g(x) e^{2I(x)} dx, \tag{3.4}$$

$$I(x) = \int^x b(x) dx, \tag{3.5}$$

where the lower limit of the integral(s) are chosen arbitrary. In our case, $b(x) = -\frac{d \log V(x)}{dx}$ and $g(x) = -b(x)$, so

$$I(x) = -\log V(x), \qquad e^{2I} = \frac{1}{V^2}. \tag{3.6}$$

After some straightforward calculations, the general standard-type Lagrangian becomes

$$L_{st}(x, \dot{x}) = \frac{1}{2}\left(\frac{\dot{x}}{V(x)}\right)^2 + \frac{1}{2}\frac{1}{V(x)^2}. \tag{3.7}$$

By introducing a new variable (field) Y and the corresponding new potential $W(Y)$ via a local change of variable

$$Y = \int^x \frac{dx}{V(x)}, \quad W(Y) = \frac{1}{2V(x(Y))^2}, \tag{3.8}$$

we can rewrite this Lagrangian in the canonical (i.e. standard) form

$$L_{st}(Y, \dot{Y}) = \frac{1}{2}\dot{Y}^2 + W(Y). \tag{3.9}$$

Having in mind the presented procedure, it is straightforward to obtain the standard-type Lagrangian for various tachyonic potentials, which for the case $V(x) = 1/\cosh(\beta x)$ has the form (3.2).

Now, a general Euler-Lagrange equation for the Lagrangian of the form (3.2) is

$$\ddot{Y}(t) - \beta^2 Y(t) = 0, \tag{3.10}$$

which is the equation of motion for an inverted harmonic oscillator (here with constant mass and frequency). The solution of this equation is a well known one

$$Y(t) = e^{t\beta} C_1' + e^{-t\beta} C_2'. \tag{3.11}$$

For the initial and final conditions $Y(0) = y_1$ and $Y(\tau) = y_2$, respectively, the integration constants are

$$C_1' = \frac{y_2 e^{\beta\tau} - y_1}{e^{2\beta\tau} - 1}, \tag{3.12}$$

$$C_2' = y_1 - \frac{y_2 e^{\beta\tau} - y_1}{e^{2\beta\tau} - 1}, \tag{3.13}$$

and the solution is

$$Y(t) = \frac{1}{2} e^{-\beta t} (\coth(\beta\tau) - 1) \left[y_1 \left(e^{2\beta\tau} - e^{2\beta t} \right) + y_2 \left(e^{\beta(2t+\tau)} - e^{\beta\tau} \right) \right]. \tag{3.14}$$

The Lagrangian (3.2) for the solution (3.14) takes the form

$$\begin{aligned} L &= \frac{1}{8} \beta^2 e^{-2\beta t} (\coth(\beta\tau) - 1)^2 \left(\left(-y_2 e^{\beta(2t+\tau)} + y_1 e^{2\beta t} + y_1 e^{2\beta\tau} - y_2 e^{\beta\tau} \right)^2 \right. \\ &\left. + \left(-y_2 e^{\beta(2t+\tau)} + y_1 e^{2\beta t} - y_1 e^{2\beta\tau} + y_2 e^{\beta\tau} \right)^2 \right), \end{aligned} \tag{3.15}$$

while the classical action S_c is a quadratic one with respect to y_1 and y_2

$$S_c(y_2, \tau, y_1, 0) = \frac{\beta}{2} \left(\left(y_1^2 + y_2^2 \right) \coth(\beta\tau) - 2 y_1 y_2 \mathrm{csch}(\beta\tau) \right), \tag{3.16}$$

where $\mathrm{csch}(\beta\tau) = 1/\sinh(\beta\tau)$. So, starting from a very "strange"-nonlinear forms (1.1), (1.2) and (3.2), we ended up with a locally equivalent system with a quadratic Lagrangian and action, quite suitable for the quantization of our (toy) model.

4. Transition Amplitude in the Real Case

We are now in a position to calculate, i.e. write down transition amplitude (propagator) for the action (3.16), which is now quadratic with respect to y_1 and y_2 [20],

$$\mathcal{K}_\infty(y_2, \tau; y_1, 0) = \sqrt{-\frac{1}{2\pi i\hbar} \frac{\partial^2 S_c}{\partial y_1 \partial y_2}}\, e^{i\frac{S_c}{\hbar}}. \tag{4.1}$$

It can be also written in the form [21]

$$\begin{aligned}\mathcal{K}_\infty(y_2,\tau;y_1,0) &= \lambda_\infty\left(-\frac{1}{2h}\frac{\partial^2 S_c}{\partial y_1\partial y_2}\right)\left|\frac{1}{h}\frac{\partial^2 S_c}{\partial y_1\partial y_2}\right|_\infty^{1/2}\times \\ &\quad \chi_\infty\Big(-\frac{1}{h}S_c(y_2,\tau;y_1,0)\Big),\end{aligned} \tag{4.2}$$

where an arithmetic λ-function and additive character χ_∞ are defined as

$$\lambda_\infty(b) = e^{-\frac{i\pi}{4}\,\mathrm{sgn}(b)}, \quad \chi_\infty(a) = e^{-2\pi i a}. \tag{4.3}$$

