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Survey Paper

A COMPREHENSIVE REVIEW OF ORTHOGONAL POLYNOMIALS AND FUNCTIONS WITH APPLICATION IN FILTER DESIGN

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Abstract. *This paper presents some of the results and contributions to the theory of orthogonal polynomials, orthogonal functions and corresponding filters, achieved in the last fifteen years. This theory is based on new definitions and specific generalisations of orthogonal functions and polynomials derived directly in the complex domain. The main objective of this paper is to enable some new applications of orthogonal polynomials in the identification, modelling, signal processing and control of dynamical systems. Accordingly, the paper is divided into seven sections. All central chapters, which describe specific classes of orthogonal polynomials and functions, begin with a brief mathematical background and proposed forms for filter design. In this paper, we give some main results for classical, almost, improved almost, quasi-, generalised and their applications in filter design.*

Key words: *Almost orthogonal polynomials, quasi-orthogonal polynomials, generalised orthogonal polynomials, Legendre type polynomials, orthogonal filters, bilinear transformation.*

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1. INTRODUCTION

In this century, great progress has been made in the field of orthogonal rational functions, orthogonal algebraic and trigonometric polynomials, and orthogonal systems in general. In particular, there are a large number of papers dealing with the application of orthogonal systems in electronics, circuit theory, digital signal processing and telecommunications. One of the most important applications of orthogonal functions is the design of orthogonal filters, which can be successfully used for design of orthogonal signal generators, for the modelling and identification of dynamical systems, and for the practical implementation of optimal and adaptive systems.

Legendre polynomials and their orthogonal properties were intensively researched at the end of the eighteenth century to determine the attraction between the celestial bodies during their orbits [1]. Hermite polynomials were invented as a result of solving differential equations on infinite and semi-infinite intervals, and development of arbitrary functions on these intervals [1]. The theory of continuous fractals gave a strong impetus to the later study of orthogonal polynomials by Szegö, and thus the mathematicians of the mid-nineteenth century laid the foundation for Laguerre polynomials [2]. All the mentioned types of polynomials which are orthogonal on the real axis with respect to the defined weight function have been effectively used for the numerical evaluation of the integral value using the quadrature formulas since the Gaussian era. In 1807, while solving partial differential equations related to the heat conduction, Fourier noticed that the solution can be represented as a series of sine functions with exponential weights. He later transferred the same idea to the representation of arbitrary function as the final sum of sine and cosine functions. In his researches, Chebyshev established that of all the approximation polynomials of an arbitrary function over the interval [-1, 1], the best minimization of the maximum error can be achieved with a linear combination of certain polynomials, which today are referred to as Chebyshev. Somewhat later, Jacobi and Gegenbauer polynomials were developed. A certain class of orthogonal functions, the so-called part by part constant basis functions, founded application in the identification of dynamical systems. These families of functions include Haar and Walsh functions [3]. In the first half of the twentieth century, generalised orthonormal rational basis functions were studied in separate papers by Takenaka [4] and Malmquist. The applications of these functions to the approximation of functions defined on the unit circle (analysis of discrete systems) and the semi-plane (continuous systems) were developed by Walsh in the middle of the last century. During this period, we can also recognize significant research of Szegö and Geronimus [5], [6] on the analysis of polynomials orthogonal in the different domains [7], [8], as well as work of Djrbashian on orthogonal rational functions on the unit circle with fixed poles [7]. Due to certain special properties, the works on orthogonality on the real axis and the unit circle are most numerous [8].

Authors of this paper have paid significant attention in recent years to investigating new types of orthogonal polynomials and their possible applications in control systems and other applications. The main contributions of these researches are new types of orthogonal functions and polynomials with the use of several different types of transfer functions for their generating [9]-[20]. Based on these functions we obtained new classes of improved almost and quasi-orthogonal polynomials as well as practically implemented orthogonal filters in the form of printed circuit boards [11], [12]. As some possible application of the newly derived almost orthogonal filters can be found in [12], [21]. It should be mentioned that in [11], new types of filters based on orthogonal Legendre and Malmquist quasi-orthogonal polynomials have been proposed. For these polynomials, we derived a complete mathematical apparatus in terms of the definition of the inner products, calculating the appropriate norms, as well as the deriving significant relations (Christoffel and Rodrigues formula). In [16], a new class of orthogonal Legendre type filters with complex poles and zeroes were designed with application in the modelling of a real industrial system.

Most of the polynomials and functions mentioned in this paper were used to design filters that were practically realised and later used in modelling, identification and control of systems. Sometimes it was necessary to adapt the obtained transfer functions to a specific filter design.

The rest of this paper is organised as follows. In Sections 2-5, classes of orthogonal polynomials are presented (shifted, almost orthogonal, quasi-orthogonal and finally orthogonal polynomials based on symmetric transformations). Appropriate orthogonal filters practically designed from these polynomials and functions are described in Section 6. Concluding remarks and further work can be found in Section 7.

2. Shifted Orthogonal Polynomials

Considered polynomials are orthogonal over certain interval. To allow an analysis over arbitrary intervals, we can introduce shifted polynomials [12], [14], [22] which are suitable for describing the signal over any considered interval.

