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OPTIMAL SLIDING MANIFOLD DESIGN FOR LINEAR SYSTEMS SUBJECTED TO A CLASS OF UNMATCHED DISTURBANCES*

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Abstract. *This paper offers the optimal sliding manifold design for the traditional sliding mode and integral sliding mode control of linear systems that minimizes the impact of unmatched constant or slowly-varying external disturbance vector. System sensitivity upon the unmatched disturbances is assessed by the steady-state dependent criterion function. The ability and efficiency of the adopted control strategies in solving the given optimization problem are analyzed. The proposed approach has been demonstrated and verified on numerical examples by computer simulations.*

Key words: *sliding mode control, matching conditions, unmatched disturbances, sliding manifold design*

1. Introduction

Theoretical invariance to parameter perturbations and exogenous disturbances in ideal sliding mode (SM) [1] is the most important feature of SM control (SMC) systems. In order to gain such valuable property, system disturbances need to meet the invariance conditions [2], as originally termed by Draženović in 1969, i.e. may enter the system through the control channels only. SM dynamics is insensitive to such 'matched' disturbances, but is affected by the unmatched ones. However, in reaching of the sliding manifold, a system is sensitive both to matched and unmatched disturbance. To completely eliminate the reaching phase, integral SM (ISM) method was introduced in [3], where SM exists from the initial

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time, thus making the system sensitive only to unmatched disturbances. ISM control has been thoroughly studied (see [4] and the references therein).

Depending on their generic nature, unmatched disturbances have various detrimental effects on SMC systems, so that a significant attention is directed to alleviating their impact to SM motion. When unmatched disturbances have parametric nature or are state dependent, asymptotic stability can be achieved by adequate sliding manifold design [5-7], since disturbance diminishes as the system state approaches the origin.

If unmatched disturbances are both parametric and external, asymptotic stability cannot be attained. A non-vanishing unmatched disturbance forces the system trajectory to wander along the sliding manifold in the neighborhood of the origin. The only possibility to alleviate this impediment is to design a sliding manifold that in some sense minimizes the impact of unmatched external disturbances onto SM, which was considered in the subsequent papers. A sliding manifold that guarantees convergence of the linear system trajectory into a minimal invariant ellipsoid is given in [8]. Another optimization criterion, proposed in [9-10] for integral sliding manifold design in case of nonlinear systems, is to minimize the equivalent disturbance that affects SM dynamics. It is shown that the sliding manifold needs to be designed so that the ISM controller leaves the unmatched disturbance untouched. Any attempt in compensating it would actually amplify the unmatched disturbance.

Another sliding manifold design method for linear systems in a conventional SM was developed in [11], for constant and slowly varying unmatched external disturbances. This class of disturbances is frequent in practice and therefore important enough to deserve special treatment. A new steady-state dependent quadratic optimization criterion was employed in order to minimize the impact of the considered class of disturbances onto system accuracy. However, it was revealed that the optimization could impose constraints upon arbitrary SM dynamics selection in certain cases.

This paper considers the same optimization problem from [11] for linear systems in a more general form in the presence of the aforementioned class of unmatched disturbances. A new optimization procedure is developed and possibilities and ways to achieve optimal behavior by SMC and ISMC are investigated. Traditional SMC exhibits even greater restrictions than those in [11], while ISMC offers the ability to achieve both minimization of the selected criterion and the desired SM dynamics. The feasibility and efficiency of the proposed sliding manifold design methods have been demonstrated on numerical examples and computer simulations.

The remainder of this paper is organized as follows. Optimization of the adopted criterion is conducted in Section 2. Sections 3 and 4 present SMC and ISMC with corresponding optimal sliding manifolds design methods, respectively. Illustrative examples and simulation results are given in Section 5. The paper ends with some concluding remarks and references.

