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METHODOLOGICAL APPROACH TO CONGRUENCE OF QUADRILATERALS IN HYPERBOLIC GEOMETRY

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Abstract. *In this paper we will prove new criteria for the congruence of convex quadrilaterals in Hyperbolic geometry and consequently, display the appropriate methodological approach in teaching the same. There are seven criteria for the congruence of hyperbolic quadrilaterals, while there are five for the congruence of Euclidean quadrilaterals. Using a comparative geometric analysis of quadrilateral congruence criteria in Euclidean and Hyperbolic geometry we described all possible cases and made a methodological approach to the problem. The obtained results can influence the approaches to the study of these contents with students in the hyperbolic geometry teaching.*

Key words: *congruence of convex quadrilaterals, Euclidean geometry, Hyperbolic geometry, hyperbolic quadrilaterals, methodological approach*

1. INTRODUCTION

The use of technology tools creates new situations and new dynamics in geometry's teaching in the classroom, enhancing the ways of its understanding. Moreover, our research revealed considerable evidence that techniques from hyperbolic geometry motivated students and offered them fuller participation in the teaching process, especially to visualize the Poincaré's disk and through it understand key elements of hyperbolic geometry. Models help students with their visualization while they are learning new mathematical concepts.

Absolute geometry is a geometry based on an axiom system for Euclidean geometry without the parallel postulate or any of its alternatives. Hyperbolic geometry is built from absolute geometry, and hyperbolic postulate.

Parallel Postulate: *A line and a point not on it fully define the point through that point.*

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Hyperbolic Postulate: *Through a point not on a line, at least two lines can be drawn that do not intersect the given line.*

The following theorem and its corollary are a direct consequence of the hyperbolic postulate:

Theorem 1.1 *The sum of the measures of the angles of any hyperbolic triangle is less than a straight angle, or two right angles.*

Corollary 1.2 *The sum of the measures of the angles of any convex hyperbolic quadrilateral is less than two straight angles, or four right angles.*

In the following the right angles are denoted ρ and straight angles are denoted σ. Also, when stating a congruence, the order of the letters follows the order of the congruence. For instance, we write, "$\overline{JE_1H} \cong \overline{REH}$ by AAS," implies $\angle J \cong \angle R$ and $\angle E_1 \cong \angle E$ and$\overline{E_1H} \cong \overline{EH}$.

In the Euclidean plane, there is one and only one regular n-gon with all right angles; namely, the square. In hyperbolic geometry, there is no regular n-gon like this.

How much less the sum of the measures of the angles of a triangle is than the sum of two right angles, σ, is called the triangle's defect. A hyperbolic triangle (h-triangle) has a positive defect, while a Euclidean triangle has a defect of zero.

Definition 1.3 *For a triangle $\overline{EFG}$, the following function is the defect of the triangle:*

$$defect(\overline{EFG}) = \sigma - (\angle E + \angle F + \angle G) \tag{1.1}$$

Theorem 1.4 ***(Defect Addition Theorem)*** *If a triangle is partitioned into smaller triangles, their defects must sum to the defect of the outer triangle.*

Corollary 1.5 ***(Defect Addition Corollary)*** *If the defect of even one triangle is zero, then all triangles have zero defect. If the defect of even one triangle is positive, then all triangles have a non-zero positive defect.*

Theorem 1.6 *There exists a constant k such that area $(\overline{EFG}) = k^2$defect $(\overline{EFG})$ with the defect measured in radians.*
Proof is credited to Gauss and is beyond the scope of this paper.

Thus, two h-triangles with the same angle sum have the same area. This implies that the area of a triangle can be considered either as a function of the angle sum of the triangle or as a function of the defect of the triangle.

The congruence of triangles is an important and significant topic. Necessary and sufficient conditions for the congruence of triangles, widely known as congruence criteria, are used in almost all parts of geometry. Considering that analogous theorems for the congruence of quadrilaterals are rarely mentioned, and the fact that students will not have the opportunity to study them again, it is extremely important for students to understand these concepts. One way to do this is to use a comparative analysis of congruence criteria in hyperbolic plane and Euclidean geometry which is in the focus of this paper. But quadrilaterals are made of triangles so, in this introductory section, we will discuss triangles. The rest of the paper will be about quadrilaterals.

The congruence of both Euclidean and hyperbolic triangles is defined as follows:

Definition 1.7 *Triangles $\overline{EFG}$ and $\overline{E'F'G'}$ are congruent, $\overline{EFG} \cong \overline{E'F'G'}$ if and only if the following equalities hold:*

$$\overline{EF} = \overline{E'F'} \text{ and } \overline{FG} = \overline{F'G'} \text{ and } \overline{GE} = \overline{G'E'} \text{ and}$$
$$\angle E = \angle E' \text{ and } \angle F = \angle F' \text{ and } \angle G = \angle G' \tag{1.2}$$

$\angle E$ means the interior angle at vertex$E$; that is,$\angle GEF$ and analogously for the other angles.

So, two triangles are congruent if the sides and angles of one triangle are congruent to the corresponding elements of the other. Therefore, it implies the above six equalities. However, as is well known, these six equalities are not independent. Some three of them imply the remaining three, and thus the congruence of the triangles. Statements of which three are called *the congruence criteria of triangles*. Since hyperbolic geometry is more abstract than Euclidean, to students properly understand the congruence criteria for h-triangles, it is necessary to compare them with the well-known congruence criteria in Euclidean geometry, which we deal with below. There are six congruence criteria for h-triangles, which we will now list and compare to the well-known five congruence criteria for Euclidean triangles.

