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NOVEL EXPONENTIAL TYPE APPROXIMATIONS OF THE *Q*-FUNCTION

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Abstract. *In this paper, we propose several solutions for approximating the Q-function using one exponential function or the sum of two exponential functions. As the novel Q-function approximations have simple analytical forms and are therefore very suitable for further derivation of expressions in closed forms, a large number of applications are feasible. The application of the novel exponential type approximations of the Q-function is especially important for overcoming issues arising in designing scalar companding quantizers for the Gaussian source, which are caused by the non-existence of a closed form expression for the Q-function. Since our approximations of the Q-function have simple analytical forms and are more accurate than the approximations of the Q-function previously used for the observed problem in the scalar companding quantization of the Gaussian source, their application, especially for this problem is of great importance.*

Key words: *Gaussian source, Q-function, exponential type approximations*

1. INTRODUCTION

Estimating the performance of digital communication systems, typically, signal to quantization noise ratio (SQNR) and the symbol error probability (SEP), plays a very important role in designing these systems. Performance evaluation of the average SEP of digital modulations in additive white Gaussian noise as well as fading channels and performance (distortion and SQNR) evaluation of scalar quantizers for a Gaussian source includes calculation of the improper integral whose solution cannot be obtained in a closed form [1]-[3]. Namely, in these calculations, the Gaussian *Q*-function and the erfc function, given by [4]:

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$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt = 1/2 \,\mathrm{erfc}(x/\sqrt{2}), \tag{1}$$

cannot be expressed in the form of elementary integrals. Motivated by this fact, numerous researches have been conducted with the aim of determining approximations of the *Q*-function having simple analytical forms and preferably high accuracies.

The infinite upper limit of the integral given in Eq. (1) poses the restriction on the application of numerical integration methods. The presence of the *Q*-function argument as the lower limit of the integral causes additional difficulties in analytical derivations in which this argument depends on some other random parameters so that statistical averaging is necessary [2]-[13]. For this reason, *Q*-function approximations that have simple analytical forms suitable for further derivations [5], [6], [13]-[17] are especially important. It is also preferable to determine the *Q*-function approximations of high approximation accuracies. However, from the requirement for simple analytical forms of the *Q*-function approximations, approximations of relatively low accuracies that is, of relatively high relative errors are usually outputted [13]. By taking into account these conflicting requirements, numerous approximations of the *Q*-function have been specified in literature, which, depending on the research goals and application, strive for a compromise between the simplicity of analytical form and accuracy (see for example, [2], [3], [5]-[26]).

In general, absolute and relative error (*Relative Eror*-RE) of approximating the *Q*-function with $F(x)$, is calculated from:

$$\Delta_Q(x) \triangleq \left|F(x) - Q(x)\right|, \tag{2}$$

$$\delta_Q(x) \triangleq \frac{\left|F(x) - Q(x)\right|}{Q(x)}. \tag{3}$$

It is important to point out that some of the approximations of the *Q*-function provide small RE for small values of the argument, but high RE for large argument values, while for some other approximations of the *Q* function, the opposite conclusion holds. Due to the mentioned shortcomings of the *Q*-function approximations, this problem is still very current.

This interesting research topic is especially important in the scalar quantization of a memoryless Gaussian source, since, as shown in [18], [21], [27]-[37], to design and evaluate the performance of these quantizers, closed form expressions cannot be derived for the presumed Gaussian source. Therefore, analytically simple approximate expressions for the *Q*-function that do not degrade greatly the accuracy of the *Q*-function are very necessary. The authors of papers in the field of design and performance evaluation of scalar quantizers for a memoryless Gaussian source commonly use approximations of the *Q*-function from [15]. To apply the approximations of the *Q*-function having the simplest analytical forms, they utilize the approximations of the *Q*-function from [15] determined by a relatively small number of terms of the asymptotic expansion of the erfc function. Although relatively simple closed-form expressions for distortion and SQNR of scalar quantizers designed for a Gaussian memoryless source have been derived by using approximations from [15], the impact of the *Q*-function approximation accuracy on the performance of the considered quantizer has not been analyzed. This inspired the research presented in [18], in which, in addition to proposing a novel analytical form of the *Q*-function approximation, a very important analysis of the influence of the *Q*-function approximation accuracy on the assessment of Gaussian source scalar quantization performance is presented.