2. Steady-State Vector Norm Minimization

Consider a linear time-invariant dynamic system given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t)\ , \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ ($m<n$) is the control and $d \in \Re^n$ is a norm-bounded external disturbance vector, i.e. $\|d(t)\| \le d_m, \forall t$. The system matrices are of appropriate dimensions and have full ranks. The pair (A,B) is assumed to be controllable and rank(B)=m. Let the vector d(t) be an unknown constant or slowly varying disturbance, i.e. $d(t)=d$. Also, the matching condition [2] is not satisfied, i.e. $\mathrm{rank}[B] < \mathrm{rank}[B \,\vdots\, d]$.

The control task is to bring the system state from an arbitrary initial condition into the origin with the desired dynamics. This task is easy to accomplish in the absence of disturbance by conventional linear state feedback control. However, in the presence of a constant or slowly varying immeasurable external disturbance, system trajectory will reach some nonzero steady-state $x(\infty) = \lim_{t\to\infty} x(t) \neq 0$. This raises the question of which attainable steady-state is the best under the given circumstances when a criterion is defined. The following analysis gives the answer.

In case of constant or slowly varying disturbances, it may be assumed that $\lim_{t\to\infty} \dot{x}(t) = 0$, for sufficiently faster closed loop dynamics. By letting $t \to \infty$ in (1) it holds:

$$Ax(\infty) + Bu(\infty) + d = 0\,, \tag{2}$$

Let the optimization criterion be proposed in general form:

$$J = x^{\mathrm{T}}(\infty)Qx(\infty)\,, \tag{3}$$

where Q is a positive definite symmetric matrix. For $Q=I$, (3) becomes $J = \| x(\infty) \|^2$, i.e. squared steady-state Euclid vector norm. Minimization of J provides minimal distance of the steady-state $x_{\min}(\infty)$ from the origin, i.e. minimal steady-state error. Algebraic constraint (2) is to be taken into consideration in minimization of (3). Hence, the extended criterion function becomes:

$$J = x^{\mathrm{T}}(\infty)Qx(\infty) + l^{\mathrm{T}}[Ax(\infty) + Bu(\infty) + d]\,. \tag{4}$$

Constraint (2) is incorporated into (4) by means of a Lagrange multiplier vector $l \in \Re^n$. Steady-state $x_{\min}(\infty)$ and the corresponding control $u_{\min}(\infty)$ that minimizes J are solutions of the following set of equations:

$$\frac{\partial J}{\partial x(\infty)} = 2Qx(\infty) + A^{\mathrm{T}}l = 0, \quad \frac{\partial J}{\partial u(\infty)} = B^{\mathrm{T}}l = 0, \quad \frac{\partial J}{\partial l} = Ax(\infty) + Bu(\infty) + d = 0. \tag{5}$$

Minimization of the proposed optimization criterion is described in the forthcoming theorem.

Theorem 1: A unique steady-state $x(\infty)$ and the associated control $u(\infty)$ of system (1) that minimize criterion (3) under given assumptions are given as:

$$u_{\min}(\infty) = -(B^{\mathrm{T}}A_s B)^{-1}B^{\mathrm{T}}A_s d, \quad A_s = (AQ^{-1}A^{\mathrm{T}})^{-1}\,, \tag{6}$$

$$x_{\min}(\infty) = -A^{-1}P_1 d, \quad P_1 = I_n - B(B^{\mathrm{T}}A_s B)^{-1}B^{\mathrm{T}}A_s\,. \tag{7}$$

Proof: The optimal values are reached by successive solving of (5). $x(\infty) = -\tfrac{1}{2}Q^{-1}A^{\mathrm{T}}l$ follows from the first equation, which is used in the third equation to find l as

$l = 2A_s[Bu(\infty) + d]$. Multiplying l from the left by B^T, the second equation of (5) gives the control in the form (6). Finally, substitution of the obtained l and control (6) into $x(\infty)$yields (7).■

Remark 1: From (7) it follows that if disturbance d in (1) is matched, i.e. $d=Bj$ (where j is some vector), then steady state $x_{\min}(\infty) = 0$ is feasible.

For the forthcoming analysis and design, it is required to establish property and rank of matrix P_1.

Lemma 1: Matrix P_1 is a projector, having $\mathrm{rank}(P_1) = n - m$.