Theorem 1.8 *(SAS) Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, respectively.*

Theorem 1.9 *(SSS) Two triangles are congruent if three sides of one triangle are equal to three sides of another triangle, respectively.*

Theorem 1.10 *(ASA) Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of another triangle, respectively.*

Theorem 1.11 *(AAS) Two triangles are congruent if two angles and a non-included side of one triangle are equal to two angles and a non-included side of another triangle, respectively.*

In Euclidean geometry, all triangles have the same angle sum. Thus, if two triangles have two corresponding angles equal, then their third angles must also be equal. Thus, ASA and AAS are equivalent in Euclidean geometry. But, because h-triangles differ in their defect depending on their areas, ASA and AAS are not equivalent in hyperbolic geometry. Nevertheless, both can be proven independently using only absolute geometry postulates, as they are in Aguilar (2019).

Theorem 1.12 *(SsA) Two triangles are congruent if two sides and the angle opposite one of them in one triangle are equal to two sides and the angle opposite the same side in the other triangle, and these angles are both acute, both right, or both obtuse.*

This congruence criterion is most often used in the following special form:

Theorem 1.13 *Two triangles are congruent if two sides and the angle opposite the larger of the two sides in one triangle are equal to two sides and the corresponding angle of the other triangle.*

Thus, for students of Euclidean geometry, the only new information here is that AAS is not just an easy corollary of ASA requiring only the angle sum theorem to fill in the third angle, but, in absolute geometry, they are independent theorems with their own proofs. Also, many students have only learned of HL congruence and are not aware that it is a special case of SsA congruence.

Theorem 1.14 *(AAA) Two h-triangles are congruent if three angles of one triangle are equal to three angles of another triangle.*
Proof can be found in Stanković, Zlatanović (2016).

2. Can Four Equalities Prove Quadrilateral Congruence?

The definition of the congruence of two quadrilaterals is analogous with that for the congruence of two triangles.

Definition 2.1 *Quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent, $\overline{EFGH} \cong \overline{E'F'G'H'}$ if and only if the following equalities hold:*

$$\overline{EF} = \overline{E'F'} \text{ and } \overline{FG} = \overline{F'G'} \text{ and } \overline{GH} = \overline{G'H'} \text{ and } \overline{HE} = \overline{H'E'} \text{ and}$$
$$\angle E = \angle E' \text{ and } \angle F = \angle F' \text{ and } \angle G = \angle G' \text{ and } \angle H = \angle H' \tag{2.1}$$

These eight equalities (2.1) are not independent. Some number of them imply the remaining ones, and thus the congruence of quadrilaterals. It is not yet clear what that number is. Analogy with triangles superficially suggests that the number is four. However, this is not true if any one of the equalities is of lengths. It is also not true for AAAA in Euclidean geometry – the square and the rectangle being an obvious counterexample. Taking this into account, students can try by using the appropriate theorems from Euclidean geometry to show whether four equalities can prove the congruence of quadrilaterals, which will be discussed below.

Theorem 2.2 *No four of the eight equalities (2.1) are sufficient to prove congruence of two h-quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$.*

Proof It can be shown that, for any four of the eight equalities (2.1), there are two non-congruent quadrilaterals that meet these four equalities. Such quadrilaterals are counterexamples to the claim that these four equalities are sufficient to prove congruence. We will consider all possible combinations. In the following diagrams, sides labeled with English letters and angles labeled with Greek letters are the ones given to be equal.

1. ***Four sides.*** Such quadrilaterals need not be congruent. For example, a square and a rhombus with equal sides are not congruent in either Euclidean or hyperbolic geometry.

2. ***Three sides and one angle.*** There are two possible positions for the angle.

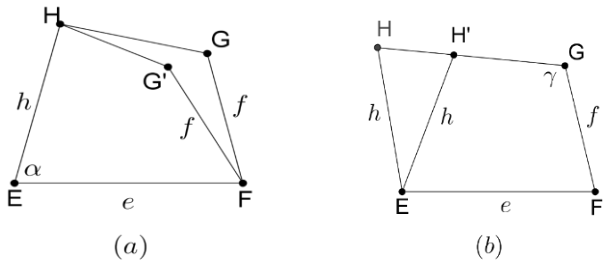

Fig. 2.1

   a. SASS. In figure 2.1(a), SASS implies $\overline{EFGH} \cong \overline{EFG'H}$, which is clearly not the case. Analogously, SSAS does not work.
   b. SSSA. In figure 2.1(b), SASS implies $\overline{EFGH} \cong \overline{EFGH'}$, which is clearly not the case. Analogously, ASSS does not work.

3. ***Two adjacent sides and two angles.*** There are three possible positions for the two angles.