Finally, the transition amplitude for our model on real space reads

$$\begin{aligned}\mathcal{K}(y_2,\tau;y_1,0)_\infty &= \lambda_\infty\left(\frac{\beta}{2h\sinh(\beta\tau)}\right)\left|\frac{\beta}{h\sinh(\beta\tau)}\right|_\infty^{1/2}\times \\ &\quad \chi_\infty\left(-\frac{1}{2h}\left(\beta\left(y_1^2+y_2^2\right)\coth(\beta\tau)-2\beta y_1 y_2 \mathrm{csch}(\beta\tau)\right)\right),\end{aligned} \tag{4.4}$$

or written in a rather explicit form

$$\begin{aligned}\mathcal{K}(y_2,\tau;y_1,0)_\infty &= \sqrt{-\frac{i\beta\mathrm{csch}(\beta\tau)}{2\pi\hbar}}\times \\ &\quad \exp\left(\frac{i}{2\hbar}\left(\beta\left(y_1^2+y_2^2\right)\coth(\beta\tau)-2\beta y_1 y_2 \mathrm{csch}(\beta\tau)\right)\right).\end{aligned} \tag{4.5}$$

It describes a nonrelativistic particle moving in an inverted (harmonic) oscillator potential $V(Y) = -\frac{1}{2}\beta Y^2$.

It is important to stress the fact that inverted and harmonic oscillators are mathematically very much alike. However, a quantum inverted oscillator system has an energy spectrum, varying from minus to plus infinity [22]. So, the state with the lowest energy corresponds to negative infinite energy, $E = -\infty$.

The general solution of Schroedinger equation for the inverted oscillator can be presented as a linear combination of solutions with definite parity

$$\Psi(x) = C\Psi_{even}(x) + D\Psi_{odd}(x), \tag{4.6}$$

C and D are real constants, and Ψ_{even} and Ψ_{odd} are expressed in terms of confluent hyperbolic functions (see [23] for more details).

Introducing "annihilation" and "creation" operators as it was done for the harmonic oscillator, one ended up with the theory with the so-called generalized eigenstates belonging to the complex energy eigenvalues. As it is known, the energy eigenvalue E can be a complex number for an unstable system in which the potential energy does not have a stable stationary point, which is the case here (see [24] and reference therein for the discussion about mathematical formulations of continuous spectrum or complex eigenvalues).

5. Transition Amplitude in the p-Adic Case

One of formulations of p-adic quantum mechanics deals with the complex valued wave functions (of the p-adic argument). It is based on the triple [4]

$$\{L_2(Q_p), W(z), U(t)\}, \tag{5.1}$$

where $W(z)$ is a unitary representation of the Heisenberg-Weyl group in the Hilbert space $L_2(Q_p)$ and $U(t)$ is a unitary dynamics (see [25] and reference therein). Generalization of p-adic and ordinary quantum mechanics (and at the same time their unification), i.e. adelic quantum mechanics, was introduced in [26], as well as adelic path (functional) integral approach.

In the p-adic case, the transition amplitude $\mathcal{K}_p$ for an action quadratic in y_1 and y_2 (we take $h = 1$ for simplicity), as it was shown in [21] is

$$\mathcal{K}_p(y_2,\tau;y_1,0) = \lambda_p\Big(-\frac{1}{2}\frac{\partial^2 S_c}{\partial y_2 \partial y_1}\Big)\Big|\frac{\partial^2 S_c}{\partial y_2 \partial y_1}\Big|_p^{1/2}\chi_p\Big(-S_c(y_2,\tau;y_1,0)\Big), \tag{5.2}$$

where the p-adic additive character χ_p is defined as [4]

$$\chi_p(a) = e^{2\pi i\{a\}_p}, \tag{5.3}$$

$\{a\}_p$ is the fractional part of the p-adic number a, while λ_p is an arithmetic complex-valued function (here with a p-adic variable), with the following basic properties

$$\lambda_p(0) = 1, \quad \lambda_p(a^2 b) = \lambda_p(b), \quad |\lambda_p(b)|_\infty = 1, \tag{5.4}$$

$$\lambda_p(a) = 1, \quad |a|_p = p^{-\mathrm{ord}(a)} = p^{2\gamma}, \quad \gamma \in Z. \tag{5.5}$$

For the p-adic model with the action (3.16), the corresponding transition amplitude (5.2) has the form

$$\mathcal{K}_p(y_2,\tau;y_1,0) = \lambda_p\Big(\frac{\beta}{2\sinh(\beta\tau)}\Big)\Big|\frac{\beta}{\sinh(\beta\tau)}\Big|_p^{1/2}\chi_p\Big(-S_c(y_2,\tau;y_1,0)\Big), \tag{5.6}$$

or more explicitly

$$\begin{aligned}\mathcal{K}_p(y_2,\tau;y_1,0) &= \lambda_p\Big(\frac{\beta}{2\sinh(\beta\tau)}\Big)\Big|\frac{\beta}{\sinh(\beta\tau)}\Big|_p^{1/2}\times\\ &\quad \chi_p\Big(-\frac{\beta}{2}\left(\left(y_1^2+y_2^2\right)\coth(\beta\tau) - 2y_1y_2\mathrm{csch}(\beta\tau)\right)\Big),\end{aligned} \tag{5.7}$$

where $\mathrm{csch}(\beta\tau) = 1/\sinh(\beta\tau)$. It is worth reminding about the validity of the group property for the transition amplitude

$$\int_{Q_p}\mathcal{K}_p(y_3,\tau_3;y_2,\tau_2)\mathcal{K}_p(y_2,\tau_2;y_1,\tau_1)dy_2 = \mathcal{K}_p(y_3,\tau_3;y_1,\tau_1), \tag{5.8}$$

where we (re)introduced appropriate time points (τ_3,τ_2,τ_1). Validity of (5.8) is checked and confirmed in the Appendix.

