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A UNIFIED APPROACH FOR DIGITAL REALIZATION OF FRACTIONAL ORDER OPERATOR IN DELTA DOMAIN

Arindam Mondal[1], Sujay Kumar Dolai[2], Prasanta Sarkar[3]

[1]Electrical Engineering Dept, Dr BC Roy Engineering College, Durgapur, India
[2]Electrical Engineering Dept, Dream Institute of Technology, Kolkata, India
[3]Electrical Engineering Dept, National Institute of Technical Teachers Training and Research, Kolkata, India

Abstract. *The fractional order operator ($s^{\pm\alpha}$,0 < α <1) plays the pivotal role for the realization of fractional orders systems (FOS). For the realization of the FOS, fractional order operator (FOO) needs to be realized either in discrete or continuous time domain. Discrete time rational approximation of FOO in the z-domain fails to provide meaningful information at fast sampling interval. Moreover, z domain rational transfer function becomes highly sensitive with respect to its coefficients variation resulting to the poor finite word length effects for digital realization. In the other hand delta operator parameterized system allows to develop unification of continuous and discrete time formulations leading to the development of a unified framework for digital realization at fast sampling interval. The discrete time approximation of the FOO in delta domain is found to be robust to its coefficient variation in comparison to the shift operator based discretization of FOO. In this paper, discrete δ-operator parameterization is proposed for the digital realization using direct discretization of FOO. As a result, superior finite word length effect is observed for the realization of the FOO in discrete delta domain. Fractional order operator with different orders (α) are considered for the realization purpose using the proposed method and the results obtained using MATLAB are presented for validation.*

Key words: *Delta domain, delta operator parameterization, finite word length effects, fractional order operator (FOO), fractional order system (FOS)*

1. INTRODUCTION

Non-integer order controllers (NIC) are also known as fractional order controller (FOC), have received an increased attention for the last few decades to the researchers particularly in the field of system theory and control [1, 2]. Fractional Calculus (FC) are

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Corresponding author: Arindam Mondal
Electrical Engineering Dept, Dr BC Roy Engineering College, Durgapur, India
E-mail: arininstru@gmail.com

the backbone of designing fractional order controllers (FOC) and FOCs are showing better response than that of integer order controller [3] for controlling the fractional order plants. For the last few decades FC has proven its applicability in the versatile areas of research such as signal processing [4], chaos electromagnetic fractional poles [5], dielectric polarization [6] etc. Improved dynamic properties of the control loops as well as the robustness of the controller are the prime features of FOC [7]. Fractional order differentiator (FOD) or fractional order integrator (FOI) are commonly known as fractional order operator (FOO), and are symbolized as $s^{\pm\alpha}$, with $0 < \alpha < 1, \alpha \in \Re$. FOO is the fundamental component for the realization of the FOC/FOS. In order to implement any FOC or FOS, the irrational infinite dimensional operator ($s^{\pm\alpha}$) is to be rationally approximated traditionally, either in continuous time domain ($s - TF$) or discrete time domain ($z - TF$). For the digital realization of the FOC, the fractional order operator is to be discretized using shift operator parameterization ($z - TF$) either by indirect or direct discretization methods. In case of indirect discretization method [7], the FOO is first fitted into continuous time rational transfer function ($s - TF$) in frequency domain and then discretized using Tustin discretization method. But in case of direct discretization, the FOO is directly converted to the z-domain rational transfer function ($z - TF$) by the use of generating function and its expansion through the continued fraction expansion method (CFE) [8], [9]. Some of the popular direct discretization methods in z-domain are described in [10], [11], [12].

As per the Shannon's theorem, for the digital realization of the FOS or FOC, the continuous time system ($s - TF$) should be sampled at a sampling rate of at least 10 times that of the system bandwidth but for the practical implementation of the system, the sampling rate is recommended to be 50 to 70 times of bandwidth [13]. It is the need of the hour to get higher bandwidth in closed loop system for physical implementation like in low budget consumer products [14] and digital controller design for invertors [15]. At a very fast sampling interval (Δ), the resultant shift operator parameterized system ($z - TF$) fails to provide meaningful information. As the sampling interval (Δ) is increasing, the resultant poles of the $z - TF$ come close to each other and found to have been concentrated near (1,0) point making the system highly sensitive to its coefficient variation [16].This will lead to serious finite word length issue for the realization of FOS or FOC in digital domain.

At a very fast sampling interval (Δ),when the digital shift operator parametrized system fails to provide meaningful information ,the delta operator [17] parameterized system is devoid of ill conditioning at fast sampling rate ($\Delta \to 0$) [18] and it is suitable for the high speed realization with improved finite word length characteristics [19]. Therefore, the problems with the shift operator parameterization at very fast sampling interval can be circumvented by the introduction of the δ-operator for the system realization in digital domain. Nowadays, delta operator is getting attention for the researchers in the field of control and signal processing [20 -32] . In this work, a direct discretization method in delta domain for discretizing the FOO is systematically presented and it has been shown that the rational approximation of FOO obtained in delta domain is robust to parameter variation and less sensitive to coefficient changes. Moreover, at fast sampling rate ($\Delta \to 0$), the delta operator parameterized system is producing similar results as can be obtained in continuous time domain. This is the motivation for working with the delta operator parameterized system and using the superior property of the δ-operator to implement the fractional order operator digitally with the better finite word length effect.