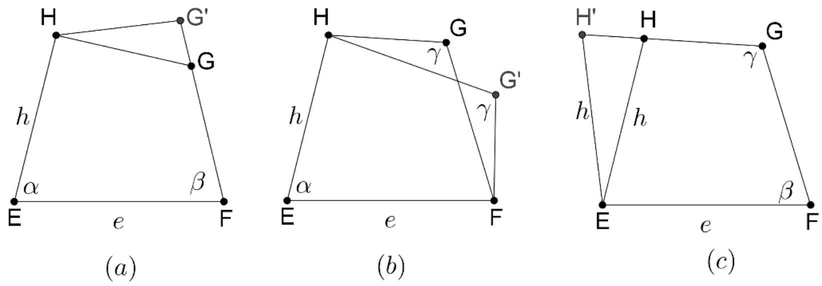

Fig. 2.2

   a. SASA. In figure 2.2(a), SASA implies $\overline{EFGH} \cong \overline{EFG'H}$, which is clearly not the case. Analogously, ASAS does not work.
   b. SAS-A. In figure 2.2(b), SAS-A implies $\overline{EFGH} \cong \overline{EFG'H}$, which is clearly not the case. Analogously, A-SAS does not work.
   c. SSAA. In figure 2.2(c), SSAA implies $\overline{EFGH} \cong \overline{EFGH'}$, which is clearly not the case. Analogously, AASS does not work.

4. ***Two non-adjacent sides and two angles.*** There are three possible positions for the two angles.

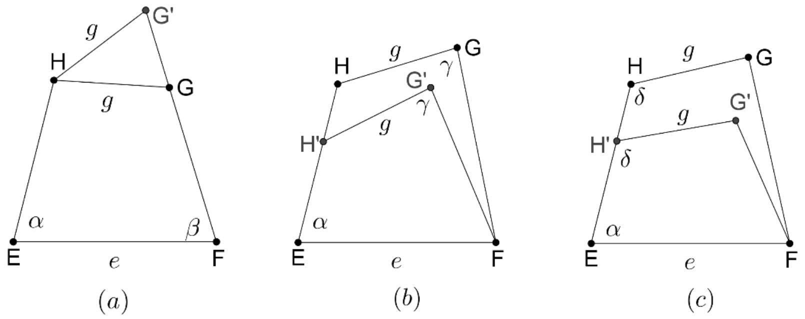

Fig. 2.3

a. S-ASA. In figure 2.3(a), S-ASA implies $\overline{EFGH} \cong \overline{EFG'H}$, which is clearly not the case. Analogously, ASA-S does not work.
b. S-AS-A. In figure 2.3(b), S-AS-A implies $\overline{EFGH} \cong \overline{EFG'H'}$, which is clearly not the case. Analogously, A-SA-S does not work.
c. SAAS. In figure 2.3(c), SAAS implies $\overline{EFGH} \cong \overline{EFG'H'}$, which is clearly not the case.

5. ***One side and three angles.*** There are two possible positions for the side.

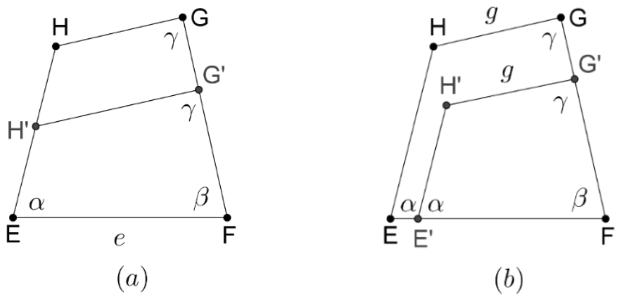

Fig. 2.4

a. ASAA. In figure 2.4(a), ASAA implies $\overline{EFGH} \cong \overline{EFG'H'}$, which is clearly not the case. Analogously, AASA does not work.
b. AAAS. In figure 2.4(b), AAAS implies $\overline{EFGH} \cong \overline{EFG'H'}$, which is clearly not the case. Analogously, SAAA does not work.

6. ***Four angles.*** There is one characteristic case and two constructible counterexamples, though not in a finite number of steps. Constructing these counterexamples will be deferred to section four. ■

3. CAN FIVE EQUALITIES PROVE QUADRILATERAL CONGRUENCE?

Now, let us consider congruence of two h-quadrilaterals with five equal elements. It turns out that some five of the eight equalities (2.1) prove congruence of two h-quadrilaterals. We will systematize the cases and consider them each as we did for the previous theorem. In the following diagrams, letters with no subscripts are the ones given to be equal while subscripted letters are proven equal in intermediate steps. Students can try by using the appropriate congruence theorems in Euclidean geometry to show whether four equalities can prove the congruence of quadrilaterals.

1. ***Four sides and one angle.*** There is only one characteristic case:

Theorem 3.1 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{FG} = \overline{F'G'} = f$ *and* $\overline{GH} = \overline{G'H'} = g$ *and*
$\overline{HE} = \overline{H'E'} = h$ *and* $\angle E = \angle E' = \alpha$ (3.1)

Proof By SAS, $\overline{HEF} \cong \overline{H'E'F'}$ (See Figure 3.1), which holds these equalities:
$\overline{FH} = \overline{F'H'}$ and $\angle EFH = \angle E'F'H' = \beta_1$ and $\angle EHF = \angle E'H'F' = \delta_1$

