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ANALYTICAL DESIGN OF QUASILINEAR CONTROL SYSTEMS*

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Abstract. *This paper presents a method for adequate transformation of the nonlinear plants and control systems models into quasilinear forms and also analytical methods for the design of nonlinear control systems on the basis of these quasilinear models. The solution of control analytical design problem for the quasilinear model of plants exists, if the controllability functional matrix is non singular. The suggested analytical design methods provide asymptotical stability of the equilibrium point in a limited area of the state space or it's globally asymptotically stability and also desirable performance of transients. There methods can be applied to design control systems for nonlinear plants with differentiable nonlinearity. Examples of nonlinear control systems design resulted in concrete plants.*

Key words: *nonlinear, plant, control system, quasilinear model, controllability functional matrix, algebraic system.*

1. INTRODUCTION

The equations of nonlinear plants and control systems frequently contain differentiable nonlinear functions. Differentiability of the functions enables transformation of equations into a kind that is convenient for the design of nonlinear control systems. Nonlinear feedbacks are used for linearization and transformations of the plants nonlinear equations to the controllable canonical form [1-3]. The method of transformation is applied at the solution of a disturbance rejection problem [3, 4]; for the design of the pacifying feedback and solution of the adaptive control and synchronization problems [5, 6]. Fussy

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control for various plants is usually designed on the basis of the linear equations. However, the nearness of linear models result in the necessity of application of the hybrid control [7].

The purpose of this paper is the representation of a rather effective approach to the design of nonlinear control systems on the basis of transformation of the plants and systems equations to the quasilinear form [8-10]. The mathematical basis of this transformation is independence from the integration way of the curvilinear integral of many variables function [11]. The quasilinear form of nonlinear equations is close to linear, therefore the well developed analytical methods of the linear control theory can be applied to the problem solution of nonlinear systems design.

This paper is organized as follows. The problem of nonlinear control systems design is given in section 2. Transformation of the nonlinear equations of dynamic plants and systems to the quasilinear form is presented in section 3. Features of this transformation are shown on a concrete example of a nonlinear plant. Suggested method of analytical nonlinear control systems design on the basis of quasilinear forms and a corresponding example are given in section 4. The design method considered in this section is very simple, but it allows providing stability of nonlinear systems equilibrium point only in the limited area of the system state space. The fifth section is devoted to conditions of globally asymptotical stability of quasilinear systems. In the final section the more complex method for analytical design of the globally asymptotically stable nonlinear control systems is offered; the corresponding example of the design is the result.

2. Statement of The Design Problem

Let some nonlinear controlled plant be described by the equation in deviations

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in R^n$ – measured state vector; $u = u(x)$ – scalar control;

$f(x, u) = [\, f_1(x, u) \ldots f_n(x, u)]^T$ – nonlinear differentiable vector-function so that

$$f(0, 0) = 0\,, \quad \frac{\partial f_i(x, u)}{\partial u} = f_{iu}(x)\,, \ x \in R^n, \quad \|x\| < \infty\,. \tag{2}$$

The design problem consists in the definition of the control $u = u(x)$ so that equilibrium point of the plant (1) was asymptotically stable, at least, in the limited area, i.e.

$$\lim_{t \to \infty} x(t, x_0, u(x)) = 0\,, \ x_0 \in \Omega_0 \in R^n\,, \tag{3}$$

where Ω_0 – limited attraction area of the equilibrium point $x \equiv 0$.

Before passing to the solution of the statement problem, we shall define the term «quasilinear form» of nonlinear functions and nonlinear vector-functions.

3. QUASILINEAR FORM OF NONLINEARITIES

If some nonlinear function $f(x) = f(x_1, ..., x_n)$ of variables $x_1, ..., x_n$ is differentiable it always can be presented as follows:

$$f(x) = a^T(x)x + f(0) = [a_1(x) \quad \ldots \quad a_n(x)]x + f(0), \tag{4}$$

where $a^T(x)$ – some functional n-vector and $a_i(x)$ – its components determined by integration of partial derivatives $f_i(x) = \partial f(x) / \partial x_i$ by some way from the point $x \equiv 0$ to a point x [8, 11]. Various ways of integration give various quasilinear representations of the same nonlinear function. We shall look at the result of three variants of the definition of the components $a_i(x)$:

$$a_i^I(x) = \int_0^1 f_i(x_1, \ldots x_{i-1}, \theta x_i, 0, \ldots 0) d\theta, \quad i = \overline{1, n}, \tag{5}$$

$$a_i^{II}(x) = \int_0^1 f_i(0, \ldots 0, \theta x_i, x_{i+1}, \ldots x_n) d\theta, \quad i = \overline{1, n}, \tag{6}$$

$$a_1^{III}(x) = \int_0^1 f_i(\theta x) d\theta, \quad i = \overline{1, n}. \tag{7}$$

