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# ON A DEFUZZIFICATION PROCESS OF FUZZY CONTROLLERS

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**Abstract**. *In this paper, innovations in the field of automatic control systems with fuzzy controllers have been considered. After a short introduction on fuzzy controllers, four different ways of a defuzzification process were introduced, and verified on the simulation of nuclear reactor fuzzy controller. The default Matlab fuzzy toolbox solution is timely most demanding, while two solutions based on the defuzzification on trapezoidal fuzzy numbers have the advantage in the process of crisp numbers calculation. Also, a solution based on the determination of the line dividing the obtained polygon into two parts of equal areas is presented.*

**Key words**: *Fuzzy logic, fuzzy controller, defuzzification*

## 1. INTRODUCTION

The solutions to various problematic situations are sought in the ability to accurately and abstractly express thoughts and interpret sensory stimuli (movement, speech, image), thus defining the basis of human intelligence. Experience teaches us that even when input information is not precise enough, people can process a large amount of information and make adequate, effective decisions. Certainly, the level of education and experience have a great impact on the real-world success of people's actions. Engineers use many methods of artificial intelligence to imitate the model of human thinking and decision-making and implement it in various practical solutions to technical problems. The use of insufficiently known or inaccurately defined terms, with frequent reliance on intuition or subjective feeling, has brought the theory of fuzzy logic to many opponents, especially among Western countries,

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arguing that this theory is without the potential for practical application, with the explanation that all vagueness and imprecision can be described by the probability theory. Even in the field of automatic control, scientists have argued that traditional management techniques are more powerful than the logic stages. Nowadays, by changing the basic approach to scientific analysis, engineers understand that the classical set theory is only a marginal case of the fuzzy set theory, and that by replacing crisp sets with fuzzy sets, any theory can be fuzzified. Thus, the classic proportional-integral regulator is only one form of fuzzy controller, and some classical control methods are analyzed by applying the fuzzy logic [1-3].

The inability to identify all possible events of a system, their incomplete knowledge and the unpredictable frequency of their occurrence make it much more difficult to describe the system and impose the use of approximate system models. For this purpose, in systems control practice there are tools for approximate modeling and the design of analytically based control algorithms such as second-order linear models for the design of PI and PID controllers. There is a direct proportion between the matching process and model and the response of a system regulated by control algorithms designed according to an approximate system model. The problem of impractical design of the controller caused by an unknown model of the system, its complexity, or the high degree of parameter changes, can be solved by the use of adaptive control methods that, due to the complicated mathematical apparatus, usually require a large number of computational iterations [4, 5]. The fuzzy logic theory can also be helpful in the optimization process in the model predictive control, possibly describing constraints. For solving highly nonlinear processes that are greatly influenced by external factors, the crucial role is played by many years of experience and knowledge of the operator, especially in the field of static and dynamic characteristics of the system, where the operator, by monitoring the state of important variables and deviations from the reference values, decides where and how much action should be taken on the process in order to achieve the goal, thus executing its driver. The operator's decision is implemented according to the rule IF (the states of variables are such...) – THEN (such a control action is required...). The experience and knowledge of operators are invaluable, but as a possible problem, their application to the control algorithm appears. Using multivalued logic linguistic expressions in IF-THEN rules describing operator actions can be effectively converted into a fully structured fuzzy-based regulation algorithm applicable to controllers [6-8]. As the fuzzy algorithm has the characteristics of a universal approximator, a set of IF-THEN rules can model an unknown process, even describing the state when operators are unable to describe the rules they enforce [9]. Another application of fuzzy logic in the design of controllers concerns a significant reduction in the time required for design and application [10].

The fuzzy number is a special fuzzy set $F\{(x,\mu_F(x)), x\in \mathbb{R}\}$ with a continuous membership function $\mu_F(x):\mathbb{R}\to[0,1]$. In this paper, trapezoidal fuzzy numbers denoted as $\tilde{A}=(a,m,n,b)$ are used, with membership function defined as

$$\mu_F(x)=\begin{cases}\dfrac{x-a}{m-a}, & x\in(a,m)\\ 1, & x\in(m,n)\\ \dfrac{b-x}{b-n}, & x\in(n,b)\\ 0, & otherwise.\end{cases} \tag{1}$$

For any two trapezoidal fuzzy numbers $\tilde{A}_1 = (a_1, m_1, n_1, b_1)$ and $\tilde{A}_2 = (a_2, m_2, n_2, b_2)$, basic unary and binary arithmetic operations are defined below:

$$\text{addition } (+): \tilde{A}_1 + \tilde{A}_2 = (a_1 + a_2, m_1 + m_2, n_1 + n_2, b_1 + b_2) \tag{2}$$

$$\text{subtraction } (-): \tilde{A}_1 - \tilde{A}_2 = (a_1 - b_2, m_1 - n_2, n_1 - m_2, b_1 - a_2) \tag{3}$$

$$\text{multiplication } (\cdot): \tilde{A}_1 \cdot \tilde{A}_2 = (a_1 \cdot a_2, m_1 \cdot m_2, n_1 \cdot n_2, b_1 \cdot b_2) \tag{4}$$

$$\text{division } (/): \tilde{A}_1 / \tilde{A}_2 = (a_1 / b_2,\ m_1 / n_2,\ n_1 / m_2,\ b_1 / a_2) \tag{5}$$

$$\text{scalar multiplication: } k\tilde{A}_1 = (ka_1, km_1, kn_1, kb_1) \tag{6}$$

$$\text{rooting: } \sqrt[n]{\tilde{A}_1} = (\sqrt[n]{a_1}, \sqrt[n]{m_1}, \sqrt[n]{n_1}, \sqrt[n]{b_1}) \tag{7}$$

The defuzzification process of trapezoidal fuzzy number $\tilde{A} = (a, m, n, b)$ is carried out applying the Center of Area method formula:

$$\text{CV } (\tilde{A}) = \frac{1}{4}(a + m + n + b) \tag{8}$$

Some of the defuzzification methods use $\lambda$, the optimism index, that is, the risk-taking attitude of the decision-maker. Pessimistic point of view is characterized taking the value $\lambda = 0$, while the value $\lambda = 1$ depicts an optimistic attitude. The most commonly used value $\lambda = 0.5$ represents the balanced (moderate) point of view.

