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AN ALTERNATIVE APPROACH TO FINITE DEFORMATION

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Abstract. *In elastoplasticity formulation constitutive relations are usually given in rate form, i.e. they represent relations between stress rate and strain rate. The adopted constitutive laws have to stay independent in relation to the change of frame of reference, i.e. to stay objective. While the objectivity requirement in a material description is automatically satisfied, in an Eulerian description, especially in the case of large deformations, the objectivity requirement can be violated even for objective quantities. Thus, instead of a material time derivative in the Eulerian description objective time derivatives have to be implemented. In this work the importance of the objective rate implementation in the constitutive relations of finite elastoplasticity is clarified. Likewise, it shows the overview of the most frequently used objective rates nowadays, their advantages and shortcomings, as well as the distinctive features of the recently introduced logarithmic rate.*

Key words: *finite deformations, objective rates, logarithmic rate, elastoplasticity, finite deformation decomposition*

1. EULERIAN DESCRIPTION OF FINITE DEFORMATION

The process of deformation of a deformable body *B* from its reference, B_0, to a current configuration, B_t, represents the change in shape and position of the observed body, as it is shown in Fig. 1. While the former leads to a varying distance between the arbitrary pairs of particles of the body *B* (here particles P and Q), the latter reflects a rigid body motion, i.e. translation and rotation. In this article an Eulerian description is adopted. Therefore, in order to define the position of the particle of interest, instead of material coordinates as independent variables spatial coordinates are used.

The distance between particles P and Q in the reference configuration, represented by a material vector d**X**, is changing during the process of deformation in a corresponding space vector d**x** according to law

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$$\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}, \tag{1}$$

where **F** is a two-point tensor, called a *deformation gradient*. Its determinant, entitled the *Jacobian determinant* or shortly *Jacobian*, meets the rule

$$\mathrm{J} = \det(\mathbf{F}) > 0. \tag{2}$$

In large deformation problems rotation plays an important role. A *polar decomposition theorem* elucidates this role. It postulates a unique multiplicative decomposition of the deformation gradient into a positive definite symmetric 2^{nd}-order Eulerian tensor **V** or Lagrangian tensor **U** known as *left* or *right stretch tensor*, respectively, and a proper orthogonal *rotation tensor* **R**, i.e.

$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}. \tag{3}$$

In the Eulerian description the *left polar decomposition* is to be applied; it assumes that the deformation is composed of the rigid body motion, represented by the rotation tensor **R**, and the stretching, defined by the Eulerian left stretch tensor **V**, which follows the rotation.

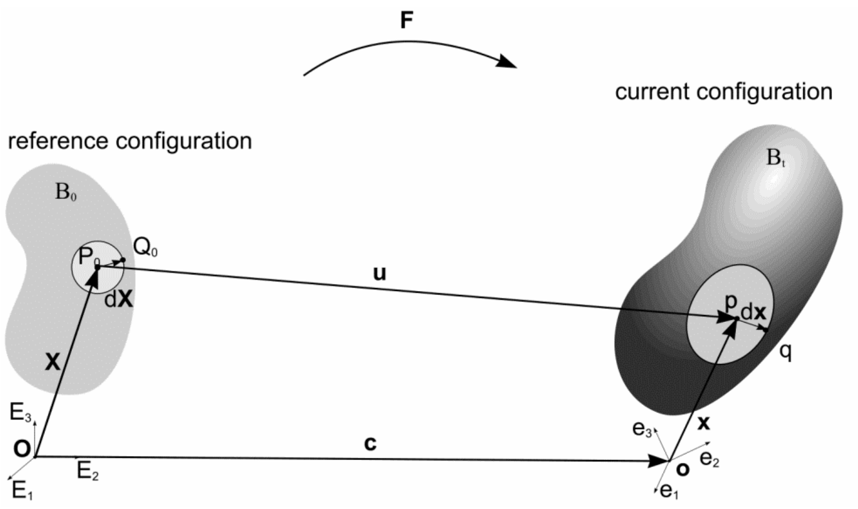

Fig. 1 Reference and current configuration of a deformable body

The squares of the left and right stretch tensors are called *left* and *right Cauchy-Green tensors* and are determined as

$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \quad \text{and} \quad \mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \tag{4}$$

are more convenient for numerical purpose. Relations between Cauchy-Green tensors are given as

$$\mathbf{B} = \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^{\mathrm{T}} \quad \text{and} \quad \mathbf{C} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{B} \cdot \mathbf{R}. \tag{5}$$

2. ANALYSIS OF MOTION

Since most of formulations of finite elastoplasticity are given in a rate form, which is well-suited for the numerical implementation of the latter into finite element based programs, it is necessary to express the previously introduced quantities as functions of time.

