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OPTIMIZATION OF THE 3P KEYS KERNEL PARAMETERS BY MINIMIZING THE RIPPLE OF THE SPECTRAL CHARACTERISTIC

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Abstract. *The ideal interpolation kernel is described by the sinc function, and its spectral characteristic is the box function. Due to the infinite length of the ideal kernel, it is not achievable. Therefore, convolutional interpolation kernels of finite length, which should better approximate the ideal kernel in a specified interval, are formed. The approximation function should have a small numerical complexity, so as to reduce the interpolation execution time. In the scientific literature, great attention is paid to the polynomial kernel of the third order. However, the time and spectral characteristic of the third-order polynomial kernels differs significantly from the shape of the ideal kernel. Therefore, the accuracy of cubic interpolation is lower. By optimizing the kernel parameters, it is possible to better approximate the ideal kernel. This will increase the accuracy of the interpolation. The first part of the paper describes a three-parameter (3P) Keys interpolation kernel, r. After that, the algorithm for optimizing the parameters of the 3P Keys kernel, is shown. First, the kernel is disassembled into components, and then, over each kernel component, Fourier transform is applied. In this way the spectral characteristic of the 3P Keys kernel, H, was determined. Then the spectral characteristic was developed in the Taylor series, H_T. With the condition for the elimination of the members of the Taylor series, which greatly affect the ripple of the spectral characteristic, the optimal kernel parameters (α_{opt}, β_{opt}, γ_{opt}) were determined. The second part of the paper describes an experiment, in which the interpolation accuracy of the 3P Keys kernel, was tested. Parametric cubic convolution (PCC) interpolation, with the 3P kernel, was performed over the images from the Test database. The Test database is created with standard Test images, which are intensively used in Digital Image Processing. By analyzing the interpolation error, which is represented by the Mean Square Error, MSE, the accuracy of the interpolation was determined. The results (α_{opt}, β_{opt}, γ_{opt}, MSE_{min}) are presented on tables and graphs. Detailed comparative analysis showed higher interpolation accuracy with the proposed 3P Keys interpolation kernel, compared to the interpolation accuracy with, 1P Keys and 2P Keys interpolation kernels. Finally, the numerical values of the optimal kernel parameters, which are determined by the optimization algorithm proposed in this paper, were experimentally verified.*

Key words: *convolution, interpolation, interpolation kernel, PCC interpolation, Keys kernel*

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1. INTRODUCTION

Interpolation is the process of estimating intermediate values between discrete samples of a continuous signal. Among other things, interpolation can be realized by applying a convolution between a discrete signal and a continuous interpolation kernel. The interpolation kernel significantly affects the accuracy and time execution of interpolation [1]. For interpolation of band-limited signals, the ideal interpolation kernel is of the form *sin*(*x*)/*x* (in the notation *sinc*) where $-\infty \leq x \leq +\infty$ [1, 2]. The spectral characteristic of the *sinc* interpolation kernel is a rectangular function, H_{sinc}. The *sinc* kernel cannot be practically realized because it has infinite limits. For this reason, there is a need to truncate the *sinc* interpolation kernel to a finite length. As a consequence of the truncated *sinc* kernel, its spectral characteristic deviates from the ideal, rectangular, characteristic, which leads to: a) ripple in the passband and stopband, and b) finite slope in the transition band.

The idea is to approximate the truncated *sinc* interpolation kernel with a low-degree polynomial function. In this way, the interpolation kernel has a small numerical complexity, and thus, allows a higher interpolation speed. These features of kernel are especially important when implemented in real-time systems. Signal interpolation using finite length interpolation kernels is realized by applying convolution. A polynomial zeroth-degree kernel allows interpolation by rounding to the nearest-neighbor [3, 4]. Nearest-neighbor interpolation is the most efficient in terms of computational speed, but in doing so, the largest interpolation error is generated. A linear, first-degree interpolation kernel is described in [5]. A quadratic, second-degree interpolation kernel is described in [3, 6]. A cubic, third-degree interpolation kernel, intended for parametric cubic convolution, PCC, is described in [1, 5].

Using numerical examples, it has been shown that cubic convolution is more precise than nearest-neighbor and linear interpolation [7 - 9]. The parameterization of the cubic interpolation kernel, by introducing the kernel parameter α, is shown in [1]. The paper [1] is one of the basic papers in the field of interpolation in digital image processing. Later, in the scientific literature, the parametric interpolation kernel from [1] was named, in honor of the author, the 1P Keys interpolation kernel. By changing the value of the kernel parameter α, the characteristics of the kernel can be changed and, in this way, adjusted to the corresponding signal that is interpolated. The process of changing the kernel parameter for customization is called parameter optimization. In [1], the optimization of the parameter α was performed by minimizing the interpolation error by developing the error function into a Taylor series in $f = 0$ (Maclaurin series). In this way, it is shown that the optimal value of the parameter $\alpha_{opt} = -0.5$. The ripple of the spectral characteristic is reduced by eliminating the members of the Taylor series that predominantly influence on the ripple. In [10], the ripple of the spectral characteristic was reduced by eliminating the members of the Taylor series that affect on the concavity of the spectral characteristic. In [11], the reduction of ripple of the spectral characteristic was achieved with $\alpha = -0.5$.

The construction of a two-parameter interpolation kernel is shown in [12, 13]. This kernel is based on the extended parameterization of the 1P Keys kernel [1]. In the scientific literature, this kernel is called the 2P Keys kernel. Optimal values of kernel parameters (α_{opt} = 0.1, β_{opt} = 0.2975) in estimating the fundamental frequency of the speech signal determined in [14]. Further expansion of parameterization, in order to improve the characteristics of the kernel, led to the construction of 3P Keys kernel [15]. The optimal values of kernel parameters in the estimates of the fundamental frequency of the speech signal are α_{opt} = -1.7, β_{opt} = -4.7, γ_{opt} = -3.8. A detailed analysis of the error estimate, presented using MSE, shows a higher accuracy of estimation using 3P kernels compared to the use of 1P Keys and 2P Keys kernels [15].