All orthogonal polynomials over a finite range (Legendre, Chebyshev of first and second type, Jacobi and Gegenbauer) are defined over the interval [-1, 1]. For the purpose of analysing and processing real signals which can have values over arbitrary intervals, classical polynomials can be redefined, i.e. shifted to the desired interval $[\tau_a, \tau_b]$. Shifted orthogonal polynomials $\{\psi_k(\tau)\}$are defined over an arbitrary interval $[\tau_a, \tau_b]$ and can be obtained from regular (unshifted) $\{\phi_k(x)\}$ by substitution $x = \tau^*$, where:

$$\tau^* = \frac{2(\tau-\tau_a)}{\tau_b-\tau_a} - 1 = A\tau + B\ . \tag{1}$$

$A = \dfrac{2}{\tau_b-\tau_a}$ and $B = -\dfrac{(\tau_b+\tau_a)}{(\tau_b-\tau_a)}$, with the condition that the interval of orthogonality is finite. Then, the definition of orthogonality is also changed:

$$(\psi_n, \psi_k) = \int_{\tau_a}^{\tau_b} \psi_n(\tau)\psi_k(\tau)\omega(\tau)d\tau \ = \begin{cases} 0, & k \neq n, \\ \|\psi_k\|^2 = \|\phi_k\|^2 / A, & k = n, \end{cases} \tag{2}$$

where $\omega(\tau) = w(\tau^*)$ and $\phi_k(\tau) = \psi_k(\tau^*)$, and norms $\|\phi_k\|$ can be found in [7] for different types of polynomials.

Classical Laguerre polynomials are orthogonal over the interval $[0, +\infty]$ with the weight function $w(x) = e^{-x}$, and Hermite over $[-\infty,+\infty]$ with $w(x) = e^{-x^2}$. It can be noticed that in both cases the following relations are valid: $w(0) = 1$ and $w(x\rightarrow\infty)\rightarrow 0$. That means that these weight functions differently sample large and small values of x. So, in order to better represent the function $f(x)$, $x \in [\tau_a, \tau_b]$, we have to move (shift) the centre ($x = 0$) into τ_a by substituting $x = \tau - \tau_a$, so that the new weight function becomes 1 in both cases for $\tau = \tau_a$. Obtained polynomials are also shifted, but we have to emphasize that described shifting of Laguerre and Hermite polynomials has completely different meaning than shifting polynomials orthogonal over the finite interval [23].

3. ALMOST ORTHOGONAL POLYNOMIALS AND FUNCTIONS

For simplicity, we now introduce the concept of generalised orthogonal functions using the well-known Legendre polynomials for the sake of simplicity (the same concept can be used for other classical orthogonal polynomials as well) [13], [15]. Our design is based on shifted Legendre polynomials that are orthogonal over the interval (0, 1). On the other hand, technical systems operate in real time, so we need the corresponding approximation over the interval (0, ∞). The solution is to use the substitution $x = e^{-t}$. In this way, the polynomial sequence orthogonal over (0, 1), becomes an exponential polynomial sequence orthogonal over the interval (0, ∞).

Let us consider the orthogonal Legendre polynomials in their explicit form [12]:

$$P_n(x) = \sum_{i=0}^{n} A_{n.i} x^i \ , \tag{3}$$

where:

$$A_{n.i} = \frac{1}{n!}(-1)^{n-i}\binom{n}{i}\frac{(n+i)!}{i!} \, . \tag{4}$$

These polynomials are orthogonal over the interval (0, 1), with the weight function $w(x) = 1$, and the following definition of orthogonality based on the inner product:

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ N_i\,, & m = n, \end{cases} \tag{5}$$

and they can be successively used for modelling, and control of dynamical systems as well as for identification of specific systems parameters.

Corresponding almost orthogonal polynomials $P_n^{(\varepsilon)}(x)$ can be defined as [13], [24]:

$$\int_0^1 P_m^{(\varepsilon)}(x)P_n^{(\varepsilon)}(x)dx = \begin{cases} \varepsilon, & m \neq n \\ N_n^{(\varepsilon)}, & m = n, \end{cases} \tag{6}$$

where ε represent a very small positive constant $(0 < \varepsilon << 1)$. The connection between classical orthogonal and almost orthogonal polynomials is proved in [12] with the following relation:

$$P_n^{(\varepsilon)}(x) = P_n(x) + \sum_{k=1}^{n-1} \frac{b_k}{\|P_k\|^2} P_k(x) \, , \tag{7}$$

where $||P_k||^2$ represents the square of the norm and b_k are polynomials dependent on ε. Obviously, for ε=0 almost orthogonal polynomials become strictly orthogonal, i.e. $\lim_{\varepsilon \to 0} P_n^{(\varepsilon)}(x) = P_n(x)$. For design of almost orthogonal filters, it is very convenient to use the following three term recurrent relation, derived in [12], [14]:

$$W_{n+1}^{(\varepsilon)}(s) = W_n^{(\varepsilon)}(s) + W_{n+1}(s) + W_n(s)k_n , \tag{8}$$

where: $k_n = \dfrac{b_n(\varepsilon)}{\|P_n\|^2} - 1,\ k_0 = \varepsilon$. The first few members of the almost orthogonal polynomials $\{P_n^{(\varepsilon)}(x)\}$ over the interval (0, 1) with the weight function $w(x)$=1 sequence are:

$$\begin{aligned} &P_0^{(\varepsilon)}(x) = 1,\\ &P_1^{(\varepsilon)}(x) = 2x - (1-2\varepsilon),\\ &P_2^{(\varepsilon)}(x) = 6x^2 - 6(1-12\varepsilon+12\varepsilon^2)x + (1-30\varepsilon+36\varepsilon^2),\\ &\vdots \end{aligned} \tag{9}$$

