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MAXIMIZING SALES UNDER CONDITIONS OF NONLENARITY

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Abstract. *This paper deals with the problem of maximizing the sales of a particular product when the revenue function is nonlinear in dependence of the demand for that product. This type of problem is usually solved by the nonlinear programming method which has been sufficiently described in mathematical theory; however, its use is not that simple. Solving functions of more than two variables is rather complicated and requires an appropriate mathematical model as well as suitable software for computer solving of the given problem, which sometimes involves team work.*

Key words: *Nonlinear programming, Kuhn-Tucker conditions, revenue function, demand*

INTRODUCTION

Problems in nonlinear programming appear when the objective function, that is, a corresponding system of constraints, is defined by nonlinear dependencies. Unfortunately, there is no universal solution to these problems, as there is for linear models, the majority of which are solved, for example, by the Simplex method. Moreover, the LP model is a special case of the general NLP model and a number of different methods and procedures have been developed for its solving. In most cases, all of them depend on the type of nonlinearity that exists in the specific NLP problem, which means that a large number of these problems have not been solved yet. This is exactly the reason why we have chosen to describe the most important terms, mathematical models and procedures for solving these types of problems in this paper.

a. Formulating NLP problems

The general task of nonlinear programming on which we will focus our attention here is the following:

Minimize the function $f(x)$, ie .find min $f(x)$ with constrains –inequalities:

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$$g_i(x) \leq 0 \qquad i = 1, \ldots m,$$

Whereby f, $g_1,\ldots g_m$ functions are defined in R^n, and $x \in R^n$.

As with the LP problem, the task is to find the vector

$$x \;\; = (x_1, \ldots, x_n)^T \in R^n$$

which satisfies the given constraints and is, at the same time, consistent with the minimum value of function $f(\mathbf{x})$. This function is called **the objective function**, and every condition $g_i\,(x) \leq 0$ $i = 1,\ldots,m$, refers to its **constraints**. Vector $x \in R^n$ which satisfies all constraints is called **feasible solution** or **feasible point**. A set of all feasible solutions makes a **feasible region or a feasible set** $S \subseteq R^n$**.**

So, the problem in nonlinear programming is finding the feasible solution x^*, whereby $f(x^*) \leq f(x)$ for each feasible solution x. A vector x^* is called **optimal** or a **solution** of the NLP problem.

The NLP problem, of course, may be defined as function maximizing $f(x)$ or defining max $f(x)$ with constrains – inequalities in the following form:

$$g_i(x) \geq 0, \qquad i = 1, \ldots, m.$$

In special cases when the objective function is linear and all the constraints are in the form of linear equations, inequations or their combination, the problem discussed above will become a linear programming problem, i.e. the standard maximum and minimum problems or some combination of the two.

b. Kuhn-Tucker optimization condition

As already mentioned the results of Nonlinear Programming Theory in mathematics are well-known and described. They are of course used in this paper on the example we had studied and which is related to an increase in the demand of a company. Therefore, it may be useful to give a brief description of these results so that we can understand all the procedures applied in the problem solving process.

The analysis of the functions that are a result of the direct and Lagrangian dual problem (well-known in the NLP theory) leads to a set of facts which primarily offer the needed and sufficient optimization conditions for the solution of both problems. The application of Lagrangian principle is based on the famous Kuhn-Tucker Theorems which occupy an important place in the convex programming theory and as such will be in the center of our attention. Assume that X is a nonempty open set from R^n, and that f, g_1, ..., g_m are formerly defined real n-dimensional functions. Now let us consider again the problem of function minimization $f(x)$ under the following conditions $x \in X$ and $g_i(x) \leq 0$, $i = 1,\ldots m$. To accomplish this, let us fix an arbitrary permissible point $x_0 \in X$, and present it as follows

$$I = \{i \,|\, g_i(x_0) = 0\}.$$

Assume now that functions f and g_i are differentiable at x_o, and vectors $\nabla g_i(x_o)$. for $i \in I$ linearly independent.