In the paper [16] the results of precision of the interpolation of audio signals, which was realized using the 3P Keys kernel [15], are presented. Audio test signals were acquired by recording G tones (G1 - G7) on a Steinway B concert piano. A detailed comparative analysis showed that the interpolation error, when the 3P Keys kernel was used, was compared to the following: a) 1P Keys kernel, 7.374 times smaller, and b) 2P Keys kernels, 2.4166 times smaller. Encouraged by the results of the papers, which unequivocally indicate the fact that increasing the number of interpolation kernel parameters reduces the interpolation error, the authors of this paper performed optimization of 3P Keys kernel parameters, in order to increase similarity with the ideal kernel, *sinc*. Thus created, optimized kernel, will further reduce interpolation error. In this paper, the process of optimization of parameters of the 3P Keys kernel [15] in the spectral domain, is presented. Optimization of kernel parameters was performed by minimizing the ripple of the spectral characteristic. The first part of the paper describes the algorithm for optimizing kernel parameters. First, by applying the Fourier transform on the 3P kernel, r, the analytical form of the spectral characteristic, H, was determined. After that, the spectral characteristics were approximated using the Taylor series, H_T. The ripple reduction was achieved by eliminating the members of the Taylor series, H_T, which have a dominant effect on the ripple increase. Then, the degree of similarity of the spectral characteristics of the ideal *sinc* kernel, H_{sinc}, and the optimized kernel, H_{opt}, was determined by comparative analysis. MSE were used as a measure of similarity [11]. Finally, the optimal parameters, (α_{opt}, β_{opt}, γ_{opt}), were determined based on the minimum of the MSE. The second part of the paper presents the results of an experiment in which the optimal parameters for 1P Keys, 2P Keys and 3P Keys kernels were determined. An algorithm for interpolation Test images, error interpolation estimation, and determination of experimental optimal parameters, is described. For the purposes of the experiment, the Image Test base was formed. Image Test base consists of: a) standard Test images for Digital Signal Processing (*Lena*, *Barbara*, *Cameraman*, *Peppers*, *Boats*, *Tulips*, and *Watch*), and b) images from the BSDS500 Image base [17]. Test images from the BSDS500 base have numeric labels, so they will be named in the same way later in this paper. By applying the algorithm for each image, the optimal parameters and the corresponding estimate errors were determined. The results are presented in tables and graphs. Finally, a comparative analysis of the experimental results with the results obtained by optimizing the spectral characteristic, was performed. Comparative analysis will determine: a) the accuracy of interpolation using MSE and b) the accuracy of estimating kernel parameters using absolute error. Finally, in the last part of the paper, an analysis of the execution time of all analyzed kernels was performed. Testing of the execution time was performed on a computer DESKTOP - S2AC43P, Processor: Intel (R) Pentium (R), CPU: G3220 3 GHZ, RAM: 8 GB and a Windows 10 operating system. The Matlab R2017b program was applied (to determine the execution time, the *tic* and *toc* functions are used). It should be emphasized that the realized experiment, within which the algorithm for PCC interpolation is described, is intended, exclusively, for the comparative analysis of the interpolation accuracy of the 1P Keys, 2P Keys and 3P Keys kernels. It was implemented using the Matlab. Therefore, the time of interpolation execution, in this case, is not of primary importance, because the condition for real-time is not set.

The paper is organized as follows: Section 2 describes 3P Keys kernel. Section 3 describes the 3P Keys kernel parameterization algorithm. Experimental results and comparative analysis are presented in Section 4. Section 5 is the Conclusion.

2. KEYS PARAMETRIC INTERPOLATION KERNELS

In paper [1], for the field of convolutional interpolation fundamental paper, the author defined a parametric interpolation kernel. The kernel was intended to image interpolation. Later, in the scientific literature, the interpolation kernel from [1] was called the 1P Keys kernel.

2.1. 1P Keys kernel

The proposed 1P Keys kernel [1] is defined as:

$$r(x)=\begin{cases}(\alpha+2)\,|x|^3-(\alpha+3)\,|x|^2+1, & |x|\le 1,\\ \alpha\,|x|^3-5\alpha\,|x|^2+8\alpha\,|x|-4\alpha, & 1<|x|\le 2\\ 0, & |x|>2\end{cases}, \tag{1}$$

where α is parameter of the 1P Keys kernel. The length of this kernel is L = 4.

2.2. 2P Keys kernel

A modification of the 1P Key kernel, with the introduction of the second kernel parameter, with length L = 6, is shown in [13]. The analytical form of the 2P Keys kernel is:

$$r(x)=\begin{cases}(\alpha-\beta+2)\,|x|^3-(\alpha-\beta+3)\,|x|^2+1, & |x|\le 1\\ \alpha\,|x|^3-(5\alpha-\beta)\,|x|^2+(8\alpha-3\beta)\,|x|-(4\alpha-2\beta), & 1<|x|\le 2\\ \beta\,|x|^3-8\beta\,|x|^2+21\beta\,|x|-18\beta, & 2<|x|\le 3\\ 0, & |x|>3\end{cases}, \tag{2}$$

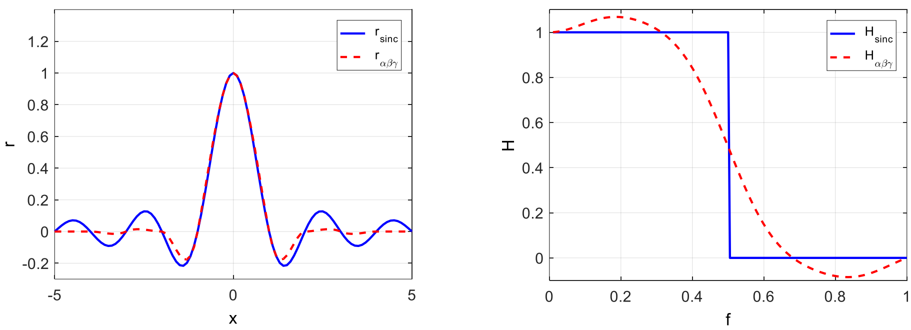
where α and β are the parameters of the kernel. For β = 0 is obtained 1P Keys kernel. In [12 - 14], it was shown that the precision of the PCC interpolation with the 2P Keys kernel was increased compared to the interpolation of the PCC interpolation with the 1P Keys kernel.

2.3. 3P Keys kernel

The results in [12 - 14] show that the precision of the PCC interpolation with 2P Keys kernel, compared to interpolation with 1P Keys kernel, is increased. With the idea of further increasing the interpolation accuracy, the parameterization of the 1P kernel, using three parameters, was performed [15]. The three-parameter kernel is called the 3P Keys kernel. The analytical form of the Keys 3P kernel is:

$$r(x)=\begin{cases}(\alpha-\beta+\gamma+2)\,|x|^3+(-\alpha+\beta-\gamma-3)\,|x|^2+1, & |x|\le 1\\ \alpha\,|x|^3+(-5\alpha-\beta-\gamma)\,|x|^2+(8\alpha-3\beta+3\gamma)\,|x|+(-4\alpha+2\beta-2\gamma), & 1<|x|\le 2\\ \beta\,|x|^3+(-8\beta+\gamma)\,|x|^2+(21\beta-5\gamma)\,|x|+(-18\beta+6\gamma), & 2<|x|\le 3\\ \gamma\,|x|^3-11\gamma\,|x|^2+40\gamma\,|x|-48\gamma, & 3<|x|\le 4\\ 0, & |x|>4\end{cases}, \tag{3}$$

where α, β and γ are the parameters of the 3P Keys kernel. As an example, Fig. 1.a shows the time characteristics of the ideal interpolation kernel, r_{sinc}, and the 3P Keys kernel, $r_{\alpha\beta\gamma}$, for kernel parameters α = -1.2, β = -0.1 and γ = -0.1.



a) b)

Fig. 1 Characteristics of the ideal *sinc* and 3P Keys kernels (α = -1.2, β = -0.1, γ = -0.1): a) time characteristics (r_{sinc}, $r_{\alpha\beta\gamma}$) and b) spectral characteristics (H_{sinc}, $H_{\alpha\beta\gamma}$)

3. Optimization of the Keys 3P kernel parameters

The spectral characteristic, H, of the 3P Keys kernel (Eq. 3) is different from the spectral characteristic, H_{sinc} of the ideal interpolation kernel r_{sinc} (Fig. 1.b). The deviation of the spectral characteristic H from H_{sinc} is described as the ripple of the spectral characteristic. The optimization process minimizes the difference between the spectral characteristics of H and H_{sinc}. Optimization involves selecting the kernel parameters α, β, and γ, so as to minimize the Mean Square Error between H and H_{sinc}. In this way, the optimal parameters of the 3P Keys kernel α_{opt}, β_{opt} and γ_{opt} are obtained.