6. p-Adic Quantum-Mechanical Ground State - Vacuum

The necessary condition for the existence of a p-adic quantum model, and rather general - an adelic model [4, 21] is the existence of a p-adic quantum-mechanical ground (vacuum) state $\Psi_p^{vac}(y)$ in the form of a characteristic function $\Omega(|y|_p)$,

$$\Omega(|y|_p) = \begin{cases} 1, & \text{if} \quad |y|_p \leq 1 \\ 0, & \text{if} \quad |y|_p > 1. \end{cases} \tag{6.1}$$

Having in mind one of the basic properties of the (p-adic) propagator and the vacuum state, also used as a definition of the vacuum state

$$\int_{Q_p} \mathcal{K}_p(y_2,\tau;y_1,0)\Psi_p^{vac}(y_1)dy_1 = \Psi_p^{vac}(y_2), \tag{6.2}$$

we get for $\Psi_p^{vac}(y) = \Omega(|y|_p)$

$$\int_{|y_1|_p\leq 1} \mathcal{K}_p(y_2,\tau;y_1,0)dy_1 = \Omega(|y_2|_p), \tag{6.3}$$

i.e.

$$\begin{aligned} &\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\Big|\frac{\beta}{\sinh(\beta\tau)}\Big|_p^{1/2} \quad \times \\ &\int_{|y_1|_p\leq 1} \chi_p\Big(-S_c(y_2,\tau;y_1,0)\Big)dy_1 \quad = \quad \Omega(|y_2|_p). \end{aligned} \tag{6.4}$$

Written more explicitly, using (3.16), the last expression becomes

$$\begin{aligned} &\int_{|y_1|_p\leq 1} \chi_p\Big(-\frac{\beta}{2}\left((y_1^2+y_2^2)\coth(\beta\tau) - 2y_1y_2\mathrm{csch}(\beta\tau)\right)\Big)dy_1 \quad \times \\ &\qquad \lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\Big|\frac{\beta}{\sinh(\beta\tau)}\Big|_p^{1/2} \quad = \quad \Omega(|y_2|_p). \end{aligned} \tag{6.5}$$

Using the properties of p-adic analytic functions sinh and cosh [2]

$$|\sinh(a)|_p = |a|_p, \quad |\cosh(a)|_p = 1, \tag{6.6}$$

and p-adic Gauss integrals (for $p \neq 2$, [4])

$$\int_{|y|_p\leq 1} \chi_p(ay^2+by)dy = \begin{cases} \Omega(|b|_p), & |a|_p \leq 1 \\ \frac{\lambda_p(a)}{|a|_p^{1/2}}\chi_p(-\frac{b^2}{4a})\Omega(|\frac{b}{a}|_p), & |a|_p > 1, \end{cases} \tag{6.7}$$

the integral in (6.3) is reduced to the form

$$\frac{\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)}{|\tau|_p^{1/2}}\chi_p\Big(-\frac{\beta}{2}y_2^2\coth(\beta\tau)\Big) \times I = \Omega(|y_2|_p), \tag{6.8}$$

where

$$I = \begin{cases} \Omega(|b|_p), & |a|_p \le 1 \\ \frac{\lambda_p(a)}{|a|_p^{1/2}}\chi_p(-\frac{b^2}{4a})\Omega(|\frac{b}{a}|_p), & |a|_p > 1. \end{cases} \tag{6.9}$$

In our case, one can read a and b from (6.5)

$$a = -\frac{\beta}{2}\coth(\beta\tau), \quad |a|_p = \frac{1}{|\tau|_p}, \tag{6.10}$$

$$b = \beta\frac{y_2}{\sinh(\beta\tau)}, \quad |b|_p = \left|\frac{y_2}{\tau}\right|_p \tag{6.11}$$

so that expression (6.8), putting (6.10) and (6.11) in (6.9), is reduced to

$$\frac{\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)}{|\tau|_p^{1/2}}\chi_p\left(-\frac{\beta}{2}y_2^2\coth(\beta\tau)\right)\Omega\left(|\frac{y_2}{\tau}|_p\right) = \Omega(|y_2|_p), \tag{6.12}$$

for $|\tau|_p \ge 1$, and

$$\frac{\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)}{\lambda_p\left(\frac{\beta}{2}\coth(\beta\tau)\right)}\chi_p\left(-\frac{\beta}{2}y_2^2\tanh(\beta\tau)\right)\Omega(|y_2|_p) = \Omega(|y_2|_p), \tag{6.13}$$

for $|\tau|_p < 1$.