In this paper, using a default Matlab fuzzy toolbox, the solution for an example of fuzzy controller will be presented. Using the polygon obtained in the process of aggregation (which could be divided into trapezoids), two solutions with different ideas of defuzzification process will be explained, and calculation times compared. At the end, a geometrical solution of the line $x = x_0$ dividing the polygon into two parts (with equal areas) will be presented, and some formulas generalized.

## 2. Fuzzy Controllers and Their Characteristics

The structure of the fuzzy controller depends on the object of control (plant) and the required degree of the control. As the possibility of applying the fuzzy controller is large, they will differ by the number of inputs and outputs, form of the membership functions, form of rules or even defuzzification process. For all of them, what is common is that the fuzzy controller, the most important part of the automatic control system, can be presented as an artificial decision maker that operates in a closed-loop system in real time [11, 12]. The structure of the fuzzy controller shown in the form of a block diagram is presented in Figure 1.

In the sequel, four parts of the fuzzy controller will be explained [13].

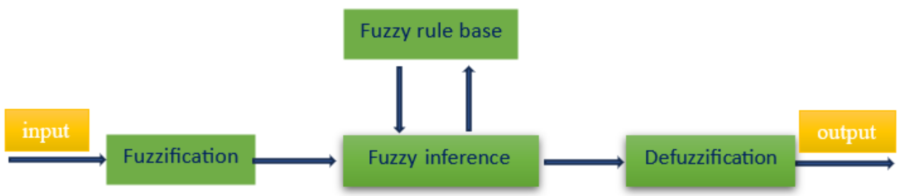


**Fig. 1** The structure of fuzzy controller [12]

The role of the fuzzification block is presented in mapping the physical values of input data into corresponding normalized values of domain and converting input data into linguistic sets and fuzzy sets. As an example, we could use the car speed of *v=70km/h* which should be transformed so the fuzzy controller could understand it. The input value, car speed, should be transformed into fuzzy set determined by the domain: very low speed $\mu_{VL}$, low speed $\mu_L$, medium speed $\mu_M$, high speed $\mu_H$, and very high speed $\mu_{VH}$. As we can see from the Figure 2, for the fuzzification value $v_0$ = *70km/h,* the membership functions are $\mu_L = 0.7$ and $\mu_M = 0.3$, as well as $\mu_{VL} = \mu_H = \mu_{VH} = 0$. There are also membership functions $\mu_M = 0.4$ and $\mu_H = 0.625$ corresponding the car speed $v_0$= *110km/h*.