The validity of the expressions (4)-(7) will be shown on the concrete examples. Let us consider function $f_*(x) = x_1^2 x_2 + x_1 x_2^3 + \eta$, where x_1, x_2 there are independent variables and η – some constant. The function $f_*(x)$ is differentiable; therefore there are its partial derivatives:

$$f_{*1}(x) = 2x_1 x_2 + x_2^3, \quad f_{*2}(x) = x_1^2 + 3x_1 x_2^2, \tag{8}$$

and $f_*(0) = \eta$. The formula (5) with reference to the derivatives (8) gives: $a_{*1}^I(x) = 0$, $a_{*2}^I(x) = x_1^2 + x_1 x_2^2$, i.e. the vector $a_*^{I,T}(x) = [0 \quad x_1^2 + x_1 x_2^2]$. Substituting the received expressions in the formula (4), we shall find $0 \cdot x_1 + (x_1^2 + x_1 x_2^2) x_2 + \eta = f_*(x)$. Similarly, formulas (6) and (7) with account (8) at $i = 1, 2$, give $a_*^{II,T}(x) = [x_1 x_2 + x_2^3 \quad 0]$ and $a_*^{III,T}(x) = [8x_1 x_2 + 3x_2^3 \quad 4x_1^2 + 9x_1 x_2^2]/12$. The received vectors and the formula (4) in view of value $f_*(0) = \eta$, also give a harder nonlinear function $f_*(x)$.

The expressions (4)-(8) are fair and in relation to differentiable vector-functions with replacement of a vector $a^T(x)$ by a corresponding functional matrix. Let, for example, $x^T = [x_1 \quad x_2 \quad x_3]$ and $f_{**}^T(x) = [2x_2 + 3x_1^2 \quad 5x_2 x_3 + \sin x_1 \quad 1,2x_3 x_1 + x_3^3]$. The expressions (5) applied to the components of this vector-function give a matrix

$$A(x) = \begin{bmatrix} 3x_1 & 2 & 0 \\ \omega(x_1) & 0 & 5x_2 \\ 0 & 0 & 1,2x_1 + x_3^2 \end{bmatrix},$$

where $\omega(x_1) = (\sin x_1)/x_1$. It will easily prove the validity of the expression $f_{**}(x) = A(x)x + f_{**}(0)$.

The expressions (4) also $f(x) = A(x)x + f(0)$ refer to the quasilinear forms of nonlinear functions and vector-functions accordingly [8, 9]. Apparently, the quasilinear form of some nonlinearity is not unique. However, any quasilinear form describes the given nonlinearity precisely, in difference, for example, from «first approximation models» [7, 12].

The vector-function $f(x,u)$ from the equation (1) satisfies the conditions (2), therefore according to expressions (4), (5) equation (1) can be submitted as follows:

$$\dot{x} = A(x)x + b(x)u\,, \tag{9}$$

where $A(x) = [a_{ij}(x)]$ – $n \times n$-functional matrix, $b(x) = [b_i(x)]$ – n-vector and $b_i(x) = f_{iu}(x)$. The equation (9) is the quasilinear form of the equation (1).

In summary, we shall emphasize: the right parts of the equations (1) and (9) are completely identical with all $x \in R^n, \|x\| < \infty$, i.e. the quasilinear form (9) is the exact representation of the nonlinear differential equations such as (1), satisfying conditions (2). The most convenient expression for the calculation of matrix $A(x)$ and vector $b(x)$ from equation (9) is the formula (5) [9].