In [13], we defined almost orthogonality in a different manner in order to accomplish further simplifications and improvements in filters design. First, we defined almost orthogonal Legendre polynomials $P_n^{(\delta)}(x)$:

$$P_n^{(\delta)}(x) = \sum_{i=0}^{n} A_{n,i}^{\delta} x^i , \tag{10}$$

where $A_{n,i}^{\delta}$ represents coefficients defined by:

$$A_{n,i}^{\delta} = (-1)^{n+i} \frac{\Gamma(n\delta+i+1)}{\Gamma(n\delta+1)i!(n-i)!} , \tag{11}$$

δ is a constant near to one: $\delta=1+\varepsilon\approx1$ and Γ is a symbol for the gamma function [5], [7]. Parameter δ is an uncertain quantity, which describes the imperfection of the system. Variations of δ contain cumulative impacts of all imperfect elements, model uncertainties, and measurement noise on the system output. The range of variations can be determined by conducting several experiments. Hence, it is expected that responses obtained from different experiments are mutually different. The responses are in certain boundaries, which depend on parameter δ i.e., on the real system components quality, or the noise level present in signal.

After applying the substitution $x = e^{-t}$ to (10) and Laplace transform, we obtain a transfer function suitable for designing almost orthogonal filters:

$$W_n^{(\delta)}(s) = \frac{\prod_{i=1}^{n}(s-i\delta)}{\prod_{i=0}^{n}(s+i)} = \frac{(s-\delta)(s-2\delta)\cdots(s-n\delta)}{s(s+1)(s+2)\cdots(s+n)} . \tag{12}$$

As a mapping function use the transformation $\bar{s}+s=0$, i.e., $\bar{s}=-s$. In this case, the left semi plane of the complex plane s is being transformed into the right semi plane [21], [25]. Almost orthogonality can be analysed from the following inner product:

$$N_{nm} = \oint_C W_n^{(\delta)}(s)\overline{W}_m^{(\delta)}(s)w(s)ds , \tag{13}$$

with weight $w(s) = 1$, $m > n$. Now, applying the Cauchy theorem, we obtain:

$$N_{nm} = 2\pi j \sum_{k=1}^{m} \text{Res}\left[W_n^{(\delta)}(s) \bar{W}_m^{(\delta)}(s) \right]. \tag{14}$$

$$N_{nm} = 2\pi j \sum_{k=1}^{m} \frac{(-1)^{n+1} \prod_{i=1}^{n} (k+i\delta) \prod_{i=1}^{m} (k-i\delta)}{k^2 \prod_{i=1}^{n} (k-i) \prod_{i=1}^{m} (k+i)}. \tag{15}$$

The first few members of the improved almost orthogonal polynomials $\{P_i^{(\delta)}(x)\}$ over the interval (0, 1) with weight function $w(x)=1$ sequence are:

$$\begin{aligned} &P_0^{(\delta)}(x) = 1, \\ &P_1^{(\delta)}(x) = (\delta+1)x - \delta, \\ &P_2^{(\delta)}(x) = (\delta+1)(\delta+2)x^2 - (\delta+1)(2\delta+1)x + \delta^2, \\ &\vdots \end{aligned} \tag{16}$$

4. Quasi-orthogonal Polynomials and Functions

The final generalisation of the concept of orthogonality can be introduced by the following definition of quasi-orthogonality for the polynomial set $P_n(x)$ [11], [18], [21]:

$$(P_n^k(x), P_m^k(x)) = \int_a^b w(x) P_n^k(x) P_m^k(x) dx = \begin{cases} 0, & 0 \le m \le n-k-1, \\ N_{n,m}^k \ne 0, & n-k \le m \le n, \end{cases} \tag{17}$$

where k represents the order of quasi-orthogonality, a and b are the limits of quasi-orthogonality interval, and $w(x)$ is the weight function. In our case of Legendre quasi-orthogonal polynomials of the first order [11]:

$$\int_0^1 P_n^k(x) P_m^k(x) x dx = \begin{cases} = 0, & 0 \le m \le n-k-1 \\ \ne N_{n,m}^k, & n \ge k+1, \end{cases} \tag{18}$$

$$P_n^k(x) = \sum_{i=0}^{n} A_{n,i}^k x^i = x \sum_{i=0}^{n} A_{n,i}^k x^{i-1}, \tag{19}$$

$$A_{n,i}^k = \frac{(-1)^{n-k+i} (n+i-k-1)!}{i!(i-1)!(n-i)!}. \tag{20}$$

In the similar way as in the case of almost orthogonal polynomials we obtain orthogonal functions (poles are integer) [11], [21]:

$$W_n^k(s) = \frac{(s-1)(s-2)\cdots[s-(n-k-1)]}{(s+1)(s+2)\cdots(s+n)}, \tag{21}$$

or in the following form (shifted and more suitable for filter design):

$$W_n^k(s) = \frac{1}{s(s+1)\cdots(s+k)} \cdot \frac{(s-2)}{(s+k+1)} \cdot \frac{(s-i)}{(s+k+(i+1))} \cdots \frac{(s-(n-k))}{(s+n-1)} . \tag{22}$$