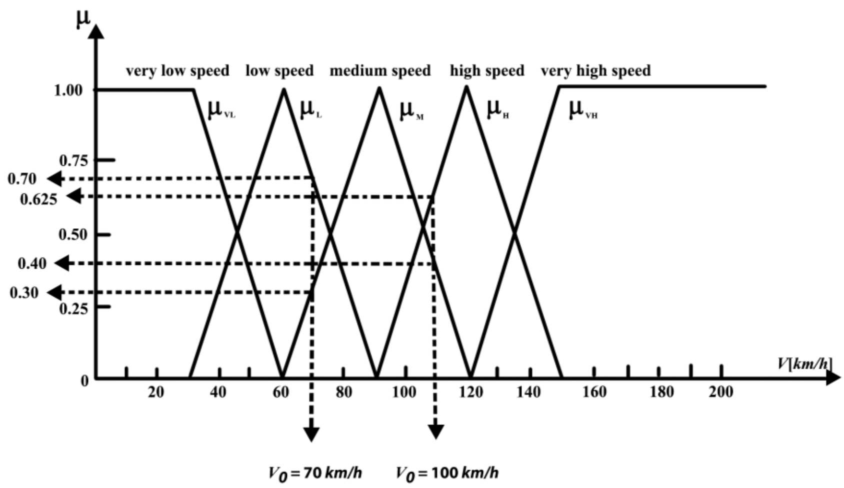


**Fig. 2** The fuzzification of a car speed [11]

All the knowledge and objectives of control are in the fuzzy rule base consisting of a database and a database of rules. Database elements are membership functions and scaling factors. Often, because of the small memory usage and easy way of describing parameters, triangular and trapezoidal membership functions are used, yielding a great impact on the performance of the fuzzy logic operator. For two given membership functions $\mu_1$ and $\mu_2$, describing two different values of the linguistic variable *x*, the intersection point represents the value of $x^*$ which is assumed to be $\mu_1(x^*)= \mu_2(x^*)> 0$. The value $\mu_1(x^*)$ represents the membership function of $x^*$. In Figure 2 it can be seen that $x^*=$ 70, where $\mu_L = 0.7$. When mapping the set of input values into memebrship functions, one must take care that for every element of the set of input values, there is at least one membership function with a positive degree. This ensures the activation of all rules and the smooth flow of control. In order for control to be optimal, it is recommended to use symmetric membership functions (except at the ends of the domain) with an intersection level of two adjacent functions at 0.5. Also, it is advised that the number of linguistic values, due to symmetry around zero, be odd and not more than 7, not to unnecessarily complicate the process of defuzzification. Scaling factors play a major role in determining the performance and stability of the system. They can be determined analytically, by establishing a link between scaling factors and system behavior, or by the trial and error method. In order for the base of the rules of the fuzzy controller to be correctly formed, attention should be paid to the choice of changing system states and output control variables, as well as the content of premises and consequences of the rules. The aim of the fuzzy controller is to mimic the reasoning of the operator and, using knowledge of the control system, to make decisions similar to human ones. This is achieved by fuzzy rules that generate the base of the rules. The fuzzy rules form a central component of fuzzy controller and represent the intelligence of each stage of the control algorithm [14]. Based on the knowledge and experience of the operator, a set of rules must be formed correctly. The fuzzy rule is of the form IF-THEN, where the premise (IF...) describes the conditions, and the consequence (THEN...) explains the consequential actions of the control activity. This form of fuzzy rules allows the definition of various nonlinear control functions enabling fuzzy controllers to successfully deal with non-linear control problems as well. The most common form of a fuzzy rule contains two premises compound by a relationship and one output that gives an activity preposition. The organization of the rule base is considered the most demanding step in the process of defining the fuzzy controller because all other parts, the number of input elements, the selection of the membership function and the procedure for determining the output of the controller are less significant than the base itself. The size of the rule base depends on the number of fuzzy rules, which in turn is determined by the number of input and output variables and their values. Each rule base must be consistent - that there are no rules that have the same premises, and give different consequences. The consequent part of the rule stage may contain a function that determines the connection between the input and output of the controller. This type of regulator, based on Takagi – Sugeno type of fuzzy inference, takes the form of

$$IF\ (x_1\text{- property}), \ldots, \text{and/or } IF\ (x_n\text{-property}),\ THEN\ (u = f(x_1, \ldots, x_n)),$$

where $f$ is the function and $x_1, \ldots, x_n$ are the numerical (quantitative) values of the input. If the function $f$ is linear, $f = a_0 + a_1x_1 + \ldots + a_nx_n$, and the coefficients $a_1 = a_2 = \ldots = a_n=0$, then the rules of Takagi - Sugeno controllers become equal to the rules of the fuzzy controller containing a singleton in the consequent part, i.e.,

$$IF\ (x_1\text{- property}), \ldots, \text{and/or } IF\ (x_n\text{-property}),\ THEN\ (u = a_0).$$

Fuzzy inference logic will be explained on the airconditioning system example.

Example 1. [12] Let the heating and cooling system dependable on the outside and inside temperature is given.
The following linguistic variables could be given:

OT = outside temperature = {low, medium, high}.
IT = inside temperature ={cold, warm, hot}.
AC= air conditioner operation ={cooling, stand by, heating}.

Premise, connected with fuzzy relation, will be representing the connection between input variables OT and IT, while AC stands for the fuzzy rule output variable.

In the sequel, 9 possible fuzzy relations (FR) will be stated.

$FR_1$= OT is low and IT is cold.
$FR_2$= OT is low and IT is warm.
$FR_3$= OT is low and IT is hot.
$FR_4$= OT is medium i IT is cold.

$FR_5$= OT is medium and IT is warm.
$FR_6$= OT is medium and IT is hot.
$FR_7$= OT is high and IT is cold.
$FR_8$= OT is high and IT is warm.
$FR_9$= OT is high and IT is hot.

Let us state 3 possible values of output variable (OV):
$OV_1$= AC is cooling.
$OV_2$= AC je stand by.
$OV_3$= AC is heating.

After defining IF and THEN parts, a fuzzy rule base can be established. Logic is that, if outside temperature is low and inside temperature is cold, the air condition should heat. This condition is presented in the fuzzy rule 1 followed by some other rules.
$Frule_1$ = IF $FR_1$, THEN $OV_3$.
$Frule_2$ = IF $FR_5$, THEN $OV_2$.
$Frule_3$ = IF $FR_3$, THEN $OV_2$.
$Frule_4$ = IF $FR_4$, THEN $OV_3$.
$Frule_5$ = IF $FR_9$, THEN $OV_1$.
$Frule_6$ = IF $FR_8$, THEN $OV_1$.
$Frule_7$ = IF $FR_2$, THEN $OV_3$.

Now we should determine how the IF part affects the THEN part, using the fuzzy implication process. Since the meaning of premise and fuzzy rules is explained using the membership functions, the same way could be also interpreted fuzzy implication. The most commonly used types of fuzzy implication are *product-based,* with a membership function determined as $\mu Frule_i = \mu FR_i \cdot \mu OV_i$ and *Mamdani, or min implication,* with a membership function $\mu Frule_i = \min \{\mu FR_i, \mu OV_i\}$.

To apply fuzzy set, the output of fuzzy controller, to a plant, it has to be understandable (readable), i.e., the defuzzification process should be applied. There are several procedures explaining the way for obtaining the output value $u$, $u \in C$, from the membership function $\mu_C(u)$.

One of the often used defuzzification methods is Center Of Area method defined by

$$u_0 = \frac{\sum_i u_i \cdot \mu_C(x,y,u_i)}{\sum_i \mu_C(x,y,u_i)}, \tag{9}$$

where $u_0$ represents the fuzzy controllers output value, $u_i \in C$ discrete values of output fuzzy set, while corresponding membership functions are marked as $\mu_C(x,y,u_i)$. In the case of fuzzy set's $C$ continuity, sums from the previous formula should be replaced by integrals.

Another defuzzification method applied in the control systems is the Center Of Gravity method defined by

$$u_0 = \frac{\sum_i u_i \sum_{j=1}^{r} \mu_{Frule}(x,y,u_i)}{\sum_i \sum_{j=1}^{r} \mu_{Frule}(x,y,u_i)}, \tag{10}$$

where r represents the number of fuzzy rules.