We will now inspect the conditions under which relation (6.8), equivalently (6.12) and (6.13), is valid for $|y_2|_p \le 1$. Before that, let us once again stress that potentially quite important consequences of the (non)existence of the p-adic (adelic) vacuum state. Conditions for existence of the Ω-function "shed" light on possible values for "time", "energy", free parameters of the theory (such as β), etc.

6.1. Case $|\tau|_p \ge 1$

Case $|\tau|_p > 1$ (and $|y_2|_p \le 1$) reduces (6.8) to (6.12) which, obviously, is never possible, while for $|\tau|_p = 1$ gives

$$\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\chi_p\left(-\frac{\beta}{2}y_2^2\coth(\beta\tau)\right)\Omega(|y_2|_p) = \Omega(|y_2|_p). \tag{6.14}$$

Because $|\frac{\beta}{2}y_2^2\coth(\beta\tau)|_p = |y_2^2|_p \le 1$, then $\chi_p\left(-\frac{\beta}{2}y_2^2\coth(\beta\tau)\right) = 1$ (due to $\{\frac{\beta}{2}y_2^2\coth(\beta\tau)\}_p = 0$), reducing (6.14) to the form

$$\lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right) = 1. \tag{6.15}$$

Arithmetic function $\lambda_p\bigg(\frac{\beta}{2\sinh(\beta\tau)}\bigg)$ is equal to 1 for $ord\bigg(\frac{\beta}{2\sinh(\beta\tau)}\bigg)$-even (recall (5.5)). In this case, $\bigg|\frac{\beta}{2\sinh(\beta\tau)}\bigg|_p = \frac{1}{|\tau|_p} = 1$ and it means that $ord\bigg(\frac{\beta}{2\sinh(\beta\tau)}\bigg) = 0$, i.e. even. Thus, relation (6.8) is valid for $|\tau|_p = 1$.

6.2. Case $|\tau|_p < 1$

Case $|\tau|_p < 1$ gives us the relation (6.13), i.e.

$$\frac{\lambda_p\bigg(\frac{\beta}{2\sinh(\beta\tau)}\bigg)}{\lambda_p\bigg(\frac{\beta}{2}\coth(\beta\tau)\bigg)}\chi_p\bigg(-\frac{\beta}{2}y_2^2\tanh(\beta\tau)\bigg)\Omega(|y_2|_p) = \Omega(|y_2|_p). \tag{6.16}$$

Now, $|\frac{\beta}{2}y_2^2\tanh(\beta\tau)|_p = |\beta^2 y_2^2\tau|_p$, and the case $|\beta^2 y_2^2\tau|_p \leq 1$ leads the additive character to be equal to one, $\chi_p(-\frac{\beta}{2}y_2^2\tanh(\beta\tau)) = 1$. The last expression then reduces to

$$\lambda_p\bigg(\frac{\beta}{2\sinh(\beta\tau)}\bigg)\lambda_p\bigg(-\frac{\beta}{2}\coth(\beta\tau)\bigg) = 1. \tag{6.17}$$

To prove this equality is valid, we use series expansion of the cosh function [4]

$$\cosh(\beta\tau) = \sum_{k=0}^{\infty}\frac{(\beta\tau)^{2k}}{(2k)!}. \tag{6.18}$$

In other words, $\cosh(\beta\tau)$ can be represented in the canonical form (as any p-adic number)

$$\cosh(\beta\tau) = p^{\gamma}(c_0 + c_1 p + ...) = p^0(1 + ...). \tag{6.19}$$

In the last expression $\gamma = 0$ and we explicitly wrote only the first term (digit c_0) from the series expansion (which is equal to 1), i.e we omitted the second and all higher terms in (6.18).

This p-adic function is the square of another p-adic function (in other way, for fixed β and τ, the p-adic number $\cosh(\beta\tau)$ is the square of another p-adic number) as long as it is analytic, i.e. for $|\beta\tau|_p \leq 1/p$, because the necessary and sufficient conditions for the existence of the solution $D \in Q_p$ of equation [4]

$$\cosh(\beta\tau) = D^2, \tag{6.20}$$

are

i) γ *is even*,

$$ii) \quad \left(\frac{c_0}{p}\right) = 1.$$

In our case, both conditions i) and ii) are satisfied,

$$\gamma = 0, \tag{6.21}$$

$$\left(\frac{c_0}{p}\right) = \left(\frac{1}{p}\right) = +1, \quad p \neq 2. \tag{6.22}$$

In this way, the left hand side of the expression (6.17) can be written

$$\begin{aligned} \lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\lambda_p\left(-\frac{\beta}{2}\coth(\beta\tau)\right) &= \\ \lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\lambda_p\left(-\frac{\beta}{2\sinh(\beta\tau)}\cosh(\beta\tau)\right) &= \\ \lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\lambda_p\left(-\frac{\beta}{2\sinh(\beta\tau)}D^2\right) &= \\ \lambda_p\left(\frac{\beta}{2\sinh(\beta\tau)}\right)\lambda_p\left(-\frac{\beta}{2\sinh(\beta\tau)}\right) &= 1, \end{aligned} \tag{6.23}$$

where we used properties (5.4).

