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INFLUENCE OF LOCAL CURVATURE ESTIMATION ON THE 3D MESH TOPOLOGICAL AND GEOMETRIC STABILITY

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Abstract: *The influence of curvature estimation on the 3D mesh topological and geometric stability is considered in this paper. It can be shown that, in the case of the most commonly used methods for local curvature estimation, the nontrivial calculus of curvature can lead to mistakes and treat as noise holders of shape. Therefore, new algorithms for solving problems in every step of these processes are proposed. The proposed algorithms are simple and quick from the viewpoint of the mathematical calculations, because they assume only a few extra steps in the algorithm of curvature evaluation. Besides, the new approach provides more accurate results then other approaches, and also reduces the possibility of 3D mesh geometry damages.*

Key words: *Computer animation, 3D models, 3D mesh geometric curvature*

1. INTRODUCTION

One of the most complicated processes in the field of geometry of three-dimensional (3D) models is the curvature estimation and calculation. Ambiguous discrete representation of the 3D model has caused a number of different approaches in the evaluation of these important characteristics. All approaches basically rely on Gauss [1] and Riemann's [2] work in the field of geometry of smooth curved surfaces. However, different topologies require specific mathematical representation of discrete geometric formulation.

Earlier definitions described continuous closed surfaces without boundaries, but the considerations of these areas exactly, as well as sharp edges fields conclude that the

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curvature estimation of generalized closed surfaces gives disastrous results in the above mentioned regions. This paper explains error in Gaussian curvature calculation in sharp regions and proposes a new method for its minimization based on accurate calculation of the neighboring area at the point of calculations.

3D model, as opposed to one-dimensional audio, two-dimensional digital image [3] or video [4], is essentially defined by the position of vertices and faces, and the space between them is an empty space. This space is, only by virtual connections of points, shown to user as a solid body. This is a difficult obstacle if we want to apply curvature estimation method of two-dimensional media to 3D mesh.

The practical implementation of curvature evaluation can usually be seen in the processes of optimization and simplification [5], triangulation, smoothing and noise filtering of meshes obtained by 3D scanning, as well as 3D representation and precise definition of robotic space. Better estimations of 3D structure are crucial in the sense of space objects recognition. In modern automatic systems, as robotic surgery e.g., it is absolutely clear that successive control and precision of robotic motion are strictly dependent on 3D surface characterization. More generally, we have proven the multidisciplinary usability of 3D mesh characterization using a theoretical aspect in the field of the data hiding [6]. In all the mentioned fields noise filtering process is actually a subject of our research. It shows the consequence of incorrect local curvature evaluation to results of triangulated surface mesh noise filtering.

The paper is organized as follows: the second section explains standard 3D formats, as well as comparison of the basic stability of a certain primitive in relation to formats of a 3D object. The third section gives the definitions of curvature notion and describes mathematical formulation of different curvature estimation methods. In this section we also explain errors of maximal curvature computations at discrete set of points. Section 4 indicates the stability of vertices with the proper choice of local curvature estimation method, and gives a good solution for precise calculation.

2. Background of 3D model representation

We have already mentioned that the 3D model format is represented with two basic sets of data: the Euclidean spatial coordinates of vertices and their order sequence for faces forming. It can be experimentally shown that the order of vertices coordinates does not affect the geometric features, but the order of vertices in faces forming significantly changes the topology of 3D mesh, and these faces, such as primitives, are classified as unstable carriers of information.

In Fig. 1 one may see different arrangements of faces in 3D format of the pyramid. Changing the order of faces and vertices does not affect the geometry of the model, but the clockwise direction of faces formation inverts the direction of normal vectors and makes a topological error.

Hence, faces are not eligible for any kind of calculations and evaluation, because the data format changes may include re-triangulation and destroy previously formed surfaces. Opposed to that, vertices are stable primitives. The change of format will not affect their coordinates. Since the vertices are specified only by their coordinates, and not by sequence order in data files, it can easily be proved that even a change of coordinates due to the whole mesh rotation or a change of the position and direction of the coordinate axes, will not interfere with geometric and topological structure of the mesh. But, following this logic, vertices may also be hosts of noise and any noise filter may cause a significant changes of its position in space.