3.1. Algorithm for minimizing of the ripple of the spectral characteristic

This part of the paper conducts the optimization of Keys 3P kernel parameters by minimizing the ripple of the spectral characteristic. The algorithm for parameters optimization consists of the following steps:

Input: r - 3P Keys kernel

Output: α_{opt}, β_{opt} and γ_{opt} - kernel parameters.

Step 1: Decomposition 3P Keys kernel r to its components r_0, r_1, r_2 and r_3.

Step 2: Determining the spectral characteristic H(f) by applying the Fourier transform over the kernel components r_0, r_1, r_2 and r_3.

Step 3: The expansion of the spectral characteristic $H(f)$ into Taylor series $H_T(f)$.

Step 4: Eliminating coefficients of the members of the spectral characteristic $H_T(f)$ which dominantly affect on the ripple of the spectral characteristic. Determining the optimal kernel parameters α_{opt}, β_{opt} and γ_{opt}.

A more detailed explanation of the algorithm steps (*Step* 1 - *Step* 4) is shown below.

3.2. Kernel components (Step 1)

The 3P Keys kernel r (Eq. (3)) can be represented as the sum of the kernel components:

$$r(x) = r_0(x) + \alpha r_1(x) + \beta r_2(x) + \gamma r_3(x), \tag{4}$$

where

$$r_0(x) = \begin{cases} 2|x|^3 - 3|x|^2 + 1. & |x| \le 1 \\ 0, & |x| > 1 \end{cases}, \tag{5}$$

$$r_1(x) = \begin{cases} |x|^3 - |x|^2, & |x| \le 1 \\ |x|^3 - 5|x|^2 + 8|x| - 4, & 1 < |x| \le 2 \\ 0, & |x| > 2 \end{cases}, \tag{6}$$

$$r_2(x) = \begin{cases} -|x|^3 + |x|^2, & |x| \le 1 \\ |x|^2 - 3|x| + 2, & 1 < |x| \le 2 \\ |x|^3 - 8|x|^2 + 21|x| - 18, & 2 < |x| \le 3 \\ 0, & |x| > 3 \end{cases}, \tag{7}$$

and

$$r_3(x) = \begin{cases} |x|^3 - |x|^2, & |x| \le 1 \\ -|x|^2 + 3|x| - 2, & 1 < |x| \le 2 \\ |x|^2 - 5|x| + 6, & 2 < |x| \le 3 \\ |x|^3 - 11|x|^2 + 40|x| - 48, & 3 < |x| \le 4 \\ 0, & |x| > 4 \end{cases}, \tag{8}$$

are components of the 3P Keys kernel. Fig. 2 shows the components of 3P Keys kernel r_0, r_1, r_2 and r_3.

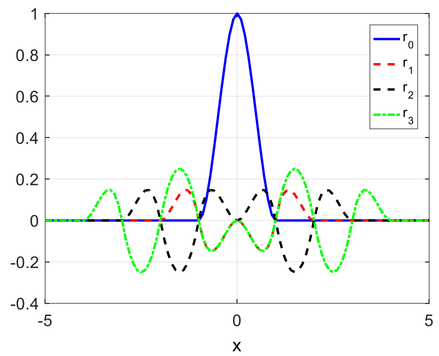


Fig. 2 3P Keys kernel components: r_0, r_1, r_2 and r_3

3.3. Spectral characteristic of the 3P Keys kernel (Step 2)

In order to optimize the parameters α, β, and γ, of the 3P Keys kernel r in the spectral domain, by using the Fourier transform (FT) the spectral characteristic of the kernel H was obtained:

$$\begin{aligned} H(f) &= FT(r(x)) = FT(r_0(x) + \alpha r_1(x) + \beta r_2(x) + \gamma r_3(x)) = \\ &H_0(f) + \alpha H_1(f) + \beta H_2(f) + \gamma H_3(f) \end{aligned} \tag{9}$$

where H_0, H_1, H_2 and H_3 are spectral components of the 3P Keys kernel:

$$H_0(f)=\int_{-\infty}^{\infty} r_o(x)\, e^{-2\pi xfi}dx\,, \tag{10}$$

$$H_1(f)=\int_{-\infty}^{\infty} r_1(x)\, e^{-2\pi xfi}dx\,, \tag{11}$$

$$H_2(f)=\int_{-\infty}^{\infty} r_2(x)\, e^{-2\pi xfi}dx\,, \tag{12}$$

and

$$H_3(f)=\int_{-\infty}^{\infty} r_3(x)\, e^{-2\pi xfi}dx\,. \tag{13}$$

By substituting Eq. (5) in Eq. (10) is obtained:

$$H_0(f)=\int_{-1}^{0}(-2x^3-3x^2+1)\, e^{-2\pi xfi}dx+\int_{0}^{1}(2x^3-3x^2+1)\, e^{-2\pi xfi}dx\,, \tag{14}$$

By substituting Eq. (6) in Eq. (11) is obtained:

$$\begin{aligned}H_1(f)=&\int_{-2}^{-1}(-x^3-5x^2-8x-4)\, e^{-2\pi xfi}dx+\int_{-1}^{0}(-x^3-x^2)\, e^{-2\pi xfi}dx+\\&\int_{0}^{1}(x^3-x^2)\, e^{-2\pi xfi}dx+\int_{1}^{2}(x^3-5x^2+8x-4)\, e^{-2\pi xfi}dx\end{aligned}\,, \tag{15}$$

By substituting Eq. (7) in Eq. (12) is obtained:

$$\begin{aligned}H_2(f)=&\int_{-3}^{-2}(-x^3-8x^2-21x-18)\, e^{-2\pi xfi}dx+\int_{-2}^{-1}(x^2+3x+2)\, e^{-2\pi xfi}dx+\\&\int_{-1}^{0}(x^3+x^2)\, e^{-2\pi xfi}dx+\int_{0}^{1}(-x^3+x^2)\, e^{-2\pi xfi}dx+\int_{1}^{2}(x^2-3x+2)\, e^{-2\pi xfi}dx\\&+\int_{2}^{3}(x^3-8x^2+21x-18)\, e^{-2\pi xfi}dx\end{aligned}\,, \tag{16}$$