Expression (6.16) is not valid for the case $|\beta^2 y_2^2 \tau|_p > 1$. Thus, relation (6.13) is valid for $|\tau|_p < 1$ and $|\beta^2 y_2^2 \tau|_p \leq 1$. In other words, our p-adic quantum system is in its vacuum state, during the "p-adic time" $|\tau|_p < 1$, as long as $|\beta^2 y_2^2 \tau|_p \leq 1$ holds (for $p \neq 2$).

Let us add that fixing the condition for Ω state for the case $p = 2$ is a straightforward, but time consuming task, and will be presented elsewhere.

7. Discussion and Conclusion

We investigate inverse cosh potential for the system initially described by the DBI tachyon Lagrangian. We are to consider or "mimic" a higher-dimensional tachyon field as a nonrelativistic quantum particle, and discuss its evolution on: real space, p-adic space (in principe for any prime p), and adelic space, to be presented elsewhere in details. In order to calculate the propagator on real and p-adic spaces we chose a more convenient classically and locally equivalent Lagrangian, and used the Feynman approach for obtaining the propagator. We found in the p-adic case (for $p \neq 2$) that necessary conditions for the existence of ground states in the form of the characteristic Ω-function:

$$\Psi_p^{vac}(y) = \Omega(|y|_p), \quad \text{for} \begin{cases} |\tau|_p = 1, & \\ |\tau|_p < 1, & |\beta^2 y_2^2 \tau|_p \leq 1. \end{cases} \tag{7.1}$$

At a glance, there is no restriction on the (values of) parameter β in the case $|\tau|_p = 1$, while the case $|\tau|_p < 1$ leads to the condition $|\beta|_p^2 \leq |y_2^{-2}\tau^{-1}|_p$.

However, there is another important condition we should have in mind: analyticity of p-adic $\sinh(\beta\tau)$ and $\cosh(\beta\tau)$ functions, which appear in our calculations from the very beginning, in the expression for the solution of the equation of motion. These functions consist of a p-adic exponential function $\exp(\beta\tau)$, which is analytic for $|\beta\tau|_p \leq 1/p$ ($p \neq 2$) [4], and this analyticity condition also holds for sinh and cosh. This means that there is one more condition to be included.

The existence of the Ω-function is unavoidable for the construction of any adelic model. Note that the Ω-function is a counterpart of the Gaussian $\exp(-\pi y^2)$ in the real case ($y \in R$), since it is invariant with respect to the Fourier transform [2].

For the theory in consideration, the adelic ground states would be of the form

$$\Psi(y_a) = \Psi_\infty(y_\infty) \prod_{p \in M} \Psi_p(y_p) \prod_{p \notin M} \Omega(|y_p|_p), \tag{7.2}$$

where M is a finite set of primes p, while $y_\infty \in R$ and $y_p \in Q_p$ defines an adele y_a, i.e. a sequence of the form

$$y_a = (y_\infty, y_2, y_3, ...y_p, ...), \tag{7.3}$$

$\Psi_\infty(y_\infty)$ are the corresponding real (counterparts of the) wave functions of the theory, and $\Psi_p(y_p)$ are our p-adic ground state wave function (7.1).

The usual probability interpretation of the wave function (7.2) will lead to

$$|\Psi(y_a)|_\infty^2 = |\Psi_\infty(y_\infty)|_\infty^2 \prod_{p \in M} |\Psi_p(y_p)|_\infty^2 \prod_{p \notin M} |\Omega(|y_p|_p)|_\infty^2. \tag{7.4}$$

Using the p-adic solution in the form of Ω-function and solution in the real case, it follows

$$|\Psi(y_a)|_\infty^2 = \begin{cases} |\Psi_\infty(y_\infty)|_\infty^2, & y_a \in Z \\ 0, & y_a \in Q \setminus Z. \end{cases} \tag{7.5}$$

This leads to some discretization of coordinates y_∞, because for all rational points density probability is nonzero only in the integer points of y_∞. This depends on the adelic quantum state of the theory and is a generic feature of adelic models for the theories with quadratic Lagrangians.

At the end, from the p-adic sector of the model it is clear that there are a few strong conditions (for $p \neq 2$ and $|y_2|_p \leq 1$) for the possible values of:

"time" τ

$$|\tau|_p \leq 1, \tag{7.6}$$

parameter β

$$|\beta|_p \leq \frac{1}{p|\tau|_p}, \tag{7.7}$$

and, in general

$$|\beta^2 y_2^2 \tau|_p \leq 1, \tag{7.8}$$

in order to exist p-adic ground state in the form of the Ω-function. An implicit discretization of space-time attached to the model is present, however, its explicit form needs further investigation, in particular the case $p = 2$. Also, the question regarding the existence of a ground state for complex eigenvalues and the (non)existence of a ground state for real energy eigenvalues in the real case also needs further investigation and discussion. A good understanding of the tachyon field dynamics would allow us to compare theoretical predictions for tachyonic inflation (with $\cosh^{-1}(\beta\tau)$ or another potential) with the newest result of PLANCK satelite and other forthcoming "missions".

