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OPTIMAL COMPRESSOR FUNCTION APPROXIMATION UTILIZING Q-FUNCTION APPROXIMATIONS

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Abstract. *In this paper, we have proposed two solutions for approximating the optimal compressor function for the Gaussian source. Both solutions are based on approximating Q-function with exponential functions. These solutions differ in that the second one is given in parametric form and can be considered as a more general solution compared to the first one, which is a special case of the second solution for a specific value of the mentioned parameter. The approximated functions proposed in the paper facilitate designing scalar companding quantizers for the Gaussian source since with the application of these functions main difficulties occurred in designing the observed quantizers for the Gaussian source can be overcome.*

Key words: *compressor function, Gaussian source, Q-function approximation, scalar companding quantization*

1. Introduction

It is well known that the software implementation of optimal companding quantizers encounters many difficulties when Gaussian source is assumed in designing [1]-[5]. This is a consequence of nonexistence of the closed-form formula for the optimal compressor function for the Gaussian source. Motivated with this drawback in designing optimal companding quantizers for the Gaussian source, we have focused our research toward solving the observed problem, i.e. toward determining some closed-form formula for the function which approximates the optimal compressor function the closest possible. What we propose is some helpful solutions for overcoming this problem. We study the Gaussian source because it arises in numerous applications. For instance, discrete Fourier transform coefficients are often considered to be the output of a Gaussian source [1].

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Moreover, the first approximation to the short-time-averaged probability density function (PDF) of speech amplitudes is provided by the Gaussian PDF.

The need for simplifying the design procedure of scalar companding quantizers for the Gaussian source, where the goal is to preserve performance as much as possible, has been the driving force behind numerous papers (for instance [2]-[5]), including this one. In [3]-[5], it has been shown that one of the manners to achieve this goal is based on the linearization of the compressor function and the resulting quantizers are known as piecewise linear scalar companding quantizers. Unlike the previous papers addressing this problem, we have focused our research toward finding some nonlinear approximations of the optimal compressor function for the Gaussian source, where the goal is to provide closed-form formulas for designing quantizers having compressor function similar the optimal compressor function. In what follows we describe in detail two manners for achieving this goal.

2. Problem Observation

In this section we recall in brief the theory of scalar companding quantization of the Gaussian source. In particular, we highlight the problem encountered in designing scalar companding quantizers for the Gaussian source. We assume, as in [2], that information source is the Gaussian source with memoryless property, zero mean value and variance σ^2

$$p(x,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{1}$$

and that compressor function is of the form [1], [6]

$$c(x,\hat{\sigma}) = x_{\max}\,\mathrm{sgn}(x)\frac{\int_0^{|x|} p^{1/3}(t,\hat{\sigma})dt}{\int_0^{x_{\max}} p^{1/3}(t,\hat{\sigma})dt}, \quad |x| \le x_{\max}\ , \tag{2}$$

so that for the given support region threshold $x_{\max}$ [7]

$$x_{\max} = \hat{\sigma}\sqrt{6\ln N}\left[1 - \frac{\ln(\ln N)}{4\ln N} - \frac{\ln(3\sqrt{\pi})}{2\ln N}\right], \tag{3}$$

and variance $\hat{\sigma}^2$, the MSE (mean squared error) distortion of the observed N-level companding quantizer is minimized. By substituting (1) in (2) one can easily determine the optimal compressor function of the companding qunatizer designed for the Gaussian source of variance $\hat{\sigma}^2$

$$c(x,\hat{\sigma}) = x_{\max}\frac{\mathrm{erf}\left(\frac{x}{\sqrt{6}\hat{\sigma}}\right)}{\mathrm{erf}\left(\frac{x_{\max}}{\sqrt{6}\hat{\sigma}}\right)}\ \mathrm{sgn}(x),\ \ |x| \le x_{\max}\ , \tag{4}$$

or equally

$$c(x,\hat{\sigma}) = x_{\max} \frac{1-2Q\left(\dfrac{x}{\sqrt{3}\hat{\sigma}}\right)}{1-2Q\left(\dfrac{x_{\max}}{\sqrt{3}\hat{\sigma}}\right)} \ \mathrm{sgn}(x), \ |x| \le x_{\max}, \tag{5}$$

where

$$\mathrm{erf}\,(u) \equiv 1-2Q(\sqrt{2}u), \tag{6}$$

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} \exp(-t^2/2)dt, \tag{7}$$

are well known erf and Q function [1]. So defined compressor function further determines the representation levels

$$y_i = y_i(\hat{\sigma}) = c^{-1}\left((2i-1)\ \frac{x_{\max}}{N}, \hat{\sigma}\right), \ i = 1, 2, \ldots, N/2, \tag{8}$$

and the decision thresholds of the observed quantizer

$$t_i = t_i(\hat{\sigma}) = c^{-1}\left(2i\ \frac{x_{\max}}{N}, \hat{\sigma}\right), \ i = 0, 1, \ldots, N/2. \tag{9}$$