By substituting Eq. (8) in Eq. (13) is obtained:

$$\begin{aligned}H_3(f)=&\int_{-4}^{-3}(-x^3-11x^2-40x-48)\, e^{-2\pi xfi}dx+\int_{-3}^{-2}(x^2+5x+6)\, e^{-2\pi xfi}dx+\\&\int_{-2}^{-1}(-x^2-3x-2)\, e^{-2\pi xfi}dx+\int_{-1}^{0}(-x^3-x^2)\, e^{-2\pi xfi}dx+\int_{0}^{1}(x^3-x^2)\, e^{-2\pi xfi}dx+\\&\int_{1}^{2}(-x^2+3x-2)\, e^{-2\pi xfi}dx+\int_{2}^{3}(x^2-5x+6)\, e^{-2\pi xfi}dx+\\&\int_{3}^{4}(x^3-11x^2+40x-48)\, e^{-2\pi xfi}dx\end{aligned}\,, \tag{17}$$

After applying Euler's formula and partial integration, the spectral components of the kernel can be written in the following form:

$$H_0(f)=\frac{6\sin^2(\pi f)-3\pi f\sin(2\pi f)}{2\pi^4 f^4}, \tag{18}$$

$$H_1(f)=\frac{3\sin^2(2\pi f)-4\pi f\sin(2\pi f)-\pi f\sin(4\pi f)}{2\pi^4 f^4}, \tag{19}$$

$$H_2(f)=\frac{-3\sin^2(\pi f)-3\sin^2(2\pi f)+3\sin^2(3\pi f)}{2\pi^4 f^4}+ \\ \frac{3\pi f\sin(2\pi f)-3\pi f\sin(4\pi f)-\pi f\sin(6\pi f)}{2\pi^4 f^4}, \tag{20}$$

and

$$H_3(f)=\frac{3(\sin^2(\pi f)-\sin^2(3\pi f)+\sin^2(4\pi f))}{2\pi^4 f^4}- \\ \frac{-\pi f(3\sin(2\pi f)+2\sin(4\pi f)-3\sin(6\pi f)-\sin(8\pi f))}{2\pi^4 f^4}. \tag{21}$$

Spectral components H_0 (Eq. (18)), H_1 (Eq. (19)), H_2 (Eq. (20)) and H_3 (Eq.(21)), are shown in Fig. 3.

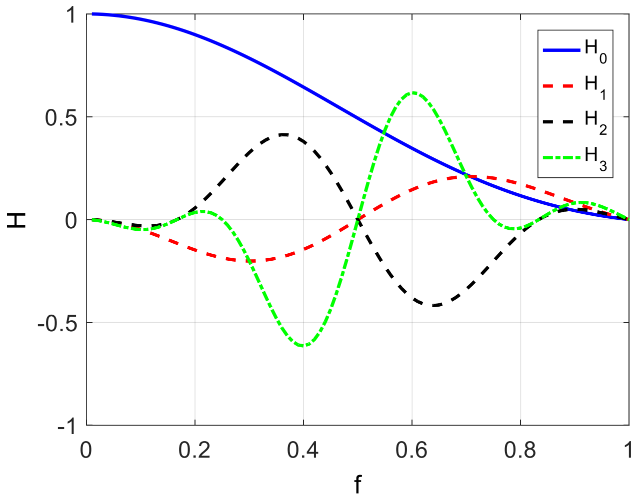


Fig. 3 Spectral components H_0, H_1, H_2 and H_3. of the 3P Keys kernel

3.4. Optimal kernel parameters (Step 3, Step 4)

In order to determine the optimal parameters of the 3P Keys kernel r in the spectral domain, the Taylor expansion H_T of spectral characteristic H (Eq. (9)) was determined.

(*Step 3*) By expansion into Taylor series in the neighborhood $f = 0$ (Maclaurin series), spectral components of the kernel were obtained:

$$H_{T0}(f)=1-\frac{4}{15}(\pi f)^2+\frac{1}{35}(\pi f)^4-\frac{8}{4725}(\pi f)^6+\frac{2}{31185}(\pi f)^8+\dots, \tag{22}$$

$$H_{T1}(f)=-\frac{8}{15}(\pi f)^2+\frac{16}{35}(\pi f)^4-\frac{232}{1575}(\pi f)^6+\frac{4112}{155925}(\pi f)^8+\dots, \tag{23}$$

$$H_{T2}(f)=-\frac{8}{15}(\pi f)^2+\frac{272}{105}(\pi f)^4-\frac{4232}{1575}(\pi f)^6+\frac{205808}{155925}(\pi f)^8+\dots, \tag{24}$$

and

$$H_{T3}(f)=-\frac{16}{15}(\pi f)^2+\frac{256}{35}(\pi f)^4-\frac{25904}{1575}(\pi f)^6+\frac{2640832}{155925}(\pi f)^8+\dots. \tag{25}$$

By substituting Eq. (22)-(25) in Eq. (9) is obtained:

$$\begin{aligned} &H_T(f)=H_{T0}(f)+\alpha H_{T1}(f)+\beta H_{T2}(f)+\gamma H_{T3}(f)= \\ &1-\frac{4}{15}(1+2\alpha+2\beta+4\gamma)(\pi f)^2+\frac{1}{105}(3+48\alpha+272\beta+768\gamma)(\pi f)^4-. \\ &\frac{8}{4725}(1+87\alpha+1587\beta+9714\gamma)(\pi f)^6+O(\pi f)^8 \end{aligned} \tag{26}$$

(*Step 4*) The minimization of the spectral characteristic (Eq. (26)) ripple is carried out by eliminating the dominant members of the spectral characteristic:

$$\begin{cases} 1+2\alpha+2\beta+4\gamma=0 \\ 3+48\alpha+272\beta+768\gamma=0 \\ 1+87\alpha+1587\beta+9714\gamma=0 \end{cases}. \tag{27}$$

After calculating the system of equations Eq. (27) is obtained:

$$\begin{aligned} \alpha_{opt}&=-\frac{4945}{8064}\approx-0.6132 \\ \beta_{opt}&=\frac{409}{2688}\approx 0.1522 \\ \gamma_{opt}&=-\frac{157}{8064}\approx-0.0195 \end{aligned}. \tag{28}$$

By substituting the optimal parameter α_{opt} = -0.5 [11], the optimal interpolation 1P Keys kernel, $r_{\text{opt\_1P}}$, was obtained:

$$r_{opt\_1P}(x)=\begin{cases} 1.5|x|^3-2.5|x|^2+1, & |x|\le 1, \\ 0.5|x|^3-2.5|x|^2-4|x|+2, & 1<|x|\le 2 \\ 0, & |x|>2 \end{cases}. \tag{29}$$

The spectral characteristic of the 1P Keys kernel, $H_{\text{opt\_1P}}$, is shown in Fig. 4. By substituting the optimal parameters α_{opt} = -0.5938, β_{opt} = 0.0938 [12] the optimal interpolation 2P Keys kernel, $r_{\text{opt\_2P}}$, was obtained:

$$r_{opt\_2P}(x)=\begin{cases}1.3124|x|^3-2.3124|x|^2+1, & |x|\le 1\\ -0.5938|x|^3+3.0628|x|^2-5.0318|x|+2.5628, & 1<|x|\le 2\\ 0.0938|x|^3-0.7504|x|^2+1.9698|x|-1.6884, & 2<|x|\le 3\\ 0, & |x|>3\end{cases}. \tag{30}$$