The first few members of quasi-orthogonal polynomials of the order k=1, $P_n^1(x)$ over the interval (0, 1) with the weight function $w(x)$=x sequence are:

$$\begin{aligned} &P_1^1(x) = -x+1, \\ &P_2^1(x) = -2x^2+3x-1, \\ &P_3^1(x) = -5x^3+10x^2-6x+1, \\ &\vdots \end{aligned} \tag{23}$$

and a few second order (k=2) quasi-orthogonal polynomials:

$$\begin{aligned} &P_2^2(x) = \frac{1}{2}x^2 - x + \frac{1}{2}, \\ &P_3^2(x) = \frac{5}{6}x^3 - 2x^2 + \frac{3}{2}x - \frac{1}{3}, \\ &P_4^2(x) = \frac{7}{4}x^4 - 5x^3 + 5x^2 - 2x + \frac{1}{4}, \\ &\vdots \end{aligned} \tag{24}$$

If we apply the definition of quasi-orthogonality on almost orthogonal polynomials given by the following relation [11], [16]:

$$P_n^{(\delta)}(x) = \sum_{i=0}^{n} A_{n,i}^{\delta} x^i , \tag{25}$$

$$A_{n,i}^{\delta} = (-1)^{n+i} \frac{\Gamma(n\delta+i+1)}{\Gamma(n\delta+1)i!(n-i)!} , \tag{26}$$

and Γ is a symbol for the gamma function, we obtain quasi-almost orthogonal Legendre type, polynomials over the interval (0, 1) with the weight function $w(x)$ = 1:

$$P_n^{(k,\delta)}(x) = \sum_{i=0}^{n} A_{n,i}^{k,\delta} x^i , \tag{27}$$

$$A_{n,i}^{k,\delta} = (-1)^{n+i+k} \frac{\prod_{j=1}^{n-k}(i+j\delta)}{i!(n-i)!} . \tag{28}$$

In the previous relation, k is the order of quasi-orthogonality, and δ is a constant, defined as $\delta = 1 + \varepsilon \approx 1$ [11], [13], [16]. The range of the parameter δ can be determined by conducting several experiments based on the designers' experience. Polynomials defined via (25) and (26) represent generalised quasi-orthogonal polynomials.

Now, let us define the quasi-orthogonality over the interval (0, 1) with the weight function $w(x) = 1$ via the inner product in the following manner:

$$\left(P_n^{(k,\delta)}(x), P_m^{(k,\delta)}(x)\right) = \int_a^b P_n^{(k,\delta)}(x) P_m^{(k,\delta)}(x) w(x) dx = \begin{cases} Q_k(\delta), & m \neq n, \\ N_n^{(k,\delta)}, & m = n, \end{cases} \tag{29}$$

$$Q_k(\delta) = q_0 + q_1\delta + q_2\delta^2 + \cdots + q_k\delta^k . \tag{30}$$

In the similar way as in the previous classes of orthogonal polynomials we obtain appropriate orthogonal functions:

$$W_n^{(k,\delta)}(s) = \frac{\prod_{i=1}^{n-k}(s - i\delta)}{\prod_{i=0}^{n}(s+i)} = \frac{(s-\delta)(s-2\delta)\cdots(s-(n-k)\delta)}{s(s+1)(s+2)\cdots(s+n)}, \tag{31}$$

or in the form adjusted for the filter design:

$$W_n^{(k,\delta)}(s) = \frac{1}{s(s+1)\cdots(s+k)} \cdot \frac{s-\delta}{s+k+1} \cdots \frac{s-(i-1)\delta}{s+k+(i-1)} \cdots \frac{s-(n-k)\delta}{s+n} . \tag{32}$$

A few first order ($k = 1$) generalised quasi-orthogonal polynomials of this sequence are:

$$\begin{aligned} &P_1^{(1,\delta)}(x) = -x + 1, \\ &P_2^{(1,\delta)}(x) = -\frac{(\delta+2)}{2}x^2 + (\delta+1)x - \frac{\delta}{2}, \\ &P_3^{(1,\delta)}(x) = -\frac{(\delta+3)(2\delta+3)}{6}x^3 + (\delta+1)(\delta+2)x^2 - \frac{(\delta+1)(2\delta+1)}{2}x + \frac{\delta^2}{3}, \\ &\vdots \end{aligned} \tag{33}$$

and a few second order (k=2) generalised quasi-orthogonal polynomials:

$$\begin{aligned} &P_2^{(2,\delta)}(x) = \frac{1}{2}x^2 - x + \frac{1}{2}, \\ &P_3^{(2,\delta)}(x) = \frac{(\delta+3)}{6}x^3 - \frac{(\delta+2)}{2}x^2 + \frac{(\delta+1)}{2}x - \frac{\delta}{6}, \\ &P_4^{(2,\delta)}(x) = \frac{(\delta+2)(\delta+4)}{12}x^4 - \frac{(\delta+3)(2\delta+3)}{6}x^3 + \\ &+ \frac{(\delta+1)(\delta+2)}{2}x^2 - \frac{(\delta+1)(2\delta+1)}{6}x + \frac{\delta^2}{12}, \\ &\vdots \end{aligned} \tag{34}$$

5. ORTHOGONAL POLYNOMIALS AND FUNCTIONS BASED ON SYMMETRIC TRANSFORMATIONS

By using Legendre orthogonal functions:

$$W_n(x) = \prod_{j=0}^{n-1} \frac{s+\alpha_j+1}{s-\alpha_j} \frac{1}{s-\alpha_n} . \tag{35}$$

