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LINEAR RECURRENCE RELATONS AND ORDINARY GENERATING FUNCTIONS APPLIED ON MODELING PROCESSES IN CONTROL THEORY

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Abstract. *In this paper we apply multistep recurrence relations, as one of very simple and useful mathematical models. It is an efficient tool for solving many problems in mathematics, science, and technics. We also use generating functions, as a connection between real number sequences and real functions, and as a very smooth and efficient connection between the discrete mathematics and (continual) mathematical analysis. We present an application of multistep homogenous linear recurrence relations for modelling some processes in the control theory. Further on, we use the ordinary generating function aiming to find appropriate formulae for calculating members of an appropriate recurrence sequence. Finally, we show the application of this novel mathematical approach on one real example in the control theory.*

Key words: *Recurrence relation, generating function, control theory*

1. INTRODUCTION

There are a lot of problems and processes in mathematics, science, technology and other fields, where the recurrence relation is the most appropriate mathematical model for describing it (see [1]-[3]). It is sufficient that some problem or process can be described

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or valued with some sequence of real or complex numbers, for example values calculated, or sampled or measured in discrete moments in time, and we can use such a mathematical model. Those values must be mutually connected via some relation (see [4]-[6]), expressed with an explicit mathematical formula. This relation usually involves $k(k \in N)$ consecutive members of a real sequence, and it can be used for calculating the next value in a sequence, based on k previous values (see [6]-[8]).

A typical problem with recurrence relations is to determine an explicit formula for calculation of any member of sequence $a_n = a(n)$ (see [4]-[6]). The most common approach is using the characteristic equation of a given recurrence relation, which is precise, but not always easy to implement and not appropriate enough for the algorithmic approach and for programming.

In [10], we already introduced and showed another approach, using the ordinary generating function for a sequence of numbers (see [4], [9], [11]-[13]).

2. Mathematical Background

The recurrence relation for some real or complex sequence of numbers $(a_n)_{n\in N}$ is a mathematical term, given with

$$F(a_{n+k},\ a_{n+k-1},\ a_{n+k-2},\dots,\ a_n) = 0, \tag{1}$$

(see, for example, [1], [6]) which is the relation in an implicit form, or

$$a_{n+k} = f\left(a_{n+k-1}, a_{n+k-2}, \dots, a_n\right), \tag{2}$$

which is the relation in an explicit form. In both formulas, n are index of this sequence and k is order of this relation. If mappings F in (1) and f in (2) are linear, then we have a linear recurrence relation. Without loss of generality, as in [10], in the rest of this research paper, we will consider only linear recurrence relations (see [1], [5]- [6]).

The algorithm for calculating members of the real sequence is shown on Fig. 1:

```
read (a0, a1… ak-1);
for (i=k, i<=n, i++)
{ai = f(ai-1, ai-2, …, ai-k)=0;     % find next sequence member
  write (ai);
  for (j=0, j<i, j++)
       aj = aj+1;                    % relocate sequence members
}
```

Fig. 1 Algorithm in meta language (see [13])

The recurrence relation is a very useful model in both cases, for real and for complex sequences. In this paper, without loss of generality, we will consider only real sequences.

In order to show our method and main results, also without loss of generality, we will use recurrence relations with small values $k = 2$ (two-step recurrence relation) or $k = 3$ (three-step recurrence relation). For a successful use of the recurrence relation, it is necessary to have k starting values, i.e., to know values of $a_0, a_1, \dots a_{k-1}$.

The most common approach for obtaining an explicit formula for calculating members of a recurrence relation is **by using the characteristic equation** of a given recurrence relation. For example, if a linear recurrence relation

$$\beta_{n+k}a_{n+k} + \beta_{n+k-1}a_{n+k-1} + \beta_{n+k-2}a_{n+k-2} + \ldots + \beta_n a_n = 0, \quad (3)$$

with starting conditions

$$a_0 = A_0,\ a_1 = A_1,\ \ldots\ a_{k-1} = A_{k-1},\ (A_0,\ A_1,\ \ldots\ A_{k-1} \in R), \quad (4)$$

is given, characteristic equation is

$$\beta_{n+k}\lambda^{n+k} + \beta_{n+k-1}\lambda^{n+k-1} + \beta_{n+k-2}\lambda^{n+k-2} + \ldots + \beta_n \lambda^n = 0,$$

i.e.,

$$\beta_{n+k}\lambda^{k} + \beta_{n+k-1}\lambda^{k-1} + \beta_{n+k-2}\lambda^{k-2} + \ldots + \beta_n = 0,$$

Solutions of this algebraic equation are real or complex numbers $\lambda_1, \lambda_2, \ldots \lambda_k$ and the formula for calculating any member of this sequence have a form

$$a_n = C_1\lambda_1{}^n + C_2\lambda_2{}^n + \ldots C_k\lambda_k{}^n. \quad (5)$$

Constants $C_1, C_2, \ldots C_k$ can be obtained from starting conditions (4).