The spectral characteristic of the 2P Keys kernel, H_{opt_2P}, is shown in Fig. 4. By substituting the optimal parameters α_{opt} = -0.6132, β_{opt} = 0.1522, γ_{opt} = -0.0195 (Eq. (28)) in Eq. (3), the optimal interpolation 3P Keys kernel, r_{opt_3P}, was obtained:

$$r_{opt\_3P}(x)=\begin{cases}1.2151|x|^3-2.2151|x|^2+1, & |x|\le 1\\ -0.6132|x|^3+3.2377|x|^2-5.4207|x|+2.7962, & 1<|x|\le 2\\ 0.1522|x|^3-1.2371|x|^2+3.2937|x|-2.8566, & 2<|x|\le 3\\ -0.0195|x|^3-0.2145|x|^2-0.78|x|-0.936, & 3<|x|\le 4\\ 0, & |x|>4\end{cases}. \tag{31}$$

The spectral characteristic of the 3P Keys kernel, is shown in Fig. 4. Moreover, Fig. 4 shows the spectral characteristics of the ideal r_{sinc} kernel (H_{sinc}). Paper [11] presents the total mean square error, MSE_T, i.e. the difference between the spectral characteristic H and the ideal box characteristic H_{sinc}:

$$MSE_T=\frac{1}{K}\sum_{k=0}^{K-1}\left|H_{sinc}(f_k)-H(f_k)\right|^2 . \tag{32}$$

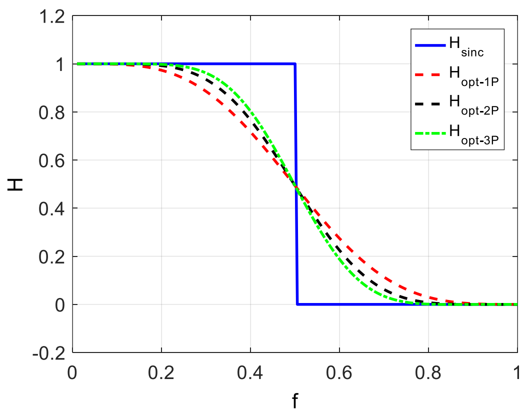


Fig. 4 Spectral characteristic of the ideal interpolation kernel H_{sinc} and optimal spectral characteristics H of: a) 1P (α_{opt} = -0.5), b) 2P (α_{opt} = -0.5938, β_{opt} = 0.0938) and c) 3P (α_{opt} = -0.6132, β_{opt} = 0.1522, γ_{opt} = -0.0195) Keys kernel

4. Experimental Results and Analysis

4.1. Experiment

An experiment, with the aim of determining: a) interpolation accuracy with the 3P Keys kernel, and b) interpolation execution time with the 3P Keys kernel, T_E, in relation to interpolations with the 1P and 2P Keys kernels, was realized. Interpolations were performed on Test Images, T_I, from the Image base. The Image base is created from some: a) standard Test images used in Digital Image Processing, and b) Test images from the BSDS500 base. Some Test images from the Image base are in color (*RGB*) and some are in black-white (*Y*). In this Experiment, interpolations were performed on black-white images. Therefore, color images were transformed into black-white images in accordance with the colorimetric equation $Y = 0.3 \cdot R + 0.59 \cdot G + 0.11 \cdot B$. The Experiment was performed as follows. First, the experimental optimal values of the kernel parameters for: a) 1P Keys (α_{opt}), b) 2P Keys (α_{opt}, β_{opt}) and c) 3P Keys (α_{opt}, β_{opt}, γ_{opt}), using the algorithm described below, were determined. After that, a comparative analysis of the error estimation of the optimal parameters of the kernels obtained: a) by optimizing the ripple of the spectral characteristic and b) obtained by experiments. Finally, a comparative analysis of interpolation accuracy between the proposed 3P Keys versus 1P Keys and 2P Keys was performed. For these reasons, the Test image , T_I, which, for analysis purposes, is presented as a two-dimensional matrix, with dimensions (L x K), was transformed into a one-dimensional matrix. The transformation was performed by connecting the rows of the Test image matrix one after the other, and, in this way, a one-dimensional matrix, X, with dimensions $N = L$ x K, was obtained (these activities are realized by the Algorithm described in Section 4.3). The interpolation is organized as follows. The interpolation of the intensity of the pixel I, $X(I)$, was performed by convolution between the interpolation kernel and intensity of the pixels $X(I- K)$, $X(I - K + 1)$, ..., $X(I + K)$, where K is the length of the interpolation kernel. The interpolated value of pixel I is $\hat{x}_I$. On the other hand, intensity of the pixel I is known ($X(I)$), and, in the Experiment, it is considered to be the True value of the pixel intensity. Further analysis involved defining interpolation error. The interpolation error was defined by MSE (Eq. (32)), which was calculated between True, $X(I)$, and the interpolated intensity, $\hat{x}_I$, of the pixel I. MSE was used in a comparative analysis of interpolation accuracy, between interpolation results with applied 1P, 2P and 3P Keys kernels. The interpolation results (MSE_{min}) are presented using graphs and tables. By comparative analysis of MSE_{min}, the precision of interpolation with the 3P Keys kernel, in relation to the precision of interpolation with the 1P and 2P keys kernels, was determined. In addition, the executions time of the PCC interpolation, T_E, was determined. Testing of the of the execution time was performed on a computer DESKTOP - S2AC43P, Processor: Intel (R) Pentium (R), CPU: G3220 3 GHZ, RAM: 8 GB and a Windows 10 operating system. The Matlab R2017b program was applied (to determine the execution time, T_E, the *tic* and *toc* functions are used). Execution time was measured for: a) complete convolution with kernels (Eq. (1), Eq. (2) and Eq. (3)), where, based on the kernel parameters α, β and γ, the coefficients of third order polynomials are calculated, and then the value of the polynomials were calculated, b) convolution with the optimized kernel parameters (Eq. (29), Eq. (30) and Eq. (31)), where the coefficients of the polynomial were previously calculated, and, after that, the value of the polynomial is were calculated, and c) convolutional kernel execution time, without interpolation. All interpolation execution times, as the arithmetic mean of the value of the results for 100000 interpolations, were determined.

4.2. Image base

For the purpose of realizing the Experiment, in which the accuracy of PCC interpolation with image interpolation, is tested, an Image base was created. Image base consists of: a) standard Test images for Digital Signal Processing, and b) images from the BSDS500 Image base [17]. Standard Test images are: *Lena* (512 x 512, *RGB*) (fig. 5.a), *Barbara* (225 x 675, *RGB*) (fig. 5.b), *Cameraman* (225 x 675, *Y*) (fig. 5). c), *Peppers* (225 x 675, *RGB*) (fig. 5.d), *Boats* (225 x 675, *RGB*) (fig. 5.e), *Tulips* (512 x 768, *RGB*) (fig. 5.f) , and *Watch* (768 x 1024, *RGB*) (Fig. 5.d). Test images from the BSDS500 base have numeric labels: *3096* (321 x 481, *RGB*) (fig. 5.h), *14037* (321 x 481, *RGB*) (fig. 5.i), *295087* (321 x 481, *RGB*) (fig. 5.j), *126007* (321 x 481, *RGB*) (fig. 5.k), *260058* (321 x 481, *RGB*) (fig. 5.l), *160068* (321 x 481, *RGB*) (fig. 5.m), *241004* (321 x 481, *RGB*) (fig. 5.n), *197017* (321 x 481, *RGB*) (fig. 5.o), *143090* (321 x 481, *RGB*) (fig. 5.p).