4. Control Systems Design on the Basis of Quasilinear Forms

Usually, when $u = 0$ the equilibrium position $x \equiv 0$ of the plant (1) either (9) is unstable or the processes in this plant are unsatisfactory. For the maintenance of the required properties of these processes nonlinear control $u = u(x)$ is designed. In the quasilinear form this control looks like:

$$u(x) = -k^T(x)x = -\sum_{i=1}^{n} k_i(x)x_i\,, \quad x \in R^n, \|x\| < \infty\,. \tag{10}$$

Here $k_i(x)$ are some nonlinear functions. The next equality follows from the expressions (9) and (10):

$$\dot{x} = D(x)x\,, \tag{11}$$

where

$$D(x) = A(x) - b(x)k^T(x)\,. \tag{12}$$

Thus, the design problem with the use of the quasilinear form is to define n nonlinear functions $k_i(x)$ from (10) so that the condition (3) is satisfied. The characteristic polynomial of the functional matrix $D(x)$ from equation (11), in view of the expression (12), it is possible to present as follows:

$$D(p, x) = \det(pE - D(x)) = \det[pE - A(x) + b(x)k^T(x)]\,. \tag{13}$$

Determinants of any matrixes satisfy the identity

$$\det(M + bk^T) = \det M + k^T(\text{adj } M)b\,, \tag{14}$$

where adj is a adjunct matrix [9, 12]. Therefore, the equality (12) in view of the expressions (13), (14) it is possible to copy as:

$$D(p,x)=A(p,x)+b(x)\mathrm{adj}(pE-A(x))k^T(x)$$

or

$$D(p,x)=A(p,x)+\sum_{i=1}^{n}k_i(x)B_i(p,x)\,, \tag{15}$$

where

$$A(p,x)=\det(pE-A(x))=p^n+\sum_{i=0}^{n-1}\alpha_i(x)p^i\ , \tag{16}$$

$$B_i(p,x)=e_i\,\mathrm{adj}(pE-A(x))b(x)=\sum_{j=0}^{n-1}\beta_{ij}(x)p^j\ ,\quad i=\overline{1,\ n}\,. \tag{17}$$

Here $e_1=[1\ 0\ \dots\ 0]$, $e_2=[0\ 1\ \dots\ 0]$, …, $e_n=[0\ 0\ \dots\ 1]$.

Let, according to stability and desirable performance of the closed nonlinear system, the characteristic polynomial of the matrix $D(x)$ from (11) be appointed the following kind:

$$D^*(p)=p^n+\delta^*_{n-1}p^{n-1}+\dots+\delta^*_1p+\delta^*_0\,. \tag{18}$$

The polynomial (18) satisfies the Hurwitz conditions [12, 13]. If the polynomial (18) is to substitute in the equation (15) instead of the polynomial $D(p,x)$, a polynomial equation is formed which is equivalent to the following algebraic system:

$$\begin{bmatrix}\beta_{10} & \beta_{20} & \cdots & \beta_{n0}\\ \beta_{11} & \beta_{21} & \cdots & \beta_{n1}\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{1n-1} & \beta_{2n-1} & \cdots & \beta_{n,n-1}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ \vdots\\ k_n\end{bmatrix}=\begin{bmatrix}\eta_0\\ \eta_1\\ \vdots\\ \eta_{n-1}\end{bmatrix}. \tag{19}$$

Here $\eta_i=\eta_i(x)$ are the coefficients of the polynomials difference: $D^*(p)-D(p,x)=\eta_0(x)+\eta_1(x)p+\dots+\eta_{n-1}(x)p^{n-1}$. In the system (19) an arguments of the functions are lowered for brevity. The solution of the algebraic system (19) defines the functions $k_i(x)$ from the control (10) of the closed system (1), (10) or the system (11) equivalent to it.