Proof: A projection matrix P is a square idempotent matrix ($P^2 = P$) that separates space into two complementary subspaces. Since $P_1^2 = P_1 \in \Re^{n \times n}$, P_1 is a projector. Then, according to projector theory, it follows:

$$\mathrm{rank}(P_1) = n - \mathrm{rank}[B(B^T A_s B)^{-1} B^T A_s] = n - \mathrm{rank}(B) = n - m\,. \tag{8}$$

This completes the proof.■

3. Sliding Mode Control

In traditional SMC, SM is mainly organized upon linear sliding manifolds. They are represented by the intersection of m sliding hyperplanes, defined by the switching function vector s:

$$s(x) = Cx = 0, \quad C \in \Re^{m \times n}, \quad s = [s_1, s_2, \cdots, s_m]^T, \tag{9}$$

under restrictions that rank(C)=m and $\det(CB) \neq 0$. The sliding manifold includes the state space origin as the equilibrium point. To establish SM in a finite time it is sufficient, for a well-known Lyapunov function candidate $V = s^T s / 2$, to provide $\dot{V} = s^T \dot{s} < 0, \forall t \geq 0$ by an appropriate vector control usually having discontinuous nature. According to the total time derivative of V by virtue of (9) and (1), it can be proved easily that, for example, the following control:

$$u = -(CB)^{-1}[CAx + F\,\mathrm{sgn}\,s],\; F = (d_m \,||C|| + \alpha)I_m \tag{10}$$

guaranties occurrence and existence of SM, where $\alpha > 0$, $\mathrm{sgn}\,s = [\mathrm{sgn}\,s_1 \;\; \cdots \;\; \mathrm{sgn}\,s_m]^T$, I_m is identity matrix of dimension m and $\|C\|$ is induced spectral matrix norm of C.

System dynamics in SM can be obtained using equivalent control method. Linear equivalent control [1] is a fictive linear control that provides sliding along the manifold (9). It can be determined from the well-known condition:

$$s = 0 \wedge \dot{s}_{|u=u_{eq}} = 0 \quad \Rightarrow \quad u_{eq} = -(CB)^{-1} C(Ax + d)\,. \tag{11}$$

Substitution of (11) into (1) gives the full-order SM dynamics:

$$\dot{x}(t) = PAx(t) + Pd(t), \quad Cx(t) = 0, \quad P = I_n - B(CB)^{-1} C\,. \tag{12}$$

Existence of the disturbance term in (12) testifies to the sensitivity of the SM dynamics upon unmatched disturbances. System trajectories will not converge into the origin, but will be driven away somewhere along the sliding manifold by the unmatched disturbance. Also, it is obvious that the matrix *P* primarily determines SM behavior. It is shown in [12] that matrix *P* is a projector having rank(P) = $n - m$.

3.1 Optimal sliding manifold design for traditional SMC

The primary objective of the switching function matrix *C* is to provide the desired SM dynamics, which is determined by the *n*-*m* nonzero eigenvalues of the system matrix *PA* in (12). The usual additional requirement is to have a fully decupled control in which every sliding variable is controlled by its own control input. This feature is gained by the condition $CB = I_m$.

From the aspect of minimizing the impact of unmatched disturbances onto system behavior, which is taken in this paper by criterion (3), matrix *C* also has an exclusive role. In other words, *C* should be found to provide minimization of (3). This can be ensured by securing minimal stead-state $x_{\min}(\infty)$ in (7) to lie on the sliding manifold (9). Hence, $Cx_{\min}(\infty) = 0$ must hold for any unknown disturbance vector. According to (7), the previous requirement gives condition:

$$CA^{-1}P_1 = 0\,, \tag{13}$$

which imposes certain constraints on elements selection in matrix *C*. The possibility of simultaneous achievement of a desired SM dynamics and minimization of criterion (3) by traditional SMC is given in the following lemma.