Another approach (see [10]) to a problem of obtaining the formula for calculation of members of the recurrence sequence is **by using the ordinary generating function**, defined with (see [6], [9], [12])

$$F(t) = \sum_{n=0}^{\infty} a_n \cdot t^n .$$

For a linear recurrence relation (3), we can obtain ordinary generating function using following steps

$$\beta_{n+k}a_{n+k} + \beta_{n+k-1}a_{n+k-1} + \beta_{n+k-2}a_{n+k-2} + \ldots + \beta_n a_n = 0, \quad \Big/\ t^{n+k}$$

$$\beta_{n+k}a_{n+k}t^{n+k} + \beta_{n+k-1}a_{n+k-1}t^{n+k} + \beta_{n+k-2}a_{n+k-2}t^{n+k} + \ldots + \beta_n a_n t^{n+k} = 0, \quad \Big/\ \sum_{n=0}^{\infty}$$

$$\beta_{n+k}\sum_{n=0}^{\infty}a_{n+k}t^{n+k} + \beta_{n+k-1}\sum_{n=0}^{\infty}a_{n+k-1}t^{n+k} + \beta_{n+k-2}\sum_{n=0}^{\infty}a_{n+k-2}t^{n+k} + \ldots + \beta_n\sum_{n=0}^{\infty}a_n t^{n+k} = 0,$$

Then we have

$$\beta_{n+k}\sum_{n=0}^{\infty}a_{n+k}t^{n+k} + \beta_{n+k-1}t\sum_{n=0}^{\infty}a_{n+k-1}t^{n+k-1} + \beta_{n+k-2}t^2\sum_{n=0}^{\infty}a_{n+k-2}t^{n+k-2} + \ldots + \beta_n t^k\sum_{n=0}^{\infty}a_n t^n = 0,$$

$$\beta_{n+k}\sum_{n=k}^{\infty}a_n t^n + \beta_{n+k-1}t\sum_{n=k-1}^{\infty}a_n t^n + \beta_{n+k-2}t^2\sum_{n=k-2}^{\infty}a_n t^n + \ldots + \beta_n t^k\sum_{n=0}^{\infty}a_n t^n = 0,$$

$$\beta_{n+k}(F(t) - a_0 \ldots - a_{k-1}t^{k-1}) + \beta_{n+k-1}t(F(t) - a_0 \ldots - a_{k-1}t^{k-2}) + \ldots + \beta_n t^k F(t) = 0,$$

$$F(t)(\beta_{n+k}+\beta_{n+k-1}t+\ldots+\beta_n t^k)=\beta_{n+k}(a_0\ldots+a_{k-1}t^{k-1})+\ldots+\beta_{n+1}t^{k-1}a_0,$$

and the generating function is

$$F(t)=\frac{\beta_{n+k}(a_0\ldots+a_{k-1}t^{k-1})+\beta_{n+k-1}t(a_0\ldots+a_{k-2}t^{k-2})+\ldots+\beta_{n+1}t^{k-1}a_0}{\beta_{n+k}+\beta_{n+k-1}t+\ldots+\beta_n t^k},$$

where starting conditions are $a_0=A_0,\ a_1=A_1,\ \ldots\ a_{k-1}=A_{k-1},\ (A_0,\ A_1,\ \ldots\ A_{k-1}\in R)$. This function is rational and after the decomposition on simple fractions, using well-known summation of geometric series

$$\sum_{n=0}^{\infty}t^n=\frac{1}{1-t}, \qquad (6)$$

we can obtain an explicit formula for calculating any member of this sequence, that will have the same form like in (5) (see [10]).