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Appendix

In this Appendix we will check the validity of the group property (for $p \neq 2$)

$$\int_{Q_p} \mathcal{K}_p(y_3,\tau_3;y_2,\tau_2)\mathcal{K}_p(y_2,\tau_2;y_1,\tau_1)dy_2 = \mathcal{K}_p(y_3,\tau_3;y_1,\tau_1), \tag{8.1}$$

for the p-adic propagator

$$\begin{aligned}\mathcal{K}_p(y_f,\tau_f;y_i,\tau_i) = \lambda_p\bigg(\frac{\beta}{2\sinh(\beta(\tau_f-\tau_i)}\bigg)\Big|\frac{\beta}{\sinh(\beta(\tau_f-\tau_i))}\Big|_p^{1/2}\times \\ \chi_p\bigg(-\frac{\beta}{2}\left(\left(y_f^2+y_i^2\right)\coth(\beta(\tau_f-\tau_i))-2y_f y_i \mathrm{csch}(\beta(\tau_f-\tau_i))\right)\bigg).\end{aligned} \tag{8.2}$$

Using properties (6.6), the last expression for the propagator becomes

$$\begin{aligned}\mathcal{K}_p(y_f,\tau_f;y_i,\tau_i) = \lambda_p\bigg(\frac{\beta}{2\sinh(\beta(\tau_f-\tau_i)}\bigg)\Big|\frac{1}{\tau_f-\tau_i}\Big|_p^{1/2}\times \\ \chi_p\bigg(-\frac{\beta}{2}\left(\left(y_f^2+y_i^2\right)\coth(\beta(\tau_f-\tau_i))-2y_f y_i \mathrm{csch}(\beta(\tau_f-\tau_i))\right)\bigg).\end{aligned} \tag{8.3}$$

Now, the left hand side of the group property expression reads

$$\lambda_p\bigg(\frac{\beta}{2\sinh(\beta(\tau_3-\tau_2)}\bigg)\lambda_p\bigg(\frac{\beta}{2\sinh(\beta(\tau_2-\tau_1)}\bigg)\Big|\frac{1}{(\tau_3-\tau_2)(\tau_2-\tau_1)}\Big|_p^{1/2}\times$$
$$\int_{Q_p}\bigg[\chi_p\bigg(-\frac{\beta}{2}\left((y_3^2+y_2^2)\coth(\beta(\tau_3-\tau_2))-2y_3y_2\mathrm{csch}(\beta(\tau_3-\tau_2))\right)\bigg)$$
$$\chi_p\bigg(-\frac{\beta}{2}\left((y_2^2+y_1^2)\coth(\beta(\tau_2-\tau_1))-2y_2y_1\mathrm{csch}(\beta(\tau_2-\tau_1))\right)\bigg)\bigg]dy_2. \tag{8.4}$$

It is more suitable to introduce

$$\triangle_{ij}=\frac{2}{\beta}\sinh(\beta(\tau_i-\tau_j)), \tag{8.5}$$

with the p-adic norm (recalling $p\neq 2$)

$$|\triangle_{ij}|_p=\Big|\frac{2}{\beta}\sinh(\beta(\tau_i-\tau_j))\Big|_p=\Big|\frac{1}{\tau_i-\tau_j}\Big|_p. \tag{8.6}$$

The product of two additive characters can be written

$$\begin{aligned}&\chi_p\bigg(-\frac{\beta}{2}\left((y_3^2+y_2^2)\coth(\beta(\tau_3-\tau_2))-2y_3y_2\mathrm{csch}(\beta(\tau_3-\tau_2))\right)\bigg)\times\\&\chi_p\bigg(-\frac{\beta}{2}\left((y_2^2+y_1^2)\coth(\beta(\tau_2-\tau_1))-2y_2y_1\mathrm{csch}(\beta(\tau_2-\tau_1))\right)\bigg)=\\&\qquad\chi_p\bigg(-\frac{y_3^2\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{y_1^2\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\bigg)\times\\&\qquad\chi_p\bigg(-\bigg[\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}+\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\bigg]y_2^2\bigg)\times\\&\qquad\chi_p\bigg(2\bigg[\frac{y_3}{\triangle_{32}}+\frac{y_1}{\triangle_{21}}\bigg]y_2\bigg).\end{aligned} \tag{8.7}$$

Introducing A and B (which do not depend on y_2) as

$$A=-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}, \tag{8.8}$$

$$B=2\bigg(\frac{y_3}{\triangle_{32}}+\frac{y_1}{\triangle_{21}}\bigg), \tag{8.9}$$

expression (8.7) takes simpler form

$$\chi_p\bigg(-\frac{\beta}{2}\left((y_3^2+y_2^2)\coth(\beta(\tau_3-\tau_2))-2y_3y_2\mathrm{csch}(\beta(\tau_3-\tau_2))\right)\bigg)\times$$

$$\chi_p\Bigg(-\frac{\beta}{2}\left(\left(y_2^2+y_1^2\right)\coth(\beta(\tau_2-\tau_1))-2y_2y_1\mathrm{csch}(\beta(\tau_2-\tau_1))\right)\Bigg)=$$
$$\chi_p\Bigg(-\frac{y_3^2\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{y_1^2\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Bigg)\times$$
$$\chi_p\Bigg(Ay_2^2+By_2\Bigg). \tag{8.10}$$