Since we have assumed symmetrical PDF (1), the symmetry in designing the observed quantizer follows, so that it holds $y_i = -y_{-i}$, $i = 1, 2, \ldots, N/2$, $t_i = -t_{-i}$, $i = 1, 2, \ldots, N/2$. From Eqs. (4)-(9) it follows

$$y_i = \sqrt{6}\hat{\sigma}\ \mathrm{erf}^{-1}\left(\frac{(2i-1)}{N}\ \mathrm{erf}\left(\frac{x_{\max}}{\sqrt{6}\hat{\sigma}}\right)\right), \ i = 1, 2, \ldots, N/2, \tag{10}$$

$$t_i = \sqrt{6}\hat{\sigma}\ \mathrm{erf}^{-1}\left(\frac{2i}{N}\ \mathrm{erf}\left(\frac{x_{\max}}{\sqrt{6}\hat{\sigma}}\right)\right), \ i = 0, 1, \ldots, N/2. \tag{11}$$

As highlighted in the papers [2]-[5], and, as one can see from Eqs. (4)-(7) and (10), (11), to preform companding quantization for a Gaussian source, the two basic blocks of companding quantizer, named compressor and expandor, should preform numerical integration (observe Eqs. (4)-(7)) and should solve integral equations (observe Eqs. (10), (11)), which is not that simple nor from the software nor from the hardware point of view. In what follows we propose two solutions for overcoming the observed problem.

3. The First Solution to the Observed Problem

Let us define an approximation of the optimal compressor function as follows

$$c_{\mathrm{app}}(x,\hat{\sigma}) = x_{\max}\,\frac{1-2F\left(\dfrac{x}{\sqrt{3}\hat{\sigma}}\right)}{1-2F\left(\dfrac{x_{\max}}{\sqrt{3}\hat{\sigma}}\right)}\ \mathrm{sgn}(x),\ \ |x|\le x_{\max}\,. \tag{12}$$

One can easily notice that the last equation is derived from Eq.(5), where the *Q*-function, $Q(\cdot)$, is directly substituted with some approximation function $F(\cdot)$. Among the available *Q*-function approximations having relatively simple analytical form [8]-[14], let us chose the one from [14], which is of the simplest analytical form

$$F^{\mathrm{GU}}(x) = \frac{1}{2}\exp\{-x^2/2\}\,. \tag{13}$$

Note that, the simplicity is here an evident need. The notation of the index GU follows from the abrivations of the authors surnames from [14] (Gasull and Utzet). Substituting Eq. (13) in Eq. (12) results in

$$c_{\mathrm{app}}^{\mathrm{GU}}(x,\hat{\sigma}) = x_{\max}\,\frac{1-\exp\left\{-\dfrac{x^2}{6\hat{\sigma}^2}\right\}}{1-\exp\left\{-\dfrac{x_{\max}^2}{6\hat{\sigma}^2}\right\}}\ \mathrm{sgn}(x),\ \ |x|\le x_{\max}\,. \tag{14}$$

For so obtained compressor function, similarly as in Eqs. (8) and (9), we further determine the representation levels and the decission thresholds from

$$y_i^{\mathrm{GU}} = c_{\mathrm{app}}^{\mathrm{GU}\,-1}\left((2i-1)\ \frac{x_{\max}}{N},\hat{\sigma}\right),\ \ i=1,2,\ldots,N/2\,, \tag{15}$$

$$t_i^{\mathrm{GU}} = c_{\mathrm{app}}^{\mathrm{GU}\,-1}\left(2i\ \frac{x_{\max}}{N},\hat{\sigma}\right),\ \ i=0,1,\ldots,N/2\,, \tag{16}$$

so that we end up with

$$y_i^{\mathrm{GU}} = \sqrt{6\hat{\sigma}^2\left[\ln(N)-\ln\left(N-(2i-1)\ \left(1-\exp\left\{-\frac{x_{\max}^2}{6\hat{\sigma}^2}\right\}\right)\right)\right]}\,,\ \ i=1,2,\ldots,N/2\,, \tag{17}$$

$$t_i^{\mathrm{GU}} = \sqrt{6\hat{\sigma}^2\left[\ln(N)-\ln\left(N-2i\ \left(1-\exp\left\{-\frac{x_{\max}^2}{6\hat{\sigma}^2}\right\}\right)\right)\right]}\,,\ \ i=0,1,\ldots,N/2\,. \tag{18}$$

Here, one can observe that with the straightforward application of the approximation from [14], we have managed to derive the closed form formulas for determining the representation levels and the decission thresholds. In the numerical results section we will discuss about the usefulness of the observed solution.

