FACTA UNIVERSITATIS
Series: **Automatic Control and Robotics** Vol. 17, N° 3, 2018, pp. 191 - 204
https://doi.org/10.22190/FUACR1803191D

A NEW METHOD FOR DESIGN OF SELECTIVE DIGITAL IIR FILTERS WITH ARBITRARY PHASE*

UDC (621.372.54:004.382)

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Abstract. *In this paper the design of selective digital filters that consists of a parallel connection of two all-pass sub-filters is presented. The phase of these filters has given an arbitrary shape $\phi(\omega)$ in both pass-band and stop-band. The proposed method allows the calculation of selective filters with elliptic-like magnitude characteristic. Equations given in the paper are general and suitable for design of filters with arbitrary phase. The efficiency of the method is demonstrated on design of filters with piecewise linear and quadratic phases.*

Key words: *all-pass filter, arbitrary phase, complementary pair, iterative method, phase corrector, phase approximation*

1. INTRODUCTION

In many practical digital filter applications in addition to the desired magnitude characteristic, a preferred feature is linear or approximately linear phase to minimize the signal distortion. This is for this reason that in existing literature majority of the papers refer to constant or linear phase filter design [1, 2]. In that case the finite impulse response (FIR) filters are a natural solution, because they can provide exactly linear phase if coefficients have symmetry. Compared with the FIR filters [3, 4], the infinite impulse response (IIR) filters can realize the same magnitude specifications and enable high-speed signal processing with considerably lower order. It is important to emphasize that the phase of the IIR filter is nonlinear, so the phase corrector is required to achieve the prescribed phase characteristic. To obtain a desired phase or group delay characteristic very often the order of corrector (all-pass filter which does not affect an already realized

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Received October 26, 2018
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* **Acknowledgement**: Research is supported by Ministry of Education, Science and Technological Development, Republic of Serbia (Grant No. III044006 and Grant No. TR33035).

magnitude characteristic) is significantly higher than the order of selective filter itself. The IIR filter design is commonly observed as a phase or magnitude approximation problem [5] or as a group delay approximation task [6].

High selectivity digital filters, like the Chebyshev and elliptic filters, demand great number of bits for presentation of the transfer function coefficients to preserve designed characteristics in the realization step. The obtained digital filter transfer function can be realized as a direct, parallel or cascaded structure.

Besides mentioned standard structures for filter implementation, selective filter could be also realized by two all-pass sub-filters connected in parallel [7-9] (see Fig. 1). The parallel all-pass structure exhibits small sensitivity in all pass-bands [10-12], that makes them very suitable for practical implementation. At the same time, it is important to note that a real benefit will be achieved in the case when both of two complementary filters are of interest because only one additional adder is enough to realize the second filter. For that reason and because of their high computational efficiency [13-15], both one-dimensional and two-dimensional filters still remain in the focus of researchers [16-20].

In this paper the effective procedure for calculation of coefficients of the all-pass filters from parallel branches is proposed, in such a way that the phase of these filters has a shape given by $\phi(\omega)$, in the whole frequency domain $\omega \in [0, \pi]$. The magnitude of a selective filter realized by a parallel connection of two all-pass filters depends on all-pass phase difference. At the same time, the phase of selective filter is an average value of phases of all-pass sub-filters. It is a very convenient property of this structure, because it gives an opportunity to achieve a predefined magnitude characteristic with the simultaneously obtained desired phase characteristic. That means parallel all-pass filters provide the desired magnitude and phase or group delay simultaneously with no need for correctors. As previously mentioned, relationship between magnitude and phase is straightforward which allow design procedure to be defined as a phase approximation problem. We choose equiripple approximation of the phase error. As a consequence, the magnitude characteristic of the resulting filter is elliptic-like i.e. it is also equiripple in the case of approximately linear phase filters.

The procedure for calculation of the coefficients of an all-pass function is based on solving the system of linear equations. The proposed method allows designing of low-pass and high-pass filters as well as filters with arbitrary number of bands. A common property of all these filters is, as previously mentioned, that a phase is given by $\phi(\omega)$ in the entire frequency domain including stop-bands.

In the case of a quadratic phase approximation [12], group delay of such filters is either linearly decreasing or linearly increasing, which makes them suitable for achieving a chirp signal compression or expansion. Thanks to this property, they can be implemented in radar and satellite systems, as well as in a wide range of telecommunication systems.

2. APPROXIMATION

The all-pass digital transfer function is presented as

$$H(z) = z^n \frac{P_n(z^{-1})}{P_n(z)} = z^n \frac{\sum_{i=0}^{n} a_i z^{-i}}{\sum_{i=0}^{n} a_i z^{i}} \tag{1}$$

where n is the degree of the polynomial P_n(z), and a_i, $i = 0,1,2,...n$ are its coefficients.