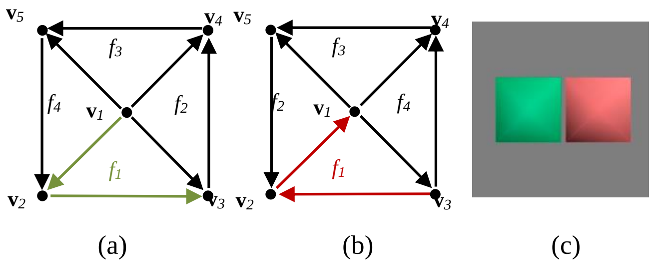

Fig. 1 (a) Faces forming by original vertices order. (b) Clockwise forming vertices order. (c) The top view rendering of both cases.

3. Survey of Curvature Estimation Methods

The notion of curvature originates from Gauss work in differential geometry [1]. To begin, we define the Euler characteristic [4]

$$\chi(M,T) = V - E + F \tag{1}$$

for a given triangulation T of a surface M in three-dimensional space $\mathbb{R}^3$ with V vertices, E edges, and F faces. Now we may write a generalized Gauss-Bonnet theorem, which gives implicit expression to the Gaussian curvature calculus. For a surface with boundary M, made by triangulation T if $\epsilon_k, k = 1,\ldots,l$ are exterior angles of boundary δM, then [7]:

$$\int_{\partial M} \kappa_G ds + \iint_M KdA + \sum_{k=1}^{l} \epsilon_k = 2\pi\chi(M,T) \tag{2}$$

where dA is unit area of surface M, and K denotes normal local Gaussian curvature of dA. The special case when M is a compact, oriented surface without boundary is:

$$\iint_M KdA = 2\pi\chi(M,T) \tag{3}$$

From differential geometry we also know that two *principal curvatures* κ_1 and κ_2 of the surface *M*, with their associated orthogonal directions $\mathbf{e}_1$ and $\mathbf{e}_2$ are the extreme values of all normal curvatures. The *mean curvature* κ_H is defined as the average of the normal curvatures $\kappa_H = (\kappa_1 + \kappa_2)/2$. The *Gaussian curvature* κ_G is defined as the product of the two principle curvatures $\kappa_G = \kappa_1\kappa_2$. Today, the most common methods for curvature estimations are given by the algorithm of Mayer *et. al* [8] and quadric fitting method for estimating surface curvature by Petitjean [9]. Discrete curvature estimation based on differential geometry approach by Mayer *et. al* [8]:

$$\kappa_{G_i} = \frac{1}{A}\left(2\pi - \sum_j \theta_j\right)$$
$$2\kappa_{H_i}\mathrm{n}_i = \frac{1}{A}\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})\left\|x_i - x_j\right\|^2 \tag{4}$$

For the given quadric form $Z' = aX'^2 + bX'Y' + cY'^2$ expressions for the principal curvature and Gaussian and mean curvature calculus are [9]:

$$\begin{aligned}\kappa_1 &= a + c + \sqrt{(a-c)^2 + b^2}\\ \kappa_2 &= a + c - \sqrt{(a-c)^2 + b^2}\\ \kappa_G &= 4ac - b^2\\ \kappa_H &= a + c\end{aligned} \tag{5}$$

where coefficients of the quadric a, b and c are solvations of last squares equations:

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n y_n & y_n^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{6}$$

However, the estimates of the curvature using a quadric are sensitive to estimates of the surface normal method based on differential geometry estimates of maximal curvature at points which are actually topological errors. The solution for eliminating the deficiency of the first method is suggested through an extended quadric by Garimela and Swartz [10]:

$$Z' = aX'^2 + bX'Y' + cY'^2 + dX' + fY' \tag{7}$$

$$\kappa_G = \frac{4ac - b^2}{(1 + d^2 + e^2)^2}$$
$$\kappa_H = \frac{a + c + ae^2 - bde}{\sqrt{(1 + d^2 + e^2)^3}} \tag{8}$$

We recommend improving the estimation accuracy of methods based on discrete differential geometry, because the previous method does not give accurate results in estimating the local curvature even with the improvements. However, note that global estimation with the extended quadric method is more accurate [11].