Function $f(x)$ which is differentiable at x_o, can have only one vector – gradient $\nabla f(x_o) = \left(\frac{\partial f(x_o)}{\partial x_1}, \frac{\partial f(x_o)}{\partial x_2}, \ldots \frac{\partial f(x_o)}{\partial x_n} \right)^T$, whereby $\frac{\partial f(x_o)}{\partial x_i}$, $i = 1,\ldots,n$ are partial derivatives of $f(x)$ at $x_{o.}$ At this point we introduce

Theorem 1.1. (Kuhn-Tucker Theorems - Necessary Conditions). *Suppose that $x_0 \in X$ is the local optimum of the given optimization problem, then there are numbers u_1, u_2,... u_m that lead to the following*

$$\nabla f(x_0) + \sum_{i=1}^{m} u_i \nabla g_i(x_0) = 0 \tag{1}$$

Whereby $u_i g_i(x_0) = 0$ *and* $u_i \geq 0$ *for each i=1,……,m.* □

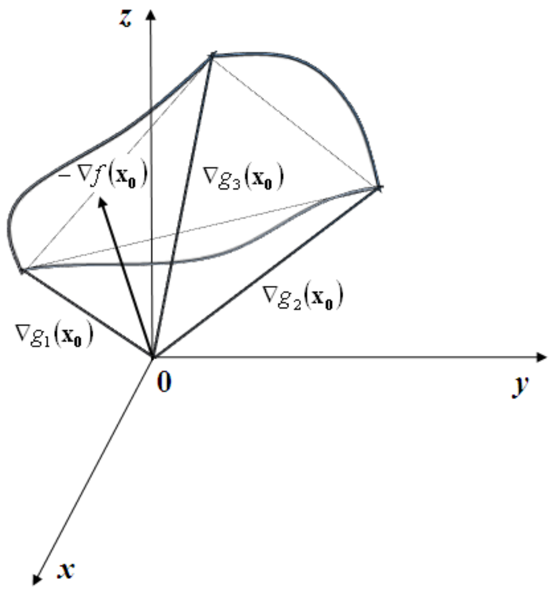

Fig. 1 Geometric interpretation of Kuhn-Tucker optimization conditions

The geometric interpretation of Kuhn-Tucker optimization conditions are shown in Figure 1. The arbitrary vector as a linear combination is as follows:

$$\sum_{i \in I} u_i \nabla g_i(x_0), \qquad u_i \geq 0,$$

and must lie in the cone defined by the gradient vectors $g_i(x)$ which define the constraints at x_0. So, equation (1) leads to the following:

$$-\nabla f(x_0) = \sum_{i \in I} u_i \nabla g_i(x_0), , \tag{2}$$

So vector $-\nabla f(x_0)$ belongs to the cone defined by the gradient vectors of active constraints $g_i(x)$ at x_0 if it meets Kuhn-Tucker optimization conditions. Here, as elsewhere, the numbers $u_1,...u_m \geq 0$... are called **Lagrangian parameters or multipliers**, while the following equations

$$u_i g_i(x_0) = 0, \qquad i = 1,...m, \tag{3}$$

are called **complementary elasticity conditions**. Kuhn-Tucker conditions may be expressed by forming a vector:

$$\nabla f(x_0) + u^T \nabla g(x_0) = 0 \quad \wedge \quad u^T g(x_0) = 0 \quad \wedge \quad u \geq 0$$

whereby $\nabla g\ (x_0)$ is the $n\ x\ m$ matrix in which i-column equals the gradient $\nabla g_i\ (x_0)$, and $u = (u_1,...,u_m)^T$ is m-dimensional vector of the Lagrangian multipliers. However, in practice the vector coefficient is usually calculated as follows:

$$u_i = \begin{cases} 0, i \notin I \\ \alpha_i, i \in I \end{cases}$$

whereby $\alpha_i > 0$, are solutions for a system of linear equations (2). Obviously, this linear system is equivalent to the system (1); therefore, on the basis of the linear independence of vectors $\nabla g_i\ (x_0)$, $i \in I$, we can conclude that its solution α_i, $i \in I$ is unique.