Müntz-Legendre polynomials [18], [26], which are orthogonal on the interval (0, 1), were obtained by [26], [27]:

$$P_n(x) = \frac{1}{2\pi i} \oint_\Gamma \prod_{j=0}^{n-1} \frac{s+\alpha_j+1}{s-\alpha_j} \frac{x^s}{s-\alpha_n} ds , \tag{36}$$

where the contour Γ surrounds all the poles of the integrand. Functions $W_n(x)$ are used for designing orthogonal Legendre filters. Müntz–Legendre polynomials are used for obtaining outputs from these filters. Let us notice that in these filters the zeroes are obtained by linear transformation of the poles. Orthogonal Laguerre functions where the zeroes have reciprocal values of the poles:

$$W_n(s) = \frac{\sqrt{1-\alpha^2}}{1-\alpha} \left(\frac{1-\alpha s}{s-\alpha} \right)^{n-1} , \tag{37}$$

are used for design of orthogonal Laguerre filters [7]. The Takenaka-Malmquist rational function system [28]-[30]:

$$W_n(s) = \frac{\sqrt{1-\alpha_n{}^2}}{1-\alpha_n s} \prod_{k=0}^{n-1} \frac{(\alpha_k - s)}{(1-\alpha_k s)}, \ n = 0,1,2,\ldots \tag{38}$$

is used for designing appropriate orthogonal filters (Takenaka-Malmquist filters), [7], [21]. Let us notice that the zeroes of $W_n(s)$ have reciprocal values of the poles. The generalisation of Malmquist functions was performed in [31] by using the mapping $s \to b/s$:

$$W_n(s) = \frac{\prod_{k=0}^{n-1}(s+\alpha_k^*)}{\prod_{k=0}^{n}(s+\alpha_k)} = \frac{1}{s+\alpha_0} \prod_{k=1}^{n} \frac{s+\alpha_{k-1}^*}{s+\alpha_k}, \ \ \alpha_k^* = \frac{b}{\alpha_k}, \ \ \alpha_k \in R, \ \ \alpha_k \ge 0 . \tag{39}$$

By using these functions and the following relation:

$$P_n(x) = \frac{1}{2\pi i} \oint_\Gamma W_n(s) x^s ds , \tag{40}$$

a class of orthogonal Müntz polynomials which are orthogonal with respect to the special inner product was derived. In this way, generalised Takenaka-Malmquist filters are designed. Outputs of these filters are obtained using (40). A new class of Müntz polynomials derived from a sequence of orthogonal Malmquist functions, is introduced by [22]. These Müntz polynomials are orthogonal with respect to the special inner product. In [31] we have applied them for design of a new class of filters based on reciprocal transformation (generalised Malmquist type) (39) which are orthogonal with respect to a new special inner product.

In this paper, the focus is on a more general class of orthogonal rational functions and filters, to which the above belong. Namely, the zeroes in the all above mentioned rational orthogonal basis functions are obtained either by the linear transformation $s \to as + b$ or the reciprocal transformation $s \to b/cs$ of the poles. The transfer functions of the orthogonal filters are $W_n(s)$, and the outputs are obtained using Müntz polynomials derived by (38). A class of orthogonal cascade filters which represents a generalisation of all the mentioned filters obtained by using linear and reciprocal mapping poles to zeroes is realised by using the symmetric bilinear transformation:

$$s \to \frac{as+b}{cs-a}, \; (a^2+bc>0), \; (a,b,c \in R) \; . \tag{41}$$

By using this transformation for mapping poles α_k to zeroes α_k^* we obtain:

$$\alpha_k^* = \frac{a\overline{\alpha_k}+b}{c\overline{\alpha_k}-a} \; . \tag{42}$$

For design of orthogonal cascade filters with real poles, with taking into account that the bilinear transformation is symmetric, we have:

$$\alpha_k^* = \frac{a\alpha_k+b}{c\alpha_k-a}, \quad \alpha_k = \frac{a\alpha_k^*+b}{c\alpha_k^*-a} \; . \tag{43}$$

The sequence of the appropriate rational functions has the following form now:

$$W_n(s) = \frac{\prod_{k=0}^{n-1}\left(s-\frac{a\alpha_k+b}{c\alpha_k-a}\right)}{\prod_{k=0}^{n}(s-\alpha_k)} \; . \tag{44}$$

If we apply the transformation (41) onto $W_n(s)$ we obtain:

$$W_n^*(s) = \frac{\prod_{k=0}^{n-1}(s-\alpha_k)}{\prod_{k=0}^{n}\left(s-\frac{a\alpha_k+b}{c\alpha_k-a}\right)} \; . \tag{45}$$

Let us consider the inner product:

$$(W_n, W_m) = \frac{1}{2\pi i}\oint_{\Gamma} W_n(s)W_m^*(s)ds \; , \tag{46}$$

where the contour Γ surrounds all the poles of $W_n(s)$. If $m \neq n$ due to the symmetry of the bilinear transformation, all the poles of the integrand (46) that lie inside the contour Γ are annulled with appropriate zeroes of $W_m^*(s)$, so the contour integral (46) is equal to zero. In the case of $m = n$, there exists one first-order pole inside the contour Γ. After applying the Cauchy theorem, we can obtain the following expression: $(W_n(s), W_m(s)) = N_n^{\,2} \neq 0$. Finally, all the expressions stated above imply:

$$(W_n, W_m) = N_n^{\,2}\delta_{n,m}\,, \tag{47}$$

where $\delta_{n,m}$ represents Kronecker symbol, and poles α_k lie inside the contour Γ, while all the zeroes α_k^* (43) lie outside the contour Γ. Thus, the sequence of the rational functions $W_n(s)$ is orthogonal in the complex domain bordered by the contour Γ with respect to the inner product (43).