4. THE SECOND SOLUTION TO THE OBSERVED PROBLEM

Let us propose a novel function of the form

$$F^{\mathrm{n}}(x) = \frac{1}{2}\exp\{-a\,x^2\}\,, \tag{19}$$

which can be considered as a class of very simple exponential parametric approximations of the *Q*-function having a special case for $a = 0.5$. In other words, for $a = 0.5$, it holds the equality of the proposed two solutions to the observed problem

$$F^{\mathrm{n}}(x)_{|a=0.5} = F^{\mathrm{GU}}(x)\,. \tag{20}$$

Combining Eqs. (12) and (19) results in

$$c_{\mathrm{app}}^{\mathrm{n}}(x,\hat{\sigma}) = x_{\max}\,\frac{1-\exp\left\{-a\,\dfrac{x^2}{3\hat{\sigma}^2}\right\}}{1-\exp\left\{-a\,\dfrac{x_{\max}^2}{3\hat{\sigma}^2}\right\}}\ \operatorname{sgn}(x),\ \ |x| \le x_{\max}\,. \tag{21}$$

For the given value of the parameter $a = a^{\mathrm{n}}$, the representation levels and the decision thresholds are determined from

$$y_i^{\mathrm{n}} = {c_{\mathrm{app}}^{\mathrm{n}}}^{-1}\left((2i-1)\ \frac{x_{\max}}{N}, \hat{\sigma}\right),\ i = 1, 2, \ldots, N/2\,, \tag{22}$$

$$t_i^{\mathrm{n}} = {c_{\mathrm{app}}^{\mathrm{n}}}^{-1}\left(2i\ \frac{x_{\max}}{N}, \hat{\sigma}\right),\ i = 0, 1, \ldots, N/2\,, \tag{23}$$

so that we derive

$$y_i^{\mathrm{n}} = \sqrt{\frac{3\hat{\sigma}^2}{a^{\mathrm{n}}}\left[\ln(N) - \ln\left(N-(2i-1)\ \left(1-\exp\left\{-a^{\mathrm{n}}\ \frac{x_{\max}^2}{3\hat{\sigma}^2}\right\}\right)\right)\right]},\ i = 1, 2, \ldots, N/2\,, \tag{24}$$

$$t_i^{\mathrm{n}} = \sqrt{\frac{3\hat{\sigma}^2}{a^{\mathrm{n}}}\left[\ln(N) - \ln\left(N-2i\ \left(1-\exp\left\{-a^{\mathrm{n}}\ \frac{x_{\max}^2}{3\hat{\sigma}^2}\right\}\right)\right)\right]},\ i = 0, 1, \ldots, N/2\,. \qquad (25).$$

In a number of papers considering the observed problem, the design of the quantizer has been performed for $\hat{\sigma}^2 = 1$ [2]-[7]. As already indicated and shown, this assumption do not diminish the significance of the observed analysis because of the fact that it can be simply extended to the case where $\hat{\sigma}^2 \neq 1$. For that reason, in this paper we assume that $\hat{\sigma}^2 = 1$. In what follows, due to simplicity reasons, we actually omitt the notation of $\hat{\sigma}^2$.

It is now questionable how to determine the value of the parameter *a*. What we propose is to determine the value of the parameter *a* so that the absolute error in approximating compressor function is minimal one. Formally, we set the following mathematical problem

$$a^{\mathrm{n}} = \arg\min_{a} \frac{1}{x_{\max}} \int_0^{x_{\max}} (c(x) - c_{\mathrm{app}}^{\mathrm{n}}(x))dx\,, \tag{26}$$

which can also be written as

$$\left.\frac{\partial\left[\displaystyle\int_0^{x_{\max}} \frac{1-2Q\left(\frac{x}{\sqrt{3}}\right)}{1-2Q\left(\frac{x_{\max}}{\sqrt{3}}\right)}dx - \int_0^{x_{\max}} \frac{1-\exp\left\{-\frac{a}{3}x^2\right\}}{1-\exp\left\{-\frac{a}{3}x_{\max}^2\right\}}dx\right]}{\partial a}\right|_{a=a^{\mathrm{n}}} = 0\,. \tag{27}$$