In [21], a particularly significant problem that arises in the design of scalar companding quantizers for the Gaussian source is pointed out, which has motivated numerous studies in this field (for instance, [28], [29], [32], [34], [36]). Namely, it was emphasized that in order to perform the companding quantization for the assumed Gaussian source, it is first necessary to perform a numerical integration, and then solve integral equations to design the considered model of the companding quantizer. To simplify the process of designing this model of quantizer, various methods of linearization of the compressor function have been proposed in the literature [28], [29], [32], [34], which enabled a much simpler design and performance evaluation by means of linearization. The authors of this paper, to the best of their knowledge, are the only ones to approach the solution of the observed problem by applying the *Q*-function approximations in determining approximate compressor functions, in the manner described in [21]. Namely, in [21], we came up with the idea to assume a novel class of the *Q*-function approximations that enables the derivation of expressions in closed form for representation levels and decision thresholds and to analyze the effects achieved by applying the *Q*-function approximation from the proposed class to approximate the compressor function. Since only the first step in this direction has been made in the research presented in [21], and approximations of the *Q*-function of simple analytical forms, but of relatively low accuracies, have been proposed, further progress in this research can justifiably be expected. For this reason, the aim of this paper is to determine some novel approximations of the *Q*-function of simple analytical forms, such as ones from [21], which will be more accurate compared to those from [21]. Finally, it should be noted that in [21] (see Eq. (12) from [21]), the description of the approximate compressor function has been given for an arbitrarily selected approximation of the *Q*-function, $F(x)$. Therefore, this paper references to [21], avoiding the repetition of the description of the compressor function and the way of its approximation by means of the *Q*-function approximations. In other words, the focus of this paper is on determining some novel improved solutions of the *Q*-function approximations in terms of accuracy, which have comparable or equal complexities of analytical forms as the approximations from [21].

The rest of the paper is structured as follows: First, a short description of the related work in the field of exponential type *Q*-function approximations is provided in Section 2. Afterwards, in Section 3, novel exponential type approximations of the *Q*-function are specified, and their performance are presented and discussed in Section 4. Finally, in Section 5, the conclusions of our results are derived.

2. Related Work

The group of exponential type approximations of the *Q*-function of relatively simple analytical forms encompasses approximations of the *Q*-function that can be represented by one exponential function or by the sum of two or more exponential functions [5], [6], [14], [16]. A special importance in this section should be given to the approximations from [5], which due to their simplicity are highly cited in literature. Namely, Chiani and Simon used the following form of the erfc function in [5] and [22]:

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_0^{\pi/2} \exp\left\{-\frac{x^2}{\sin^2\theta}\right\} d\theta,\ x \ge 0. \tag{4}$$

In particular, Craig came to this form of the erfc function in [9]. It is important to notice that unlike the standard definition of the erfc function, in which the argument of the function occurs as one of the integral limits, in the last expression, the argument of the erfc function occurs in the integrand. Simon utilized the fact that the integrand in Eq. (4) is monotonically increasing function of θ for $0 \le \theta \le \pi/2$ and he simply concluded that the upper bound of the erfc function can be determined from Eq. (4) by replacing the integrand with its maximum occurring at $\theta = \pi/2$:

$$\mathrm{erfc}(x) \le \frac{2}{\pi}\int_0^{\pi/2} \exp\{-x^2\}\, d\theta = \exp\{-x^2\},\ x \ge 0\,. \tag{5}$$

Chiani also considered significant this feature of the integrand from Eq. (4). Considering that it holds $0 \le \theta \le \pi/2$, he selected arbitrarily N+1 values for θ_0, θ_1,..., θ_N, arranged in the ascending order $0 = \theta_0 \le \theta_1 \le ... \le \theta_N = \pi/2$, and then showed that it holds:

$$\mathrm{erfc}(x) \le \frac{2}{\pi}\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i} \exp\left\{-\frac{x^2}{\sin^2\theta_i}\right\} d\theta\,. \tag{6}$$