Using the sequence $W_n(s)$ we can obtain a class of orthogonal Müntz polynomials based on (40) in the following way [32], [33]:

$$P_n(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{\prod_{k=0}^{n-1}\left(s - \frac{a\bar{\alpha}_k + b}{c\bar{\alpha}_k - a}\right)}{\prod_{k=0}^{n}(s-\alpha_k)}\,ds\,. \tag{48}$$

These polynomials can be written as:

$$P_n(x) = \sum_{k=0}^{n} A_{n,k}\,x^{\alpha_k}\,, \tag{49}$$

$$A_{n,k} = \frac{\prod_{j=0}^{n-1}\left(\alpha_k - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}\right)}{\prod_{j=0,\, j\neq k}^{n}(\alpha_k - \alpha_j)}\,,\quad k = 0,1,2,\ldots,n\,. \tag{50}$$

It is shown that these Müntz polynomials are orthogonal with respect to an inner product which is defined below. First, the operation $\otimes$ on monoms x^α and x^β is defined in the following way [33]:

$$x^\alpha \otimes x^\beta = x^{c\alpha\beta - a(\alpha+\beta) - b}\,. \tag{51}$$

Using this operation, the product of two Müntz polynomials, $P_n(x) = \sum_{k=0}^{n} p_k x^{\alpha_k}$ and $P_m(x) = \sum_{j=0}^{m} q_j x^{\alpha_j}$ can be defined in the following manner:

$$P_n(x) \otimes Q_m(x) = \sum_{k=0}^{n}\sum_{j=0}^{m} p_k q_j x^{c\alpha_k\alpha_j - a(\alpha_k+\alpha_j) - b}\,. \tag{52}$$

Using this product of Müntz polynomials, a new inner product can be defined as:

$$\left(P_n(x), P_m(x)\right)_\otimes = \int_0^1 P_n(x) \otimes \overline{P_n}(x)\frac{dx}{x^2}\,. \tag{53}$$

Finally, by using (41) and (48) we obtain [33]:

$$\left(P_n(x), P_m(x)\right)_{\otimes} = \frac{(a^2+bc)^n}{\prod_{k=0}^{n-1}|c\alpha_k - a|^2} \frac{\delta_{n,m}}{c|\alpha_n|^2 - 2a\operatorname{Re}\alpha_n - b} = N_n{}^2\delta_{n,m}\,. \tag{54}$$

Hence, Müntz polynomials (40) derived from orthogonal rational functions $W_n(s)$ are orthogonal on the interval (0, 1) with respect to the inner product (18). If rational functions $W_n(s)$ have real poles, then Müntz polynomials $P_n(x)$ are with real exponents. In that case, substituting $x = e^{-t}$ into $P_n(x)$, we obtain exponential functions:

$$\varphi_n(t) = P_n(e^{-t}) = \sum_{k=0}^{n} A_{n,k} e^{-\alpha_k t}\,. \tag{55}$$

Using (47), (51), (52), and (55) we obtain:

$$\left(\varphi_n(t), \varphi_m(t)\right)_{\otimes} = \int_0^{\infty} \varphi_n(t) \otimes \varphi_m(t) e^{-t} dt = N_n{}^2 \delta_{n,m}\,, \tag{56}$$

$$\varphi_n(t) \otimes \varphi_m(t) = \sum_{k=0}^{n} A_{n,k} e^{-\alpha_k t} \otimes \sum_{j=0}^{m} A_{m,j} e^{-\alpha_j t} = \sum_{k=0}^{n}\sum_{j=0}^{m} A_{n,k} A_{m,j} e^{-[c\alpha_k\alpha_j - a(\alpha_k+\alpha_j) - b]t}\,. \tag{57}$$

The functions (44) can be written in the form more suitable for a practical filter design:

$$W_n(s) = \frac{1}{s+\alpha_0} \prod_{k=1}^{n} \frac{s + \dfrac{a\alpha_{k-1}+b}{c\alpha_{k-1}-a}}{s+\alpha_k}, \quad \alpha_k \in R, \quad \alpha_k \ge 0\,. \tag{58}$$

As we have already said, this is a generalisation of a filter based on a simple reciprocal transformation $a = 0$, $c = 1$, but also of most existing filters by choosing specific values for parameters a, b, and c. Hence, filters based on a bilinear transformation are adaptive and by adjusting the parameters we can achieve better performances.

Remark: There are also corresponding classes of digital filters where the basis for the realisation are the corresponding transfer functions in the z-domain, but the focus of this paper is on the functions in continuous s-domain with real poles and analogue filters.