On the unit circle $z = e^{j\omega}$, function (1) can be given in the next form:

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}, \tag{2}$$

where $\left|H(e^{j\omega})\right| = 1$ at all frequencies, and the phase is

$$\phi(\omega) = n\omega - 2\arctan \frac{\sum_{i=0}^{n} a_i \sin(i\omega)}{\sum_{i=0}^{n} a_i \cos(i\omega)} \tag{3}$$

The transfer functions of digital IIR filters, realized by two all-pass filters connected in parallel as given in Fig. 1, could be presented with the next expression:

$$F(z) = \frac{1}{2}[H_1(z) + (-1)^p H_0(z)], \;\; p = 0 \;\; or \;\; 1. \tag{4}$$

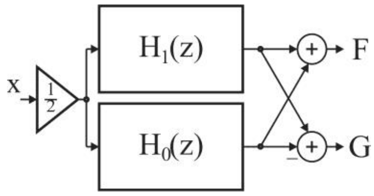


Fig. 1 Selective filter obtained by two all-pass filters in parallel

For $p = 0$ it can be written,

$$\left|F(e^{j\omega})\right| = \left|\cos \frac{\phi_1(\omega) - \phi_0(\omega)}{2}\right| \tag{5}$$

$$\phi_{F(\omega)} = \frac{\phi_1(\omega) + \phi_0(\omega)}{2}. \tag{6}$$

If the parameter p has the value $p = 1$,

$$\left|G(e^{j\omega})\right| = \left|\sin \frac{\phi_1(\omega) - \phi_0(\omega)}{2}\right| \tag{7}$$

$$\phi_{G(\omega)} = -\frac{\pi}{2} + \frac{\phi_1(\omega) + \phi_0(\omega)}{2} \tag{8}$$

is valid, where $\phi_0(\omega)$ and $\phi_1(\omega)$ are phases of all-pass filters $H_0(z)$ and $H_1(z)$, respectively.

The goal is to make the function (4) to have the phase of the shape same as the shape of the desired phase $\varphi(\omega)$. To achieve this, it is necessary to obtain adequate values for coefficients of the all-pass functions $H_0(z)$ and $H_1(z)$.

Let the function $H_0(z)$ approximate phase $\varphi(\omega)$ at all frequencies $\omega \in [0, \pi]$ in minimax sense. In that case

$$\phi_0(\omega_k) = n\omega_k - 2\arctan\frac{\sum_{i=0}^{n} a_i \sin(i\omega_k)}{\sum_{i=0}^{n} a_i \cos(i\omega_k)} = \varphi(\omega_k) + (-1)^{p+k}\varepsilon_0 \quad , \tag{9}$$

is valid, where ε_0 represents maximal phase deviation and ω_k, $k = 1, 2, ..., n$ are frequencies at which maxima and minima of the phase error curve are located. The parameter p has value 0 or 1 and defines the first phase error extremum to be minimum or maximum.

Equation (9) can also be rewritten in the form

$$\frac{\sum_{i=0}^{n} a_i \sin(i\omega_k)}{\sum_{i=0}^{n} a_i \cos(i\omega_k)} = \frac{\sin\beta_k}{\cos\beta_k}, \quad k = 1, 2, \ldots, n\ , \tag{10}$$

where values of parameters β_k are

$$\beta_k = \frac{1}{2}[n\omega_k - (\varphi(\omega_k) + (-1)^{p+k}\varepsilon_0)],\ k = 1, 2, \ldots, n\ , \tag{11}$$

or in a more convenient manner, the equation (10) becomes

$$\sum_{i=1}^{n} a_i \sin(i\omega_k - \beta_k) = \sin\beta_k, \quad k = 1, 2, \ldots, n \tag{12}$$

where $a_0 = 1$ is assumed.

The system of linear equations (12) could be solved using the following iterative approach. Let a_i^r represents the initial solution and $a_i^{r+1} = a_i^r + \Delta a_i^{r+1}$ is a new better solution. New values for transfer function coefficients a_i^{r+1} can be obtained, after minor transformation of (12), by solving the next system of linear equations

$$\sum_{i=1}^{n} a_i^{r+1} \sin(i\omega_k^r - \beta_k^r) = \sin\beta_k^r, \quad k = 1, 2, \ldots, n. \tag{13}$$

Calculation ends when condition $|a^{r+1} - a^r| \le \delta$ is fulfilled, where δ is arbitrary chosen small number (in all given examples the value $\delta = 10^{-10}$ is adopted).