It can be observed that in the case of COG method there are individual results of fuzzy rules application, as a result of which the aggregation procedure is not used. Basic characteristics of this method are simplicity and low computational difficulty, which in the control system allows frequent controls at small intervals. If max aggregation is applied and several activated rules have the same result, the COG method will count each case and contribution of each of the rules, regardless of the fact that all parts will be the same, while the COA method will count only the case whose membership function is the highest. COG method is sometimes called the Summa method because of the membership functions sum in the definition.

## 3. Defuzzification Process Of A Fuzzy Controller

In this section different defuzzification methods regarding trapezoidal fuzzy numbers, and an example of fuzzy controller will be presented.

Design a Mamdani-type fuzzy controller for a reactor in a nuclear power plant. The radioactivity of the nucleus is influenced by three main factors: the amount of water steam in the system - WS (more steam means more radioactivity), the heat of the nuclear fuel – NF (warmer fuel means less radioactivity) and xenon concentration in the nucleus – XS (more xenon means less radioactivity). The nuclear reaction is controlled by retractable control rods made from Boron - BO. The more rods are inserted into the core, the more they absorb free neutrons for fission and thus reduce the nuclear reaction.

Thus, the fuzzy regulator has 3 inputs: WS on domain [0, 100], NF on domain [0, 1000] and XS on the domain [0, 10], as well as one output BO to the domain [0, 100] (WP, KS and BO are expressed in percentages, while NG is expressed in degrees Celsius scales).

Linguistic variables are defined by their membership functions as follows:
WS: L (10, 40), Λ (20, 50, 80), Γ (60, 90),
NF: L (100, 300), Π (200, 250, 400, 500), Π (400, 600, 700, 800), Γ (650, 750),
XS: L (3, 8), Γ (2, 7),
BO: L (10, 30), Π (10, 20, 40, 50), Λ (40, 50, 60), Π (50, 60, 80, 90), Γ (70, 90).

The sets of linguistic values corresponding to the given membership functions are presented below.
L(WS) = {Small, Medium, Large},
L(NF) = {Cold, Lukewarm, Warm, Hot},
L(XS) = {Low, High},
L(BO) = {Pull_out _ Quickly, Pull_out_Slowly, No_Change,
Pull_down_Slowly, Pull_down_Quickly}.
Next rules are defined:
Rule 1: If NF is Hot and XS je High then BO is Pull_out _ Quickly,
Rule 2: If WS is Medium and XS je High then BO is Pull_out_Slowly,
Rule 3: If WS is Medium and NF not Cold then BO je No_Change,
Rule 4: If WS is Large or NF is Lukewarm or XS is Low then BO is Pull_down_Slowly,
Rule 5: If VS is Large and NF is Cold and XS is Low then BO je Pull_down_Quickly.

Operation OR is defined using MAX operator, operation AND using MIN operator, IMPLICATION using Mamdani definition, and DEFUZZIFICATION applying CENTROID method.

In the following, the graphic procedure of the fuzzy implication and calculation of the value of putting the control rods in the nuclear reactor if the values WS=70, NF= 360, and XS=4 are given, will be presented.

Several solutions of the above problem will be given: Using MATLAB functions Fuzzy logic toolbox, defuzzification methods presented in papers [15,16], and determining the line $x = x_0$ by which the obtained polygon is divided into two parts of equal areas.

*Solution 1*: Using MATLAB fuzzy toolbox functions, parameters of fuzzy controller will be determined.

Input linguistic variables are WS, NF, and XS, while variable BO represent the output. Their membership functions are presented in Figure 3.

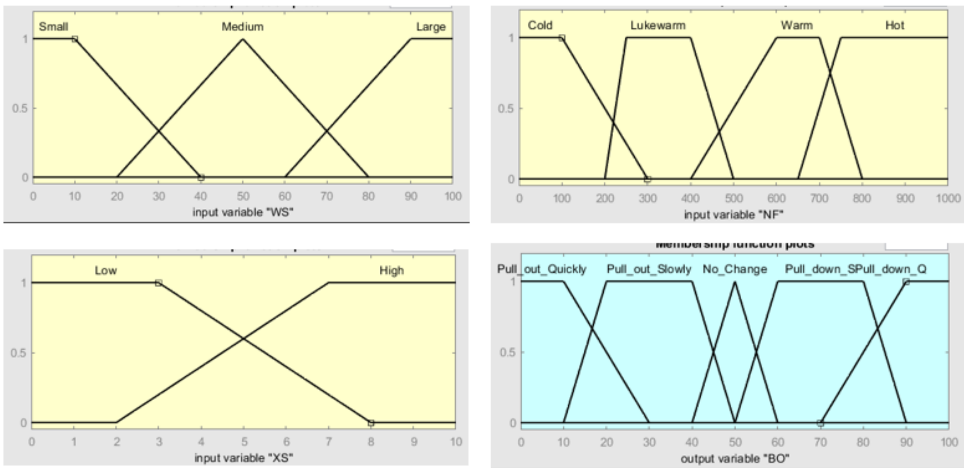


**Fig. 3** Membership functions corresponding input and output linguistic variables

A detailed presentation of fuzzification process, implication, output values and defuzzification are presented in Figure 4, in which can be seen that current output value for a given input values VS=70, NF= 360, and XS=4 is equal BO=58.2.

The procedure of getting result consists of, like it was presented in formula (9), solving the expression consisting of definite integrals:

$$\frac{\int_{10}^{40/3}(0.1x^2-x)dx+\int_{40/3}^{160/3}\frac{1}{3}x\,dx+\int_{160/3}^{60}(0.1x^2-5x)dx+\int_{60}^{80}x\,dx+\int_{80}^{90}(-0.1x^2-9x)dx}{\int_{10}^{40/3}(0.1x-1)dx+\int_{40/3}^{160/3}\frac{1}{3}dx+\int_{160/3}^{60}(0.1x-5)dx+\int_{60}^{80}dx+\int_{80}^{90}(-0.1x-9)dx}$$

The limits of these integrals are some of the points $x_1$, $x_2$, ..., $x_{11}$ presented in Figure 6. Calculations of listed integrals demand a lot of time and effort, making the procedure of hand-calculation quite challenging.