This is a new generalised inner product based on the bilinear transformation of poles to zeroes. For illustrative purposes, a sequence of polynomials of the first few polynomials for the following values of poles $\alpha_0 = -2,\ \alpha_1 = -3,\ \alpha_2 = -4,\ \alpha_3 = -5,\ \alpha_4 = -6$ and parameters of bilinear transformation $a = 1$, $b = 1$, $c = 1$ are:

$$\begin{aligned}
P_0(x) &= x^2,\\
P_1(x) &= \frac{10}{3}x^3 - \frac{7}{3}x^2,\\
P_2(x) &= \frac{39}{4}x^4 - \frac{35}{3}x^3 + \frac{35}{12}x^2,\\
P_3(x) &= \frac{1232}{45}x^5 - \frac{897}{20}x^4 + 21x^3 - \frac{91}{36}x^2,\\
&\vdots
\end{aligned} \tag{59}$$

6. Orthogonal Filters Design and Practical Realisations

One of the most important applications of orthogonal functions is the design of orthogonal filters [11]-[17]. These filters can be used for signal approximation [14], for the design of orthogonal signal generators [12], for modelling and system identification [13], [21], [22], [37], [39] as well as for the practical implementation of optimal and adaptive systems [36] and control methods [25]. The theory of classical orthogonal filters has been well studied and described in numerous papers [7], [11]-[21], [40]. In [25], the design procedures for classical, almost quasi-orthogonal filters of Legendre and Müntz-Legendre type are described [21]. To successfully design and implement certain types of filters, we need to start from the rational functions given in the form (7) and (12) for almost orthogonal, (19) and (20) for quasi-orthogonal, (23) for generalisation quasi-orthogonal and, (34) and (39) for orthogonal cascade filters based on symmetric transformations.

Figure 1 shows an almost orthogonal filter of the Legendre type [13], [14] with six sections, which was realised in the analogue technique. The transfer function of this filter corresponds to the expression (8) in which the poles and zeroes have integer values (i = 1, 2,...,6, n = 5). The following components were used for the realisation of this filter: 15 operational amplifiers (µA 741CN), 60 resistors, 12 potentiometers and 6 electrolytic capacitors.

Figure 2 presents the realised printed circuit board of the improved almost orthogonal filters [13]. The following components were used for the realisation of this filter: 8 operational amplifiers (LM 324N), 48 resistors, 9 potentiometers and 8 electrolytic capacitors.

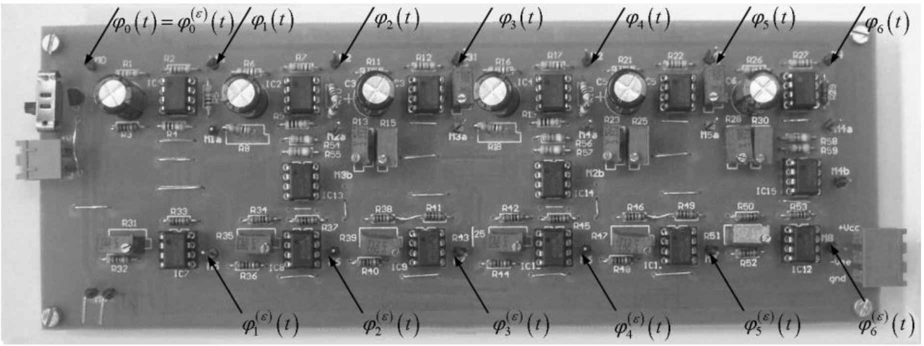

Fig. 1 Almost orthogonal Legendre type filter - printed circuit board

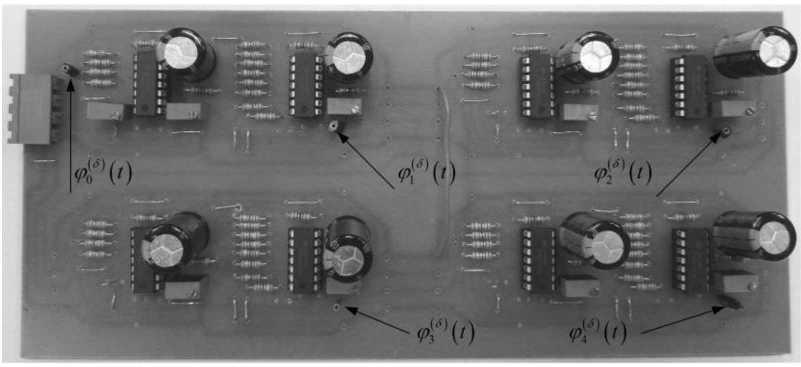

Fig. 2 Improved almost orthogonal Legendre type filter - printed circuit board

A laboratory setup for testing our quasi-orthogonal filter is given in Fig. 3 [11], [21]. The setup consists of a printed circuit board with the realised first order Legendre quasi-orthogonal filter, microprocessor unit and power supply. A practically realised experimental printed circuit board for a generalised quasi-orthogonal first-order filter with four sections (n=4) is given in Fig. 4 [21]. The following components were used for the realisation of this filter: 7 operational amplifiers (LM 324N), 55 resistors, 9 potentiometers and 7 electrolytic capacitors.