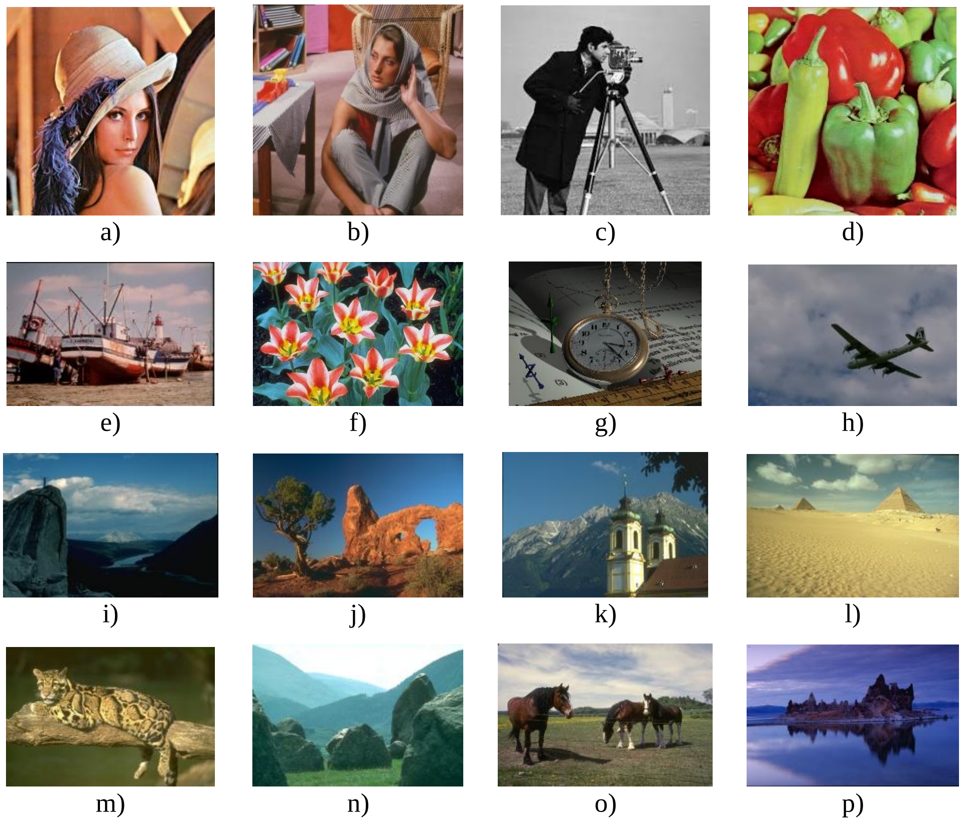


Fig. 5 Test image for Digital Image Processing: a) *Lena*, b) *Barbara*, c) *Cameraman*, d) *Pappers*, e) *Boats*, f) *Tulips*, d) *Watch*. Test images from BSDS500 database, with numeric labels: h) *3096*, i) *14037*, j) *295087*, k) *12607*, l) *260058*, m) *160068*, n) *241004*, o) *197017*, p) *143090*

4.3. Algorithm for interpolation error determining

The following Algorithm performs interpolation of the Test images, determines the interpolation error and determines the MSE depending on the parameters α, β and γ. Optimal parameters were determined by minimizing MSE. Algorithm is realized in the following steps:

Input: (r_0, r_1, r_2, r_3) – 3P Keys kernel parameters, (α_{min}, $\Delta\alpha$, α_{max}, β_{min}, $\Delta\beta$, β_{max}, γ_{min}, $\Delta\gamma$, γ_{max}) - parameter boundaries and iteration steps, L – kernel length, T_I (L x K) - Test image.
Output: α_{opt}, β_{opt}, γ_{opt}. optimal parameters. MSE_α, $MSE_{\alpha\beta}$, $MSE_{\alpha\beta\gamma}$.

Step 1: Converting a color image to a black-white image.
IF *Test image == Color Image*
$T_I = 0.3 \cdot R + 0.59 \cdot G + 0.11 \cdot B$
END
Step 2: Transformation of the image T_I (L x K) into a one-dimensional matrix X:
FOR $\ell = 1 : L$
FOR $k = 1 : K$
$X((\ell - 1) \cdot K + k) = T_I(\ell, k)$
END k
END ℓ
The dimensions of the one-dimensional matrix X are (1, N), where $N = L$ x K.
FOR $\gamma = \gamma_{min} : \Delta\gamma : \gamma_{max}$.
FOR $\beta = \beta_{min} : \Delta\beta : \beta_{max}$
FOR $\alpha = \alpha_{min} : \Delta\alpha : \alpha_{max}$
Step 3: Construction of the kernel: $r = r_0 + \alpha r_1 + \beta r_2 + \gamma r_3$,
Step 4: The length of interpolation frame is: $M = 2 \cdot L - 1$
FOR $I = 1: N\text{-}M+1$,
Step 5: Selecting the I-th frame: $X_I = X(1: I+M-1)$
Step 6: Estimation of $\hat{x}_i$ by applying PCC: $\hat{x}_I = X_I[1:2:M] \otimes r$, where the symbol $\otimes$ stands for convolution.
Step 7: Estimation error is: $e(I) = X_I(L) - \hat{x}_I$
END I
Step 8: Mean square error of estimation of 1P kernel:

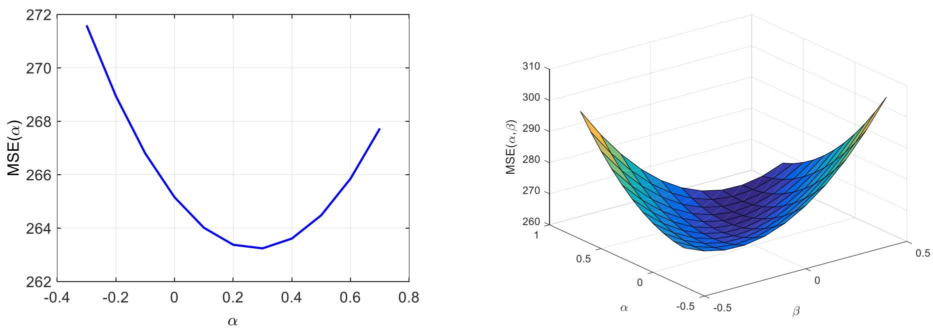
$$MSE_\alpha(\alpha) = 1/(N - M + 1) \sum_{k=1}^{N-M+1} |e(k)|^2 ,$$

END α
Step 9: Mean square error of estimation of 2P kernel: $MSE_{\alpha\beta}(\beta) = MSE_\alpha$,
END β
Step 10: Mean square error of estimation of 3P kernel: $MSE_{\alpha\beta\gamma}(\gamma) = MSE_{\alpha\beta}$,
END γ
Step 11:. Optimal values of 3P kernel parameters: $(\alpha_{opt}, \beta_{opt}, \gamma_{opt}) = \arg\min_{\alpha,\beta,\gamma}(MSE_{\alpha\beta\gamma})$.