For study invariance of geometry and topology to the surface filtering processes the local curvature estimation is more important than global. Our approach is based on a more accurate calculation of the neighboring area at the point of interest. Instead of Voronoi area, which does not give good results for obtuse triangles, we suggest a *barycentric* area.

For each point of interest x_i there are $N(i)$ triangles, adjacent to x_i, where θ_i is its interior angle that is formed by two adjacent edges e_j and e_{j-1}. At the case of finite 3D mesh, points x are marked as vertices v and angle between edges $v_i v_{j-1}$ and $v_j v_{j-1}$ is defined as α_{ij} and shown in Fig. 2. Then the Barycentric area at the point of interest x_i is given:

$$\mathcal{A}_B = \frac{1}{6} \sum_{j \in N_1(i)} \left(\cot \theta_{ij} + \cot \alpha_{ij} \right) \left\| x_i - x_j \right\|^2 \sin^2 \theta_{ij} \tag{9}$$

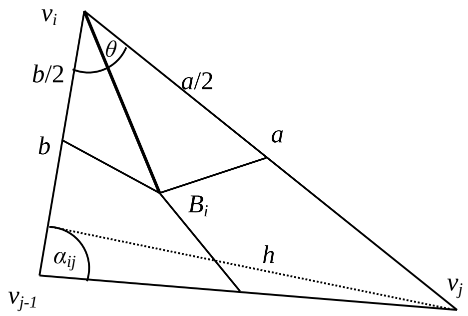

Fig. 2 The triangular face and relevant parameters at vertex v_i and its neighbors v_j and $v_{j\text{-}1}$.

Barycentric area for each point of interest v_i is calculated as a sum of all adjacent triangles of barycentric areas, which are formed by midpoints of adjacent edges and the triangle barycenter(fig 2). Thus, the shape and the accuracy of the adjacent area is converging to infinitesimal elliptical area by differential geometry definition.

In order to achieve greater convergence of the area we can introduce another term, which represents the relationship between the angle θ at vertex and length of edges around it.

$$\mathcal{A}_{Converg} = \frac{1}{8} \sum_{j \in N_1(i)} \cot \theta_{ij} l_{ij}^2 \tag{10}$$

If θ is the angle at vertex i, l^2 squared length of edge around it, using the expression (7) and (8) we obtain an expression for the final converged surface of the area:

$$\mathcal{A}_T = \mathcal{A}_B + \mathcal{A}_{Converg} \tag{11}$$

$$\mathcal{A}_T = \frac{1}{2} \sum_{j \in N_1(i)} \cot \theta_{ij} \left\| x_i - x_j \right\|^2 \left(\frac{1}{3} \sin^2 \theta_{ij} + \frac{1}{4} \right) \tag{12}$$

However, we will show that experimental results indicate that estimation accuracy is mostly dependent on the angle θ at vertex, and less than length of the edges and the theoretical region convergence. Indeed, this argument is valid for measuring the stability of certain areas, but for determining a shape of meshes and define the difference between noise and the shape, the convergence region is crucial.

4. Topological and Geometric Stability of 3D Mesh in Relation to Noise Filtering

Our improvement achieves more accurate approximation of the elliptic area, and also achieves more accurate estimation of the local curvature and the increased resistance to the filtering and fairing parts of mesh surface. These processing operations fundamentally include local curvature estimation. Minimizing curvature energy [12]:

$$E(K,V) = E_{dist}(K,V) + E_{rep}(K) + E_{spr}(K,V) \tag{13}$$

is the final step to noise filtering surfaces of the mesh.

$$\begin{gathered} E_{dist}(K,V) = \sum_{i=1}^{n} d^2(x_i, \phi_V(|K|)) \\ E_{rep}(K) = C_{rep} m \end{gathered} \tag{14}$$

Noised surfaces are actually geometric bubbles, which are usually formed by one vertex with high local curvature. Rarely (but it is not impossible) two vertices or first ring of adjacent vertices may create noise bubbles [13].

Nevertheless, curvature of the first ring of adjacent vertices affects the shape features and it is very important to precisely evaluate gradient of curvature at vertices and their first ring of neighbors. If a gradient has the same sign as the estimated curvature, then this vertex and its first ring of neighbors we have to treat as parts of a geometric shape. Otherwise, this vertex is probably noise vertex, and it may be filtered like other noise signals.