A velocity of the observed material point can be defined in the Eulerian description by the following relation

$$\mathbf{v}(t) = \dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}, \tag{6}$$

while its increment is given as

$$\mathrm{d}\mathbf{v}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial \mathbf{x}} \cdot \mathrm{d}\mathbf{x} = \nabla \mathbf{v} \cdot \mathrm{d}\mathbf{x}, \tag{7}$$

i.e.

$$\mathrm{d}\mathbf{v} = \mathrm{d}\dot{\mathbf{x}} = \mathbf{L} \cdot \mathrm{d}\mathbf{x}. \tag{8}$$

Here **L** represents the *velocity gradient*. It actually maps the material line element d**x** to its rate in the current configuration. Unlike the deformation gradient, that describes a local deformation state of the particle P and is related to the reference configuration, the velocity gradient defines a *rate of change* of a local deformation state of the particle P and it is *not* related to the reference configuration. As a function of the deformation gradient the velocity gradient can be expressed in the form

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}. \tag{9}$$

It can be decomposed into its symmetric part, related to stretching, and skew-symmetric part, related to rotation,

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{10}$$

where the *rate of deformation tensor* or the *stretching tensor* **D** and the *vorticity tensor* or the *spin tensor* **W** are given respectively as

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}) \quad \text{and} \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}), \tag{11}$$

(see e.g. Malvern, 1969, and Mićunović, 1990).

One of the key tasks in defining a reliable model of elastoplastic material behaviour is the choice of a suitable strain rate measure that will be implemented in the constitutive laws. The stretching tensor **D** can be one of these strain rate measures. But, in spite of its name as the rate of deformation tensor, until recently it has been considered that the stretching tensor cannot be defined either as a Lagrangian or as an Eulerian strain rate tensor and therefore it has not been recognized as a rate of deformation (see Ogden, 1984). However, Xiao et al. (1997, 1998b) have proved that the stretching tensor **D** can be exactly integrated to give the Hencky strain tensor **h,** defined in the Eulerian description by the relation

$$\mathbf{h} = \frac{1}{2}\ln \mathbf{B} = \ln \mathbf{V}. \tag{12}$$

How to define the natural deformation rate **D** as a direct flux of the Hencky (logarithmic) strain **h** will be explained in Section 3.3.

3. Objectivity of a Tensor Field

The deformation of the body can be observed by one or several physical observers and, thus, it can be described in different ways. For example, we have two observers that are recording the process of deformation. One observer is labelled with star and another one without star. Consequently their records are designated on the same way. The transformation between the coordinate systems associated to the space position of both observers is described by the relation

$$\mathbf{x}^* = \mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t), \tag{13}$$

where **Q** is a proper orthogonal tensor of relative rotation and **c** is a vector of relative translation of one observer relatively to another**.** Observations can be recorded in different time and therefore, using the time distance in records a, the time difference can be specified as

$$t^* = t - a. \tag{14}$$

Physical phenomena do not depend on the choice of the observer, which is not necessarily the case of their kinematical description. That reflects on the mathematical formulation of physical laws such as constitutive models.

According to Ogden (1984), scalar a_0, vector $\boldsymbol{\alpha}_0$ or 2nd-order tensor field $\mathbf{A}_0$ defined in the Lagrangian configuration, are objective if they conform to these transformation rules

$$\begin{aligned} a_0^*(\mathbf{X},t^*) &= a_0(\mathbf{X},t) \\ \boldsymbol{\alpha}_0^*(\mathbf{X},t^*) &= \boldsymbol{\alpha}_0(\mathbf{X},t) \\ \mathbf{A}_0^*(\mathbf{X},t^*) &= \mathbf{A}_0(\mathbf{X},t). \end{aligned} \tag{15}$$

Since the following relation holds

$$\dot{\mathbf{A}}_0^*(\mathbf{X},t^*) = \dot{\mathbf{A}}_0(\mathbf{X},t), \tag{16}$$

the conclusion we arrive to is that the material time derivative of the transformed Lagrangian 2nd-order tensor satisfies the objectivity requirement.