Theorem 1. The system (19) has the solution if the next condition satisfies

$$\det U(x)=\det[b(x)\ A(x)b(x)\dots A^{n-1}(x)b(x)]\neq 0\,. \tag{20}$$

Proof. The polynomials $B_i(p,x)$ (17) can be written down as the polynomial vector-column:

$$B(p,x)=[B_1(p,x)\quad B_2(p,x)\quad \dots\quad B_n(p,x)]^T=E\mathrm{adj}(pE-A(x))b(x)\,. \tag{21}$$

The adjunct matrix adj$(pE-A(x))$ in view of the formula (32) from [13, p. 88] and the polynomial (16) satisfies the expression:

$$\begin{aligned}&\mathrm{adj}(pE-A(x))=Ep^{n-1}+[A(x)+\alpha_{n-1}(x)E]p^{n-2}+\dots\\ &\dots+[A^{n-1}(x)+\alpha_{n-1}(x)A^{n-2}(x)+\dots+\alpha_1(x)E]\end{aligned} \tag{22}$$

The equality (21) in view of the expression (22) will be written down as

$$\begin{aligned} B(p,x) = \{[Ep^{n-1} + (A(x) + \alpha_{n-1}(x)E)\,p^{n-2} + ...\\ ... + (A^{n-1}(x) + \alpha_{n-1}(x)A^{n-2}(x) + ... + \alpha_1(x)E)]b(x)\}, \end{aligned}$$

or

$$\begin{aligned} B(p,x) = \{b(x)\,p^{n-1} + [A(x)b(x) + \alpha_{n-1}(x)b(x)]\,p^{n-2} + ...\\ ... + [A^{n-1}(x)b(x) + \alpha_{n-1}(x)A^{n-2}(x)b(x) + ... + \alpha_1(x)b(x)]\}. \end{aligned} \tag{23}$$

Let M_u be the matrix of the system (19). The coefficients of each column of this matrix are the coefficients of the corresponding polynomial of the vector-column (23). Therefore, the transposed matrix M_u can be presented as follows:

$$M_u^T = [M_1 \quad M_2 \quad \dots \quad M_{n-1} \quad M_n], \tag{24}$$

where

$$\begin{aligned} &M_1 = A^{n-1}(x)b(x) + \alpha_{n-1}(x)A^{n-2}(x)b(x) + ... + \alpha_1(x)b(x),\\ &M_2 = A^{n-2}(x)b(x) + \alpha_{n-1}(x)A^{n-3}(x)b(x) + ... + \alpha_2(x)b(x),\\ &\vdots\\ &M_{n-1} = A(x)b(x) + \alpha_{n-1}(x)b(x),\\ &M_n = b(x). \end{aligned}$$

It is easy to see, that the columns of the matrix M_u^T represent the linear combinations of the columns of the matrix $U(x)$ from the condition (19):

$$U(x) = [b(x) \;\; A(x)b(x) \;\; \dots \;\; A^{n-1}(x)b(x)].$$

Therefore by of the known properties of determinants there is equality: $\det M_u^T = \det U(x)$ [12, 13]. Transposing does not change the value of a determinant, therefore from here the statement of the theorem 1 follows. *The theorem 1 is proved*.

Note, the condition (20) refers to as the controllability condition of the nonlinear system (9) [9, 10]. If the matrix A and the vector b in the equation (9) are constants then the inequality (20) passes in the known Kalman controllability condition [2, 9].

So, if the vector $k(x)$ (10) is determined by the expressions (16)-(19) the elements of the matrix $D(x)$ in (11) are continuous; the matrix $D(0) = const$ and its characteristic polynomial satisfy the Hurwitz conditions. In this case the equilibrium point $x \equiv 0$ of the nonlinear closed system (10), (11) is asymptotically stable in limited area (3). Only in some cases the equilibrium point $x \equiv 0$ of this system is globally asymptotically stable [12]. Hence, the expressions (10)-(19) allow finding the nonlinear control which provides stability of the equilibrium point $x \equiv 0$ of the nonlinear plant (1), (2) and the certain performance of the control process by a choice of the coefficients of the polynomial (18). It is easy to see, that these expressions can be applied to the design of modal control for linear plants with constant parameters, as the well known J. Ackermann formula [10].

The method of nonlinear control systems design with application of the quasilinear form and the expressions (10)-(19) we shall show on an example.

Example 1. The plant is described by the next equations:

$$\dot{x}_1 = 2x_2\,,\ \dot{x}_2 = 3\sin x_1 + 5x_3\,,\ \dot{x}_3 = 1{,}2x_3 + x_3^3 + u\,. \tag{25}$$

To find the control $u = -k(x)$ by which the equilibrium point $x \equiv 0$ of the plant (25) will be asymptotically stable. The state variables x_1, x_2, x_3 are measured.