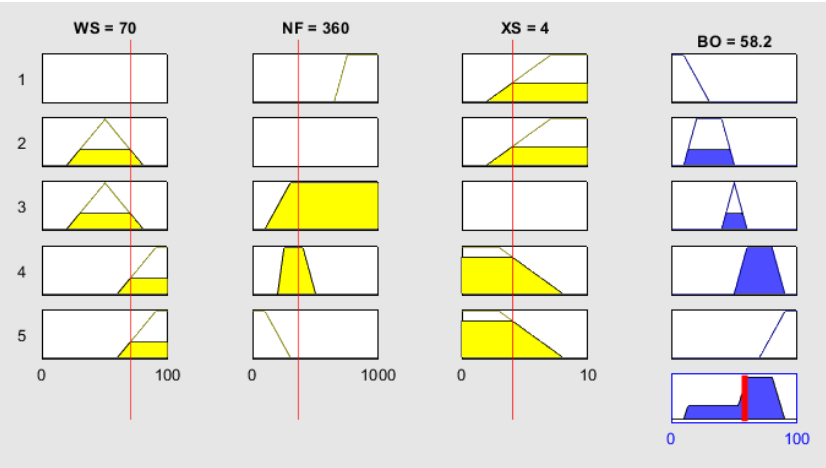


**Fig. 4** The managing process of a nuclear reactor

The controlled surface obtained for given input data can be seen in the Figure 5.

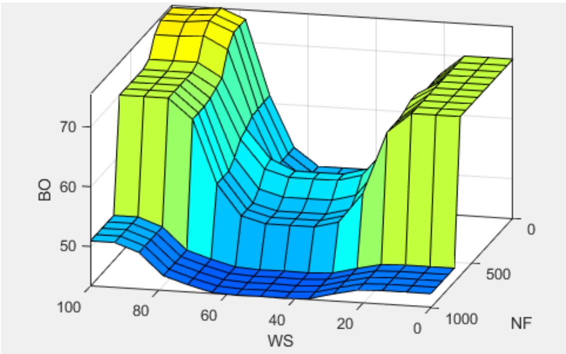


**Fig. 5** Control surface of a fuzzy controller

The time needed for the defuzzification process execution on Dell Laptop Inspirion 15, Intel Core i3-1005G1 is 0.765s.

*Solution 2*: The result of the aggregation process in Solution 1 is the polygon presented in Figure 6. Points on the x-axis in which the vertices of the longer base of the obtained trapezoids and foots of their heights are located and marked with: $x_1 = 10$, $x_2 = 40/3$, $x_3 = 40$, $x_4 = 130/3$, $x_5 = 140/3$, $x_6 = 50$, $x_7 = 160/3$, $x_8 = 170/3$, $x_9 = 60$, $x_{10} = 80$ and $x_{11} = 90$.

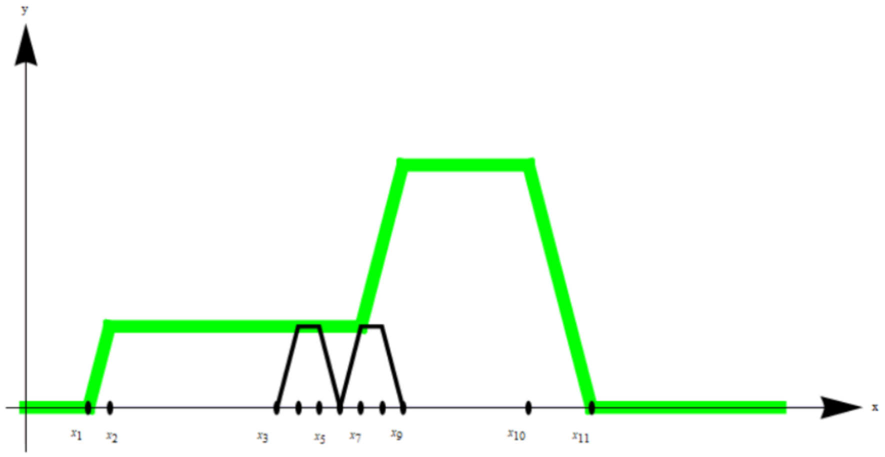


**Fig. 6** Polygon obtained in the aggregation process

Using idea of defuzzification presented in [15], where trapezoidal fuzzy number $\tilde{A}=(a,m,n,b)$ is defuzzified by formula

$$CV=\frac{1}{6}\left(a+2(m+n)+b\right) \tag{11}$$

the crisp values ($CV$) of three trapezoids $\tilde{A}_1=\left(10,\frac{40}{3},\frac{170}{3},60\right)$, $\tilde{A}_2=(50,60,80,90)$, and $\tilde{A}_3=\left(50,\frac{160}{3},\frac{170}{3},60\right)$ obtained on the polygon are equal $CV_1=35$, $CV_2=35$, and $CV_3=55$, which yields that polygonal surface is equal $P=50$.

*Solution 3*: Using idea presented in [16], where trapezoidal fuzzy number $\tilde{A}=(a,m,n,b)$ is defuzzified by formula

$$CV=\frac{1}{2}\left((1-\lambda)(A+m)+\lambda(b+n)\right) \tag{12}$$

where $\lambda\in[0,1]$ represents an optimism index, for the moderate optimism index, $\lambda=0.5$, the same crisp values as in Solution 2 are obtained, while the error obtained in comparison with Solution 1 is equal 10.09%.

On the Figure 7 the ratio of error obtained in Solution 3 for a different values of parameter $\lambda$, compared with values from Solutions 1 and 2 will be presented.