By intorducing the approximation

$$2Q\left(\frac{x_{\max}}{\sqrt{3}}\right) \approx \exp\left\{-\frac{a}{3}x_{\max}^2\right\}\Big|_{a=a^{\mathrm{n}}}\,, \tag{28}$$

whose validity we will demonstrate in the numerical results section, we simplify the problem to be solved, so that we face with the following optimization problem

$$\left.\frac{\partial\left[\displaystyle\int_0^{x_{\max}} \left(1-2Q\left(\frac{x}{\sqrt{3}}\right)\right)dx - \int_0^{x_{\max}} \left[1-\exp\left\{-\frac{a}{3}x^2\right\}\right]dx\right]}{\partial a}\right|_{a=a^{\mathrm{n}}} = 0\,. \tag{29}$$

The solution to the above mentioned optimization problem can not be obtained as closed-form solution. However, we can solve this iteratively

$$a^{(i+1)} = \left(\frac{\sqrt{3\pi}}{4} \frac{P\left(\frac{3}{2}, \frac{a^{(i)}}{3}x_{\max}^2\right)}{\displaystyle\int_0^{x_{\max}} \left(1-2Q\left(\frac{x}{\sqrt{3}}\right)\right)dx}\right)^{\frac{2}{3}}\,, \tag{30}$$

where

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^x \exp\{-t\}\ t^{a-1}dt\,, \tag{31}$$

and $\Gamma(\cdot)$ denotes Gamma function.

5. Numerical Results

This section discusses the results obtained by applying the two observed approximations of the Q-function. Let us first consider Fig. 1, where $x_{\max}/\sqrt{3}$ dependence on bit rate $R = \log_2 N$ is presented, where $x_{\max}$ is calculated from Eq. (3). As we have expected and it is already known, the value of the support region threshold increases with the bit rate. Fig. 1 has infact the purpose of illustrarting the range of arguments of the *Q*-function that is of interest for our analysis. Note that we have assumed bit rates $R \geq 5$ bit/sample, commonly observed in the asymptotic analysis of scalar companding quantizers. One can observe that for the considered bit rates, and, accordingly, for the considered arguments of the *Q*-function, the function takes very small values (see Fig. 2). Moreover, one can easily calculate that when *a* takes values near 0.5 and x_{max} is given by Eq. (3), the right hand side of Eq. (28) rapidly approaches to zero. Accordingly, our assumption given by Eq. (28) is here justified.

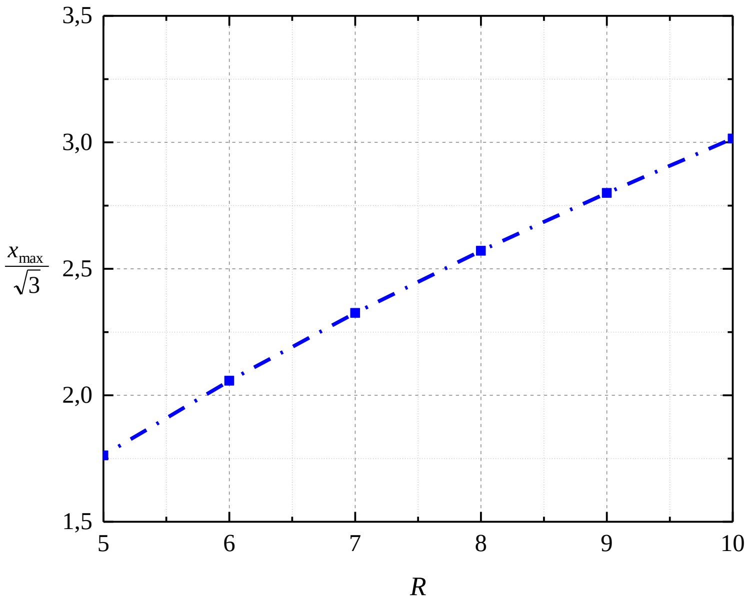

Fig. 1 $x_{max}/\sqrt{3}$ dependence on bit rate *R*

For the second solution, by examining different initial solutions $a^{(0)}$ for the iterative process given by Eq. (30) we have obtained the results approaching the ones achieved with the first solution, also described in the paper. This can be seen in Fig. 3, where the optimal compressor function and the obtained approximations are shown for the case where $N = 128$, $x_{max} = 4.0274$. Interestingly, by examining different initial values of the iterative process we have determined that after only one iteration the approximated compressor functions $c_{app}^{GU}(x)$ and $c_{app}^{n}(x)$ are similar ones. For instance, one can see $c_{app}^{GU}(x)$ and $c_{app}^{n}(x)$ obtained for $a^{(1)} = 0.4332$, where $a^{(0)} = 1$. It possibly indicates the manner of obtaining the approximation from [14], since this manner has not been explained in [14]. In fact, the approximation from [14] has been obtained heuristically. This solution can be further improved by examining two or more regions of the input signal values and by proposing more suitable approximations, which, for the above mentioned regions, are lower bound approximations as two considered solutions. This analysis we left for the future research.