Theorem 1.2. (Sufficiency of Kuhn-Tucker Conditions)
Suppose that f and g_i, $i \in I$ are convex and differentiable at $x_0 \in R^n$. If Kuhn-Tucker conditions are satisfied at the same point, ie. there are u_i, $i \in I$ such that (1) is true, then x_0 is a solution to the global minimization NLP problem.

Theorem 1.3. (Arrow - Ethoven)
Suppose that the given nonlinear program is as follows:

$$\max \xi = f(x)$$
$$\textit{Under the condition } g_i(x) \leq r_i \qquad (i = 1, 2, ...m)$$
$$x \geq 0$$

and the following conditions are satisfied:
*(a) the objective function $f(x)$ is differentiable and **quasi-concave** in the non-negative orthant.*
*(b) all constraint functions $g_i(x)$ are differentiable and **quasi-convex** in the non-negative orthant.*
(c) The point $\overline{x}$ satisfies Kuhn-Tucker maximization conditions.
(d) One of the following conditions is satisfied:
d1. $f_j(\overline{x})$ at least for one variable x_j.
d2. $f_j(\overline{x}) > 0$ for a variable x_j that takes on positive value without any loss in constraints.
d3. Not all n derivatives of function $f_j(\overline{x})$ are equal to zero, while function $f(x)$ is twice more differentiable in the neighborhood of $\overline{x}$, i.e. all second order partial derivatives are at $\overline{x}$.
d4. $f(x)$ is a concave function
then $\overline{x}$ has maximum of $\xi = f(x)$. □

c. Sales maximization

The objective of a typical micro-analysis of businesses primarily means profit maximization. However, the management may consider maximizing sales revenue a more important business objective than maximizing profit (this also depends on different organizational structures the management deals with). Total revenue P_u is one of the most important parameters that describe company's competitiveness in an industry. One of the criteria of the company's success and good management is whether the company increases its sales revenue or not. In that way and due to business results, profit as a parameter directly affects the system of rewards, i.e. the salaries of all employees including the salaries of the management.

In other words, sales maximization is certainly an alternative goal of any organization, given that the company's management, in order to avoid shareholders' dissatisfaction, continually takes care that the total income level does not go below the defined minimum, i.e.,

$$\min D_u(x) = \xi_0 \tag{4}$$

In that case the problem that the company's management has to deal with is maximizing the total revenue function $P_u = P_u(x)$ considering the following constrains condition

$$D_u = P_u(x) - T_u(x) \geq \xi_0 \tag{5}$$

Whereby $D_u(x)$ is total income, $P_u(x)$ is total revenue; $T_u(x)$ refers to total expenses and x to production volume or demand. This condition can also be shown as follows:

$$\max P_u = P_u(x)$$

under the condition $T_u(x) - P_u(x) \leq -\xi_0, \quad (\xi_0 > 0)$

$$x \geq 0$$

The question of whether Kuhn-Tucker conditions can be applied to this model or not primarily depends on the following two things: whether function $P_u(x)$ is differentiable and concave, and whether function $T_u(x)$ is differentiable and convex. If this is so, the constraints function $D_u = T_u(x) - P_u(x)$ is also differentiable and convex, which means that Kuhn-Tucker necessary conditions can be applied.

Of course, it is unlikely that we could draw more general conclusions based on assumptions about concavity, that is, convexity of quasiconcave function Pu(x) and – quasiconvex function $T_u(x)$. If we do consider such assumptions, then constraints function $D_u = T_u(x) - P_u(x)$ is the sum of two quasiconvex functions, while, at the same time, we cannot claim that the function itself is quasiconvex. If this be the case, then constraints function $D_u(x)$ can be transformed into quasiconvex, which further allows the application of sufficient conditions for extreme values.