We notice that on both sides the value $\lambda=0.5$ percent of the error compared to Solution 1 symmetrical increases by 14%, while compared to Solution 2 this percentage is equal to approximately 12%, and spreads symmetrically starting with the value obtained by the value $\lambda=0.6$ in which the percentage of error is equal to 2,062%. In the first case, the lowest percentage of error occurs for the moderate attitude of the decision maker, while the percentage of error increases when the opinion of the decision-maker approaches an optimistic or pessimistic point of view. It is a similar situation in the second case, when the smallest error occurred for the optimism index slightly higher than $\lambda = 0.6$ compared to the moderate attitude of the decision-maker, increasing the error by changing the value of the optimism index.

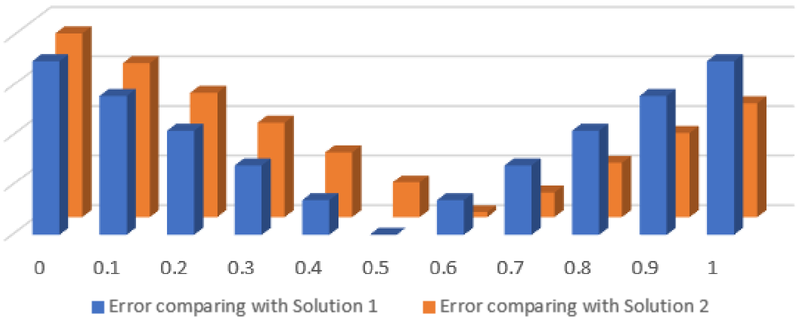


**Fig. 7** Ratio between errors expressed in percentages

These conclusions are consistent with the fact that real results in decision-making are generally obtained for a moderate value of the optimism index.

Time required to complete the necessary calculations of the defuziffication process is 0.67s.

By comparing the time of presented solutions required to perform the defuzzification process, it is observed that the least time required is in Solution 2, 0.67s, then in Solution 3, 0.686s, while the most time is spent for the procedure shown in Solution 1, 0.765s. If the time required for certain parts of the defuzzification process is compared, they are equal in terms of solutions: Solution 1 = 0.686s + 0.686s+0.701s, Solution 2 = 0.67s+0.67s+0.67s, while the Solution 3, by dividing the polygon into parts and using formula (1) to solve definite integrals whose limits are on x-axes of the corresponding polygon dividing points, required time for the numerator is 0.779s+0.785s+0.811s+0.826s+0.826s and for denominator is 0.826s+0.826s+0.826s+0.826s+0.841s., clearly emphasizing that the defuzzification procedures shown in Solution 2 and Solution 3 are significantly faster than the usual procedure presented in Solution 1, which can be of great importance in real-time critical control systems.

*Solution 4*: The idea of this solution consists determining the line x = $x_0$ to divide the polygon obtained in the fuzzification process into two parts of equal surfaces. The coordinates of points on the Figure 8 are: $A(x_1,0)$, $B(x_2,0)$, $C(x_3,0)$, $D(x_4,0)$, $E(x_5,0)$, $F(x_6,0)$, $G(x_7,0)$, $H(x_8,0)$, $K(x_9,0)$, $(x_5,\mu)$, $P(x_8,y_2)$, $Q(x_6,y_2)$, $R(x_4,y_1)$, $S(x_2,y_1)$, $U(0,\mu)$, $V(0,y_1)$ and $W(0,y_2)$.

The surface of polygon AKPQMRS is equal to the sum of surfaces $P_1 = P_{\Delta ABS}$, $P_2 = P_{\square BDRS}$, $P_3 = P_{\square DEMR}$, $P_4 = P_{\square EFQM}$, $P_5 = P_{\square FHPQ}$ and $P_6 = P_{\Delta HKP}$. That surface is equal

$$P = \frac{1}{2}AB \cdot BS + BD \cdot DR + \frac{1}{2}DE \cdot (DR + EM) +$$
$$\frac{1}{2}EF(EM + FQ) + FH \cdot HP + \frac{1}{2}HK \cdot KP \qquad (13)$$

Based on the coordinates $x_1, x_2, \ldots, x_9, y_1, y_2$ and $\mu$, the surface defined by equation (5) can be transformed into

$$P=\frac{1}{2}\left(x_4+x_5-x_1-x_2\right)y_1+\frac{1}{2}\left(x_8+x_9-x_5-x_6\right)y_2+\frac{1}{2}\left(x_6-x_4\right)\mu \tag{14}$$

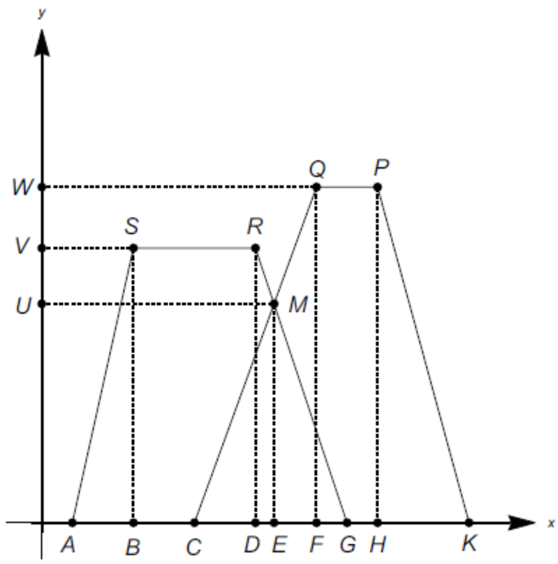


**Fig. 8** Polygon divided into trapezoids

It also holds

$$P_1=\frac{1}{2}(x_2-x_1)y_1, \quad P_2=(x_4-x_2)y_1, \quad P_3=\frac{1}{2}(\mu+y_1)(x_5-x_4),$$
$$P_4=\frac{1}{2}(\mu+y_2)(x_6-x_5), \quad P_5=(x_8-x_6)y_2, \quad P_6=\frac{1}{2}(x_9-x_8)y_2, \tag{15}$$