The Eulerian scalar quantity a, vector $\boldsymbol{\alpha}$ or 2nd-order tensor **A,** contrarily to the Lagrangian quantities**,** are objective if they transform according to the following rules

$$\begin{aligned} a^*(\mathbf{x},t^*) &= a(\mathbf{x},t) \\ \boldsymbol{\alpha}^*(\mathbf{x},t^*) &= \mathbf{Q}(t)\cdot\boldsymbol{\alpha}(\mathbf{x},t) \\ \mathbf{A}^*(\mathbf{x},t^*) &= \mathbf{Q}(t)\cdot\mathbf{A}(\mathbf{x},t)\cdot\mathbf{Q}(t)^{\mathrm{T}}. \end{aligned} \tag{17}$$

The last relation clearly shows that the transformation of the Eulerian quantities is dependent of the rotation tensor **Q** and accordingly of the change of frame.

The material time derivative of an objective Eulerian 2nd-order tensor changes as

$$\dot{\mathbf{A}}^* = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^{\mathrm{T}}) = \dot{\mathbf{Q}}\cdot\mathbf{A}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\mathbf{A}\cdot\dot{\mathbf{Q}}^{\mathrm{T}} \neq \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^{\mathrm{T}}. \qquad (18)$$

The last relation shows that the material time derivative of an objective 2nd-order tensor does not obey the transformation rule $(17)_3$, and, therefore, it can be concluded that *the material time derivative in the Eulerian description is not an objective quantity*. Therefore, instead of the material time derivative an *objective time derivative* has to be used in order to preserve the objectivity requirement in the Eulerian description. Then, the following relation holds

$$\dot{\mathbf{A}}^* = \mathbf{Q}\cdot\overset{\diamond}{\mathbf{A}}\cdot\mathbf{Q}^{\mathrm{T}}, \qquad (19)$$

where $\overset{\diamond}{\mathbf{A}}$ is the objective time derivative of the objective Eulerian quantity $\mathbf{A}$.

3.1. Corotational and convective frame

One of the observers, O, can be detected at the fixed point of space $\mathbf{o}$, while the second, observer O^*, is located on the moving body at point $\mathbf{o}^*$ and it moves and rotates together with the deformable body (see Fig. 2). The point $\mathbf{o}$ is the origin of a fixed *background frame*, while $\mathbf{o}^*$ is the origin of a so-called *co-deforming frame*. In such a way, relation (13) shows the transformation between the background and the co-deforming frame. That means that the pair ($\mathbf{x}$, t) represents the point in the Galilean space-time, occupied by the particle P, observed by O from the background frame, while the pair ($\mathbf{x}^*$, t) represents the *same* point in the space observed by O^* from the transformed moving frame. It will be assumed that both observers record the position of the particle at the same time; thus the time difference a vanishes.

From relations (13) and (14) the transformation between the frames can be defined as

$$\mathbf{x}^* = \mathbf{K}(t)\cdot\mathbf{x} + \mathbf{c}(t) \qquad \text{with} \qquad t^* = t, \qquad (20)$$

where the time dependent tensor $\mathbf{K}$ is not the proper orthogonal but a general asymmetric 2nd-order tensor determined by the following 1st-order differential system with a prescribed initial value

$$\dot{\mathbf{K}} = \mathbf{\Psi}\cdot\mathbf{K}, \qquad \mathbf{K}\,|_{t=0} = \mathbf{1}. \qquad (21)$$

In the previous relation $\mathbf{1}$ is the unit 2nd-order tensor while $\mathbf{\Psi}$ is the asymmetric 2nd-order tensor given by

$$\mathbf{\Psi} = \mathbf{\Omega} + \mathbf{\Gamma}, \qquad (22)$$

where $\mathbf{\Omega}$ is the antisymmetric part and $\mathbf{\Gamma}$ is the symmetric part of $\mathbf{\Psi}$. The skew-symmetric tensor $\mathbf{\Omega}$ is called the *spin*.

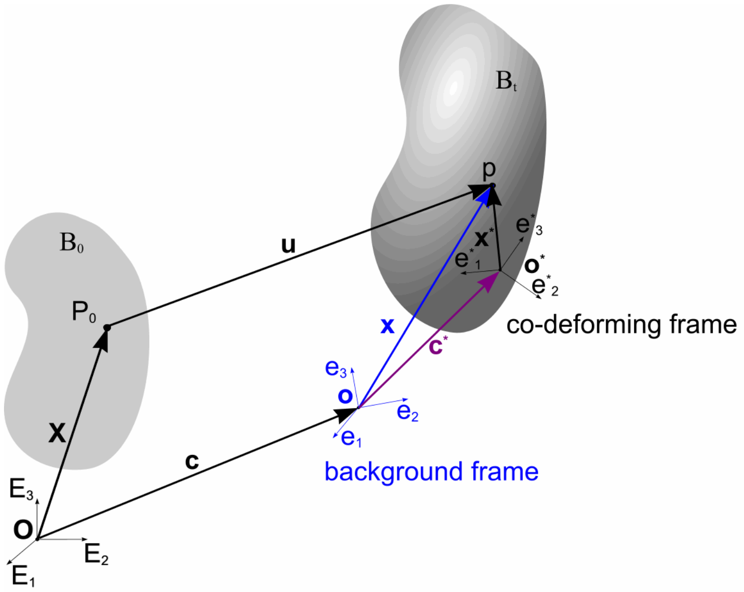

Fig. 2 Background and co-deforming frame

Transformation of an objective Eulerian tensorial quantity **A**, defined in the background frame, to $\mathbf{A}^*$ in the co-deforming frame can be determined by the transformation rule