The described algorithm had the purpose of testing the interpolation error with the 3P Keys kernel in relation to the interpolation error with the 1P and 2P Keys kernels. The algorithm was implemented in Matlab, and, except for testing, is not intended for real-time systems. Therefore, the execution time of the algorithm is not of dominant importance. However, in the Experiment, using the Matlab function *tic* and *toc*, for the case of applying 1P, 2P and 3P Keys kernels, the interpolation execution time, T_E, is determined. Based on the execution time, a comparative analysis was performed.

4.4. Experimental Results

Using the Test Algorithm described in Section 4.3, interpolation of the Test images was performed. Interpolation for some values of α, β and γ parameters from the specified range has been performed. In addition, interpolation with the 1P, 2P and 3P Keys kernels with all parameters from the range was performed. For each interpolation, the interpolation error, MSE, is determined. Based on the minimum interpolation error, MSE_{min}, the optimal interpolation kernel parameter was determined. Figure 5.a shows the dependence of the MSE_α on the parameter α, for the 1P Keys kernel (Test image *Boats*). The optimal parameter, α_{opt}, was determined as $(\alpha_{opt}) = \arg\min_{\alpha}(MSE_\alpha)$. Figure 5.b shows the dependence of $MSE_{\alpha\beta}$ on the parameters α and β for the 2P Keys kernel (Test image *Boats*). The optimal parameters α_{opt} and β_{opt}, were determined as $(\alpha_{opt}, \beta_{opt}) = \arg\min_{\alpha,\beta}(MSE_{\alpha\beta})$. The minimum interpolation errors, MSE_{min}, and the corresponding optimal kernel parameters, when interpolating all Test images from the Image base, are shown in: a) Table 1 (1P Keys, α_{opt}, MSE_{1P}), b) Table 2 (2P Keys, α_{opt}, β_{opt}, MSE_{2P}) and c) Table 3 (3P Keys, α_{opt}, β_{opt}, γ_{opt}, MSE_{3P}). Table 4 shows the execution time of PCC convolution for: a) complete convolution with kernels (label in the table: Int_1), (Eq. (1), Eq. (2) and Eq. (3)), b) convolution with the optimized kernel parameters (label in the table: Int_2) (Eq. (29), Eq. (30) and Eq. (31)) and c) convolutional kernel execution time, without interpolation (label in the table: Ker_T). All interpolation execution times, as the arithmetic mean of the value of the results for 100000 interpolations, were determined.



a) b)

Fig. 5 Dependence of MSE on kernel parameters for the Test image *Boats*: a) 1P Keys kernel and b) 2P Keys Kernel

Table 1 Optimal parameter *α* and minimum MSE for 1P Keys kernel.

Image base	Image	α_{opt}	MSE_{1P}
DSP test base	Lena	-0.3000	11.3234
	Barbara	-0.1000	247.0271
	Cameraman	-0.5000	0.3133
	Pappers	-0.6200	75.7521
	Boats	-0.3000	263.2390
	Tulips	-0.7000	14.5797
	Watch	-0.4000	49.9283
BSDS500 baze	3096	0.200	0.7933
	14037	-0.6000	10.0185
	295087	0.1000	2.3780
	126007	-0.4000	19.1678
	260058	-0.300	4.8327
	160068	0.6000	0.5835
	241004	-0.300	6.4673
	197017	0.3000	6.4499
	143090	-0.01	18.2042
		$\overline{\alpha_{opt\_1P}}$	$\overline{MSE_{1P}}$
		-0.1706	45.6911

Table 2 Optimal parameters *α* and *β,* and minimum MSE for 2P Keys kernel

Image base	Image	α_{opt}	β_{opt}	MSE_{2P}
DSP test base	Lena	-0.3000	-0.1000	11.3137
	Barbara	-0.1000	0	247.0271
	Cameraman	-0.3000	-0.2000	0.3114
	Pappers	-0.5400	0.1000	75.2829
	Boats	-0.4000	0.1000	262.7854
	Tulips	-0.6000	0.2000	14.1536
	Watch	0	0.3000	49.2893
BSDS500 baze	3096	0.2100	0.0030	0.6346
	14037	-0.300	0.200	7.9427
	295087	0.0400	-0.0100	1.9018
	126007.	-0.4000	0.0100	15.3342
	260058	-0.300	-0.010	3.8658
	160068	0.7000	0.1000	0.4663
	241004	-0.3000	-0.0300	5.1729
	197017	0.4000	0.0900	5.1557
	143090	-0.0100	0.0040	14.5632
		$\overline{\alpha_{opt\_2P}}$	$\overline{\beta_{opt\_2P}}$	$\overline{MSE_{2P}}$
		-0.1375	0.0473	44.7000

Table 3 Optimal parameters *α, β* and *γ* , and minimum MSE for 3P Keys kernel

Image base	Image	α_{opt}	β_{opt}	γ_{opt}	MSE_{3P}
DSP test base	Lena	-0.3000	-0.1000	-0.0500	11.3130
	Barbara	-0.1000	-0.3000	-0.3000	242.1622
	Cameraman	0.3000	-0.1000	0.1000	0.3113
	Pappers	-0.5200	0.1000	-0.0200	75.2664
	Boats	0.5000	0.2000	0.0500	262.7747
	Tulips	-0.6000	0.2000	-0.0500	14.1407
	Watch	-0.1000	0.1000	-0.1500	49.2107
BSDS500 baze	3096	0.2000	-0.007	-0.005	0.4760
	14037	-0.300	0.2000	-0.010	5.9566
	295087	0	0	0.0400	1.4259
	126007.	-0.400	0.1000	0.0800	11.4961
	260058	-0.400	-0.060	0.0001	2.8970
	160068	0.7000	0	-0.110	0.3491
	241004	-0.300	0	0.0400	3.8780
	197017	0.400	0.0800	-0.014	3.8666
	143090	-0.020	0.0900	0.1000	10.9091
		$\overline{\alpha_{opt\_3P}}$	$\overline{\beta_{opt\_3P}}$	$\overline{\gamma_{opt\_3P}}$	$\overline{MSE_{3P}}$
		-0.1587	0.0314	-0.0187	43.5271

Table 4 Execution time for PCC interpolation.