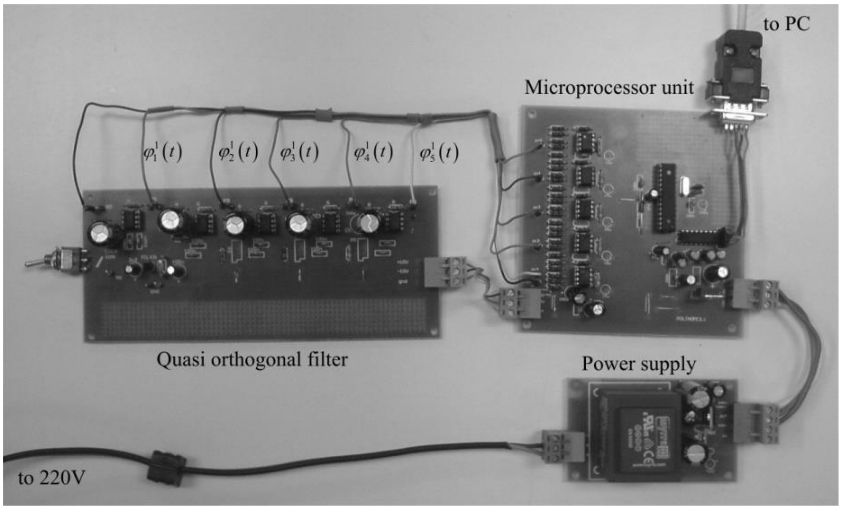

Fig. 3 Laboratory setup for a first-order Legendre quasi-orthogonal filter

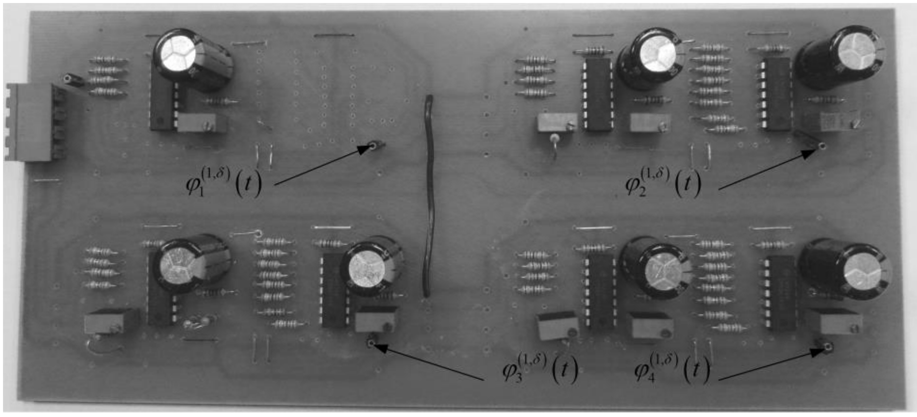

Fig. 4 Generalised Legendre first-order quasi-orthogonal filter - printed circuit board

Relations (34) and (39) represent transfer functions of the new orthogonal cascade filter based on symmetric transformations, and it is a base for its practical realisation (Fig. 5). The designed filter in Fig. 5 is based on the bilinear transformation which is a generalisation of the reciprocal one so the printed circuit board is for both.

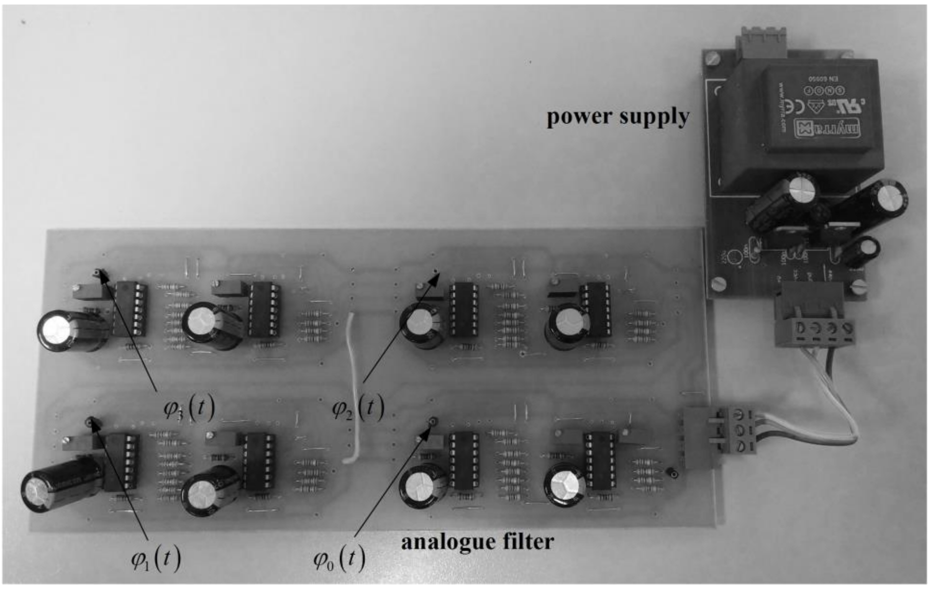

Fig. 5 An orthogonal filter based on bilinear transformation, a printed circuit board

7. CONCLUSION AND FUTURE WORK

In this paper we have made a survey study of new classes of orthogonal polynomials (almost, improved almost, quasi-, generalised quasi-, and polynomials obtained by symmetric transformations) derived in recent years. These polynomials and functions can have a wide range of applications in various scientific and engineering fields (modelling [9], [14], [19], [31], identification [21], [39], sensitivity analysis [15], [38], model predictive control [41]-[43], control systems theory [18], [21], [34], neural networks [17], [18], [35], [37], [38], fuzzy systems [21], ANFIS [21], [36], DPCM system [20], [31] etc.) [19], [20]. Some of the systems and devices where these orthogonal functions and filters have found application are modular DC servo drive [14], [36], [42], [43], magnetic levitation system [17], two rotor aero-dynamical system (TRAS) [37], tower crane [38], multi tank system [15], [39], [41], protector cooling system [9], [19], [31], anti-lock braking system (ABS) [18], differential pulse-code modulation (DPCM) system [20], [31].

The aim is to synthesize all these polynomials and functions that are important to us in some way and compare them using certain measures of quality evaluation. Of course, the further development and generalisation of already existing functions and filters, especially the discrete ones, which have been talked about the least here, and the continuation of their application in various fields is also planned.

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