Solution. In the equations (25) the vector-function $f(0,0) = 0$, therefore according to the formula (5) the quasilinear equation (9) corresponds to this nonlinear equations with

$$A(x) = \begin{bmatrix} 0 & 2 & 0 \\ 3\omega(x_1) & 0 & 5 \\ 0 & 0 & 1{,}2 + x_3^2 \end{bmatrix},\ b(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{26}$$

It is easy to see that the equation (9) in view of the expressions (26) is the exact representation of the equations of the nonlinear plant (25). In this case $\det U(x) = -50$, the condition (20) is carried out, i.e. the solution of the design problem exists.

Passing to its definition, we find by the formulas (16) and (17) the polynomials: $A(p,x) = p^3 - (1{,}2 + x_3^2)p^2 - 6\omega(x_1)p + (7{,}2 + 6x_3^2)\omega(x_1)$, $B_1(p,x) = 10$, $B_2(p,x) = 5p$, $B_3(p,x) = p^2 - 6\omega(x_1)$. The desirable polynomial $D^*(p)$ (18) undertakes on the basis of a standard polynomial with coefficients: $\Delta_0 = 1$, $\Delta_1 = 2{,}2$, $\Delta_2 = 1{,}9$, $\Delta_3 = 1$. In this case the transient lasts 4,04 seconds in the corresponding linear system [9]. As the designed system is nonlinear, we shall put the time scale equal to 2. Then the desirable characteristic polynomial of the quasilinear matrix (12) is $D^*(p) = p^3 + 3{,}8p^2 + 8{,}8p + 8$. This polynomial, obviously, satisfies to the Hurwitz criterion, and the system (19) becomes:

$$\begin{bmatrix} 10 & 0 & -6\omega(x_1) \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 - (7{,}2 + 6x_3^2)\,\omega(x_1) \\ 8{,}8 + 6\,\omega(x_1) \\ 5 + x_3^2 \end{bmatrix}.$$

The solution of this system leads to the control: $u(x) = -0{,}8x_1 - 2{,}28\sin x_1 - -1{,}76x_2 - 1{,}2x_2\omega(x_1) - 5x_3 - x_3^3$ according to the expression (10). It is easy to establish the characteristic polynomial $D(p,x)$ calculated under the formula (13) is equal to the desirable polynomial $D^*(p)$. Hence, the equilibrium point $x \equiv 0$ of the closed system is asymptotically stable in some limited area [12].

5. Nonlinear Systems Globally Asymptotically Stable

In some cases the plants (1) have such properties that the quasilinear forms of their equations allows to design the control systems whose equilibrium point is globally asymptotically stable. Systems of this type further are referred to as «the nonlinear systems globally asymptotically stable». The state vector $x(t, x_0)$ of the nonlinear systems globally asymptotically stable satisfies the condition (3) at $x_0 \in \Omega_0 = R^n$, $\|x_0\| < \infty$ [9, 12].

To formulate the stability conditions of the nonlinear systems globally asymptotically stable, we shall consider any system (11) with order *n*, where the matrix $D(x)$ is $n - 2$ time differentiable. Let us assume, some limited, $n - 1$ time differentiable on x vector $b^*(x)$ exists in R^n for all $\|x\| < \infty$. The matrix $L(x) = [l_1(x), \ldots, l_n(x)]$ is determined as follows:

$$l_1(x) = b^*(x)\,,\; l_i(x) = D(x) l_{i-1}(x) - \dot{l}_{i-1}(x)\,,\; i = \overline{2, n}\,. \tag{27}$$

Notice, the derivative on time $\dot{l}_i(x)$ of the column $l_i(x)$, $i = \overline{1, n-1}$ in (27) are determined on the trajectories of the above mentioned system (11) [9, 10]. Therefore the matrix $L(x)$ and its derivative on time $\dot{L}(x)$ are functions only of the state vector x of the system (11) or some functions of time.

The following lemma is used in the definition of the conditions of globally asymptotical stability of the nonlinear systems (11).

Lemma. If the matrix $L(x)$ is determined by the expressions (27) on the solution of the system (11); there is a matrix $L^{-1}(x)$ and the matrix $G(x) = D(x) - \dot{L}(x) L^{-1}(x)$, then the matrix $D_1(x) = L^{-1}(x) G(x) L(x)$ is the transposed companion matrix of the characteristic polynomial of the matrix $G(x)$.