3. METHOD

There are various systems and processes in control theory (as well as in other areas and fields), that we usually divide into two main groups: continual systems and discrete systems. We should also notice a big difference between discrete systems (discrete by their nature) and discretized models of continual systems (see [14]). Anyway, the mathematical apparatus for analysis of discrete systems and analysis of discretized systems are the same. Mathematical models of continual systems are discretized when applying a discrete control [14]. The identification of the process and discretization can be done in two ways: by applying some method for identification of continual processes, and then to discrete it; or to discrete continual model during identification of process. Discrete processes are identified to mathematical models directly. Discrete processes in industry are, in fact, very rare. Their discrete character is a consequence of some discretization mechanism, for example by embedding some discrete measure instruments [14]. The result of such identification is the linear discrete mathematical model

$$y(n)=-\sum_{i=1}^{n}a_i\cdot y(k-1)+\sum_{i=1}^{n}b_i\cdot x(k-1-m)+\eta(k)$$

This model is with noise $\eta(k)$ and delays, where m is the number of delay cycles. Further on, we will deal with processes and systems in an ideal case, so models will be without noise and without delay, in the simplified form

$$y(n)=\sum_{i=1}^{n}a_i\cdot y(k-1)\ .$$

Suppose that we have some process (P) and that this process is described with some real values, calculated in discrete moments of time. Let those values be states and/or outputs of this process. This process can be shown in a form of diagram (see Fig. 2).

State and output of this model in each moment is described with a number a_n, $n=k$, where $k=0,\ 1,\ 2...$ in particular successive moments t_k. Each subsequent state is determined by k previous states and mathematical model of this process are described with some relation

$$a_n = f\left(a_{n-1}, a_{n-2}, \ldots, a_{n-k}\right), \tag{7}$$

with starting conditions (4), that is of same type as relation (2).

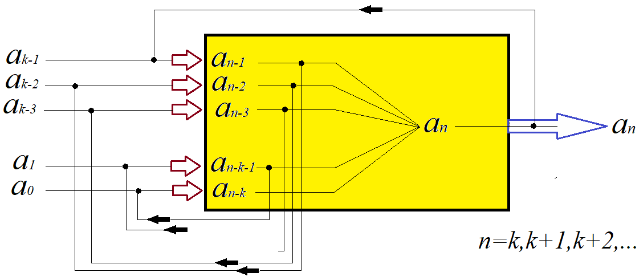

Fig. 2 Block diagram for system characterized with relation (7)

Without loss of generality, we will take $k = 2$, aiming to have the simpler form of recurrence relation, so called "two-step recurrence relation".

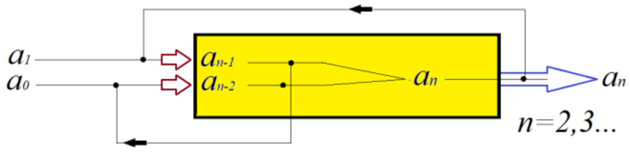

Fig. 3 Block diagram for system characterized with relation with $k=2$

Mathematical model of this process is

$$a_n = f(a_{n-1}, a_{n-2}), a_0 = A_0,\ a_1 = A_1 (A_0, A_1 \in R),$$

or with "shifted" indexes,

$$a_{n+2} = f(a_{n+1}, a_n), a_0 = A_0,\ a_1 = A_1 (A_0, A_1 \in R),$$

or in the implicit form

$$\beta_{n+2} a_{n+2} + \beta_{n+1} a_{n+1} + \beta_n a_n = 0, a_0 = A_0,\ a_1 = A_1 (A_0, A_1 \in R),$$

For a linear recurrence relation (7), we can obtain an ordinary generating function using following steps

$$\beta_{n+2} a_{n+2} + \beta_{n+1} a_{n+1} + \beta_n a_n = 0, \qquad \Big/\ t^{n+2}$$

$$\beta_{n+2} a_{n+2} t^{n+2} + \beta_{n+1} a_{n+1} t^{n+2} + \beta_n a_n t^{n+2} = 0, \qquad \Big/\ \sum_{n=0}^{\infty}$$

$$\beta_{n+2} \sum_{n=0}^{\infty} a_{n+2} t^{n+2} + \beta_{n+1} \sum_{n=0}^{\infty} a_{n+1} t^{n+2} + \beta_n \sum_{n=0}^{\infty} a_n t^{n+2} = 0,$$

Then we have

$$\beta_{n+2}\sum_{n=0}^{\infty} a_{n+2}t^{n+2} + \beta_{n+1}t\sum_{n=0}^{\infty} a_{n+1}t^{n+1} + \beta_n t^2\sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\beta_{n+2}\sum_{n=2}^{\infty} a_n t^n + \beta_{n+1}t\sum_{n=1}^{\infty} a_n t^n + \beta_n t^2\sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\beta_{n+2}(F(t) - a_0 - a_1 t) + \beta_{n+1}t(F(t) - a_0) + \beta_n t^2 F(t) = 0,$$

$$F(t)(\beta_{n+2} + \beta_{n+1}t + \beta_n t^2) = \beta_{n+2}(a_0 + a_1 t) + \beta_{n+1}ta_0,$$

and generating function is

$$F(t) = \frac{\beta_{n+2}(a_0 + a_1 t) + \beta_{n+1}ta_0}{\beta_{n+2} + \beta_{n+1}t + \beta_n t^2}, \tag{8}$$

where starting conditions are $a_0 = A_0,\ a_1 = A_1, (A_0,\ A_1 \in R)$.