Very often in practice the linear phase characteristic is desirable. In such a case design process is simpler because $H_0(z)$ is pure delay line of the order n. It brings no error in approximation of the desired linear phase $\varphi(\omega)$. As a consequence, total approximation error is influenced only by phase approximation error of the all-pass sub-filter $H_1(z)$. Therefore, the resulting filter will have the elliptic magnitude characteristic, taking into account the straightforward relationship between the magnitude of selective filter and phases of all-pass sub-filters, given by (5) and (7). In case of the arbitrary shape phase, the all-pass function $H_0(z)$ approximates the desired phase $\phi(\omega)$ in whole frequency band $\omega \in [0, \pi]$. As a good starting point, poles $z_i = \rho_i\, e^{j\theta i}$ positioned equidistantly inside the unit circle (for example $\rho_i = 0.8$, $i = 1,2,...,n$ and

$$\theta_i = \begin{cases} \pm\dfrac{2(i-1)}{n}\pi,\ i=1,2,\dots,(n+1)/2 & \text{for odd } n \\ \pm\dfrac{2i-1}{n}\pi,\ i=1,2,\dots,n/2 & \text{for even } n \end{cases} \tag{14}$$

can be chosen. This approach gives equiripple phase error and enough extrema for iterative procedure in which the phase $\phi_0(\omega)$, starting from a linear shape, deforms and converges to desired $\varphi(\omega)$ with as small as possible approximation error for the given filter order. At the end of this process, the phase $\phi_0(\omega)$ will approximate the desired phase $\varphi(\omega)$ in mini-max sense and with the optimal maximal deviation ε_0.

According to (6) and (8) it is possible to create a selective filter with a phase that approximates the given ideal phase $\varphi(\omega)$ in all pass-bands and stop-bands, if coefficients of all-pass function $H_1(z)$ are adequately determined. If the phase $\varphi_1(\omega)$ approximates the given ideal phase $\varphi(\omega)$ in mini-max sense, with the maximal allowed phase error ε_1, with offset of π radian in adjacent regions, the resulting filter will have in the worst case maximal phase error of value $\varepsilon_0 + \varepsilon_1$. In this way, the desired phase shape is achieved with complete control of maximal allowed phase error. At the same time, according to (5) and (7), magnitude of selective filter is uniquely defined and also under control. The parallel configuration gives an opportunity for both complementary filters to be realized at the same time, with only one additional adder. In accordance with equations (5) and (7), in the regions where phase difference between $\phi_0(\omega)$ and $\phi_1(\omega)$ is close to $2k\pi$ radians, pass-bands are obtained. On the other hand, in the regions where $\phi_0(\omega)$ and $\phi_1(\omega)$ differ for $(2k+1)\pi$ radians, stop-bands are realized.

We need to point out the existence of two different cases. If only one selective filter is of interest, the procedure is more or less straightforward. Starting from the given allowed attenuations in pass-bands and stop-bands, maximal phase errors ε_k can be calculated. To reach given limitations, more poles of the transfer function $H_1(z)$ will be located in stop-bands, compared to pass-bands. For minimal attenuation of $a_{\min} = 40\ dB$ in the stop-band, a very small phase error is acceptable ($\varepsilon = 0.02\ rad$). The phase error $\varepsilon = 0.0063\ rad$ corresponds to the attenuation of $a_{\min} = 50\ dB$. In the pass-band, the situation is a little bit different. Maximal pass-band attenuation $a_{\max} = 3\ dB$ could be achieved with $\varepsilon = 1.57\ rad$ phase error. Going further $\varepsilon = 0.3\ rad$ phase error corresponds to $a_{\max} = 0.1\ dB$ pass-band attenuation, which is still a significant phase error compared to the stop-band error. If both complementary selective digital filters are of interest, in the design process we take care only about stop-bands information. Thus, very small phase error values are allowed in all bands. This leads to very small maximal attenuations in pass-bands of complementary filter. For example, if minimal attenuation in a stop-band of one filter is $a_{\min} = 50\ dB$, the corresponding maximal attenuation in pass-band of complementary filter will be $a_{\max} = 4.3\cdot 10^{-5}\ dB$. Even if minimal attenuation in a stop-band of one filter is just $a_{\min} = 35\ dB$, the complementary filter will have $a_{\max} = 0.0014\ dB$ maximal attenuation in the corresponding pass-band. This confirms the validity of the approach of focusing only on information from stop-bands. The group delay of a filter depends directly on the phase $\varphi(\omega)$ shape. If $\varphi(\omega)$ is the linear function of frequency, the group delay is constant at all frequencies except in transition zones between pass-bands and stop-bands. If a phase is of the quadratic shape $\varphi(\omega) = -k_2\omega^2 - k_1\omega$, the group delay is linearly increasing for $k_2 < 0$ and linearly decreasing for $k_2 > 0$, in all frequency bands excluding transition zones. This kind of filters can be used for chirp pulse expansion ($k_2 < 0$), or chirp pulse compression ($k_2 > 0$) in radar systems.