Under these conditions we use θ to denote the Lagrangian:

$$\theta = P_u(x) + y(-\xi_0 - T_u(x) + P_u(x))$$

Kuhn-Tucker conditions are composed of boundary conditions:

$$\frac{\partial \theta}{\partial x} = P_u^{'}(x) - yT_u^{'}(x) + yP_u^{'}(x) \leq 0$$

$$\frac{\partial \theta}{\partial y} = -\xi_0 - T_u(x) + P_u(x) \geq 0 \tag{6}$$

$$x > 0$$

In the case of $P_u(0) = 0$ and $T_u(0) > 0$, i.e. that production is equal to zero, $x = 0$, then the following will be the case:

$$\frac{\partial \theta}{\partial y} = -\xi_0 - T_u(0) < 0$$

which shows that the second boundary condition is not fulfilled. Instead, we have to assume that $x > 0$, the condition which is absolutely in accordance with the fact that the production level that is equal to zero cannot be an element of the optimal solution set $[x_1, x_2]$.

Due to the non-negativity condition $x > 0$, we can say that $\partial\theta / \partial x = 0$, which means that the first weak inequation (6) has to be fulfilled as an equation. The solution of that equation refers to the rule according to which we can define production as the one which maximizes sales with the following constraints:

$$P_u^{'}(x) = \frac{y}{1+y} \cdot T_u^{'}(x) \tag{7}$$

Where y can be equal to or greater than zero, i.e. $y \geq 0$, if $y = 0$, this rule boils down to $P_u'(x) = 0$ while the company will tend to achieve production whose marginal revenue is as a follows:

$$P_g(x) = P_u^{'}(x) = 0$$

Since the company would make maximum profit possible under these circumstances this would be output under ideal conditions. However, bearing in mind our assumptions, such an extreme situation is not possible because demand, x_i which makes possible the aforementioned conditions is outside the set of possible solutions, i.e., $x_i \notin [x_1, x_2]$. In that case we have to assume that $y > 0$; however, this further means that $\partial\theta / \partial y = 0$ on the basis of which we can conclude that profit constraints hold with equality while the company tries to make at least minimum income ξ_0. Assume that $y > 0$, this is the case when the output level maximizes sales, or in other words when marginal revenue is less than marginal cost, i.e.

$$P_u^{'}(x) < T_u^{'}(x) \qquad \text{because} \qquad \frac{y}{1+y} < 1 \tag{8}$$

which would generally lead to higher output levels than the profit maximization rule, i.e., $P_u'(x) = T_u'(x)$, the case when marginal revenue is equal to marginal cost.

The particular problem we are dealing with here is the possibility of increasing sales of, i.e., demand for Ariel, the washing powder, sold at **Maxi** supermarket in Zajecar, (Belgian international food retailer **Delhaize Group**). The gained discrete data set shows nonlinear dependence among the parameters we were interested in. Therefore, we opted for nonlinear programming in order to solve the optimization problem – in this particular case, maximum sales.

The paper further lists research methodology including the problem solution. Since revenue function $P_u(x)$ is unknown as well as total expenses $T_u(x)$ and total income $D_u(x)$ functions, the first step is to compute these functions on the basis of statistical discrete data set, i.e. the empirical data we gained after having conducted a survey in the above mentioned company. The approximation method was used to compute functions as shown in Table 1.

Table 1 Discrete Data Set

x	$P_u(x)$
1	89.00
3	261
5.50	464.75
6.80	565.76
8.40	685.44
11.00	869.00
12.50	968.75
14.70	1106.90
16.20	1193.56
18.50	1322.75
20.00	1400

x	$T_u(x)$
1	5.5
3	207,00
5.5	295.75
6.80	386.72
8.40	445.68
11.	623.00
12.	743.75
14.70	945.27
16.20	1099.32
18.50	1361.75
20.00	1550.00

Table 1 shows the survey results carried out at **Maxi** supermarket in Zajecar, (**Delhaize Group**). What we did was to monitor the sales of the washing powder Ariel in the period of 30 days. In the very table, *x* stands for demand and refers to *kg* $\cdot 10^2$ and functions $P_u(x)$ and $T_u(x)$ stands for RSD $\cdot 10^2$.