This function is rational and after the decomposition on simple fractions, using (6), we have *F*(*t*) in an explicit form. We will assume, without loss of generality, that poles of function (8) are real (not complex) and unique, and that all coefficients in (7) are real. Other cases could be a topic for some further research.

4. Application

In order to validate the proposed method, G.U.N.T. Flow Control Trainer RT522 (Fig. 4) is a comprehensive structure equipped with modern industrial components. The pump delivers water from the tank through a piping system. The fluid flow is measured using an electromagnetic sensor, which allows further processing of the measured quantity by giving a standardized current signal at the output. The flow indicator is a rotometer. An industrial digital controller is used for control. The actuator, connected in a closed non-return loop, is an electromotive valve. The manual ball valve, allows defining the disturbances that are introduced into the system. The controlled parameter K_s and the size to be manipulated and written directly to the two-channel line recorder.

The system also contains management software (RT650.50) connected to a computer. The tank has a capacity of 30*l*, the centrifugal pump has a power of 250*V* with a maximum flow rate of 150 *l/min* and a speed of 2800 *rpm*. The maximum flow rate of the electromagnetic sensor is 6000 *l/h*. The control cabinet contains a power switch, a safety STOP button, a pump start button, a control terminal for monitoring output variables and manual control of the system. The control cabinet also has a printer that automatically prints the values of the output variables. A closer view of the entire system with the tank is shown in Fig. 5. The system is connected via a computer, which supports the *LabView* software package, and with auxiliary applications it is possible to set, control and monitor the operation of the system.

Fig. 4 G.U.N.T. Flow Control Trainer RT522

Fig. 5 A closer view of the entire system

Several experiments have been performed to identify the system in the form of a transfer function. The idle system is excited by setting the desired system response to a value of 1400 *l/h*. Experimental results are given in Fig. 6.

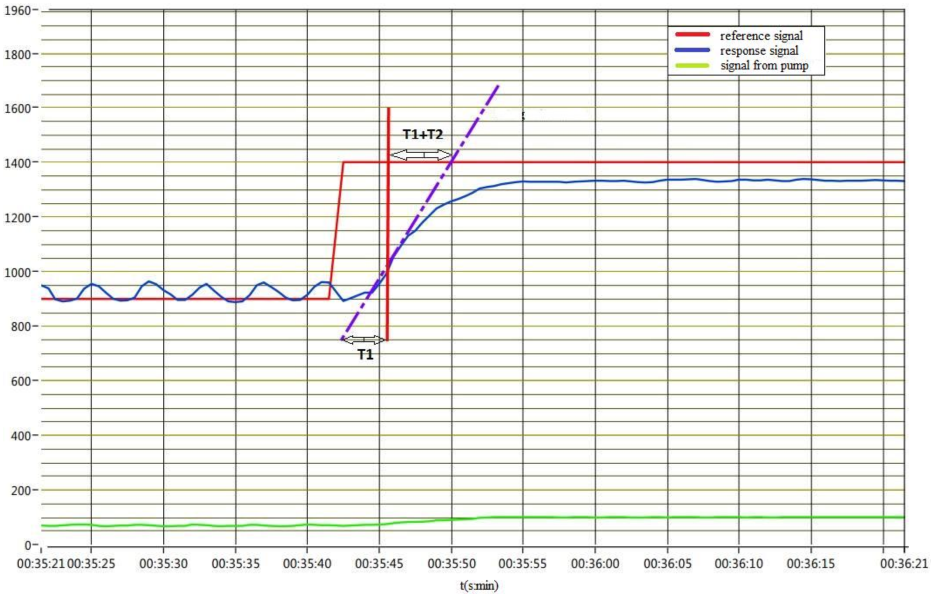

Fig. 6 Experimental results

Using the graph-analytical method [14] of the bounce response of the system for the transfer function, the following function was obtained:

$$W(s) = \frac{1}{5.20386s^2 + 4.59700s + 1}$$
$$5.20386s^2 Y(s) + 4.59700sY(s) + Y(s) = X(s)$$

which leads us to a recurrence relation

$$5.20386a_{n+2} + 4.59700a_{n+1} + a_n = 1. \tag{9}$$

Now we will apply the algorithm introduced in [10]. Having in mind that tanks are empty at the beginning, we will assume that $a_0 = 0$, $a_1 = 0$, as start conditions. Although our problem is non-homogenous the linear recurrence relation (right side of equation (9) is not equal to 0), we can apply same method as with homogenous relations, to obtain an explicit formula for a_n.