Using (8.10), the expression (8.4) becomes

$$\lambda_p\Bigg(\frac{1}{\triangle_{32}}\Bigg)\lambda_p\Bigg(\frac{1}{\triangle_{21}}\Bigg)\Bigg|\frac{1}{(\tau_3-\tau_2)(\tau_2-\tau_1)}\Bigg|_p^{1/2}\times$$
$$\chi_p\Bigg(-\frac{y_3^2\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{y_1^2\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Bigg)\times$$
$$\int_{Q_p}\chi_p\Bigg(Ay_2^2+By_2\Bigg). \tag{8.11}$$

This integral can be calculated, with the solution [4]

$$I^*=\int_{Q_p}\chi_p\Bigg(Ay_2^2+By_2\Bigg)=\lambda_p(A)\frac{1}{|A|_p^{1/2}}\chi_p\Bigg(-\frac{B^2}{4A}\Bigg),\quad A\neq 0. \tag{8.12}$$

Having in mind (8.8) and (8.9)

$$|A|_p=\Bigg|\frac{\tau_3-\tau_1}{(\tau_3-\tau_2)(\tau_2-\tau_1)}\Bigg|_p, \tag{8.13}$$

$$\frac{B^2}{4A}=\frac{(\triangle_{32}y_1+\triangle_{21}y_3)^2}{\triangle_{31}\triangle_{32}\triangle_{21}}, \tag{8.14}$$

so that (8.12) becomes

$$I^*=\lambda_p\Bigg(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Bigg)\times$$
$$\Bigg|\frac{(\tau_3-\tau_2)(\tau_2-\tau_1)}{\tau_3-\tau_1}\Bigg|_p^{1/2}\chi_p\Bigg(-\frac{(\triangle_{32}y_1+\triangle_{21}y_3)^2}{\triangle_{31}\triangle_{32}\triangle_{21}}\Bigg), \tag{8.15}$$

and (8.11) turns to

$$\frac{1}{|\tau_3-\tau_1|_p^{1/2}}\lambda_p\Bigg(\frac{1}{\triangle_{32}}\Bigg)\lambda_p\Bigg(\frac{1}{\triangle_{21}}\Bigg)\times$$
$$\lambda_p\Bigg(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Bigg)\times$$
$$\chi_p\Bigg(-\frac{y_3^2\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{y_1^2\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Bigg)\times$$
$$\chi_p\Bigg(-\frac{(\triangle_{32}y_1+\triangle_{21}y_3)^2}{\triangle_{31}\triangle_{32}\triangle_{21}}\Bigg). \tag{8.16}$$

Using some basic properties of sinh (which are the same as in the real case), $\triangle_{21}$ and $\triangle_{32}$ can be put in the forms

$$\begin{aligned}\triangle_{21} = \frac{2}{\beta}\sinh(\beta(\tau_2-\tau_1)) &= \frac{2}{\beta}\sinh(\beta(\tau_2-\tau_3+\tau_3-\tau_1)) = \\ &\triangle_{31}\cosh(\beta(\tau_3-\tau_2)) - \triangle_{32}\cosh(\beta(\tau_3-\tau_1)),\end{aligned} \tag{8.17}$$

$$\begin{aligned}\triangle_{32} = \frac{2}{\beta}\sinh(\beta(\tau_3-\tau_2)) &= \frac{2}{\beta}\sinh(\beta(\tau_3-\tau_1+\tau_1-\tau_2)) = \\ &\triangle_{31}\cosh(\beta(\tau_2-\tau_1)) - \triangle_{21}\cosh(\beta(\tau_3-\tau_1)),\end{aligned} \tag{8.18}$$

so that the product of additive characters in (8.16) gives exactly

$$\chi_p\Big(-\frac{1}{\triangle_{31}}(y_3^2+y_1^2)\cosh(\beta(\tau_3-\tau_1)) - 2\frac{y_3y_1}{\triangle_{31}}\Big), \tag{8.19}$$

i.e.

$$\chi_p\Big(-\frac{\beta}{2}\left((y_3^2+y_1^2)\coth(\beta(\tau_3-\tau_1)) - 2y_3y_1\mathrm{csch}(\beta(\tau_3-\tau_1))\right)\Big). \tag{8.20}$$

Now, the left hand side of (8.1) (i.e. (8.4)) looks like

$$\begin{aligned}&\frac{1}{|\tau_3-\tau_1|_p^{1/2}}\lambda_p\Big(\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(\frac{1}{\triangle_{21}}\Big)\lambda_p\Big(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Big)\times \\ &\chi_p\Big(-\frac{\beta}{2}\left((y_3^2+y_1^2)\coth(\beta(\tau_3-\tau_1)) - 2y_3y_1\mathrm{csch}(\beta(\tau_3-\tau_1))\right)\Big).\end{aligned} \tag{8.21}$$

The part which was not simplified and calculated until now is the product of three λ-functions

$$\begin{aligned}&\lambda_p\Big(\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(\frac{1}{\triangle_{21}}\Big)\times \\ \lambda_p\Big(&-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Big).\end{aligned} \tag{8.22}$$