$$\mathbf{A}^* = \mathbf{K}\cdot\mathbf{A}\cdot\mathbf{K}^{\mathrm{T}}. \tag{23}$$

Accordingly, the material time derivative of the transformed quantity $\mathbf{A}^*$ can be presented as

$$\dot{\mathbf{A}}^* = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{K}\cdot\mathbf{A}\cdot\mathbf{K}^{\mathrm{T}}) = \dot{\mathbf{K}}\cdot\mathbf{A}\cdot\mathbf{K}^{\mathrm{T}} + \mathbf{K}\cdot\dot{\mathbf{A}}\cdot\mathbf{K}^{\mathrm{T}} + \mathbf{K}\cdot\mathbf{A}\cdot\dot{\mathbf{K}}^{\mathrm{T}} = \mathbf{K}\cdot\overset{\diamond}{\mathbf{A}}\cdot\mathbf{K}^{\mathrm{T}}, \tag{24}$$

where the objective time derivative, introduced in the previous Section, is determined by the tensor $\boldsymbol{\Psi}$ as

$$\overset{\diamond}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A}\cdot\boldsymbol{\Psi} + \boldsymbol{\Psi}^{\mathrm{T}}\cdot\mathbf{A}. \tag{25}$$

The kinematical property of relation (24) can be understood in such a way that the material time derivative of the counter part of the Eulerian quantity **A** in the co-deforming frame is the co-deforming counter part of the objective time derivative of the same quantity **A** in the background frame.

The objective rates of the symmetric Eulerian 2$^{\mathrm{nd}}$-order field have been so far given in a general form. Since the objective rates play an essential role in modelling of various material behaviours in the Eulerian description, the type of the objective rate should be carefully chosen. Depending on the choice of the tensor $\boldsymbol{\Psi}$ the objective rates can be generally classified in two categories of *corotational* and *non-corotational objective rates* (cf. Bruhns et al., 2004).

Every time the symmetric part of $\boldsymbol{\Psi}$, given by relation (22), vanishes, i.e. $\boldsymbol{\Psi}$ becomes equal to the spin $\boldsymbol{\Omega}$, the frame determined by Eqs. (20) and (21) is a *spinning* or *corotating frame* and **K** is equal to the rotation tensor. Otherwise, $\boldsymbol{\Psi}$ determines a *convective frame*. While the former experiences only constant rotation the latter can deform and rotate continuously during the deformation process. As for the convective frame, the coordinate system in $\mathbf{o}^*$ is no longer a Cartesian coordinate system and when it undergoes the change of frame, a physical or kinematical quantity can lose some important features. For example, the eigenvalues of the quantity of interest can be modified during the process of deformation. If we want to preserve the physical or kinematical features of a physical or a kinematical tensor, the tensor $\boldsymbol{\Psi}$ *must be* a skew-symmetric tensor, which means that $\boldsymbol{\Psi} = \boldsymbol{\Omega}$. That leads to **K** = **Q**, i.e. **K** is the proper orthogonal rotation tensor. For details see Bruhns et al. (2004) and Xiao et al. (2005).

Integration of (24) leads to the *generalised objective time integration* of the objective rate

$$\mathbf{A} = \mathbf{K}^{-1} \cdot \int_t \mathbf{K} \cdot \overset{\diamond}{\mathbf{A}} \cdot \mathbf{K}^{\mathrm{T}} \mathrm{d}t \ \cdot \mathbf{K}^{\text{-T}}, \tag{26}$$

and it is applied in the co-deforming frame. This relation is found to be useful for the numerical implementation of the objective time derivatives in the constitutive relations of elastoplasticity.