The proof of this lemma is given in [10], therefore here only the numerical example is the result. Let the matrix $D(x) = D(t)$ and the vector $b^*(x) = b^*(t)$ look like:

$$D(t) = \begin{bmatrix} -1 & t \\ 2 & -t \end{bmatrix},\; b^*(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

In this case a matrix $L(t)$ determined by the expressions (27) and also matrixes $\dot{L}(t)$, $L^{-1}(t)$ are equal:

$$L(t) = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix},\;\; \dot{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; L^{-1}(t) = \frac{1}{t^2+1} \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

The matrix $G(t) = D(t) - \dot{L}(t) L^{-1}(t)$ and its characteristic polynomial look like:

$$G(t) = \frac{1}{t^2+1} \begin{bmatrix} -t^2 - t - 1 & t^3 + t - 1 \\ 2t^2 + 3 & -t^3 - 2t \end{bmatrix},\; G(p,t) = \det(pE - G(t)) = p^2 + \delta_1(t) p + \delta_0(t)\,,$$

where

$$\delta_1(t) = \frac{t^3 + t^2 + 3t + 1}{t^2 + 1}\,,\;\; \delta_0(t) = \frac{-t^5 + t^4 - 2t^3 + 4t^2 - t + 3}{(t^2+1)^2}\,.$$

Calculating the matrix $D_1(t)$ under the formula $D_1(t) = L^{-1}(t) G(t) L(t)$, we shall receive the companion matrix

$$D_1(t) = \begin{bmatrix} 0 & -\delta_0(t) \\ 1 & -\delta_1(t) \end{bmatrix}.$$

This expression, obviously, fully complies with the statement of the lemma.

The conditions of globally asymptotically stability of the nonlinear systems (11) are determined by the following theorem.

Theorem 2. Let the matrix $L(x)$ be determined by the expressions (27) on solutions of system (11), and this matrix and its derivative on time $\dot{L}(x)$ are continuous and limited, and $\det L(x) \neq 0$ at all $x \in R^n$, $\|x\| < \infty$. Then, if the coefficients of the characteristic polynomial of the matrix

$$G(x) = D(x) - \dot{L}(x)L^{-1}(x), \tag{28}$$

are constant and satisfy the Hurwitz criterion at all $x \in R^n$, $\|x\| < \infty$, the equilibrium point $x \equiv 0$ of the system (11) is globally asymptotically stable.

Proof. The matrix $L^{-1}(x)$ exists on conditions of the theorem. If $x = L(x)\tilde{x}$ then the system (11) is transformed to

$$\dot{\tilde{x}} = D_2(x)\tilde{x}, \tag{29}$$

where $\tilde{x}$ is a new state vector, and the matrix $D_2(x)$ is determined by the following expression

$$D_2(x) = [L^{-1}(x)D(x)L(x) - L^{-1}(x)\dot{L}(x)]. \tag{30}$$

The right part of the expression (30), in view of the equality (28), is possible to present as follows:

$$D_2(x) = L^{-1}(x)G(x)L(x).$$

According to the lemma the matrix $D_2(x)$ is the transposed companion matrix of the characteristic polynomial of the matrix $G(x)$ (28) (it is similar to matrix $D_1(t)$). But in conditions of the theorem 2 the coefficients of the characteristic polynomial of the matrix $G(x)$ are constant and satisfy the Hurwitz criterion. Hence, in the system (29) the matrix $D_2(x)$ is constant ($D_2(x) = D_2 = const$) and stable. Therefore, any solution of the system (29) are globally asymptotically stable, i.e. $\lim_{t\to\infty} \tilde{x}(t, \tilde{x}_0) = 0$ for all $\tilde{x}_0 \in R^n$, $\|\tilde{x}_0\| < \infty$ [12].

On the other hand, the expression $x = L(x)\tilde{x}$ is Lyapunov's transformation by virtue of the mentioned above properties of a matrix $L(x)$ also at all $x \in R^n$, $\|x\| < \infty$ [12]. Hence the statement of the theorem 2 follows from here. *The theorem 2 is proved*.