The coefficients of the all-pass filter $H_1(z)$

$$H_1(z) = z^m \frac{P_m(z^{-1})}{P_m(z)} = z^m \frac{\sum_{i=0}^{m} b_i z^{-i}}{\sum_{i=0}^{m} b_i z^{i}} \tag{15}$$

can be determined by a similar procedure as explained for $H_0(z)$, just introducing phase offset of another π *rad* in every new band starting from $\omega = 0$ up to $\omega = \pi$. The idea will be confirmed by several examples of design of filters with linear phase, quadratic phase and combination of linear and quadratic phase, but it is applicable to any other shape.

The coefficients of the all-pass filter $H_1(z)$ can be determined by equiripple approximation of the given phase $\varphi(\omega)$,

$$\phi_1(\omega_k) = m\omega_k - 2\arctan\frac{\sum_{i=0}^{m} b_i \sin(i\omega_k)}{\sum_{i=0}^{m} b_i \cos(i\omega_k)} = -(l-1)\pi + \varphi(\omega_k) + (-1)^{p+k}\varepsilon_l ,$$
$$l = 1, 2, ..., l_1, k = 1, 2, ..., m \tag{16}$$

where the parameter $p = 0$ or 1 determines whether the first extremum will be minimum or maximum and m is the order of the polynomial $P_m(z)$ and b_i, $i = 0,1,2,...,m$ are the polynomial coefficients. The total number of bands is given with l_1.

It can be rewritten

$$\frac{\sum_{i=0}^{m} b_i \sin(i\omega_k)}{\sum_{i=0}^{m} b_i \cos(i\omega_k)} = \tan\left\{\frac{1}{2}[m\omega_k + (l-1)\pi - \varphi(\omega_k) - (-1)^{p+k}\varepsilon_l]\right\}$$
$$l = 1, 2, ...l_1, k = 1, 2, ..., m \tag{17}$$

and after additional simplification

$$\frac{\sum_{i=0}^{m} b_i \sin(i\omega_k)}{\sum_{i=0}^{m} b_i \cos(i\omega_k)} = \tan\beta_l$$
$$l = 1, 2, ...l_1, k = 1, 2, ..., m \tag{18}$$

where the values of β_l are given by

$$\beta_l = \frac{1}{2}[m\omega_k + (l-1)\pi - \varphi(\omega_k) - (-1)^{p+k}\varepsilon_l]\,. \tag{19}$$

As in equation (12), the equation (18) can be presented in the following form

$$\sum_{i=1}^{m} b_i \sin(i\omega_k - \beta_l) = \sin\beta_l, \quad k = 1, 2, \ldots, m$$
$$l = 1, 2, ..., l_1, k = 1, 2, ..., m. \tag{20}$$

where $b_0 = 1$.

For example, for $l_1 = 5$, that is the number of magnitude characteristic bands, we have

$$\begin{aligned}
\beta_1 &= \frac{1}{2}[m\omega_k - \varphi(\omega_k) - (-1)^{p+k}\varepsilon_1],\ p = 0 \text{ or } 1,\ k = 1,2,\ldots,m_1 \\
\beta_2 &= \frac{1}{2}[m\omega_{m_1+k} + \pi - \varphi(\omega_{m_1+k}) - (-1)^k\varepsilon_2],\ k = 1,2,\ldots,m_2 \\
\beta_3 &= \frac{1}{2}[m\omega_{m_1+m_2+k} + 2\pi - \varphi(\omega_{m_1+m_2+k}) - (-1)^k\varepsilon_3],\ k = 1,2,\ldots,m_3 \\
\beta_4 &= \frac{1}{2}[m\omega_{m_1+m_2+m_3+k} + 3\pi - \varphi(\omega_{m_1+m_2+m_3+k}) - (-1)^k\varepsilon_4],\ k = 1,2,\ldots,m_4 \\
\beta_5 &= \frac{1}{2}[m\omega_{m_1+m_2+m_3+m_4+k} + 4\pi - \varphi(\omega_{m_1+m_2+m_3+m_4+k}) - (-1)^k\varepsilon_5],\ k = 1,2,\ldots,m_5\ ,
\end{aligned} \tag{21}$$

where m_i represents number of phase error extrema in i-th band and $\sum_{i=1}^{l_1} m_i = m.$

Substituting β_l values into the equation (20), a filter with three pass-bands and two stop-bands can be achieved. Similarly, for $l_1 = 2$, a complementary low-pass/high-pass pair of filters is realized. For $l_1 = 3$, the pass-band and its complementary stop-band filter are obtained. So it is obvious that with increasing the l_1 by two, filter with one more pass-band and stop-band is obtained.