With further introducing

$$a_i = \frac{2(\theta_i - \theta_{i-1})}{\pi}\,, \tag{7}$$

$$b_i = \frac{1}{\sin^2\theta_i}\,, \tag{8}$$

Eq.(6) was transformed into:

$$\mathrm{erfc}(x) \le \sum_{i=1}^{N} a_i \exp\{-b_i x^2\}\,. \tag{9}$$

Although a more accurate approximation of the upper bound of the erfc function is obtained by increasing the value of N, the complexity of the analytical form of the obtained approximation certainly increases. For this reason, in [5], only the following values for N were assumed: $N = 1$, $N = 2$, $N = 3$ and $N = 6$, and the resulting approximations of the Q-function were determined from Eqs. (6)-(8) for equispaced points $\theta_i = i\pi/(2N)$, i =1,2,...,N. Also, for $N = 2$, Chiani determined the following upper bound approximation of the erfc function in [5]:

$$\mathrm{erfc}(x) \simeq g(x, \theta_{\mathrm{opt}}) = \frac{1}{6}\exp\{-x^2\} + \frac{1}{2}\exp\left\{-\frac{4}{3}x^2\right\}, \tag{10}$$

that is, the following upper bound approximation of the Q-function:

$$\overline{F}^{[5]}(x) = \frac{1}{12}\exp\left\{-\frac{x^2}{2}\right\} + \frac{1}{4}\exp\left\{-\frac{2x^2}{3}\right\},\ x > 0.5\,. \tag{11}$$

Namely, he applied the trapezoidal rule of numerical integration for an arbitrarily chosen point θ from $0 \le \theta \le \pi/2$

$$\operatorname{erfc}(x) \simeq g(x,\theta) = \left(\frac{1}{2} - \frac{\theta}{\pi}\right)\exp\{-x^2\} + \frac{1}{2}\exp\left\{-\frac{x^2}{\sin^2\theta}\right\}, \tag{12}$$

and performed a numerical optimization procedure to determine the particular value of the parameter θ

$$\theta_{\text{opt}} = \arg\min_{\theta} \frac{1}{R}\int_0^R \frac{\left|g(x,\theta) - \operatorname{erfc}(x)\right|}{\operatorname{erfc}(x)}\,dx, \tag{13}$$

so that the minimum of the integral of the absolute relative error for the interval [0, *R*] was achieved, where he assumed $20\log_{10}R = 13$ dB. As a result, $\theta_{\text{opt}} \approx \pi/3$ was determined. With further substitution of this value for θ in Eq. (12), Eq.(10) was than derived.

Chiani showed that the approximation of the erfc function or *Q*-function determined in this way is much more accurate than the Chernoff-type approximation [16]

$$F^{[16]}(x;a,b) = a\exp\{-b\,x^2\},\ a,b \in \mathbb{R}. \tag{14}$$

The function from Eq. (14), depending on the choice of real values of parameters *a* and *b*, as well as the interval of the argument values to which it is applied, can represent either the lower or upper bound approximation of the *Q*-function or the *Q*-function approximation itself. For comparative purposes, Chiani used the following upper bound approximation of the *Q*-function:

$$Q(x) \le \overline{F}^{[16]}(x) = \exp\left\{-\frac{x^2}{2}\right\},\ x \ge 0, \tag{15}$$

that is, the following upper bound approximation of the erfc function:

$$\operatorname{erfc}(x) \le 2\exp\{-x^2\},\ x \ge 0. \tag{16}$$

As highlighted in [5] and [8], the tightest possible Chernoff-type upper bound approximation of the *Q*-function can be determined if the right side of the inequality from Eq. (15) is multiplied by 0.5

$$Q(x) \le \overline{F}^{[6]}(x) = \frac{1}{2}\exp\left\{-\frac{x^2}{2}\right\},\ x \ge 0. \tag{17}$$

In [6], mathematical proof was given that the upper bound approximation of the *Q*-function, specified as in Eq. (17), is the most accurate bound approximation of the Chernoff-type for $x \ge 0$. Note that the same form of the upper bound approximation of the *Q*-function was reported many years later in [17], where a somewhat different approach to determining this approximation was applied.

Research from [14], published after one from [5], continued to address the problem of optimizing Chernoff-type approximations of the *Q*-function. In particular, by narrowing the interval of argument values and by performing the numerical optimization of parameters *a* and *b* of the Chernoff-type approximation, the following upper and lower bound approximations of the *Q*-function are specified in [14]:

$$Q(x) \le \overline{F}^{[14]}(x) = 0.28 \exp\left\{-1.275\frac{x^2}{2}\right\},\ x > 0.5\ , \tag{18}$$

$$Q(x) \ge \underline{F}^{[14]}(x) = 0.3 \exp\left\{-1.01\frac{x^2}{2}\right\},\ x > 0.535\ . \tag{19}$$

The authors of [14] pointed out that the bound approximations of the Q-function of higher accuracies can be determined if the interval of argument values in which these approximations are indeed bound approximations of the Q-function is narrowed. Note that the dashes placed above and below the $F(\cdot)$ function are introduced to distinguish the upper and lower bound approximations of the Q-function.