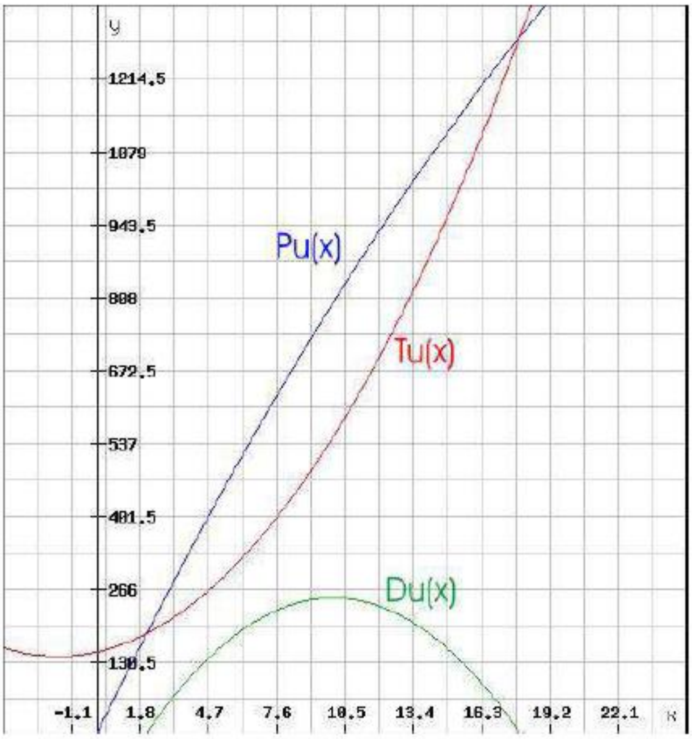

Fig. 2 Graph of approximate functions of total revenue, expenses and income

On the basis of the empirical data we gathered during the research, and which are shown in Table 1 where the **approximation method** is used[1], see [6], we can now compute the necessary functions; i.e. total revenue

$$P_u(x) = -1{,}0005x^2 + 89{,}988x \tag{9}$$

total expenses

$$T_u(x) = 3x^2 + 10x + 150 \tag{10}$$

and total income $D_u(x) = P_u(x) - T_u(x)$ all shown in Figure 2.

Now, we can apply the described Kuhn-Tucker optimization conditions to sales maximization, which as a consequence has an increase in the total revenue.

The optimization problem that arises here is how to maximize the following:

$$\max P_u = P_u(x)$$

$$\text{under the condition} \quad T_u(x) - P_u(x) \le -\xi_0, \quad \xi > 0$$

$$x \ge 0$$

Also, total income $D_u(x)$ cannot be less than $50 \cdot 10^2$ i.e., one of the constraints conditions is the following:

$$\xi_0 \ge 50$$

If we apply Kuhn-Tucker conditions, our starting point is the following:

$$\frac{\partial\theta}{\partial x} = P_u' - yT_u'(x) + yP_u'(x) \le 0 \tag{11}$$

$$\frac{\partial\theta}{\partial y} = -\xi_0 - T_u(x) + P_u(x) \ge 0 \tag{12}$$

Then, after the application of function differentiation rules in (9) and (10), and appropriate replacing in (11) and (12), we can arrange the inequations and finally get the following:

$$\frac{\partial\theta}{\partial x} = -2{,}001x + 89{,}988 - y\left(8{,}001x - 79{,}988\right) \le 0 \tag{13}$$

$$\frac{\partial\theta}{\partial y} = -4{,}005x^2 + 79{,}988x - 200 \ge 0 \tag{14}$$

Now, if we assume that $x = 0$ in (14), we have that $-200 \ge 0$ which is contradictory.

Therefore, condition $x > 0$ generates the following relation:

$$x > 0 \Rightarrow \frac{\partial\theta}{\partial x} = 0 \tag{15}$$

which further leads to the following:

$$-2{,}001x + 89{,}988 - y(8{,}001x - 79{,}988) = 0 \tag{16}$$

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[1] The approximation method is discussed in (6), *Operational research*, with original software solutions, ***Methods***, which enable the computer application of this method.