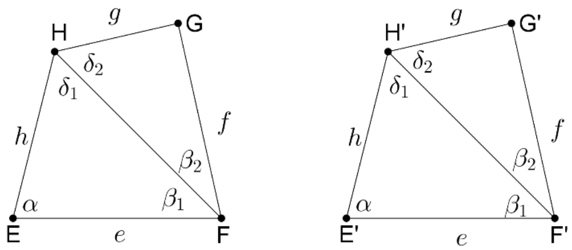

Fig. 3.1

From (3.1) and $\overline{FH} = \overline{F'H'}$, by SSS, we obtain $\overline{FGH} \cong \overline{F'G'H'}$. Thus,
$\angle HFG = \angle H'F'G' = \beta_2$ and $\angle FHG = \angle F'H'G' = \delta_2$
$\angle FGH = \angle F'G'H'$ (3.2)
Now we have,
$\angle EFG = \angle E'F'G' = \beta_1 + \beta_2$ (3.3)
$\angle EHG = \angle E'H'G' = \delta_1 + \delta_2$ (3.4)
The eight equalities (2.1) needed for congruence are satisfied with the given equalities and (3.2), (3.3) and (3.4). Thus, $\overline{EFGH} \cong \overline{E'F'G'H'}$. ■

This quadrilateral congruence criterion we will call SASSS. Students can observe that only SAS and SSS was cited, which are absolute geometry theorems, so this criterion is also Euclidean.

2. ***Three sides and two angles.*** There are three characteristic cases. Two of them guarantee a congruence of quadrangles and one not. Let us consider the first two.

Theorem 3.2 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{FG} = \overline{F'G'} = f$ *and* $\overline{HE} = \overline{H'E'} = h$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$

Proof By SAS, $\overline{HEF} \cong \overline{H'E'F'}$ (See Figure 3.2), which holds these equalities: $\overline{FH} = \overline{F'H'}$ and $\angle EFH = \angle E'F'H' = \beta_1$ and $\angle EHF = \angle E'H'F' = \delta_1$

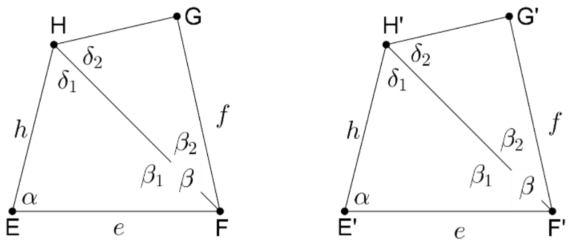

Fig. 3.2

$\angle HFG = \angle H'F'G' = \beta_2 = \beta - \beta_1$, so, by SAS, $\overline{HFG} \cong \overline{H'F'G'}$, which holds the equalities $\angle FGH = \angle F'G'H'$ and $\angle GHF = \angle G'H'F' = \delta_2$. $\angle GHE = \angle G'H'E' = \delta_1 + \delta_2$. Thus, the eight equalities (2.1) needed for congruence are satisfied and $\overline{EFGH} \cong \overline{E'F'G'H'}$. ∎

This quadrilateral congruence criterion we will call SASAS. Students can observe that only SAS and SSS was cited, which are absolute geometry theorems, so this criterion is also Euclidean.

Theorem 3.3 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{FG} = \overline{F'G'} = f$ *and* $\overline{HE} = \overline{H'E'} = h$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle H = \angle H' = \delta$ *and*
$\angle G$ *and* $\angle G'$ *are both acute, both right, or both obtuse.*

Proof By *SAS,* $\overline{HEF} \cong \overline{H'E'F'}$ (See Figure 3.3), which holds these equalities: $\overline{FH} = \overline{F'H'}$ and $\angle EFH = \angle E'F'H' = \beta_1$ and $\angle EHF = \angle E'H'F' = \delta_1$

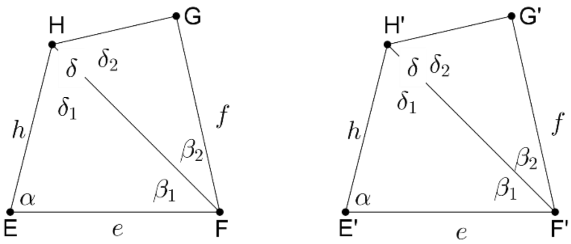

Fig. 3.3

$\angle FHG = \angle F'H'G' = \gamma_2 = \gamma - \gamma_1$. Since $\angle G$ and $\angle G'$ are both acute, both right, or both obtuse, by SsA, $\overline{GFH} \cong \overline{G'F'H'}$, which holds the equalities $\angle HGF = \angle H'G'F'$ and $\angle GFH = \angle G'F'H' = \beta_2$.
$\angle EFG = \angle E'F'G' = \beta_1 + \beta_2$. Thus, the eight equalities (2.1) needed for congruence are satisfied and $\overline{EFGH} \cong \overline{E'F'G'H'}$. ■

This quadrilateral congruence criterion we will call ASASs. Students can observe that only SAS and SsA was cited, which are absolute geometry theorems, so this criterion is also Euclidean.

Theorem 3.4 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are not necessarily congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{FG} = \overline{F'G'} = f$ *and* $\overline{HE} = \overline{H'E'} = h$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle G = \angle G' = \gamma$

Proof Quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ (figure 3.4) are a counterexample. ■

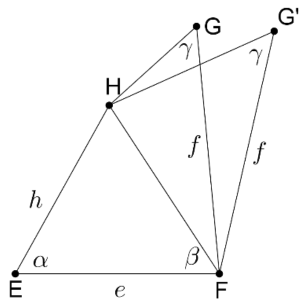

Fig. 3.4

3. ***Two sides and three angles.*** Sides can be adjacent or opposite. Let us first consider the case of adjacent sides. There are three different cases and all of them guarantee a congruence of h-quadrangles, though only the first two are absolute geometry.