Lemma 2: Dynamic system (1) will reach in SM the minimal steady-state (7) if the switching function matrix *C* is in the form:

$$C = C_0[H \,\vdots\, I_m], \quad \det([H \,\vdots\, I_m]B) \neq 0\,, \tag{14}$$

where $C_0 \in \Re^{m\times m}$ is a freely chosen matrix whereas matrix $H \in \Re^{m\times(n-m)}$ is determined from condition (13). Consequently, SM dynamics (12) is predefined by projector:

$$P = I_n - B([H \,\vdots\, I_m]B)^{-1}[H \,\vdots\, I_m] \tag{15}$$

for any C_0 and cannot be arbitrary specified.

Proof: Matrix *C* must be determined from the condition (13) that constructs sliding manifold through the minimal steady-state for any unmatched disturbance vector. Since according to Lemma 1 $\mathrm{rank}(P_1) = n - m$, equation (13) forms a system of $m\cdot(n-m)$ linearly independent equations with $m\cdot n$ unknowns, which are the elements of matrix *C*. Hence, $m\times(n-m)$ elements of *C* can be uniquely expressed in terms of the remaining $m\times m$ part of *C*, denoted as C_0. Then *C* can be decomposed as $C = [C_0H \,\vdots\, C_0]$, which can be expressed as (14). Replacement of such *C* into *P* in (12) gives the projector matrix *P* in a new form (15). *P* does not depend on C_0 due to $C_0^{-1}C_0$ cancelation, meaning that SM dynamics is fixed regardless of the selection of C_0 .■

This inability of SM dynamics adjustment during minimization is in a compliance with the results given in [11] for the case of directed disturbance vector of lower dimension, where it is shown that in certain cases constraints may arise in SM dynamic selection.

Remark 2: If $\det([H \,\vdots\, I_m]B) = 0$ SM cannot be organized along the sliding manifold that leads the system state into the minimal steady-state (7).

Proof: As mentioned earlier, $\det(CB) \neq 0$ is the SM necessary condition. *C* that provides $Cx_{\min}(\infty) = 0$ is $C = C_0[H \,\vdots\, I_m]$. Then $\det(CB) = \det(C_0)\det([H \,\vdots\, I_m]B)$. Since C_0 can be arbitrary chosen, $\det(C_0) \neq 0$ can be always fulfilled. This means that SM condition reduces to $\det([H \,\vdots\, I_m]B) \neq 0$. Otherwise, SM cannot be organized along that manifold.■

The remaining degree of freedom C_0 in *C* can be used for providing decentralized SMC.

Theorem 2: Optimal sliding manifold (9) for system (1), which minimizes criterion (3) in the sense and under conditions of Theorem 1 and Lemma 2 and ensures decentralized control, is attained by the following matrix *C*:

$$C = [0_{m\times n} \,\vdots\, I_m] \cdot [A^{-1}P_1 \,\vdots\, B]^{+}, \tag{16}$$

where superscript + denotes matrix pseudoinverse.

Proof: To provide system (1) to have minimal steady-state (7) in SM under unmatched disturbances, *C* should satisfy (13). Decentralized control condition $CB = I_m$ introduces additional $m \cdot m$ linearly independent equations to the previous set of equations. Now, the joint system of equations, presented as:

$$C[A^{-1}P_1 \,\vdots\, B] = [0_{m\times n} \,\vdots\, I_m], \tag{17}$$

consists of $m \cdot n$ linearly independent equations with $m \cdot n$ unknowns. Hence, *C* can be uniquely found as (16) by using matrix pseudoinverse.■

A way to achieve the ability of adjusting SM dynamics in the minimization of the steady-state is implementation of ISMC.

4. Integral Sliding Mode Control

ISMC [3] emerged as an effort to eliminate the reaching phase, present in traditional SMC. System is expanded with *m* integrators, whose outputs are adjusted so that the SM starts from the initial time instant. The switching function includes both the state variables and integrators outputs. Thus, the SM dynamics is of full *n*-th order, instead of the reduced *n-m* order.