as well as

$$\tilde{P}=\frac{1}{2}P=\frac{1}{4}(x_4+x_5-x_1-x_2)y_1+\frac{1}{4}(x_8+x_9-x_5-x_6)y_2+\frac{1}{4}(x_6-x_4)\mu, \tag{16}$$

$$P_{12}=P_1+P_2=\frac{1}{2}(2x_4-x_1-x_2)y_1 \tag{17}$$

$$P_{123}=P_1+P_2+P_3=\frac{1}{2}(x_5-x_4)\mu+\frac{1}{2}(x_4+x_5-x_1-x_2)y_1 \tag{18}$$

$$P_{1234}=P_1+P_2+P_3+P_4=\frac{1}{2}(x_4+x_5-x_1-x_2)y_1+\frac{1}{2}(x_6-x_5)y_2+\frac{1}{2}(x_6-x_4)\mu \tag{19}$$

Let us consider the point $X_0 = (x_0,0)$ and corresponding intersection of the line $x = x_0$ and line ASRMQPK. There are the next cases:

a) If $x_0 \in [x_1, x_2)$, then the point $\tilde{X}$ is on the segment AB. Triangles $\Delta AX_0\tilde{X}_0$ and $\Delta ABC$ are similar (with the same ninety degrees angle), and by proportion $\frac{X_0A}{BA} = \frac{\tilde{X}_0A}{SA}$ we obtain that

$$\frac{x_0 - x_1}{x_2 - x_1} = \frac{y_0}{y_1} \Rightarrow y_0 = \frac{x_0 - x_1}{x_2 - x_1} y_1 \Rightarrow \tilde{P} = \frac{1}{2}(x_0 - x_1)y_0 \Rightarrow \tilde{P} = \frac{1}{2}\frac{(x_0 - x_1)^2}{x_2 - x_1} y_1 \tag{20}$$

It follows that $x_0$ satisfies the quadratic equation

$$0 = \frac{y_1}{x_2 - x_1}(x_0)^2 - 2\frac{x_1 y_1}{x_2 - x_1} x_0 + \frac{(x_1)^2 y_1}{x_2 - x_1} - \frac{1}{2}(x_4 + x_5 - x_1 - x_2)y_1 - \frac{1}{2}(x_8 + x_9 - x_5 - x_6)y_2 - \frac{1}{4}(x_6 - x_4)\mu. \tag{21}$$

b) If $x_0 \in [x_2, x_4)$, it holds $y_0 = y_1$. Now it holds $P_1 + BX_0 \cdot BS = \tilde{P}$, i.e.

$$\frac{1}{2}(x_2 - x_1)y_1 + (x_0 - x_2)y_1 = \frac{1}{4}(x_4 + x_5 - x_1 - x_2)y_1 + \frac{1}{4}(x_8 + x_9 - x_5 - x_6)y_2 + \frac{1}{4}(x_6 - x_4)\mu. \tag{22}$$

Based on that, it can be concluded that

$$x_0 = \frac{1}{4}(x_1 + x_2 + x_4 + x_5)y_1 + \frac{1}{4}(x_8 + x_9 - x_5 - x_6)y_2 + \frac{1}{4}(x_6 - x_4)\mu. \tag{23}$$

c) If $x_0 \in [x_4, x_5)$, then $\tilde{X}_0$ is the intersection point of the lines $x = x_0$ and the segment *RM*. Triangles $\Delta GX_0\tilde{X}_0$ and $\Delta GDR$ are similar (with the same sharp angle) and $y_0 = \frac{x_7 - x_0}{x_7 - x_4} y_1$. The polygon $AX_0\tilde{X}_0RS$ surface is equal

$$P_1 + P_2 + \frac{1}{2}(x_0 - x_4)(y_0 + y_1),$$

from where rearranging the elements, one obtains that $x_0$ is the solution of the equation

$$0 = \frac{y_1}{2(x_4 - x_7)}(x_0)^2 - \frac{x_7 y_1}{x_4 - x_7} x_0 + \frac{1}{4}(x_4 - x_6)\mu + \frac{1}{4}\left(\frac{2x_4x_7}{x_4 - x_7} + x_4 - x_1 - x_2 - x_5\right)y_1 + \frac{1}{4}(x_5 + x_6 - x_8 - x_9)y_2. \tag{24}$$

If $x_0 \in [x_5, x_6)$, the line $x = x_0$ (which holds the point $X_0 = X_0(x_0,0)$) has the intersection with the segment *MQ* in the point $\tilde{X}_0 = \tilde{X}_0(x_0, y_0)$, yielding to the similarity of right-angled triangles $\Delta QFC$ and $\Delta CX_0\tilde{X}_0$ (with the same sharp angle). Based on this, it holds $y_0 = \frac{x_0 - x_3}{x_6 - x_3} y_2$. The polygon $AX_0\tilde{X}_0MRS$ surface is equal

$$P_1 + P_2 + P_3 + \frac{1}{2}(x_0 - x_5)(y_0 + \mu),$$

from where it can be concluded that $x_0$ is a solution to a quadratic equation

$$0 = \frac{y_1}{2(x_4 - x_7)}(x_0)^2 - \frac{x_7 y_1}{x_4 - x_7} x_0 + \frac{1}{4}(x_4 - x_6)\mu + \frac{1}{4}\left(\frac{2x_4x_7}{x_4 - x_7} + x_4 - x_1 - x_2 - x_5\right) y_1 + \frac{1}{4}(x_5 + x_6 - x_8 - x_9) y_2 \qquad (25)$$