Inspired by the conclusion from [21] that further improvements in the accuracy of the applied approximations are very necessary, as well as by the fact that the authors of this paper, to the best of their knowledge, were the only ones who initiated solving the observed problem of designing compressor functions, in what follows we present the concept of how to achieve this improvement, with the restriction that relatively simple approximations of the Q-function from the class of exponential type approximations are applied.

3. Novel Approximations of the Q Function

Let us first recall that the bound approximations of the Q-function from [17] and [38] are given by:

$$\overline{F}^{[17]}(x) = \frac{1}{2}\exp\left\{-\frac{x^2}{2}\right\}, \tag{20}$$

$$\overline{F}^{[38]}(x) = \exp\left\{-\frac{x^2}{2}\right\}. \tag{21}$$

These bound approximations of the Q-function are characterized by very low accuracies, which degrade with increasing the value of the argument x. As highlighted in [6], it was proved that the upper bound approximation of the Q-function given by Eq. (20) is the most accurate Chernoff-type bound approximation of the Q-function for $x \ge 0$. Let us also recall that the Q-function approximation from [17], given by Eq. (20), was utilized in [21] as the initial solution to the compressor function approximation problem.

As described in detail in the previous section, Chiani applied the trapezoidal rule of numerical integration, optimized the choice of the parameter θ, obtained $\theta_{\text{opt}} \approx \pi/3$, and determined the approximation of the Q-function that represents the sum of two exponential functions [5]. If in the derivation conducted by Chiani we assume $\theta = \pi/2$, and reduce one exponential term accordingly, we can determine our first novel exponential type (Chernoff-type) approximation of the Q-function as follows:

$$F^{\text{novel 1}}(x) = \frac{1}{4}\exp\left\{-\frac{x^2}{2}\right\}. \tag{22}$$

In order to further determine novel approximations of the Q-function, it is important to highlight that in [14], it was pointed out that the bound approximations of the Q-

function of higher accuracies can be determined for narrower interval of argument values in which these bounds are approximations. If we apply these approximations to solve the problem specified in [21], we can expect a greater accuracy of the compressor function approximation because, as we will show in the next section, a much smaller relative error is introduced compared to that from [17]. However, as the Chernoff-type bound approximation of the *Q*-function are generally characterized by relatively low accuracies, it is interesting to consider the performance of the *Q*-function approximations that are not the *Q*-function approximation bounds and that are defined as the geometric and arithmetic mean of the *Q*-function approximation bounds from [14]. In accordance with the above mentioned, we have formulated the following two novel approximations of the *Q*-function of exponential type:

$$F^{\text{novel }2}(x)=\sqrt{\underline{F}^{[14]}(x)\overline{F}^{[14]}(x)}=0.2898\exp\left\{-1.1375\,\frac{x^2}{2}\right\}, \tag{23}$$

$$F^{\text{novel }3}(x)=\frac{\underline{F}^{[14]}(x)+\overline{F}^{[14]}(x)}{2}=0.14\exp\left\{-1.275\frac{x^2}{2}\right\}+0.15\exp\left\{-1.01\,\frac{x^2}{2}\right\}. \tag{24}$$

Although for most of the considered values of argument *x*, by applying the approximations of the *Q*-function determined as the geometric and arithmetic mean of these two approximations, one can expect greater accuracies compared to the approximation of the *Q*-function given by Eq. (19), for some values of argument *x*, we will show that a further increase of the accuracy can be achieved by utilizing the benefits of applying the *Q*-function approximation given by Eq. (18). Therefore, in this paper we propose the following approximation of the *Q*-function:

$$F^{\text{novel }4}(x;k)=\frac{k\overline{F}^{[14]}(x)+\underline{F}^{[14]}(x)}{k+1},\ k\in\mathbb{N}\,, \tag{25}$$

which gives more weight in averaging to the *Q*-function approximation of the higher accuracy specified by Eq. (18) than the one specified by Eq. (19). Note that for $k = 1$, the last approximation of the *Q*-function is identically equal to the approximation specified by Eq. (24):

$$F^{\text{novel }4}(x;k)_{|k=1}=F^{\text{novel }3}(x)\,. \tag{26}$$

Based on the results presented in the following section, we will conclude that introducing such a modification of the *Q*-function approximation is very meaningful, since changing the value of parameter *k* can increase the accuracy of the *Q*-function approximation given by Eq. (24). The choice of one value of parameter *k* will depend on some additional criteria that need to be introduced, which will be left for our future research.