Starting from the system (1), integral sliding manifold *s*=0 is defined by the following switching function

$$s(t) = Dx(t) - Dx(0) + \int_0^t Ex(\tau)\mathrm{d}\tau, \tag{18}$$

where $D, E \in \Re^{m\times n}$ are design matrices and rank(*DB*)=*m*. Note that *s*(0)=0. For

$$E = -D(A - BK), \tag{19}$$

(18) becomes:

$$s = D[x - x(0) + \int_0^t (A - BK)x(\tau)\mathrm{d}\tau], \tag{20}$$

which can be understood as a scaled difference between the trajectories of actual and desired nominal system described by matrix (A-BK), [9]. Nominal stable dynamics can be selected by eigenvalues allocation of (A-BK), by feedback matrix K. Hence, (A-BK) is invertible for stable systems.

Control that provides SM can be found using the same Lyapunov function candidate $V = s^T s / 2$. It can be easily shown that:

$$u = -(DB)^{-1}[(DA + E)x + H \operatorname{sgn} s],\ H = (d_m \,||D|| + \varepsilon) I_m\,, \tag{21}$$

with $\varepsilon > 0$ provides $\dot{V} < 0, \forall t \geq 0$. Hence, control (21) enforces ISM.

SM dynamics is obtained using equivalent control:

$$s = 0 \wedge \dot{s}\big|_{u=u_{eq}} = 0 \quad \Rightarrow \quad u_{eq} = -(DB)^{-1}[(DA + E)x + Dd]\,, \tag{22}$$

whose substitution into (1) by virtue of (19) yields:

$$\dot{x}(t) = (A - BK)x(t) + [I - B(DB)^{-1}D]d(t)\ . \tag{23}$$

Disturbance term in (23), known as equivalent disturbance, shows sensitivity of the SM dynamics to unmatched disturbances. An unmatched disturbance prevents the system trajectory to converge into the origin, but to wander along the sliding manifold. In case of a constant or slowly varying disturbance, when $\lim_{t\to\infty} \dot{x}(t) = 0$ can be assumed, a nonzero steady-state $x(\infty) \neq 0$ occurs. By letting $t \to \infty$ in (23), under above presumption, the following relation is obtained:

$$(A - BK)x(\infty) = -[I - B(DB)^{-1}D]d(\infty)\,, \tag{24}$$

which can be used for the steady-state calculation.

4.1. Optimal sliding manifold design for ISMC

Matrices E and D should be determined during the in integral sliding manifold design. Since E according to (19) provides the desired SM dynamics, matrix D offers an additional degree of freedom in the manifold design. There are several recommendations how to select$D$. A suitable choice of D presented in [13] is that D should satisfy the condition $DB = I_m$, which results in a fully decupled control. A possible solution for D is left pseudo-inverse of B, i.e. $D = B^+$. This choice was derived in [9] in order to minimize the equivalent disturbance norm. It was shown that the resulting equivalent disturbance is then equal to the unmatched part of the original disturbance, i.e. no amplification of unmatched disturbances occurs.

In this paper the selection of matrix D will be derived to minimize the steady-state vector norm. Clearly from (24), D influences the impact of the unmatched disturbances onto the steady-state. In this way a properly designed ISMC can provide minimal steady-state error. Note that a multiplication of (24) from left by D gives:

$$D(A - BK)x(\infty) = 0\,. \tag{25}$$

This requirement applies to every steady-state, and therefore also to $x_{\min}(\infty)$. Hence, D should be found to satisfy (25) for $x(\infty) = x_{\min}(\infty)$. The following theorem gives the solution.

Theorem 3: Optimal integral sliding manifold (18) for system (1), which minimizes criterion (3) in the sense and under conditions of Theorem 1 and ensures decentralized control, is attained by the following matrix D:

$$D = [0_{m\times n} \;\vdots\; I_m]\cdot[(A-BK)A^{-1}P_1 \;\vdots\; B]^{+}\,. \tag{26}$$