$$5.20386a_{n+2} + 4.59700a_{n+1} + a_n = 1 \quad / \, t^{n+2}$$
$$5.20386a_{n+2}t^{n+2} + 4.59700a_{n+1}t^{n+2} + a_n t^{n+2} = t^{n+2} \quad / \sum_{n=0}^{\infty}$$

and we have

$$5.20386\sum_{n=0}^{\infty} a_{n+2}t^{n+2} + 4.59700\sum_{n=0}^{\infty} a_{n+1}t^{n+2} + \sum_{n=0}^{\infty} a_n t^{n+2} = \sum_{n=0}^{\infty} t^{n+2}$$
$$5.20386\sum_{n=0}^{\infty} a_{n+2}t^{n+2} + 4.59700t\sum_{n=0}^{\infty} a_{n+1}t^{n+1} + t^2\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} t^{n+2}$$
$$5.20386\sum_{n=2}^{\infty} a_n t^n + 4.59700t\sum_{n=1}^{\infty} a_n t^n + t^2\sum_{n=0}^{\infty} a_n t^n = t^2\sum_{n=0}^{\infty} t^n$$
$$5.20386F(t) + 4.59700tF(t) + t^2F(t) = t^2\,\frac{1}{1-t}$$
$$F(t)(5.20386 + 4.59700\,t + t^2) = \frac{t^2}{1-t},$$

so ordinary generating function is

$$F(t) = \frac{t^2}{(1-t)(5.20386 + 4.59700t + t^2)} = \frac{-1}{5.20386}\cdot\frac{t^2}{(1-t)(0.38759+t)(0.49579-t)}$$

Suppose that we can make decomposition

$$F(t) = \frac{A}{1-t} + \frac{B}{0.38759+t} + \frac{C}{0.495787-t},$$

then we will come to the system of linear equations

$$\begin{cases} -A+B-C=-0.19216, \\ 0.10820A-1.49579B+0.61241C=0 , \\ 0.19216A+0.49579B+0.38759C=0, \end{cases}$$

with solutions $A = 0.27466$, $B = 0.02355$ and $C = 0.10605$. So, we have

$$F(t) = \frac{0.27466}{1-t} + \frac{0.02355}{0.38759+t} + \frac{0.10605}{0.495787-t},$$

$$F(t) = \frac{0.27466}{1-t} + \frac{0.06076}{\left(1-\frac{-t}{0.38759}\right)} + \frac{0.21390}{\left(1-\frac{t}{0.49579}\right)}.$$

Generating function is

$$F(t) = 0.27466 \sum_{n=0}^{\infty} t^n + 0.06076 \sum_{n=0}^{\infty} \left(\frac{-t}{0.38759}\right)^n + 0.21390 \sum_{n=0}^{\infty} \left(\frac{t}{0.49579}\right)^n,$$

$$F(t) = \sum_{n=0}^{\infty} \left(0.27466 + \frac{0.06076\cdot(-1)^n}{0.38759^n} + \frac{0.21390}{0.49579^n}\right) t^n .$$

Explicit formula for calculating numbers of real sequence a_n is

$$a_n = 0.27466 + \frac{0.06076\cdot(-1)^n}{0.38759^n} + \frac{0.21390}{0.49579^n} .$$

We can use the obtained formula for the calculation of any member of this real sequence, i.e., we can calculate value that characterizes our system.

5. CONCLUSION

Multistep recurrence relation is one of the useful mathematical models and also a very simple tool for many problems in mathematics, science and technology. So, there is a possibility to apply multistep linear recurrence relations for modelling problems in the control theory, what is the goal of this paper. An ordinary generating function of real sequence is used, in order to obtain formulae for calculating members of a sequence. Generating functions are just one of mathematical tools for the connection between real number sequences and real functions.

The main purpose of our method is to obtain function, expressed with an explicit formula (continual model) that represents a recurrence sequence of real numbers, which is a problem with a completely discrete nature. This method is some kind of a “D2C” (discrete-to-continual) smooth transformation.

This approach, which is here applied on some simple problems in the control theory, is just an introduction into a wide variety of possible applications for solving problems from other fields (computer science, economy, biology, digital signal processing…), that have similar mathematical properties. It also opens wide new frontiers for further research in this field of applied mathematics.

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