4. NUMERICAL RESULTS

This section discusses the results (RE) determined by applying the proposed *Q*-function approximations and presents an adequate comparison with the *Q*-function approximations from literature, having the same or similar complexities of analytical forms as the proposed approximations. Let us first consider Figure 1, which compares REs determined for the case of applying the upper bound approximations of the *Q*-function from [17] and [38] and our novel 1 Q-function approximation (specified by Eq. (22)). Note that we compare here the Chernoff-type approximations of the Q-function, hence, of the same complexities of the analytical forms. It can be noticed that the novel 1 approximation of the Q-function achieves significantly higher accuracy compared to the ones determined by applying the upper bound approximations of the Q-function from [17] and [38], for most of the values of argument x taken into consideration.

Figure 2 confirms the conclusion from [14]. Namely, in [14], it is anticipated that higher accuracies of the Q-function approximation bounds can be achieved if the interval of argument values in which these Q-function approximations are approximation bounds is narrowed. Figure 3 compares the dependences of the relative errors on the value of argument x for the case of applying the approximations of the Q-function from [14] and [17] with our novel 2 approximations of the Q-function. Based on these dependences, it can be noticed that our novel 2 approximation of the Q-function provides a significantly higher accuracy than the approximation of the Q-function from [17], while for one interval of the argument values, the

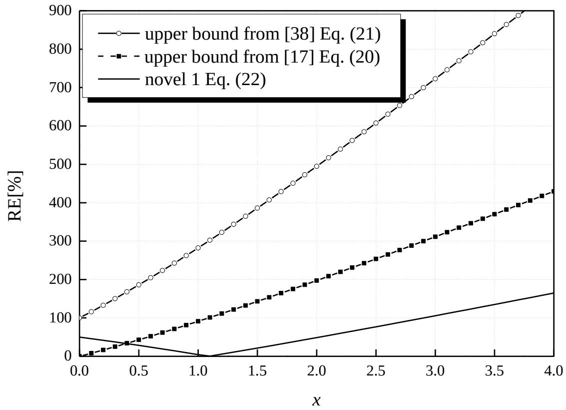

Fig. 1 Relative error: Application of the Q-function approximations from [17] and [38] and novel 1 Q-function approximation of the Chernoff-type.

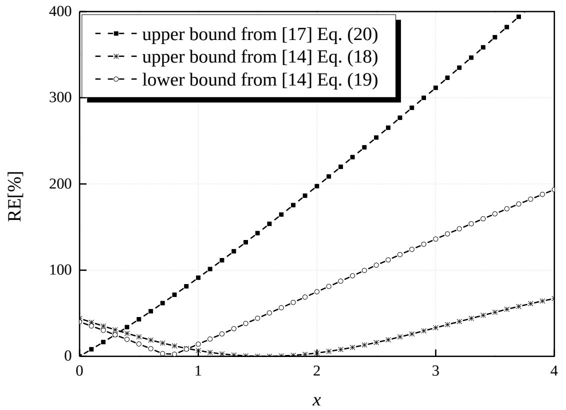

Fig. 2 Comparison of relative errors: Application of the Q-function approximations from [14] and [17].

advantage in terms of accuracy has the upper bound approximation from [14]. A similar conclusion can be derived based on the results shown in Figure 4, where the performance comparison of the novel 2 and novel 3 approximations of the *Q*-function and the approximation of the *Q*-function from [14] is presented.

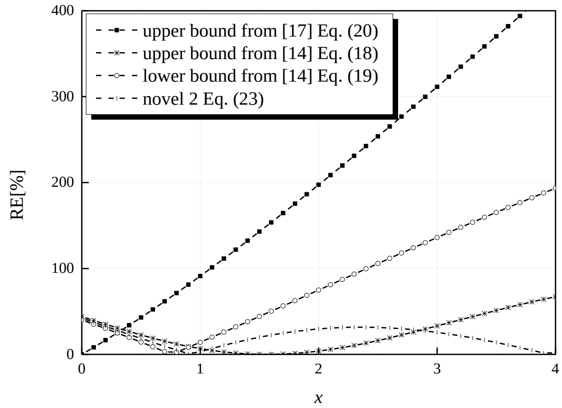

Fig. 3 Relative error: application of the approximations from [17] and [14] and novel 2 approximations determined as geometric mean of approximations from [14].