Proof: To provide system (1) to have minimal steady-state (7) in ISM under unmatched disturbances, D should satisfy (25) for $x(\infty)=x_{\min}(\infty)$. Substitution of (7) into (25) gives $D(A-BK)A^{-1}P_1 d = 0$. This condition will hold for any d if D satisfies matrix equality:

$$D(A-BK)A^{-1}P_1 = 0_{m\times n}\,. \tag{27}$$

According to Lemma 1, $\operatorname{rank}(P_1) = n-m$, while the remaining matrices in the product (27) have full rank. Hence, the obtained system of equations (27), from which D should be determined, consists of $m\cdot(n-m) = m\cdot n - m^2$ linearly independent equations with $m \cdot n$ unknowns. To obtain a unique solution for D, system (27) can be enhanced with a useful decentralized control condition $DB = I_m$ that brings another m^2 linearly independent equations. The joint system can be rewritten as:

$$D[(A-BK)A^{-1}P_1 \;\vdots\; B] = [0_{m\times n} \;\vdots\; I_m]\,. \tag{28}$$

Now, the number of unknowns corresponds to the number of linearly independent equations and unique solution of (28) comes out by help of matrix pseudoinverse, given by (26).■

5. Illustrative Examples

The developed sliding manifold design methods for SMC and ISMC have been tested on an example of a controllable system (1) with:

$$A = \begin{bmatrix} -1 & -3 & 1 & 5 \\ 1 & -2 & 1 & -2 \\ 1 & 2 & -3 & 2 \\ -3 & -4 & -1 & -5 \end{bmatrix},\quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix},\quad x_0 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \tag{29}$$

subjected to an unmatched disturbance vector. $Q = I$ is chosen in the optimization criterion (3). For traditional SMC, Theorem 2 gives:

$$C = \begin{bmatrix} 0.077 & 0.88 & -0.178 & -0.106 \\ -0.077 & 0.464 & 0.247 & -0.032 \end{bmatrix} \tag{30}$$

and SM controller (10) is realized for $F = 1.8372\cdot I_2$. SM dynamics is defined by two nonzero eigenvalues $\lambda_{1,2} = -2.084 \pm j8.525$ of the system matrix PA, and cannot be chosen. In case of ISMC, for the desired SM dynamics given by the spectrum $\lambda \in \{-6, -7, -8, -9\}$, D and E are obtained as:

$$D=\begin{bmatrix}-0.34 & 6.02 & 0.6 & -1.87\\ 0.401 & -2.189 & 0.003 & 0.695\end{bmatrix},\quad E=\begin{bmatrix}-0.342 & 24.983 & 2.273 & -2.492\\ -0.003 & -11.63 & -0.587 & 1.193\end{bmatrix} \quad (31)$$

according to (26) and (19). ISM controller (21) is realized for $H = 11.79 \cdot I_2$. Although both control laws (10) and (21), having dominant switching component, may induce chattering in real systems, reason for their application is that the attained SM should as much as possible resemble an ideal one, which is the assumption of the conducted analysis.

For a constant unmatched disturbance $d(t)=[1 \quad 0 \quad 1 \quad -1]^T \cdot h(t-4)$, Theorem 1 gives the minimal steady state (7) as $x_{\min}(\infty)=[-0.198 \quad -0.018 \quad -0.054 \quad -0.201]^{\mathrm{T}}$, whose norm is $||x_{\min}(\infty)||=0.288$. Fig. 1 shows that both controllers, SMC (dashed trace (1)) and ISMC (trace (2)), achieve analytically predicted minimal stead-state vector norm. However, SMC has a slower response than ISMC, since ISMC dynamics can always be chosen faster than the predefined dynamics resulting in SMC. Another disadvantage of SMC is the existence of the reaching phase, in which a system is sensitive to matched and unmatched disturbances.

The obtained results are compared with the performance of ISM controller whose matrices D and E are selected according to the recommendation $D = B^+$ proposed in [9], Fig. 1 trace (3). For the same desired dynamics integral sliding manifold is defined by:

$$D=\begin{bmatrix}0.223 & 0.092 & -0.248 & 0.107\\ -0.029 & 0.01 & 0.184 & 0.116\end{bmatrix},\quad E=\begin{bmatrix}12.928 & 24.42 & 7.072 & -5.586\\ -4.549 & -11.2 & -2.389 & 4.486\end{bmatrix},\quad (32)$$

providing steady-state $x(\infty)=[-0.67 \quad -0.048 \quad 1.31 \quad -0.101]^{\mathrm{T}}$ having considerably larger norm $||x(\infty)||=1.4754$. Both ISM controllers provide the same behavior until the moment of unmatched disturbance activation. Afterwards, ISMC (32) exhibits higher sensitivity to constant unmatched disturbances than the one with the optimal manifold (31).