Theorem 3.5 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{HE} = \overline{H'E'} = h$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ *and* $\angle H = \angle H' = \delta$

Proof By SAS, $\overline{HEF} \cong \overline{H'E'F'}$ (See Figure 3.5), which holds these equalities:
$\overline{FH} = \overline{F'H'}$ and $\angle EFH = \angle E'F'H' = \beta_1$ and $\angle EHF = \angle E'H'F' = \delta_1$

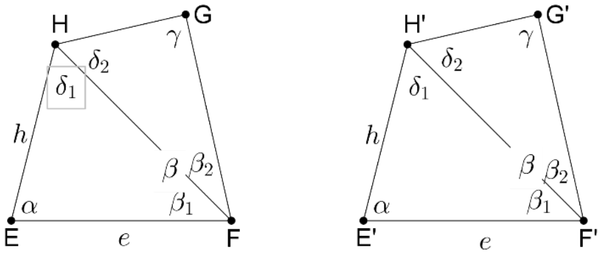

Fig. 3.5

$\angle HFG = \angle H'F'G' = \beta - \beta_1 = \beta_2$, so, by AAS, $\overline{GFH} \cong \overline{G'F'H'}$, which holds the equalities $\overline{FG} = \overline{F'G'}$, $\overline{GH} = \overline{G'H'}$ and $\angle GHF = \angle G'H'F' = \delta_2$. $\angle GHE = \angle G'H'E' = \delta_1 + \delta_2$. Thus, the eight equalities (2.1) needed for congruence are satisfied and $\overline{EFGH} \cong \overline{E'F'G'H'}$. ■

This quadrilateral congruence criterion we will call SASAA. Students can observe that only SAS and AAS was cited, which are absolute geometry theorems, so this criterion is also Euclidean.

Theorem 3.6 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{EF} = \overline{E'F'} = e$ *and* $\overline{FG} = \overline{F'G'} = f$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ *and* $\angle H = \angle H' = \delta$

Proof By SAS, $\overline{EFG} \cong \overline{E'F'G'}$ (See Figure 3.6), which holds these equalities:
$\overline{EG} = \overline{E'G'}$ and $\angle GEF = \angle G'E'F' = \alpha_1$ and $\angle FGE = \angle F'G'E' = \gamma_1$

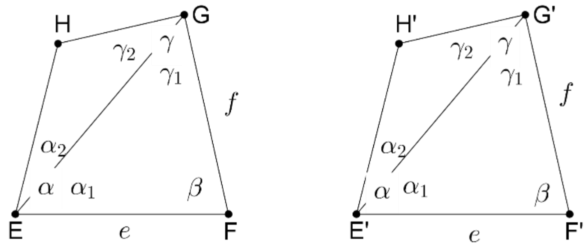

Fig. 3.6

$\angle HEG = \angle H'E'G' = \alpha - \alpha_1 = \alpha_2$ and $\angle HGE = \angle H'G'E' = \gamma - \gamma_1 = \gamma_2$ so, by ASA, $\overline{EGH} \cong \overline{E'G'H'}$, which holds the equalities $\overline{GH} = \overline{G'H'}$ and $\overline{HE} = \overline{H'E'}$ and $\angle GHE = \angle G'H'E'$.
Thus, the eight equalities (2.1) needed for congruence are satisfied and $\overline{EFGH} \cong \overline{E'F'G'H'}$. ■

This quadrilateral congruence criterion we will call ASASA. Students can observe that only SAS and ASA was cited, which are absolute geometry theorems, so this criterion is also Euclidean.

Theorem 3.7 *The h-quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if*
$\overline{GH} = \overline{G'H'} = g$ *and* $\overline{GE} = \overline{G'E'} = h$ *and*
$\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ *and* $\angle H = \angle H' = \delta$

Proof $\overline{FG}$ and $\overline{F'G'}$ are either equal or not equal. Suppose $\overline{FG} = \overline{F'G'} = f$, then $\overline{HGFE} \cong \overline{H'G'F'E'}$ by theorem 3.5, SASAA. See figure 3.7.

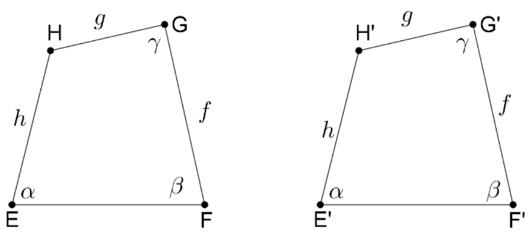

Fig. 3.7

Suppose $\overline{FG} \neq \overline{F'G'}$. $\overline{F'G'} < \overline{FG}$ or switch labels on $\overline{EFGH}$ and $\overline{E'F'G'H'}$. There exists a point F_1 between F and G such that $\overline{F_1G} = \overline{F'G'} = f_1$. Draw a ray $\overrightarrow{F_1Q}$ such that Q is on $\overline{EH}$ and $\angle GF_1Q = \beta$. See figure 3.8.