The group delay of the resulting selective filter approximates the shape of

$$\tau(\omega) = -\frac{d\varphi(\omega)}{d\omega} = \frac{1}{2}\left[\tau_0(\omega) + \tau_1(\omega)\right], \tag{22}$$

at entire interval $\omega \in (0, \pi)$ except in transition zones between pass-bands and stop-bands, where τ_0 and τ_1 denote group delays of all-pass functions $H_0(z)$ and $H_1(z)$, respectively.

3. Phase Correctors

The described method for all-pass filter synthesis can be also used for calculation of the phase corrector coefficients. The method is based on solving the system of linear equations (11), whereby the coefficients β_k have value

$$\beta_k = \frac{1}{2}[\theta(\omega_k) + N\omega_k - \varphi(\omega_k) - (-1)^k\varepsilon]\ ,\ k = 1,\ 2,\ \ldots,\ N. \tag{23}$$

In this instance, $\theta(\omega_k)$ denotes an ideal phase to be approximated, $\varphi(\omega)$ is selective filter phase and N is the order of all-pass phase corrector.

4. Examples

To confirm the efficiency of the described procedure, several examples for three different phase shapes will be given.

4.1. Quadratic Phase

In the previous work [12] the algorithm for design of selective filters with a quadratic phase is presented. This algorithm is based on approximation of an ideal quadratic phase

$$\phi(\omega) = k_2\omega^2 + k_1\omega, \tag{24}$$

in min-max sense. Such filters can be used in the radar technique for chirp pulse compression (for $k_2 < 0$) and expansion ($k_2 > 0$), as shown is Fig. 2 in the case of signal compression.

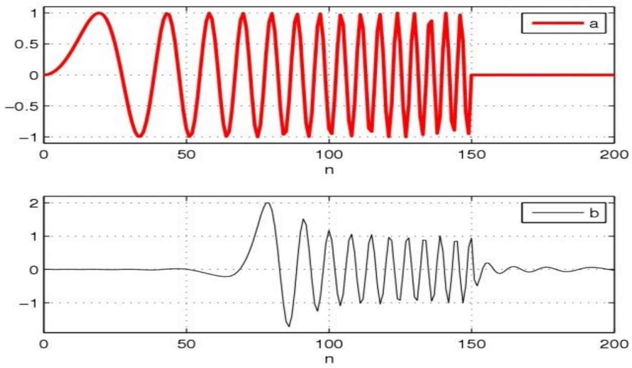
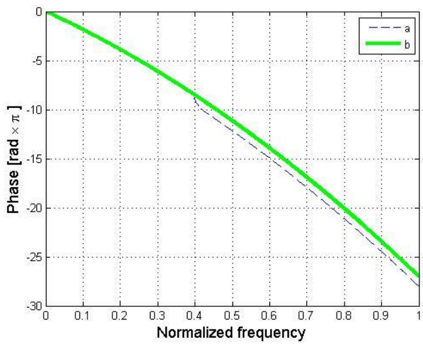


Fig. 2 (a) A input chirp signal and (b) output of filter with a quadratic phase for signal compression



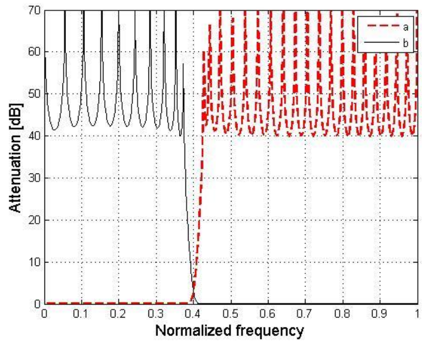


Fig. 3 The quadratic phase of all-pass sub-filters of order (a) m =28 and (b) n = 27.

Fig. 4 Attenuation of (a) low-pass and (b) complementary high-pass filter

The first example represents the quadratic phase selective filter with minimal attenuation in stop-band $a_{\min} = 40$ *dB* and maximal attenuation in pass-band $a_{\max} = 0.00043$ *dB* to achieve the same attenuation in both stop-bands. The filter is realized as a parallel connection of IIR all-pass filters of the order $m = 28$ and $n = 27$, with number of phase error local extrema $m_1 = 9$ and $m_2 = 19$ before and after phase jump of π radians, respectively, and coefficient $k_2 = -5$ from the equation (24).

The phase and attenuation of the obtained complementary filters are shown in Fig. 3 and Fig. 4, respectively. The corresponding group delay characteristics of both all-pass sub-filters are shown in Fig. 5. The phase of resulting selective filter is the average value of the presented all-pass filters phase as given with equations (6) and (8).