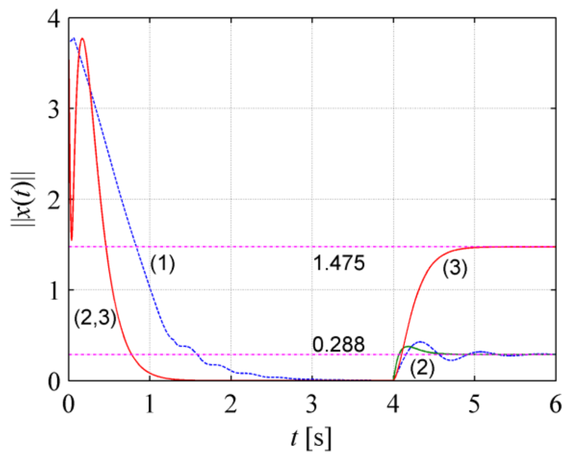

Fig. 1 State vector norms.

To emphasize limitations of traditional SMC in steady-state error minimization and efficiency of ISMC with the proposed sliding manifold design method, consider the following regulation problem. Let a control plant be given in the controllable canonical form:

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + B(u(t) + f_1(t)) + d(t),\\
&y(t) = Gx(t),\\
&A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{33}$$

The control task is to regulate the output $y(t)$ to reach and maintain a desirable value $y_r = 1$. Since $y = x_1$, error vector $e(t) = [x_1(t) - y_r \quad x_2(t) \quad x_3(t)]^{\mathrm{T}}$ is used as an input to the controllers. In traditional SMC optimal sliding manifold satisfies condition (13) and the corresponding matrix C can be decomposed as $C = C_0[I_1 \;\vdots\; H] = c_0[1 \;\vdots\; 0 \quad 0]$. However, it is obvious that $\det([I_1 \;\vdots\; H]B) = 0$ and SM cannot be attained according to Remark 2. Hence, traditional SMC is unable to provide minimization in this case and only ISMC can be applied. SM dynamics has been set by desired eigenvalues $\lambda \in \{-3,-4,-5\}$. To minimize optimization criterion (3) (Q=I) by ISMC, a sliding manifold (18) must be constructed according to the results of Theorem 3. According to (26) and (19), D and E are obtained as:

$$D = [47 \quad 12 \quad 1]\ ,\ E = [60 \quad 0 \quad 0]\,. \tag{34}$$

Let the system be subjected to a matched $f_1(t) = 2\sin(\pi \cdot t)$ and unmatched $d(t) = [-0.1\ \ -0.6\ \ 0.2]^{\mathrm{T}} \cdot f_2(t)$ disturbances. The unmatched disturbance consists of a constant and slowly varying component, i.e. $f_2(t) = [1 + \sin(0.1t)] \cdot h(t - 10)$. Approximating the unmatched disturbance as constant having maximal value, i.e. $f_2(t) = 2h(t-10)$, the expected steady-state error $e_{\min}(\infty) = [0 \quad 0.2 \quad 1.2]^{\mathrm{T}}$ ($x_{\min}(\infty) = [1 \quad 0.2 \quad 1.2]^{\mathrm{T}}$) and the corresponding minimal norm $\|e_{\min}(\infty)\| = 1.2166$ can be calculated according to (7). It is important to notice from the calculated $e_{\min}(\infty)$ that the zero steady-state error is gained for the first coordinate. The proposed integral sliding manifold design also provides this valuable feature. Namely, it can be proven that the proposed integral sliding manifold design method always guaranties zero steady-state for the first coordinate in case of systems in controllable canonical form. This means that the first coordinate, which is usually the output in many systems, becomes insensitive to any constant unmatched disturbances in steady-state for the minimizing integral sliding manifold. Therefore the system output in this example is expected to be very little sensitive to slow sinusoidal unmatched disturbance.