Therefore, we propose a new algorithm to curvature estimation in the form of an additional step in the iteration of curvature calculations. The next step includes calculating the gradient of curvature at each vertex of the first ring of neighbors, as well as comparison of the obtained values with the value of local curvature at vertex.

5. Experimental Results of the Curvature Evaluations

We have performed several tests of evaluation of Gaussian and mean curvature using discrete differential geometry and fitting quadrics methods.

Fig. 3 The plot of Gaussian and mean curvature estimation respectively. Red colored areas are regions with a high level of curvature. The blue areas have low values of curvature.

Fig. 4 The plot of Gaussian and mean curvature estimation computed by local differential geometry method.

In processes of the global curvature evaluation and shape recognition a dominance of quadrics fitting method, compared to the differential geometry method, is obvious.

In the case where 3D mesh is subjected to a simplification process, red dots mark vertices whose coordinates remained unchanged after the mentioned process.

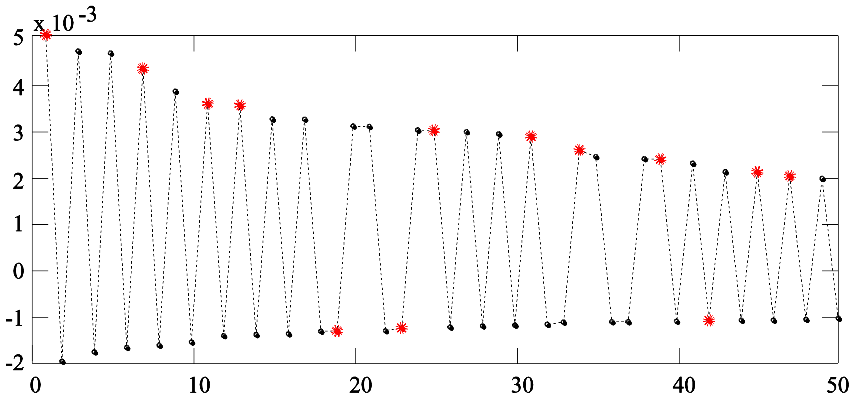

Fig. 5 Plot of stabile vertices (red dots) extracted by Gaussian curvature estimation computed by fitting quadric method.

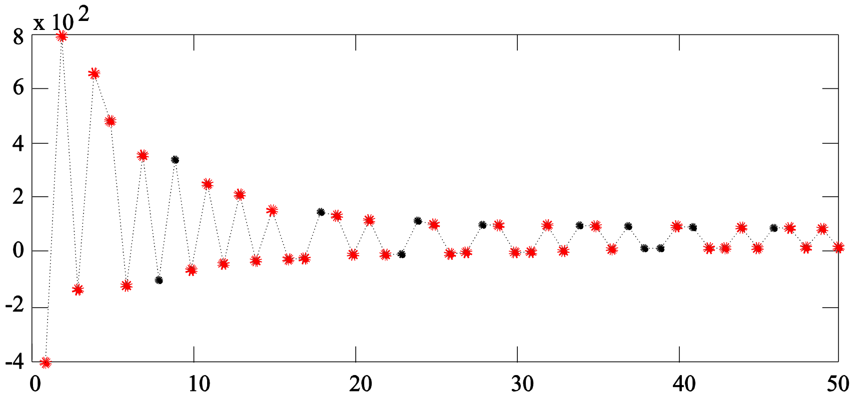

Fig. 6 Plot of stabile vertices extracted by Gaussian curvature estimation, computed by local differential geometry method [11][14].

It is obvious that the estimated regions are rather wide with the fitting quadrics method, so we can see in figures (Fig. 5) wrong selection of vertices with high curvature values. Our method of calculating local differential geometry has a significant advantage in extraction of stable vertices (Fig. 6). These vertices survived decimation during the simplification process and, at the same time, represent vertices with very important geometric features. One can see the order of magnitude of a greater number of red marked vertices shown in Fig 6, then ones that are shown in Fig 5.