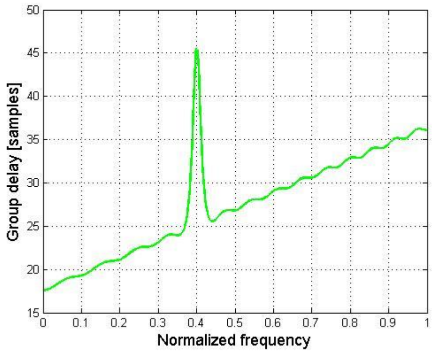


Fig. 5 The linear group delay of quadratic phase low-pass and high-pass filter

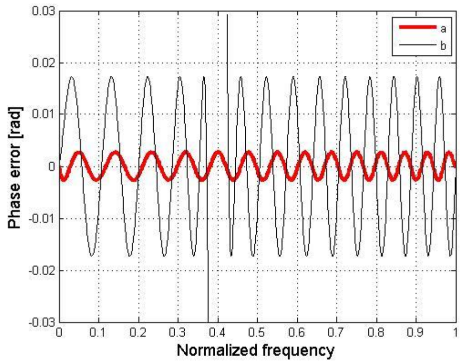


Fig. 6 Phase approximation error for all-pass filter (a) of order $n = 27$and (b) $m = 28$

The phase of all-pass filters, given in Fig. 3, approximates the ideal quadratic phase with error displayed in Fig. 6. Maximal phase error of all-pass filter of the order $n = 27$ is equal to 0.00275 radians. For the given filter order this is an optimal result, obtained by solving the system of $n + 1$ equations. The number of equations is sufficient to provide the minimal value of maximal phase deviation ε to be determined simultaneously with the coefficients of all-pass filter. The all-pass filter of the order $m = 28$ approximates an ideal quadratic phase in two bands [0, 0.375] and [0.425, 1] with maximal error 0.0175 radians, as given in Fig. 6. The boundary frequencies of two bands depend on a predetermined attenuation and filter order. In other words, a given attenuation could be achieved by all-pass filters of lower order but in that case transition zone would be wider.

4.2. Linear Phase

The novelty in this approach is that the same algorithm, presented in details in [12], can be used for design of a filter with linear phase. To achieve that, in ideal quadratic phase equation, the parameter a will be set to zero. The final solution is reached in a very small number of iterative steps with no detected problems with convergence. Through this example, the possibility of design of filters with more than two bands is shown.

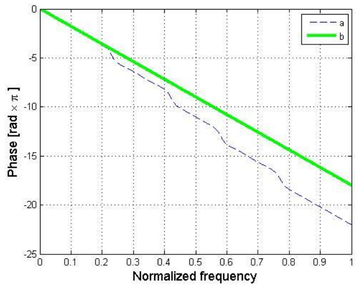


Fig. 7 The linear phase of (a) all-pass sub-filter of order $n = 22$ and (b) delay line of order $m = 18$

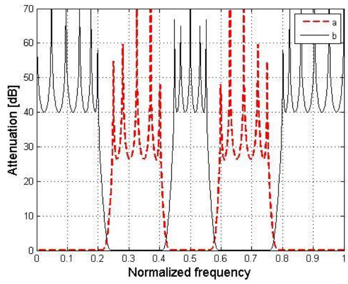


Fig. 8 Attenuation of five bands complementary filters

The second example is a linear phase selective filter with minimal attenuation in three stop-bands $a_{\min} = 40\ dB$ and maximal attenuation in two pass-bands $a_{\max} = 0.01\ dB$. The filter is realized as a parallel connection of all-pass filter of the order $n = 22$ and delay line of the order $m = 18$, with number of phase error local extrema $m_1 = m_5 = 5$, $m_2 = m_3 = m_4 = 4$ between the respective phase jumps of π radians, and $k_2 = 0$ from the equation (24). Phase and attenuation of this configuration is shown in Fig. 7 and Fig. 8, respectively. The group delay with an approximately constant value in five bands, according to linear phase, is displayed in Fig. 9. The delay line has an ideal linear phase. The overall phase error is a consequence of the existence of all-pass approximation error which is given in Fig. 10. As already mentioned, in linear phase filter design, equiripple phase error leads to equiripple magnitude error i.e. elliptic filters are obtained. During the design we take care only about specifications of one filter. The complementary filter has the minimal attenuation in a stop-band of less than 30 dB as a consequence of relatively high allowed attenuation in the pass-band $a_{\max} = 0.01\ dB$.