To see the difference, another integral sliding manifold has been designed according to [9] as:

$$D = [0 \quad 0 \quad 1]\ ,\ E = [60 \quad 47 \quad 12]\,. \tag{35}$$

The expected steady-state error vector norm can be calculated as $\|e(\infty)\| = 1.2796$, which is larger then the obtained minimal one.

The proposed sliding manifold design has been verified and illustrated by computer simulations. ISM controller (21) is realized with H obtained for $d_m = 1.2806$ and $\varepsilon = 0.1$. Error vector norms for two sliding manifolds, the optimal (34) and the other (35), are depicted in Fig. 2 by traces (1) and (2) respectively.

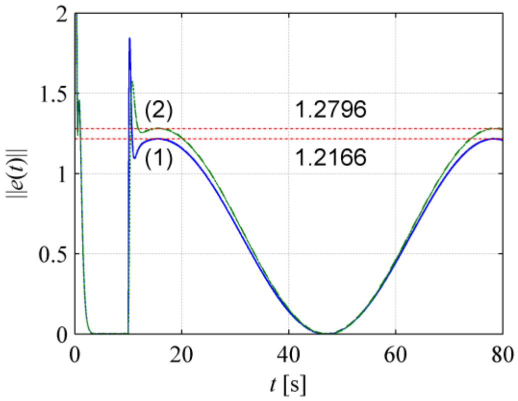

Fig. 2 Error vector norms.

It can be inferred that the state coordinates are insensitive to the matched disturbance but are affected by sinusoidal unmatched disturbance. After the transient period, the error vector norms stay below analytically predicted values for the constant disturbance.

According to the output responses for the two sliding manifolds, given in Fig. 3, low sensitivity to unmatched disturbance is confirmed in case of the optimal manifold (trace (1)). As expected, a much larger output error occurs for the non-optimal manifold (trace (2)).

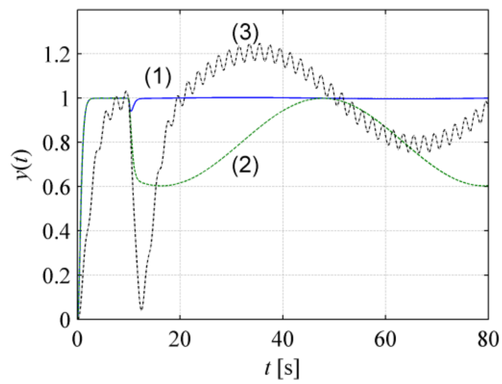

Fig. 3 Output responses.

Finally, conventional PI controller $G_r(s) = k_p(1 + 1/T_i s)$ has been also designed for comparison. To cope with the ISM controllers, the design requirement is to obtain as fast as possible critically damped response, which is ensured by $k_p = 2.1126$ and $T_i = 1$ sec.

The resulting closed-loop dynamics is described by spectrum $\lambda \in \{-0.785, -0.785, -3.43\}$, which indicates that PI controller cannot be as fast as ISM controllers. Output response (Fig. 3, trace (3)) is significantly slower and exhibits poor performance with respect to the sinusoidal matched and unmatched disturbances. Hence, ISMC with the optimal sliding manifold is the superior choice over the conventional PI controller due to much faster dynamics, invariance to the matched disturbances as well as negligible output steady-state error.

6. Conclusions

According to the conducted analysis and computer simulations, traditional SMC exhibits significant limitations in minimizing the steady-state error norm in the presence of unmatched constant or slowly-varying disturbances. The application of the traditional SMC in the considered optimization problem denies the possibility of adjusting SM dynamics and in certain cases SMC cannot be even applied.

On the other hand, ISMC eliminates all shortcomings present in SMC. The proposed integral sliding manifold design method very efficiently finds the sliding manifold that guarantees threefold goal: (i) minimize steady-state error norm in ISM under action of unmatched constant or slowly varying disturbances, (ii) ensures arbitrary predefined SM dynamics and (iii) provides fully decupled control.

The presented numerical examples and related simulations have confirmed the analytically predicted results.

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