3.2. Non-corotational rates

The objective non-corotational rate of the objective Eulerian quantity can be generally defined as

$$\overset{\triangledown}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \boldsymbol{\Psi} + \boldsymbol{\Psi}^{\mathrm{T}} \cdot \mathbf{A}. \tag{27}$$

With the specific choice of $\boldsymbol{\Psi}$ as

$$\boldsymbol{\Psi} = \mathbf{W} + m\,\mathbf{D} + c\,\mathrm{tr}(\mathbf{D})\,\mathbf{1}, \tag{28}$$

a broad class of objective non-corotational rates can be defined from (27):

$$\overset{\triangledown}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A} \cdot (\mathbf{W} + m\mathbf{D} + c\,\mathrm{tr}(\mathbf{D})\mathbf{1}) + (\mathbf{W} + m\mathbf{D} + c\,\mathrm{tr}(\mathbf{D})\mathbf{1})^{\mathrm{T}} \cdot \mathbf{A}, \tag{29}$$

where m and c are real numbers. For certain values of m and c in the last relation, well known rates can be obtained:
the (upper) Oldroyd rate

$$\overset{\triangledown}{\mathbf{A}}^{\mathrm{Ol}} = \dot{\mathbf{A}} - \mathbf{L} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}^{\mathrm{T}} \quad \text{for} \qquad m = \text{-}1 \text{ and } c = 0, \tag{30}$$

the Cotter-Rivlin rate

$$\overset{\triangledown}{\mathbf{A}}^{\mathrm{CR}} = \dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{L} \quad \text{for} \qquad m = 1 \text{ and } c = 0, \tag{31}$$

the Truesdell rate

$$\overset{\triangledown}{\mathbf{A}}^{\mathrm{Tr}} = \dot{\mathbf{A}} - \mathbf{L} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}^{\mathrm{T}} + \mathrm{tr}(\mathbf{D})\mathbf{A} \quad \text{for} \qquad m = -1 \text{ and } c = 0.5. \tag{32}$$

3.3. Corotational rates

The symmetric part of tensor $\boldsymbol{\Psi}$ may vanish, as it was pointed out in Section 3.1. It turns out that $\boldsymbol{\Psi} = \boldsymbol{\Omega}$. The rotating, or co-deforming, frame is then determined by the skew-symmetric 2nd-order Eulerian tensor **Q** instead of the general asymmetric 2nd-order tensor **K**, introduced earlier. The skew-symmetric spin tensor **Ω**, determining the rotating frame, is defined by:

$$\boldsymbol{\Omega} = \dot{\mathbf{Q}}^{\mathrm{T}} \cdot \mathbf{Q} = -\mathbf{Q}^{\mathrm{T}} \cdot \dot{\mathbf{Q}} = -\boldsymbol{\Omega}^{\mathrm{T}}. \tag{33}$$

The rotating frame becomes the *corotating frame* and the general objective time derivative $\overset{\lozenge}{\mathbf{A}}$ becomes the corotational rate $\overset{\circ}{\mathbf{A}}$, defined as

$$\overset{\circ}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \mathbf{A}. \tag{34}$$

The following relation describes transformation of the objective Eulerian 2nd-order tensor quantity from the background to the corotating frame

$$\mathbf{A}^{*} = \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^{\mathrm{T}}, \tag{35}$$

while the material time derivative of the transformed quantity $\mathbf{A}^{*}$ in the corotating frame turns into

$$\dot{\mathbf{A}}^{*} = \mathbf{Q} \cdot \overset{\circ}{\mathbf{A}} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{36}$$

One comes to the conclusion that the corotational rate of the objective Eulerian tensor **A** corresponds to the material rate of **A** in the corotating frame (cf. Xiao et al., 1998a). The last relation does not hold for tensors that are not objective.

Different corotational rates can be obtained for different choices of the spin tensor **Q**. Although the chosen spin relation (36) satisfies transformation rule $(17)_3$, the corresponding corotational rate does not need to be an objective quantity. That means that the most important demand of objectivity of the corotational rate (cf. Truesdell et al., 2004) may be violated. Bearing in mind the fact that the objective corotational rate has the crucial importance in a material behaviour description, especially of an inelastic behaviour, the significance of the proper choice of the objective rate and their defining spin tensors is again pointed out.

For different choices of a single antisymmetric real function, named a *spin function*, Xiao et al. in (1998a) and (1998b) define a general class of spin tensors and corresponding general class of objective corotational rates. In the spin function $h(z)$, z represents the ratio between n distinct eigenvalues of the left and right Cauchy-Green tensors, where $n \le 3$ (for detailed definition of the spin function see Xiao el al. 1998b).