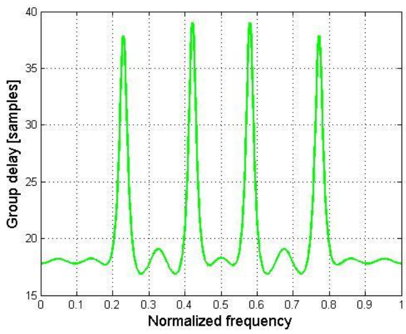


Fig. 9 The constant group delay of linear phase five bands complementary filters

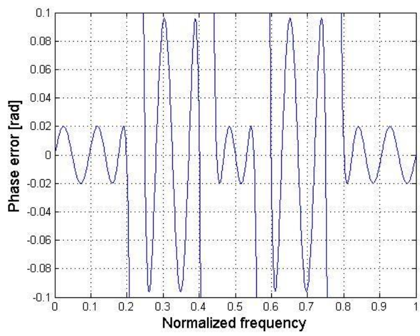


Fig. 10 Phase approximation error for all-pass filter of order $n = 22$

In complementary filter design input parameters are only stop-band attenuations because the maximal pass-band attenuation will be extremely low for stop-band attenuations higher than 30 dB.

4.3. Arbitrary Phase Shape

The third example will introduce a new class of filters with arbitrary phase shape. In this case, the phase shape is represented as a combination of quadratic and linear function.

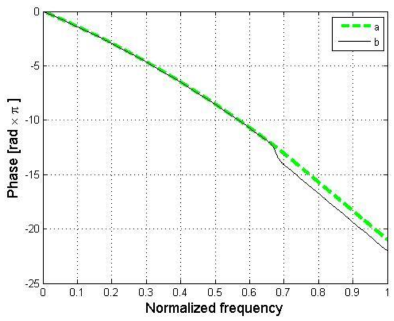


Fig. 11 The phase of all-pass sub-filter of order (a) $n = 22$ and (b) $m = 21$

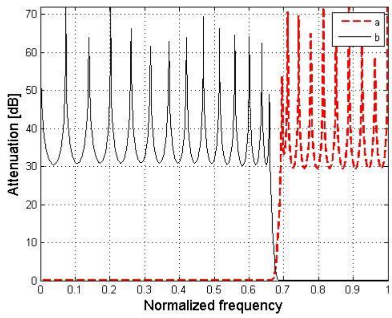


Fig. 12 Attenuation of (a) low-pass filter with quadratic phase and (b) high-pass filter with linear phase

The filter is realized as a parallel connection of all-pass filters of the order $m = 21$ without phase jump, and $n = 22$ introducing the π *rad* phase jump in every transition zone, with number of phase error local extrema $m_1 = 13$, $m_2 = 9$ before and after the phase jump, and $k_2 = -5$ from the equation (24), with minimal attenuation in the stop-band $a_{\min} = 30$ *dB* and maximal attenuation in the passband $a_{\max} = 0.01$ *dB*.

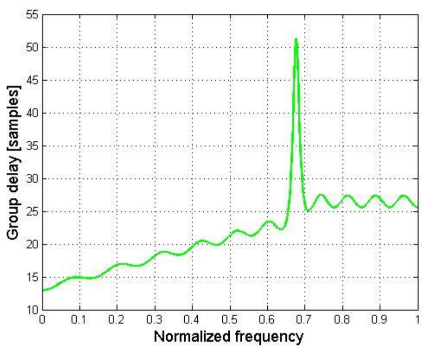


Fig. 13 The group delay of piecewise quadratic and linear phase

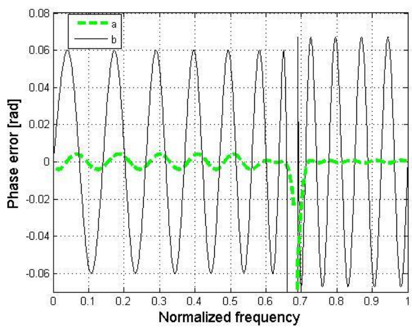


Fig. 14 Phase error of all-pass filters of order (a) $n = 21$ and (b) $m = 22$

In case of approximation of quadratic or linear phases on full frequency band, the number of extrema on the phase error curve is equal to the number of polynomial coefficients, so there are enough equations to obtain the desired filter.

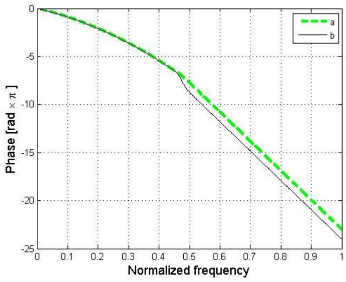


Fig. 15 The phase of all-pass sub-filter of order (a) n = 25 and (b) m = 24

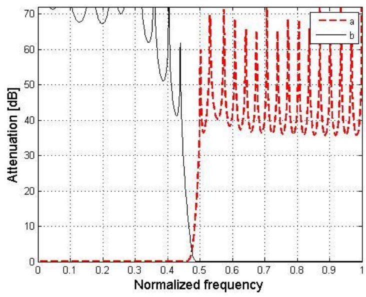


Fig. 16 Attenuation of (a) low-pass filter with quadratic phase and (b) high-pass filter with linear phase

In case of an arbitrary phase approximation the phase characteristic could be given independently for every band. In case of an all-pass sub-filter without phase jumps in transition zones, the condition of continuity of the function is fulfilled at all frequencies from 0 to π.