Furthermore, condition $y = 0$ generates the following:

$$-2,001x + 89,988 = 0 \tag{17}$$

from which we calculate x, i.e.

$$x = 44,9715$$

If value x is now used in (14), we will have the following:

$$\frac{\partial\theta}{\partial y} = -4693,57 < 0$$

which is contradictory to condition $\partial\theta / \partial y \geq 0$. Therefore, if we assume that $y > 0$ we will have the following equation:

$$\frac{\partial\theta}{\partial y} = 0 \Leftrightarrow -4,005x^2 + 79,988x - 200 = 0 \tag{18}$$

The following roots are calculated by solving the quadratic equation (18):

$$x_1 = 2,9296 \qquad x_2 = 17,0649 \tag{19}$$

Let us now check whether the solutions we arrived at, the roots x_1 and x_2 satisfy the Kuhn-Tucker conditions. If we replace root x_1 in equation (15), i.e. in (16), we will have that $y = -1,488$, i.e. $y < 0$, which is contradictory to the previously given condition $y > 0$. Thus, we can conclude that root x_1 in equation (18) does not satisfy the Kuhn-Tucker conditions. If we apply the same procedure with the other root x_2 in equation (18), we get that $y > 0$, which satisfies the Kuhn-Tucker conditions, in this case in relation to the demand.

On the other hand, the total income function D_u is as follows

$$D_u(x) = -4,0005x^2 + 79,988x - 150 \tag{20}$$

If this function's differentiation gives a derivative equal to zero, we have the following:

$$D_u^{'} = -8,0001x + 79,988 = 0 \tag{21}$$

By solving this equation (21), we get the following values for x:

$$x = 9,9984 \qquad 22)$$

The solution given at (22) stands for the demand or output volume whereby the total income D_u is maximized in relation to the product (washing powder). Furthermore, if we compare the root x_2 of the equation (18) with x, i.e. $x_2 > x$, we can conclude that $x_2 = 17,0649$ and that is the value that maximizes the total revenue function under the condition that the minimum value of the total income is as follows:

$$\min D_u = \xi_0 = 50 \cdot 10^2 .$$

CONCLUSION

Thus, the problem is solved and we also confirm our hypothesis on the possibility of sales increase, i.e., that there are certain situations, depending on the market conditions, when it is possible to increase sales of a product at the expense of decreasing the total income.

The case we illustrated also shows that the major idea in NLP numerical solving when there are two variables is this: zero is to be taken as the value of each variable, which significantly simplifies boundary conditions since a number of members vanish in the process and therefore the mathematical model is rather simplified. If it is possible in this way to find appropriate non-negative values of the Lagrangian multipliers that satisfy all the boundary conditions in inequations, the solution which equals zero is the optimal one. However, if some of the inequations are disturbed, that will indicate that one variable or *more* are positive. For each positive value of the variable it is possible to loosen the conditions by changing the inequation into the equation. Solving that equation will lead either to a solution or to a contradiction. If we end up with a contradiction, we will have to search for new ideas and repeat the process all over again.

It is also worth mentioning that if there are functions of more than two variables in nonlinear models, the very process of optimization problem solving becomes more complicated; therefore, it is necessary in such cases to write a suitable software since it will be more appropriate to use a computer for solving problems of this type. This, in most cases, especially when we deal with complex problems, requires teamwork.

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MAKSIMIZIRANJE PRODAJE U USLOVIMA NELINEARNOSTI

U ovom radu razmatramo problem maksimiziranja prodaje određenog artikla u slučaju kada je funkcija prihoda nelinearna u zavisnosti od tražnje određenog proizvoda. Ovakav zadatak se u principu rešava metodom nelinearnog programiranja, koja je u matematičkoj teoriji dovoljno opisana, ali njena primena nije tako jednostavna. U slučaju funkcija sa više od dve promenljive rešavanje ovakvih zadataka je veoma komplikovano i zahteva konstrukciju odgovarajućeg matematičkog modela, kao i pisanje softvera, kojim bi se postavljeni zadatak rešio na računaru, što ponekad zahteva timski rad.

Ključne reči: *Nelinearno programiranje, Kuhn-Tucker-ovi uslovi, funkcija prihoda, tražnja.*