e) If $x_0 \in [x_5, x_6)$, the point $X_0 = (x_0,0)$ is on the segment *FH*, while the intersection of lines $x = x_0$, and PQ is the point $\tilde{X}_0 = \tilde{X}_0(x_0, y_2)$. The polygon $AX_0\tilde{X}_0QMRS$ surface is equal

$$P_1 + P_2 + P_3 + P_4 + \frac{1}{2}(x_0 - x_6) y_2 .$$

By equating this surface with $\tilde{P}$ one obtains

$$x_0 = -\frac{1}{4y_2}\mu(x_6 - x_4) - \frac{1}{4y_2}(x_4 + x_5 - x_1 - x_2) y_1 - \frac{1}{4y_2}(x_9 - x_8 - x_6 - x_5) y_2 \qquad (26)$$

f) If $x_0 \in [x_5, x_6)$, the line $x = x_0$ (which holds the point $X_0 = X_0(x_0,0)$) has the intersection with the segment *PK* in the point $\tilde{X}_0 = \tilde{X}_0(x_0, y_0)$, yielding to the similarity of right-angled triangles $\Delta PHK$ and $\Delta HX_0\tilde{X}_0$ (with the same sharp angle). Based on this, it holds $y_0 = \frac{x_9 - x_0}{x_9 - x_8} y_2$. The line $x = x_0$ divides the polygon *AKPQMRS* surface into two equal surface parts, one obtains that the surface of the triangle $\Delta KX_0\tilde{X}_0$ is equal $\tilde{P}$, which gets

$$\frac{(x_9 - x_0)^2}{x_9 - x_8} y_2 = \frac{1}{2}(x_4 + x_5 - x_1 - x_2) y_1 + \frac{1}{2}(x_8 + x_9 - x_5 - x_6) y_2 + \frac{1}{2}(x_6 - x_4)\mu. \qquad (27)$$

By arranging this equation, it holds

$$0 = \frac{y_2}{x_9 - x_8}(x_0)^2 - \frac{2x_9 y_2}{x_9 - x_8} x_0 - \frac{1}{2}(x_4 + x_5 - x_1 - x_2) y_1 - \frac{1}{2}\left(x_8 + x_9 - x_5 - x_6 - \frac{(x_9)^2}{x_9 - x_8}\right) y_2 + \frac{1}{2}(x_6 - x_4)\mu. \qquad (28)$$

Let us consider a special case of Figure 8, when $M \equiv R$, or equivalently, $x_5 = x_4$ and $\mu = y_1$. General six cases, previously explained, become

$$\frac{y_1}{x_2-x_1}(x_0)^2-2\frac{x_1y_1}{x_2-x_1}x_0+\frac{(x_1)^2y_1}{x_2-x_1}$$
$$=\frac{1}{2}(x_4+x_6-x_1-x_2)y_1+\frac{1}{2}(x_8+x_9-x_4-x_6)y_2 \qquad (29)$$

$$x_0=\frac{1}{4}(x_1+x_2+x_4+x_6)y_1+\frac{1}{4}(x_4+x_6-x_8-x_9)y_2 \qquad (30)$$

$$\frac{y_1}{2(x_4-x_7)}(x_0)^2-\frac{x_7y_1}{x_4-x_7}x_0+\frac{1}{4}(x_4+x_6-x_8-x_9)y_2$$
$$+\frac{1}{4}\left(\frac{2x_4x_7}{x_4-x_7}+x_4-x_1-x_2-x_6\right)y_1=0 \qquad (31)$$

$$\frac{y_2}{x_3-x_6}(x_0)^2+\frac{1}{2}\left(\frac{x_4(x_3+x_6)}{x_3-x_6}+x_8+x_9-x_6\right)-$$
$$\left(y_1+\frac{(x_3+x_4)y_2}{x_3-x_6}\right)x_0+\frac{1}{2}(x_1+x_2+x_6-x_4)y_1=0 \qquad (32)$$

$$x_0=-\frac{1}{4y_2}(x_4+x_6-x_1-x_2)y_1-\frac{1}{4y_2}(x_9-x_8-x_6-x_5)y_2 \qquad (33)$$

$$\frac{y_2}{x_9-x_8}(x_0)^2-\frac{2x_9y_2}{x_9-x_8}x_0-\frac{1}{2}(3x_4-x_6-x_1-x_2)y_1$$
$$-\frac{1}{2}\left(x_8+x_9-x_5-x_6-\frac{(x_9)^2}{x_9-x_8}\right)y_2=0 \qquad (34)$$

This solution represents a standard geometrical approach to determining the surface of a polygon. Dividing polygon obtained in the aggregation process in Solution 1 into triangles and rectangles, mathematical expressions/formulas of mid-results in six cases were obtained, getting the formula for calculation the surface of starting polygon.

Further research in the field of defuzzification of fuzzy controllers could be in the direction of representation of (aggregation) obtained polygon as a sum of triangles and its defuzzification using Fuzzy AHP [17] with five degrees of optimism index, and recently presented Surface Fuzzy AHP [18].

## 4. Conclusion

In this paper, the authors presented a brief overview of several methods for a fuzzy controller defuzzification process. Firstly, they gave a solution from the MATLAB Fuzzy toolbox, then by two different ways of defuzzification trapezoidal fuzzy numbers, and last, geometrical proof of dividing the polygon obtained in the aggregation process into two parts (consisting of triangles and rectangles) of equal surfaces with the line $x = x_0$. They also generalized the approach presented in the last solution, when polygon consists only of trapezoids. It is also proven that the time required for the defuzzification process is significantly less when the process is performed using formulas for the defuzzification of trapezoidal fuzzy numbers, as it is presented in Solutions 2 and 3, than by using Fuzzy toolbox created solution, as in Solution 1. This improvement in requirement solution time can be significant in immediate-response systems and critical control systems.

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