However, the condition of continuity of the first derivative is not satisfied in general. As a consequence, the phase error curve possesses the number of extrema that is greater than the number of coefficients in all-pass filter function. Therefore, in every iterative step, before the solving of system of linear equations, the set of relevant phase error extrema has to be retained. It should be noted that the problem is solvable and that numerous filters are designed, but that the solutions are achieved in appreciably greater number of iterations compared to the pure linear or the quadratic phase.

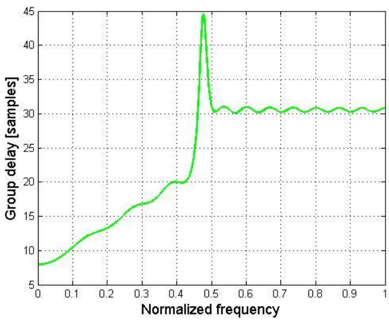


Fig. 17 The group delay of piecewise quadratic and linear phase

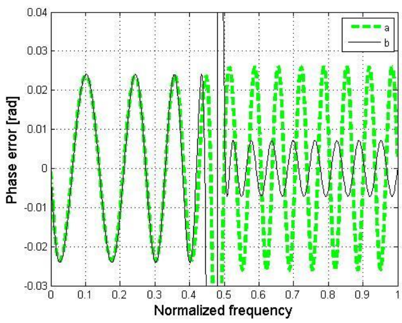


Fig. 18 Phase error of all-pass filters of order (a) n = 25 and (b) m = 24

Unlike the previous example, where even order all-pass filter has only complex conjugate poles, in the last example all-pass filter of order $m = 24$ has two simple real poles at $\omega = 0$ and $\omega = \pi$. Odd order all-pass filter of order $n = 25$ has a real pole at $\omega = 0$. The first extremum at both phase error curves, as consequence, is minimum. All extrema in the first band are almost at the same frequencies as given in Fig. 18. According to equation (7) the magnitude has extremely low value i.e. attenuation is almost 70 *dB* (Fig.

16) instead of the desired 40 *dB*. If the even order all-pass filter has no real poles, attenuation would be around 40 *dB* in the same manner as given in Fig. 12. So, keeping the same order of an all-pass filter, but introducing two real poles instead of one complex conjugate pair it is possible to improve magnitude characteristic in first band. The all-pass filter of order $n = 25$ is designed with parameters $m_1 = 8$, $m_2 = 17$ and $k_2 = 5$. The phase characteristics are shown in Fig. 15 and corresponding group delay in Fig. 17.

5. Conclusion

In this paper the method for design of complementary digital IIR filters with an arbitrary phase is proposed. The selective filters are realized as a parallel connection of two all-pass sub-filters with transfer functions $H_1(z)$ and $H_0(z)$ with the given arbitrary phase shape $\varphi(\omega)$. Given examples include a multiband filter with the linear phase shape $\varphi(\omega) = -n\omega$ (filter with five bands), all-pass networks with quadratic phase $\varphi(\omega) = -k_2\omega^2 - k_1\omega$ and an arbitrary phase given as a combination of quadratic and linear phase. At low frequencies, the phase of filter $H_1(z)$ matches the phase of $H_0(z)$ and afterwards it introduces the π *rad* phase jump in every transition zone between two bands. Thus, the order of all-pass filter $H_1(z)$ is always given as $m = n + k - 1$, where n is the order of the all-pass function $H_0(z)$ and k is the overall number of bands. The design of the selective filter is defined as a phase approximation problem. Based on the straightforward dependence between the selective filter magnitude and the all-pass filters phase difference, equiripple phase error nature generally leads to approximately equiripple magnitude (if phase is linear at all frequencies, one all-pass filter becomes a pure delay line and elliptic filter is obtained) .

These filters have low sensitivity in pass-bands and that makes them suitable for practical implementation. Their magnitude characteristic is very selective and elliptic-like. Determining the coefficients of all-pass functions $H_1(z)$ and $H_0(z)$ is based on solving the system of linear equations, and final solution is obtained in a small number of iterative steps. Filters with quadratic phase with the phase parameter $k_2 > 0$ can be used for chirp pulse compression, while the ones with $k_2 < 0$ can be used for chirp pulse expansion. Such filters can be applied in radar, satellite and high-range telecommunication systems and also in audio systems in creating different sound effects.

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