The choice of the spin function as

$$h(z) = h^{\mathrm{J}}(z) = 0 \tag{37}$$

yields the spin tensor

$$\boldsymbol{\Omega}^{\mathrm{J}} = \mathbf{W}, \tag{38}$$

which, implemented in (34), defines the well-known *Zaremba-Jaumann* rate

$$\overset{\circ}{\mathbf{A}}{}^{\mathrm{J}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \mathbf{W} - \mathbf{W} \cdot \mathbf{A}. \tag{39}$$

The Jaumann rate was the first introduced in the rate formulation of inelastic material behaviour. Since it can be easily implemented and numerical calculations are not so time-consuming it has been widely accepted and used. It is, as well, incorporated in several commercial finite element codes. Implementation of the Jaumann rate in constitutive theories gives appropriate results for the case of small deformations. However, this rate may not be an adequate choice for the case of finite deformations (Lehmann, 1972; Dienes, 1979; Simo & Pister, 1984; Khan & Huang, 1995; Bažant & Vorel, 2014). It has been shown that for the pure elastic deformation constitutive model based on the Jaumann rate gives an unstable response at simple shear, known as *shear oscillatory phenomenon.*

In order to overcome the deficiencies encountered with the Jaumann rate implementation in finite deformation formulation, numerous alternative corotational rates have been developed (cf. Xiao et al., 2000).

One of them is the *polar* or *Green-Naghdi* rate. If the spin function takes the form

$$h(z) = h^{\mathrm{R}}(z) = \frac{1-\sqrt{z}}{1+\sqrt{z}}, \tag{40}$$

the *polar spin* $\mathbf{\Omega}^{\mathrm{R}}$ will be obtained

$$\mathbf{\Omega}^{\mathrm{R}} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}}, \tag{41}$$

which, substituted in (34), defines the *Green-Naghdi rate*

$$\overset{\circ}{\mathbf{A}}{}^{\mathrm{GN}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \mathbf{\Omega}^{\mathrm{R}} - \mathbf{\Omega}^{\mathrm{R}} \cdot \mathbf{A}. \tag{42}$$

Even though the introduction of the aforementioned rates solves the problem of the unrealistic harmonic stress responses obtained for the Jaumann rate, Simo & Pister (1984) have showed that for the case of pure elastic deformation, where the recoverable elastic-like behaviour was expected, none of the constitutive relations, based on formerly used objective rates, can fulfil Bernstein's integrability condition to give an elastic relation, i.e. path-dependent and dissipative processes are detected for all rates.

Recently the new spin function, so-called *logarithmic spin function,* has been introduced as

$$h(z) = h^{\mathrm{R}}(z) = \frac{1+\sqrt{z}}{1-\sqrt{z}} + \frac{2}{\ln(z)}. \tag{43}$$

It leads to the *logarithmic spin tensor*

$$\mathbf{\Omega}^{\mathrm{Log}} = \dot{\mathbf{R}}^{\mathrm{Log}} \cdot (\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}, \tag{44}$$

whose implementation in (34) gives the *logarithmic corotational rate* or *Log-rate* of **A**

$$\overset{\circ}{\mathbf{A}}{}^{\mathrm{Log}} = \dot{\mathbf{A}} + \mathbf{A} \cdot \mathbf{\Omega}^{\mathrm{Log}} - \mathbf{\Omega}^{\mathrm{Log}} \cdot \mathbf{A}. \tag{45}$$

The previously defined symmetric stretching tensor **D** is a natural characterization of the rate of change of the local deformation state, thus we want to present it as a direct flux of a strain measure. What is more, we are interested in finding out which Eulerian strain measure **e** and which corotational time derivative satisfy the following relation

$$\overset{\circ}{\mathbf{e}} = \dot{\mathbf{e}} + \mathbf{e}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\mathbf{e} = \mathbf{D}, \tag{46}$$

where with **e** a general Eulerian strain measure has been designated.

In Xiao et al. (1997) and (1998a) the authors have proved that among all strain measures and among all corotational rates the spatial logarithmic strain measure **h** and the logarithmic rate are the unique choice that satisfies the above demand, i.e.

$$\overset{\circ}{\mathbf{h}}^{\mathrm{Log}} = \dot{\mathbf{h}} + \mathbf{h}\cdot\boldsymbol{\Omega}^{\mathrm{Log}} - \boldsymbol{\Omega}^{\mathrm{Log}}\cdot\mathbf{h} = \mathbf{D}. \tag{47}$$

In (44) it can be recognized that the logarithmic spin is determined by the proper orthogonal *logarithmic rotation* tensor $\mathbf{R}^{\mathrm{Log}}$ that is defined by the linear tensorial differential equation

$$\dot{\mathbf{R}}^{\mathrm{Log}} = -\mathbf{R}^{\mathrm{Log}}\cdot\boldsymbol{\Omega}^{\mathrm{Log}}, \qquad \mathbf{R}^{\mathrm{Log}}\big|_{t=0} = \mathbf{1}. \tag{48}$$