According to the expression (28) for the research of the stability of the nonlinear system $\dot{x} = f(x)$ it is necessary to construct its quasilinear representation (11), to choose a vector $b^*(x)$ and then a matrix $L(x)$ to find by expression (27). If the matrix $L(x)$ will be non singular and limited, then the matrix $G(x)$ is calculated under expression (28) and its characteristic polynomial is found:

$$G(p,x) = \det(pE - G(x)) = p^n + \sum_{i=0}^{n-1} \gamma_i(x)\, p^i. \tag{31}$$

If the coefficients $\gamma_i(x)$ are constants and the polynomial $G(p,x)$ satisfies the Hurwitz criterion, then the equilibrium point $x \equiv 0$ of the researched system is globally asymptotically stable.

Natural nonlinear systems satisfy the conditions of the theorem 2 rather seldom. However, the systems of such type can be designed under some conditions which have been mentioned below.

6. Design of a Nonlinear Systems Globally Asymptotically Stable

Let the equation of the plant (1) be transformed to the quasilinear form (9). Then it is submitted as (11), (12) with an unknown vector $k(x)$. In this case the matrixes $D(x)$ and $L(x)$ are determined by the formulas (12) and (27) dependent on the vector $k(x)$, i.e. in the expressions (11), (28) the matrixes $D(x) = D(x,k)$ and $L(x) = L(x,k)$. Assume, that matrixes $L(x,k)$ and $\dot{L}(x,k)$ are continuous, limited, and $\det L(x,k) \neq 0$ at all $x \in R^n$, $\|x\| < \infty$. In view of the entered designations the equality (28) will be written down as follows

$$G(x,k) = \bar{A}(x,k) - b(x)k^T(x), \tag{32}$$

where

$$\bar{A}(x,k) = A(x) - \dot{L}(x,k)L^{-1}(x,k). \tag{33}$$

Assume also, that the matrix $\bar{A}(x,k)$ and the vector $b(x)$ satisfy the controllability condition (20). Further the design method fully complies with expressions (16)-(19). However, if the design problem considered in section 4 has a solution only under condition (20), the design problem of the nonlinear systems globally asymptotically stable has a solution by the considered method, if the condition (20) is carried out and the solution of the corresponding system (19) depends only on the vector *x*.

The design method following from expressions (27), (32), (33) and (16)-(19) is considered on the example of control systems design for some nonlinear plant.

Example 2. To design the control (10) providing stability of the equilibrium point $x \equiv 0$ of the plant is described by the equations:

$$\dot{x}_1 = x_1^2 + 5x_2, \quad \dot{x}_2 = x_1^3 - 2x_3, \quad \dot{x}_3 = x_2\varphi(x_2) + u, \tag{34}$$

where $\varphi(x_2)$ is function continuous and limited at everything $x \in R^n$, $\|x\| < \infty$; the variables x_1, x_2, x_3 are measured.

Solution. According to the expression (5) the matrix and the vector of quasilinear forms (9) of the plant equations (34) look like

$$A(x) = \begin{bmatrix} x_1 & 5 & 0 \\ x_1^2 & 0 & -2 \\ 0 & \varphi(x_2) & 0 \end{bmatrix}, \; b(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{35}$$

If the vector $k(x) = [k_1(x) \;\; k_2(x) \;\; k_3(x)]^T$ the next matrixes follow from (12), (35) and (27):

$$D(x,k)=\begin{bmatrix} x_1 & 5 & 0 \\ x_1^2 & 0 & -2 \\ -k_1 & h & -k_3 \end{bmatrix},\ L(x,k)=\begin{bmatrix} 0 & 0 & -10 \\ 0 & -2 & 2k_3 \\ 1 & -k_3 & -2h+k_3^2-\dot{k}_3 \end{bmatrix}, \tag{36}$$

where $h=h(x)=\varphi(x_2)-k_2(x)$. Arguments of the functions in the right parts of these and subsequent expressions are omitted for brevity. In this case $\det L(x,k)=-20$, therefore it is possible to calculate the matrix $L^{-1}(x,\bar{k})$, and under formula (33) the matrix $\bar{A}(x,k)$:

$$L^{-1}(x,\bar{k})=\begin{bmatrix} -0,2h-0,1\dot{k}_3 & -0,5k_3 & 1 \\ -0,1k_3 & -0,5 & 0 \\ -0,1 & 0 & 0 \end{bmatrix},\ \bar{A}(x,k)=\begin{bmatrix} x_1 & 5 & 0 \\ x_1^2+0,2\dot{k}_3 & 0 & -2 \\ 0,1(k_3\dot{k}_3+2\dot{h}-\ddot{k}_3) & \varphi-0,5k_3 & 0 \end{bmatrix}. \tag{37}$$