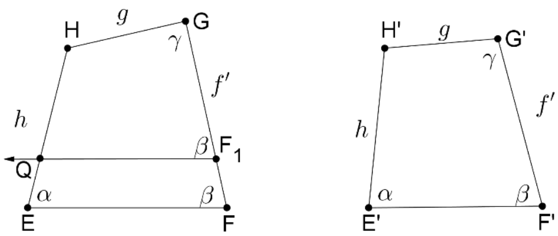

Fig. 3.8

Let the point E_1 be on ray $\overrightarrow{F_1Q}$ and such that $\overline{E_1F_1} = \overline{E'F'}$. By Theorem 3.2 (SASAS), $\overline{HGF_1E_1} \cong \overline{H'G'F'E'}$, which holds the equality $\angle HE_1F_1 = \alpha$. There are three possible positions for E_1 relative to F_1 and Q on this ray. We will label them cases (a), (b) and (c). See figure 3.9.

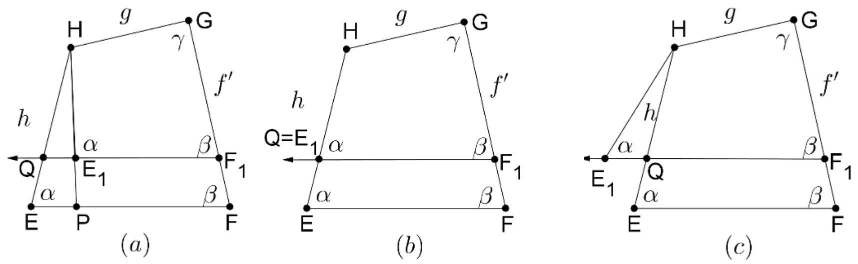

Fig. 3.9

Case (a). E_1 is between F_1 and Q. By the exterior angle inequality theorem applied to $\overline{EPH}$, $\alpha < \angle HPF$. But the interior angles of $\overline{PE_1F_1F}$ are $\angle HPF$, $\sigma - \alpha$, $\sigma - \beta$ and β. The sum of these angles is $2\sigma + \angle HPF - \alpha$. By theorem 1.1, this must be less than 2σ, which implies $\angle HPF < \alpha$. This is a contradiction, so case (a) is not true.

Case (b). E_1 coincides with Q. The sum of the interior angles of $\overline{PE_1F_1F}$ is 2σ, which contradicts theorem 1.1, so case (b) is not true.

Case (c). Q is between E_1 and F_1.

Suppose $\alpha = \rho$. $h = \overline{HE_1} < \overline{HQ} < \overline{HE} = h$, a contradiction.

Suppose $\alpha \neq \rho$. Locate J and K, the feet or perpendiculars dropped from H onto $\overline{E_1F_1}$ and $\overline{EF}$, respectively. See figure 3.10.

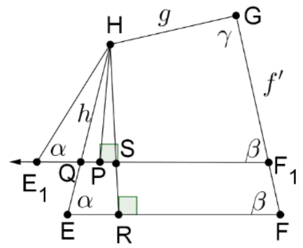

Fig. 3.10

$\overline{JE_1H} \cong \overline{KEH}$ by AAS, which holds the equality $\overline{JH} \cong \overline{KH}$. There is a point L on $\overline{KH}$ and $\overline{E_1F_1}$. Thus, $h = \overline{E_1H} < \overline{HJ} < \overline{HL} < \overline{HK} < \overline{HE} = h$. This is a contradiction, so case (c) is no more true than cases (a) and (b). $\overline{FG} \neq \overline{F'G'}$ meets only with contradiction, so $\overline{EFGH} \cong \overline{E'F'G'H'}$. ∎

This quadrilateral congruence criterion we will call SSAAA. Student can observe that we cited the angle sum of triangles being less than a straight angle and the angle sum of quadrilaterals being less than two straight angles, which are hyperbolic geometry theorems, so this criterion is not true in Euclidean geometry.

Now we will consider two opposite sides and three angles. There is only one characteristic case, and it does not guarantee congruence.

Theorem 3.8 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are not necessarily congruent if* $\overline{EF} = \overline{E'F'} = e$ *and* $\overline{GH} = \overline{G'H'} = g$ *and* $\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ *and* $\angle H = \angle H' = \delta$

Proof Quadrilaterals $\overline{\mathrm{EFGH}}$ and $\overline{\mathrm{E'F'G'H'}}$ (figure 3.11) are a counterexample. ■

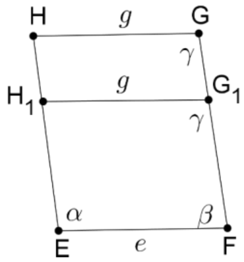

Fig. 3.11

4. ***One side and four angles.*** There is only one characteristic case.

Theorem 3.9 *The h-quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are congruent if* $\overline{EF} = \overline{E'F'} = e$ *and* $\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ $\angle G = \angle G' = \gamma$ *and* $\angle H = \angle H' = \delta$

Proof $\overline{EH}$ and $\overline{E'H'}$ are either equal or not equal. Suppose $\overline{EH} = \overline{E'H'} = h$, then $\overline{HEFG} \cong \overline{H'E'F'G'}$ by theorem 3.5, SASAA. See figure 3.12.