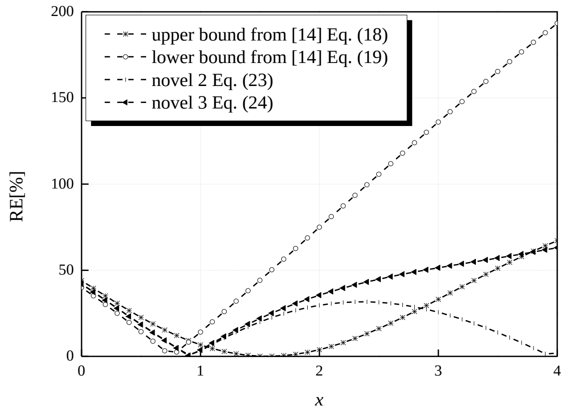

Fig. 4 Relative error: application of the approximations of the *Q*-function from [14], novel 2 and novel 3 approximations of the *Q*-function defined as the geometric mean and arithmetic mean of the approximations from [14].

Figure 5 compares the performance of novel 2, novel 3, and novel 4 approximations of the *Q*-function, with a different choice of parameter k being considered for novel 4 approximation, where it holds $k \in \mathbb{N}$. We can conclude that by increasing the value of parameter k, in the greater part of the interval of the argument values which were taken into consideration in our analysis, the relative error significantly improves, that is, RE decreases. The choice of the specific value of parameter k can be done by comparing the average RE, and choosing the lowest value among the considered ones, which will certainly depend on the specific application, i.e. the interval of argument values taken into consideration, which are important for the specific application. Note that in this paper, the interval of values of argument x is specified as in [21] for the purposes of applying the scalar companding quantization of the Gaussian source. The results show that our modification of the Q-function approximation is very meaningful, since the change in the value of parameter k can increase the accuracy of the approximation given by Eq. (25).

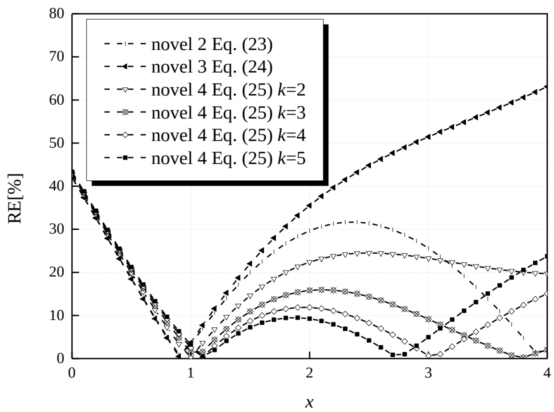

Fig. 5 Relative error: performance comparison of novel2, novel 3 and novel 4 approximations of Q-function.

In short, due to the stated properties of the proposed approximations of the Q-function of exponential type, it can be anticipated that the scope of applications of these approximations is indeed wide. The authors believe that the application of novel solutions for the approximation of the Gaussian Q-function is of great importance in neural networks as well, since the network coefficients are most often modeled by the Laplacian or Gaussian probability density function.

4. CONCLUSION

In this paper, we have highlighted one significant problem that arises in designing scalar companding quantization for the Gaussian source, which has motivated numerous studies in this field. We indicated the fact that in order to simplify the design process of this quantizer model, various methods of linearization of the compressor function have been proposed in literature, which enabled the design and performance evaluation of the linear solutions thus obtained. The authors of this paper are, to the best of their knowledge, the only ones to approach the solution of the observed problem by applying the Q-function approximations in determining the approximate compressor function. Due to early phase of this research, and due to the fact that it was noticed that it is possible to achieve further improvements, in this paper, we have defined some novel approximations of the Q-function that can be useful for overcoming the issue in designing the scalar companding quantizer for the Gaussian source. In addition to the fact that our proposed approximations are characterized by very simple analytical forms, these approximations are also characterized by higher accuracies in relation to the approximations of the *Q*-function of comparable complexities of the analytical forms. Due to the stated properties of the proposed *Q*-function approximations of exponential type, we can anticipate that the scope of applications of these approximations is indeed wide. However, the versatility of the novel proposed *Q*-function approximations in numerous engineering problems and analyses involving the computation of the *Q*-function has been left for our future research.

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