The corotating frame obtained from the background frame by the rotation $\mathbf{R}^{\mathrm{Log}}$ is named a *logarithmic corotating frame*. The material time derivative of an objective Eulerian quantity **A**, in the logarithmic corotating frame, is exactly the logarithmic rate of the same quantity, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}^{\mathrm{Log}}\cdot\mathbf{A}\cdot(\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}) = \mathbf{R}^{\mathrm{Log}}\cdot\overset{\circ}{\mathbf{A}}^{\mathrm{Log}}\cdot(\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}. \tag{49}$$

Applying the last assertion to **h** and the stretching **D**, the following relation will be obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}^{\mathrm{Log}}\cdot\mathbf{h}\cdot(\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}) = \mathbf{R}^{\mathrm{Log}}\cdot\mathbf{D}\cdot(\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}, \tag{50}$$

that is, *in the logarithmic corotating frame stretching* **D** *is a true time rate of* **h**. Integration of (49) leads to the corotational integration

$$\mathbf{A} = (\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}\cdot\int_t \mathbf{R}^{\mathrm{Log}}\cdot\overset{\circ}{\mathbf{A}}^{\mathrm{Log}}\cdot(\mathbf{R}^{\mathrm{Log}})^{\mathrm{T}}\,\mathrm{d}t\ \cdot\mathbf{R}^{\mathrm{Log}}, \tag{51}$$

and it is performed in the logarithmic corotating frame. The last relation is a particular form of the generalised objective time integration given by (26). This relation will be of tremendous importance in numerical calculations.

Following the theoretical postulates given in Bruhns et al. (1999) in Trajković-Milenković (2016) it is numerically proved that the implementation of the Log-rate in the constitutive relations of finite elastoplasticity successfully solved the aforementioned problems observed with all other corotational and non-corotational rates. On several benchmark examples it has been proved that the material behaviour prediction based on the logarithmic rate is stable and completely in accordance with experimental tests, in opposition to actually popular objective rates.

Unlike to the Jaumann rate and the Green-Naghdi rate, which are already incorporated in the commercial finite element codes through the built-in subroutines and their application is completely optimized, the logarithmic rate has to be implemented in the software using the special user subroutine which allows user to define his own material model. This kind of subroutine offers the user a lot of possibilities but it will be called in each time step and for each integration point and, therefore, the numerical calculation can be time-consuming, which could be from the numerical point of view the only disadvantage of using the logarithmic rate.

4. Decomposition of Finite Deformation

Elastoplasticity represents a combination of two completely different types of material behaviour, namely elasticity and plasticity. A great number of the modern theories of elastoplasticity are confined to the description of a rate-independent behaviour of elastoplastic materials, i.e. viscous effects are ignored. Before 1960, most contributions in the rate-independent elastoplasticity theory were dedicated to the field of small deformations. Some of the basic ideas of the theories for small deformations can be fully or partially applied for the case where finite deformations are occurring (cf. Naghdi, 1990, and Xiao et al., 2006).

One of those ideas is the composite structure of elastoplasticity. It means that a total deformation, or total deformation rate, of elastoplastic material can be decomposed into its *elastic*, or reversible, and *plastic*, or irreversible, part and then a separate constitutive relation for each part has to be established. Taking into consideration the incremental essence of elastoplastic behaviour of material, we are more interested in the strain rate than the strain itself. The rate of infinitesimal strain $\boldsymbol{\varepsilon}$ can be additively decomposed in the following form:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p. \tag{52}$$

Even though the researchers agree with the aforementioned statement for the case of small deformations, the decomposition of finite deformation into its reversible and irreversible part causes the disagreement within the members of the plasticity community dividing them into several various schools of plasticity.

The first belongs to the group which follows the idea that the classical Prandtl and Reuss formulation (52) can be extended to the finite deformation description using the *additive decomposition of the natural deformation rate* to its elastic and plastic part, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \tag{53}$$

It has been thought that the above decomposition of the stretching can hold only for certain restrictive cases of deformation and materials, such as small elastic and finite plastic deformations in metals (Simo & Hughes, 1998). Such an opinion is coming from the fact that in order to fulfil the objectivity requirement the rate type model must involve the objective rate, instead of the material time derivative, and the use of the Jaumann and some other well-known objective rates in the context of decomposition (53) produce irregular results, such as the shear oscillation phenomenon and the residual stresses occurrence for elastic closed strain path as well as non-integrability problem (see Trajković-Milenković, 2016, for details and relevant references). Thus, this decomposition has been rejected for a long time as inappropriate for a general purpose. This held true until recently when an implementation of the newly discovered logarithmic rate solved the existing problems (Xiao et al., 1997b, and Bruhns et al., 1999) which has been also numerically proved in Trajković-Milenković (2016).