Matrix $\bar{A}(x,k)$ (37) and vector $b(x)$ (35) satisfy the condition (20), therefore allows under formulas (16), (17) to find polynomials:

$$\bar{A}(p,x,k)=\det(pE-\bar{A}(x,k))=p^3+\bar{\alpha}_2(x,k)p^2+\bar{\alpha}_1(x,k)p+\bar{\alpha}_0(x,k),$$

$$\bar{B}_1(p,x,k)=4h+2\dot{k}_2,\ \bar{B}_2(p,x,k)=-2p+2x_1,\ \bar{B}_3(p,x,\bar{k})=p^2-x_1p-0,5x_2^2-\dot{k}_3(x),$$

where

$$\bar{\alpha}_2(x,k)=-x_1,\ \ \bar{\alpha}_1(x,k)=2\varphi(x_2)-2\dot{k}_3(x)-5x_1^2,$$

$$\bar{\alpha}_0(x,k)=-x_1[2\varphi(x_2)-\dot{k}_3(x)]+k_3(x)\dot{k}_3(x)+2[\dot{k}_2(x)-\dot{\varphi}(x_2)]+\ddot{k}_3(x).$$

The desirable polynomial (18) here is equal to $D^*(p)=p^3+\delta_2^*p^2+\delta_1^*p+\delta_0^*$ and satisfies the Hurwitz criterion. The system (19) in this case looks like

$$\begin{bmatrix} -10 & 2x_1 & -5x_1^2-\dot{k}_3 \\ 0 & -2 & -x_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} \delta_0^*-\bar{\alpha}_0 \\ \delta_1^*-\bar{\alpha}_1 \\ \delta_2^*-\bar{\alpha}_2 \end{bmatrix}.$$

This algebraic system has the solution which depends only on the state vector of the designing control system: $k_3(x)=\delta_2^*+x_1$, $\bar{\alpha}_1(x)=2\varphi(x_2)-7x_1^2-10x_2$, $k_2(x)=\varphi(x_2)-0,5(\delta_1^*+x_1\delta_2^*)-5x_2-4x_1^2$, $\bar{\alpha}_0(x)=10x_3-2x_1\varphi(x_2)-17x_1^3-60x_1x_2$, $k_1(x)=0,1[\delta_0^*+x_1\delta_1^*+(7x_1^2+5x_2)\delta_2^*]+3,1x_1^3+7,5x_1x_2-x_3$.

The found solution allows writing down the required control $u(x)$ (10) for the system (34). Now the matrixes $D(x)$, $L(x)$, $L^{-1}(x)$, $\bar{A}(x)$ and also $G(x)$ can be calculated under the expressions (36), (37) and (32) in view of the found functions $k_i(x)$ and $\bar{\alpha}_{i-1}(x)$, $i=1,2,3$. The characteristic polynomial of the matrix $G(x)$ coincides with the chosen polynomial $D^*(p)$, i.e. satisfies conditions of the theorem 2. The matrix $L(x)$ and its derivative $\dot{L}(x)$ are continuous and limited at all limited x. Therefore the equilibrium point of the nonlinear system (34) with control (10) and the found coefficients $k_1(x),\ k_2(x),\ k_3(x)$ is globally asymptotically stable [12].

6. CONCLUSION

Representation of the nonlinear plants and system equations in the quasilinear form is exact. Quasilinear models allow solving analytically a problem of control systems design for nonlinear plants with measured state variables. This design problem has the solution if the controllability functional matrix is non singular. Required nonlinear control is determined by the solution of the algebraic linear equations system. Equilibrium point of the closed system is asymptotically stable in the limited area of state space. Globally asymptotically stability is provided under some additional conditions. Efficiency of the suggested methods is shown by the examples of analytical design of control systems for nonlinear plants.

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