6. Conclusion

Explanations and descriptions of the most commonly used methods for local curvature estimation show that nontrivial calculus of curvature can lead to mistakes and be treated as noise holders of a shape. We conclude that the effect of local curvature estimation is essential for the proper treatment of geometry in the filtering or fairing surfaces of the given 3D mesh . We also proposed new algorithms for solving problems on every step of these processes.

Solutions which we have suggested are simple and quick from the viewpoint of mathematical calculations, because they assume only a few extra steps in the algorithm of curvature evaluation. This will speed up the later steps of curvature energy computation and filtering process. Our new approach provides more accurate results then other approaches, and also reduces the possibility of 3D mesh geometry damages.

References

[1] P. Dombrowski, 150 Years after Gauss' Disquisitiones Generales Circa Superficies Curvas: with the Original Text of Gauss, Société mathématique de France, Paris, 1979.

[2] B. Riemann, "On the hypotheses which lie at the Bases of Geometry," Nature, vol. 8, no. 183, pp. 14–17, December 1998. [Online]. Available: http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/Geom.pdf

[3] M. D. Swanson, M. Kobayashi, A. H. Tewfik, "Multimedia data-embedding and watermarking technologies," in *Proceedings of the IEEE*, vol. 86, no. 6, pp 1064–1087, 1998. [Online]. Available: http://dx.doi.org/10.1109/5.687830

[4] F. Hartung, B. Girod, "Digital watermarking of raw and compressed video," in *Proceedings of SPIE Digital Compression Technologies and Systems for Video Communication*, vol. 2945, pp. 205–213, 1996. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.53.6776

[5] P. Heckbert, M. Garland, "Survey of polygonal surface simplification algorithms," Siggraph 97 Course Notes, no. 25, ACM Press, New York, 1997. [Online]. Available: http://www.dtic.mil/cgi-bin/GetTRDoc?AD=ADA461098

[6] B. Vasic, B. Vasic, "Simplification resilient LDPC-coded sparse-QIM watermarking for 3D-meshes," IEEE Transaction on Multimedia, vol. 15, no. 7, pp. 1532–1542, 2013. [Online]. Available: http://dx.doi.org/10.1109/TMM.2013.2265673

[7] M. Spivak, *A comprehencive introduction to differential geometry*, vol. 2, ed. 2, Publish or Perish Inc., Wilmington, Delaware, 1979.

[8] M. Meyer, M. Desbrun, P. Schroder, A. H. Barr, "Discrete differential-geometry operators for triangulated 2-manifolds mesh," in *Proceedings of VisMath'02*, Berlin-Dahlem, Germany, pp. 237–247, 2002. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.24.3427

[9] S. Petitjean, "A survey of methods for recovering quadrics in triangle meshes," ACM Computing Surveys, vol. 34 no. 2, pp. 211–262, 2002. [Online]. Available: https://dl.acm.org/purchase.cfm?id=508354&CFID=416637604&CFTOKEN=86550331

[10] R. V. Garimella, B. K. Swartz, "Curvature estimation for unstructured triangulations of surfaces," Technical Report LA-UR-03-8240, Los Alamos National Laboratory, 2003. [Online]. Available: http://math.lanl.gov/Research/Publications/Docs/garimella-2003-curvature.pdf

[11] B. Vasic, "Ordered statistics vertex extraction and tracing algorithm (OSVETA)," Advances in Electrical and Computer Engineering, vol. 12, no 4, pp. 25–32, 2012. [Online]. Available: http://www.aece.ro/archive/2012/4/2012_4_4.pdf

[12] H. Hoppe, T. DeRose, T. Duchamp, J. McDonald, W. Stuetzle, "Mesh optimization," in *Proceedings of SIGGRAPH 93*, Anaheim, CA, USA, pp. 19–26, 1993. [Online]. Available: https://dl.acm.org/purchase.cfm?id=166119&CFID=416637604&CFTOKEN=86550331

[13] I. Guskov, Z. Wood, "Topological noise removal," in *Proceedings of Graphics Interface 2001 GRIN'01*, Toronto, Ontario, Canada, pp. 19–26, 2001. [Online], Available: http://dl.acm.org/citation.cfm?id=780990#

[14] B. Vasic, "Ordered statistic vertex extraction and tracing algorithm (OSVETA)," MatLab Software. [Online]. Available: http://www.batavasic.com/research/OSVETA.zip