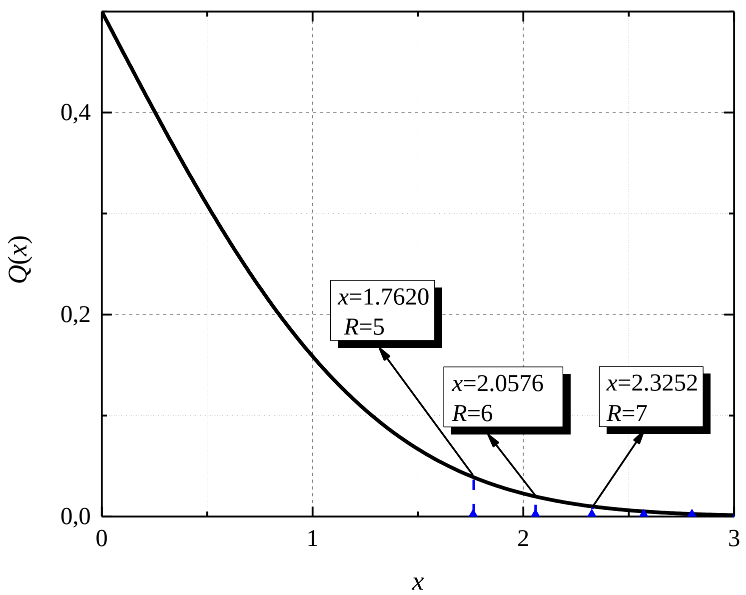

Fig. 2 *Q*-function with noted values of interest for the analysis

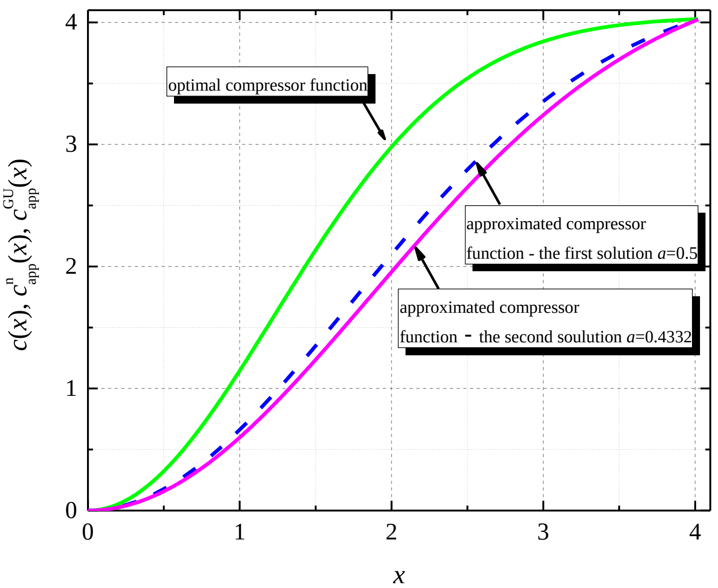

Fig. 3 Optimal compressor function $c(x)$, the novel approximated compressor function $c_{\text{app}}^{\text{n}}(x)$, the approximated compressor function with GU approximation $c_{\text{app}}^{\text{GU}}(x)$, for Gaussian PDF of unit variance, N = 128, $x_{\max}$ = 4.0274

4. Conclusion

In this paper, we have proposed two solutions for approximating the optimal compressor function for the Gaussian source. We have shown that both solutions, based on exponential approximations of the *Q*-function, provide the derivation of the closed-form formulas that specify relatively simple design of the observed quantizer. We have anticipated that the proposed solutions can be further improved by examining two or more regions of the input signal values and by proposing more suitable approximation for the above mentioned regions, which we have left for the future research. Eventually, by taking into account that with an application of a properly chosen filtering technique to non-Gaussian source the sequences, which are approximately independent and Gaussian are produced, one can conclude that our proposal can be widely applicable.

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