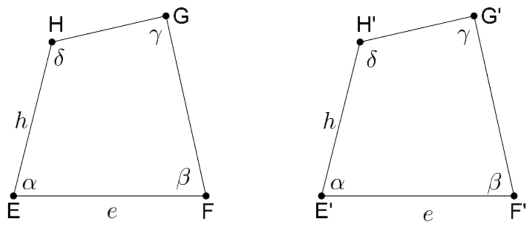

Fig. 3.12

Suppose $\overline{EH} \neq \overline{E'H'}$. $\overline{E'H'} < \overline{EH}$ or switch labels on $\overline{EFGH}$ and $\overline{E'F'G'H'}$. There exists a point H_1 between E and H such that $\overline{EH_1} = \overline{E'H'} = h_1$. Draw a ray $\overrightarrow{H_1P}$ such that P is on $\overline{FG}$ and $\angle EH_1P = \delta$. See figure 3.13.

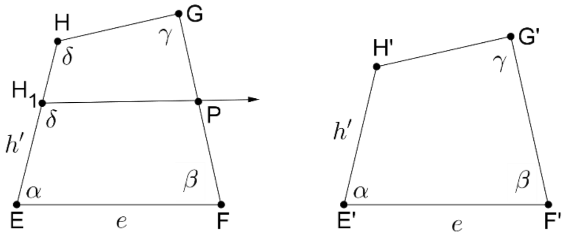

Fig. 3.13

Let the point G_1 be on ray $\overrightarrow{H_1P}$ and such that $\overline{H_1G_1} = \overline{H'G'} = h'$. By Theorem 3.2 (SASAS), $\overline{FEH_1G_1} \cong \overline{F'E'H'G'}$. There are three possible positions for G_1 relative to H_1 and P on this ray. We will label them cases (a), (b) and (c). See figure 3.14.

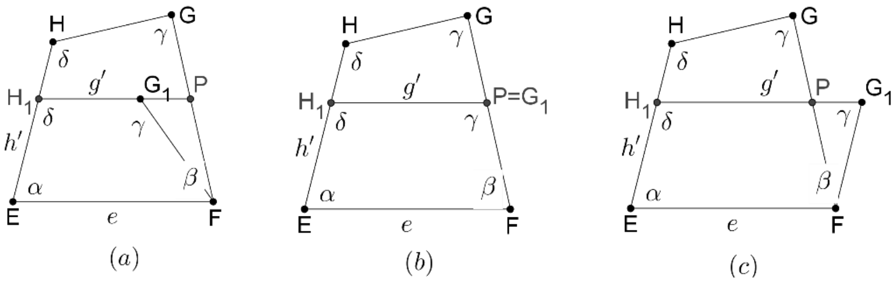

Fig. 3.14

Case (a). G_1 is between H_1 and P. $\angle EFG_1 < \angle EFP = \beta$. But $\angle EFG_1 = \beta$ because $\overline{FEH_1G_1} \cong \overline{F'E'H'G'}$. Thus, case (a) is not true.
Case (b). G_1 coincides with P. $\overline{FEH_1G_1} \cong \overline{F'E'H'G'}$, which holds the equality $\angle FG_1H_1 = \gamma$. But $\overline{EFGH}$ is larger than $\overline{EFG_1H_1}$ so it must have a greater defect, not all the same angles. Thus, case (b) is not true.
Case (c). P is between H_1 and G_1. $\angle EFG_1 > \angle EFP = \beta$. But $\angle EFG_1 = \beta$ because $\overline{FEH_1G_1} \cong \overline{F'E'H'G'}$. Thus, case (c) is not true.
Thus, $\overline{EH} \neq \overline{E'H'}$ meets only with contradiction, so $\overline{EFGH} \cong \overline{E'F'G'H'}$. ∎

This quadrilateral congruence criterion we will call SAAAA. Students can observe that case (b) could be true in Euclidean geometry, where a quadrilateral can have a similar one inside it. So this criterion only works for h-quadrilaterals.

4. Congruence of Saccheri and Lambert Quadrilaterals

In this section we will deal with special classes of quadrilaterals in hyperbolic geometry such as Saccheri and Lambert quadrilaterals. Saccheri considered a type of quadrilateral, called a Saccheri quadrilateral, as in attempting to prove the parallel postulate. In Euclidean geometry, a Saccheri quadrilateral is a rectangle. Students should observe the common properties of the Saccheri quadrilateral and a rectangle.

Definition 4.1 *A Saccheri quadrilateral is a quadrilateral with two equal sides perpendicular to the base. The top side is the summit or upper base and the angles on the summit are the summit angles.*

For a Saccheri quadrilateral $\overline{EFGH}$, the base is $\overline{EF}$ and the summit is $\overline{GH}$. The summit angles of a Saccheri quadrilateral are equal, and they are less than a right angle. If they were right, then $\overline{EFGH}$ would be a rectangle and we would be doing Euclidean geometry.

Definition 4.2 *A Lambert quadrilateral is a quadrilateral with three right-angles.*

In Euclidean geometry, three right angles imply that the fourth is also right and it is a rectangle.