The second approach is the most common in the finite deformation theories; that is the *multiplicative decomposition of the deformation gradient*, given by

$$\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p, \tag{54}$$

where $\mathbf{F}^e$ and $\mathbf{F}^p$ represent elastic and plastic part of the deformation gradient, respectively.

An *intermediate* stress-free configuration achieved by an elastic unloading from the current configuration is introduced by this formulation. Therefore, while the mapping from the reference to the current configuration is described by the deformation gradient **F**, the mapping from the reference to the intermediate configuration can be described by its plastic part $\mathbf{F}^p$ and from the intermediate to the current configuration by its elastic part $\mathbf{F}^e$. An arbitrary rigid body rotation **Q** superimposed on the intermediate configuration has no influence on the decomposition (54), thus the determination of the elastic and plastic part of deformation gradient is not unique, i.e.

$$\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p = \overline{\mathbf{F}}^e \cdot \overline{\mathbf{F}}^p, \qquad \text{with} \qquad \overline{\mathbf{F}}^e = \mathbf{F}^e \cdot \mathbf{Q} \qquad \text{and} \qquad \overline{\mathbf{F}}^p = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{F}^p. \tag{55}$$

In addition, according to Naghdi (1990), multiplicative decomposition (54) has several shortcomings. The first "lies in the fact that the stress at a point in an elastic-plastic material can be reduced to zero without changing plastic strain only if the origin in stress space remains in the region enclosed by the yield surface." This is usually not the case, except of some special cases such as isotropic hardening. However, it is observed, that the yield surface may move in the stress space during deformation. Furthermore, "even if the stress can be reduced to zero at each material point, the resulting configuration will not, in general, form a configuration for the body as a whole, but only a collection of local configurations."

A further physically admissible decomposition is established by Green & Naghdi (1965) postulating the *additive decomposition of the Green strain*, the Lagrangian strain measure. The authors introduce a strain-like variable of Lagrangian type, called plastic strain $\mathbf{E}^p$, as a primitive variable. The total Green strain is decomposed using the form

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p, \tag{56}$$

where only for the case of small deformations $\mathbf{E}^e$ can be denoted as elastic strain. Having well understood the limited applicability of the additive separation of **E** for large deformations, they do not interpret the difference $\mathbf{E} - \mathbf{E}^p$ as an elastic strain or part, but as an alternative convenient variable used for well motivated purposes.

5. CONCLUSION

Since the introduction of the logarithmic rate has successfully eliminated the aforestated shortcomings of the proposed decomposition of finite deformation, our recommendation would be the implementation of the self-consistent Eulerian finite elastoplasticity theory, based on the logarithmic rate and the additive decomposition of the natural deformation rate. As well, it will be the base of our future work. The implementation of the proposed theory into commercial finite element codes will be used for numerical calculations of homogeneous and non-homogeneous problems in order to show the advantage of the suggested approach. The obtained results will later be published elsewhere.

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ALTERNATIVNI PRISTUP KONAČNIM DEFORMACIJAMA

U savremenoj formulaciji elastoplastičnog ponašanja materijala konstitutivne relacije su uglavnom date u formi izvoda, tj. predstavljaju vezu između izvoda napona i izvoda deformacije kako u formulaciji plastičnog tako i elastičnog dela deformacije. Usvojene konstitutivne relacije moraju ostati nezavisne u odnosu na promenu koordinatnog sistema, tj. da ostanu objektivne. Dok je preduslov objektivnosti u materijalnoj deskripciji automatski zadovoljen, u Ojlerovoj deskripciji, posebno u slučaju velikih deformacija, objektivnost može biti narušena čak i za objektivne promenljive. Stoga, umesto materijalnog izvoda, u konstitutivnim relacijama datim u Ojlerovoj deskripciji moraju se implementirati objektivni izvodi. Ovaj rad doprinosi pojašnjenju važnosti implementacije objektivnih izvoda u konstitutivnim relacijama konačne elastoplastičnosti i daje pregled danas najčešće korišćenih objektivnih izvoda, njihovih prednosti i nedostataka, kao i izuzetnih karakteristika nedavno uvedenog logaritamskog izvoda.

Ključne reči: *konačne deformacije, objektivni izvodi, logaritamski izvod, elastoplastičnost, dekompozicija velikih deformacija*