	Execution time T_E (s)		
	$T_{E\_1P\_Keys}$	$T_{E\_2P\_Keys}$	$T_{E\_3P\_Keys}$
Int_1	$1.4305\cdot10^{-6}$	$2.4876\cdot10^{-6}$	$2.5225\cdot10^{-6}$
Int_2	$1.1903\cdot10^{-6}$	$2.0704\cdot10^{-6}$	$2.0990\cdot10^{-6}$
Ker_T	$5.6499\cdot10^{-7}$	$5.6492\cdot10^{-7}$	$5.6489\cdot10^{-7}$

4.5. Comparative analysis

According to the results presented in Table 1, Table 2 and Table 3, it is obvious that: a) MSE when applying 2P Keys kernel compared to 1P Keys kernel: $\overline{MSE_{1P}}$ / $\overline{MSE_{2P}}$ = 45.6911 / 44.700 = 1.0222 times smaller, b) MSE when applying 3P Keys kernel compared to 1P Keys kernel: $\overline{MSE_{1P}}$ / $\overline{MSE_{3P}}$ = 45.6911 / 43.5271 = 1.0497 times smaller, and c) MSE when applying 3P Keys kernel compared to 2P Keys kernel: $\overline{MSE_{2P}}$ / $\overline{MSE_{3P}}$ = 44.700 / 43.5970 = 1.0269 times smaller.

The optimal values of the kernel parameters, determined by minimizing the ripple of the characteristic of the 3P Keys kernel (Eq. (28)), are: α_{opt} = -0.6132, β_{opt} = 0.1522 i γ_{opt} = 0.0195. Using the experimental results (Table 3), it was shown that the mean values of the optimal kernel parameters, determined for all Test images, are: $\overline{\alpha_{opt\_3P}}$ = -0.1587, $\overline{\beta_{opt\_3P}}$ = 0.0314 and $\overline{\gamma_{opt\_3P}}$ = -0.0187. The absolute error of the kernel parameters, determined by algorithm for minimizing of the ripple of the spectral characteristic, in relation to the experimentally determined kernel parameters, are: a) $\Delta\alpha_{3P}$ = $|\alpha_{opt} - \overline{\alpha_{opt\_3P}}|$ = $|-0.6132-(-0.1587)|$ = 0.4545, b) $\Delta\beta_{3P}$ = $|\beta_{opt} - \overline{\beta_{opt\_3P}}|$ = $|0.1522-0.0314|$ = 0.1208,

c) $\Delta\gamma_{3P}$ = $|\gamma_{opt} - \overline{\gamma_{opt}}|$ = $|-0.0195-(-0.0187)|$ = 0.0008. The Total absolute error of kernel parameter estimation for all Test images is E_T = $\sqrt{\Delta\alpha_{3P}^2 + \Delta\beta_{3P}^2 + \Delta\gamma_{3P}^2}$ = 0.4703.

In accordance with the results presented in Table 4, for a complete convolution with non-optimized kernels, (Eq. (1), Eq. (2) and Eq. (3)), (label in the Table 1: Int1), it is concluded that Time execution, T_E, when applying: a) 2P Keys kernel compared to 1P Keys kernel is $T_{E\_2P\_Keys}$ / $T_{E\_1P\_Keys}$ = $2.4876 \cdot 10^{-6}$ / $1.4305 \cdot 10^{-6}$ = 1.7389 times bigger, b) 3P Keys kernel compared to 1P Keys kernel is $T_{E\_3P\_Keys}$ / $T_{E\_1P\_Keys}$ = $2.5225 \cdot 10^{-6}$ / $1.4305 \cdot 10^{-6}$ = 1.7633 times larger, and c) 3P Keys kernel compared to 2P Keys kernel is $T_{E\_3P\_Keys}$ / $T_{E\_2P\_Keys}$ = $2.5225 \cdot 10^{-6}$ / $2.4876 \cdot 10^{-6}$ = 1.014 times larger

In accordance with the results presented in Table 4, for a complete convolution with optimized kernels, (Eq. (29), Eq. (30) and Eq. (31)), (label in the Table 1: Int1), it is concluded that Time execution, T_E, when applying: a) 2P Keys kernel compared to 1P Keys kernel is $T_{E\_2P\_Keys}$ / $T_{E\_1P\_Keys}$ = $2.0704 \cdot 10^{-6}$ / $1.1903 \cdot 10^{-6}$ = 1.7393 times bigger, b) 3P Keys kernel compared to 1P Keys kernel is $T_{E\_3P\_Keys}$ / $T_{E\_1P\_Keys}$ = $2.0990 \cdot 10^{-6}$ / $1.1903 \cdot 10^{-6}$ = 1.7634 times larger, and c) 3P Keys kernel compared to 2P Keys kernel is $T_{E\_3P\_Keys}$ / $T_{E\_2P\_Keys}$ = $2.0990 \cdot 10^{-6}$ / $2.0704 \cdot 10^{-6}$ = 1.013 times larger.

The convolutional kernel execution time, T_E, without interpolation (label in the table: KerT) is approximately $5.649 \cdot 10^{-7}$ for all Keys kernels. The reason is that all kernels, after optimization, have the same numerical complexity: a third-order polynomial with constant coefficients. When a 3P Keys interpolation kernel with optimized parameters is applied, the convolution execution time, compared to non-optimized kernels, is $T_{E\ 3P\ Keys}$ / $T_{E\ 3P\ Keys\ opt}$ = $2.5225 \cdot 10^{-6}$ / $2.0990 \cdot 10^{-6}$ = 1.2017 times less.

The results from the described Experiment and the conducted detailed comparative analysis of interpolation error, which were expressed through MSE, indicated the fact that the accuracy of interpolation when the 3P Keys kernel was applied, in relation to 1P and 2P kernels, increased. The testing algorithm is implemented in the Matlab programming language. The interpolation execution times were calculated using the Matlab function *tic* and *toc*. The experiment only showed precision interpolation with 3P Keys kernels compared to precision with 1P Keys and 2P Keys kernels. In addition, the relative ratio of the interpolation execution times is determined. However, for real-time interpolation, the convolution algorithm must be written in a programming language (for example, programming language C) where, in the compilation process, optimizations can be made to reduce program execution time (Eq. 31). In this way, image processing can be provided in real-time mode and, among other things, on Personal Computers.

5. Conclusion

The paper presents an algorithm for optimizing the parameters of the 3P Keys interpolation kernel. Parameter optimization was performed in the spectral domain by minimizing the ripple of the spectral characteristic. First, the spectral characteristic was developed in the Taylor series, and, after that, the members of the Taylor series that have a great effect on increasing the riple of the spectral characteristic, were eliminated. From the conditions of elimination of the dominant members of the Taylor series, the optimal values of the parameters 3P Keys kernel (α_{opt} = -0.6132, β_{opt} = 0.1522, γ_{opt} = -0.0195) were determined. Verification of the accuracy of the 3P Keys kernel when interpolating images was performed experimentally. The interpolation accuracy is expressed through the MSE interpolation error. Detailed comparative analysis showed that the 3P Keys kernel, with experimentally determined optimal parameters, has a higher interpolation accuracy compared to the 1P Keys kernel 1.0497 times, and compared to the 2P Keys kernel 1.0269 times. Based on the presented results, it is concluded that the 3P Keys kernel is superior to the 1P Keys and 2P kernels and that the interpolation error is very small. Experimental results show that the 3P Keys kernel, with the optimal parameters, which are determined by the optimization algorithm presented in this paper, performed the interpolation of the Test images with great precision. The 3P Keys kernel with optimal parameters, compared to the ideal *sinc* kernel, has a small numerical complexity, and therefore, it is suitable for implementation in convolutional interpolations for operation in real-time systems.

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