Theorem 4.3 *Two Lambert quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$, with acute angles at H and H' are congruent if and only if one of the following sets of conditions is true:*

a) $\overline{EF} = \overline{E'F'}$ *and* $\overline{FG} = \overline{F'G'}$ *b)* $\overline{EF} = \overline{E'F'}$ *and* $\overline{EH} = \overline{E'H'}$

c) $\overline{EH} = \overline{E'H'}$ *and* $\overline{GH} = \overline{G'H'}$ *d)* $\overline{EH} = \overline{E'H'}$ *and* $\angle H = \angle H'$

e) $\overline{EF} = \overline{E'F'}$ *and* $\angle H = \angle H'$ *f)* $\overline{EH} = \overline{E'H'}$ *and* $\overline{FG} = \overline{F'G'}$ (4.1)

Proof Cases (a), (b), (c), (d) and (e) follow immediately from theorems 3.6 (ASASA), 3.5 (SASAA), 3.8 (SSAAA), 3.10 (SAAAA) and 3.10, respectively.

Since every Saccheri quadrilateral can be divided into two congruent Lambert quadrilaterals, students can observe that it immediately follows that:

Corollary 4.4 *Two Saccheri quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$, with bases $\overline{EF}$ and $\overline{E'F'}$ and with summits $\overline{GH}$ and $\overline{G'H'}$, respectively, are congruent if and only if one of the following sets of conditions is true:*

a) $\overline{EF} = \overline{E'F'}$ *and* $\overline{FG} = \overline{F'G'}$ *b)* $\overline{EF} = \overline{E'F'}$ *and* $\overline{GH} = \overline{G'H'}$

c) $\overline{FG} = \overline{F'G'}$ *and* $\overline{GH} = \overline{G'H'}$ *d)* $\overline{EF} = \overline{E'F'}$ *and* $\angle G = \angle G'$

e) $\overline{FG} = \overline{F'G'}$ *and* $\angle G = \angle G'$ *f)* $\overline{GH} = \overline{G'H'}$ *and* $\angle G = \angle G'$ (4.2)

Theorem 4.5 *The quadrilaterals $\overline{EFGH}$ and $\overline{E'F'G'H'}$ are not necessarily congruent if*
$\angle E = \angle E' = \alpha$ *and* $\angle F = \angle F' = \beta$ *and*
$\angle G = \angle G' = \gamma$ *and* $\angle H = \angle H' = \delta$

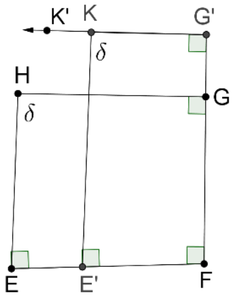

Fig. 4.1

Proof $\overline{EFGH}$ is a Lambert quadrilateral with $\angle E = \angle F = \angle G = \rho$ and $\angle H < \rho$. Find G' in $\overrightarrow{FG}$ such that $F - G - G'$ and $\angle G' = \rho$. Let E' be the foot of the perpendicular dropped onto $\overleftrightarrow{EF}$ from K. By construction, $\overline{E'FG'K}$ is a Lambert quadrilateral with $\angle K < \rho$.

If K moves infinitely from point G on $\overrightarrow{GK'}$, then $\angle K$ will be smaller and smaller and in some point $\angle K = \angle H = \delta$.

So, there exists a point K on $\overrightarrow{GK'}$ such that $\angle K = \angle H$. Thus, $\overline{EFGH}$ and $\overline{E'FGK}$ have all equal angles but they are not congruent. ■

5. Conclusion

This paper aims to develop some new congruence criteria of convex h-quadrilaterals and relate them to previous work on hyperbolic geometry and its applications. Accordingly, we presented the appropriate methodological approach in teaching using comparative geometric analysis of quadrilateral congruence criteria in Euclidean and hyperbolic geometry.

We start with six congruence criteria for triangles (SAS, SSS, ASA, AAS, SsA and AAA), the first five of which are absolute geometry and the last of which applies only to h-triangles. We then prove that no four equalities can prove quadrilateral convergence. We then prove that there are seven congruence criteria for convex quadrilaterals (SASSS, SASAS, ASASs, SASAA, ASASA, SSAAA, SAAAA), the first five of which are absolute geometry and the last two of which apply only to h-quadrilaterals. Finally, we list the six congruence criteria for Lambert quadrilaterals and the six congruence criteria for Saccheri quadrilaterals and prove the one that is not a direct consequence of the seven congruence criteria for general h-quadrilaterals.

As a final remark it should be stated that the results derived in this paper could probably be applied to the Einstein relativistic velocity model of hyperbolic geometry (Barbu, 2010).

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METODIČKI PRISTUP KONGRUENCIJI ČETVOROUGLOVA U HIPERBOLIČKOJ GEOMETRIJI

U ovom radu dokazaćemo nove kriterijume kongruencija konveksnih četvorouglova u hiperboličkoj geometriji i, u skladu sa tim, prikazati metodički pristup u izučavanju istog. Postoji sedam kriterijuma za kongruenciju hiperboličkih četvorouglova, dok ih je pet koji važe za četvorouglove u euklidskoj geometriji. Koristeći komparativnu geometrijsku analizu kriterijuma kongruencije koji važe u Euklidskoj i Hiperboličkoj geometriji, opisali smo sve moguće slučajeve i, u skladu sa tim, odgovarajući metodički pristup za svaki od njih. Dobijeni rezultati mogu uticati na pristupe izučavanja ovih sadržaja sa studentima u nastavi Hiperboličke geometrije.

Ključne reči: *kongruencija konveksnih četvorouglova, Euklidska geometrija, Hiperbolička geometrija, hiperbolički četvorouglovi, metodički pristup*