We will now transform the third λ-function in (8.22). To do that, we will use the well known properties

$$\lambda_p(a)\lambda_p(b) = \lambda_p(a+b)\lambda_p\Big(\frac{1}{a}+\frac{1}{b}\Big), \tag{8.23}$$

$$\lambda_p(a)\lambda_p(-a) = 1. \tag{8.24}$$

Having in mind (6.20), we can write

$$\cosh\left(\beta(\tau_3-\tau_2)\right) = D_1^2, \quad \cosh\left(\beta(\tau_2-\tau_1)\right) = D_2^2, \tag{8.25}$$

and using the second property from (5.4) together with (8.23) and (8.24), we get

$$\lambda_p\Big(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Big)=$$
$$\frac{\lambda_p\Big(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Big)}{\lambda_p\Big(-\frac{\triangle_{32}}{\cosh(\beta(\tau_3-\tau_2))}-\frac{\triangle_{21}}{\cosh(\beta(\tau_2-\tau_1))}\Big)}=$$
$$\frac{\lambda_p\Big(-\frac{D_1^2}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{D_2^2}{\triangle_{21}}\Big)}{\lambda_p\Big(-\frac{\triangle_{32}}{\cosh(\beta(\tau_3-\tau_2))}-\frac{\triangle_{21}}{\cosh(\beta(\tau_2-\tau_1))}\Big)}=$$
$$\frac{\lambda_p\Big(-\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{1}{\triangle_{21}}\Big)}{\lambda_p\Big(-\frac{\triangle_{32}\cosh(\beta(\tau_2-\tau_1))+\triangle_{21}\cosh(\beta(\tau_3-\tau_2))}{\cosh(\beta(\tau_3-\tau_2))\cosh(\beta(\tau_2-\tau_1))}\Big)}=$$
$$\frac{\lambda_p\Big(-\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{1}{\triangle_{21}}\Big)}{\lambda_p\Big(-\frac{\triangle_{32}\cosh(\beta(\tau_2-\tau_1))+\triangle_{21}\cosh(\beta(\tau_3-\tau_2))}{D_1^2D_2^2}\Big)}=$$
$$\frac{\lambda_p\Big(-\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{1}{\triangle_{21}}\Big)}{\lambda_p\Big(-\triangle_{32}\cosh(\beta(\tau_2-\tau_1))-\triangle_{21}\cosh(\beta(\tau_3-\tau_2))\Big)}. \tag{8.26}$$

It is easy to check that the argument of the λ-function in the denominator is equal to $-\triangle_{31}$,

$$\triangle_{31}=\triangle_{32}\cosh(\beta(\tau_2-\tau_1))+\triangle_{21}\cosh(\beta(\tau_3-\tau_2)), \tag{8.27}$$

so that expression (8.26) becomes

$$\lambda_p\Big(\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(\frac{1}{\triangle_{21}}\Big)\frac{\lambda_p\Big(-\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(-\frac{1}{\triangle_{21}}\Big)}{\lambda_p\Big(-\triangle_{31}\Big)}=\frac{1}{\lambda_p\Big(-\triangle_{31}\Big)}. \tag{8.28}$$

Using

$$\frac{1}{\lambda_p\Big(-\triangle_{31}\Big)}=\lambda_p\Big(\triangle_{31}\Big)=\lambda_p\Big(\frac{1}{\triangle_{31}}\Big). \tag{8.29}$$

we can rewrite the expression (8.22)

$$\lambda_p\Big(\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(\frac{1}{\triangle_{21}}\Big) \quad \times$$

$$\lambda_p\Big(-\frac{\cosh(\beta(\tau_3-\tau_2))}{\triangle_{32}}-\frac{\cosh(\beta(\tau_2-\tau_1))}{\triangle_{21}}\Big) \quad = \quad \lambda_p\Big(\frac{1}{\triangle_{31}}\Big), \tag{8.30}$$

or in a more compact form

$$\lambda_p\Big(\frac{1}{\triangle_{32}}\Big)\lambda_p\Big(\frac{1}{\triangle_{21}}\Big)\lambda_p\Big(-\frac{\triangle_{31}}{\triangle_{32}\triangle_{21}}\Big)=\lambda_p\Big(\frac{1}{\triangle_{31}}\Big). \tag{8.31}$$

Finally, using (8.20), (8.31) and having in mind (8.5), the left-hand side of (8.1) becomes

$$\lambda_p\Big(\frac{\beta}{2\sinh(\beta(\tau_3-\tau_1))}\Big)\Big|\frac{1}{\tau_3-\tau_1}\Big|_p^{1/2} \times$$

$$\chi_p\Big(-\frac{\beta}{2}\left((y_3^2+y_1^2)\coth(\beta(\tau_3-\tau_1))-2y_3y_1\operatorname{csch}(\beta(\tau_3-\tau_1))\right)\Big), \tag{8.32}$$

which is exactly the expression for the propagator $\mathcal{K}_p(y_3,\tau_3;y_1,\tau_1)$. Thus, the group property (8.1) is explicitly shown and confirmed.

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