In reviewing the literature regarding the direct discretization of FOO in delta domain, it can be seen that no research has demonstrated about the coefficient sensitivity and robustness of the parameter variation of the rational transfer function in delta operator parameterized systems obtained through direct discretization method. This study is essential to high speed implementation of the fractional order system/fractional order controller in discrete time domain.

This paper deals with the high speed digital design and implementation of FOC using delta operator parameterization by direct discretization method. The main scientific contribution of this paper is to prove the superiority of the rational transfer function in delta domain which is obtained by using the direct discretization of the FOO in delta domain in terms of stability, robustness with respect to parameter variation and finite word length implementation, particularly at fast sampling interval.

The paper is organized as follows: Section 2 describes the fundamentals of fractional calculus and fractional order operator. Direct discretization of the FOO in delta domain is discussed in section 3. Section 4 describes the finite word length property of delta operator transfer function. Results are analyzed in section 5 and section 6 is devoted for summary and conclusions.

2. Fractional Order Calculus and Fractional Order Operator

2.1. Fractional order Calculus

Fractional calculus (FC) is the study of the extension of non-integer order derivative and integrals. In this study a common operator, called as fractional order differ-integral is defined by (1).

$$ {}_aD_t^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}} & (\alpha > 0) \\ 1 & (\alpha = 0) \\ \int_a^t (dt)^{\alpha} & (\alpha < 0) \end{cases} \tag{1} $$

In this study the following definitions of the fractional order differ-integral is reproduced [2].

Definition - Riemann-Liouville [R-L] definition of fractional order differ-integral:

$$ {}_aD_t^{\alpha} g(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{d\tau^m} \int_a^t (t-\tau)^{m-\alpha-1} g(\tau) d\tau \tag{2} $$

where, $m-1 < \alpha < m \in N$ and $\alpha \in R$ is a fractional order of the differ-integral of function $g(t)$.

Laplace transformation of (2) gives the following equation.

$$ L[{}_0D_t^{\alpha} g(t)] = \begin{cases} s^{\alpha} G(s) \\ s^{\alpha} G(s) - G^{'}(s) \end{cases} \tag{3} $$

$G^{'}(s) = \sum_{k=0}^{n-1} s_0^k D_t^{\alpha-k-1} g(0)$ is the initial condition and $n-1 < \alpha < n \in N$.

2.2. Fractional order operator

From the definition mentioned in (2) and its Laplace transformation represented by (3), the term s^{α} is known as fractional order operator (FOO) and if the value of $\alpha < 0$, it is called as fractional order derivative, the fractional order integral is represented if the value of $\alpha > 0$. This is why the operator is called as fractional order differ-integral operator. To represent any FOC or FOS, the Laplace transform of the fractional order differential equation is done and the transfer function is a function of s^{α}.

3. Direct Discretization of Fractional Order Operator in Delta Domain

3.1. Fundamentals of Delta operator

Delta operator is an alternative operator used to describe any discrete time systems and it is represented by δ. Usually, any discrete time systems are represented using shift operator parameterization and denoted as q .The delta operator is defined by (4)[17].

$$\delta = \frac{q-1}{\Delta} \tag{4}$$

where, Δ is the sampling interval.

The following identity is obtained in a limiting case, if delta operator is applied on a differentiable signal $x(t)$ at a high sampling rate ($\Delta \to 0$)

$$\lim_{\Delta \to 0} \delta\, x(t) = \frac{x(t+\Delta) - x(t)}{\Delta} \cong \frac{d}{dt} x(t) \tag{5}$$

From (5), it may be observed that at fast sampling rate ($\Delta \to 0$), delta operator (δ) resembles to the d/dt operator in continuous time domain. One of the most important properties of the delta operator is that at fast sampling interval, the continuous time result and discrete time result can be obtained simultaneously. To establish the relationship between the frequency variables in continuous time domain (s) and discrete delta domain (γ), following intermediate steps are carried out.

The relationship between the frequency variable (z) in shift operator parameterization and the frequency variable (γ) in delta operator parameterization is given by

$$\gamma = \frac{z-1}{\Delta} \tag{6}$$

The frequency variable 's' and 'z' are related by $z = e^{s\Delta}$ and therefore (6) can be rewritten as

$$\gamma = \frac{e^{s\Delta}-1}{\Delta} \Rightarrow e^{s\Delta} = 1 + \gamma\Delta \Rightarrow s = \frac{1}{\Delta}\ln(1+\gamma\Delta) \tag{7}$$

Therefore, the frequency variables in continuous time domain (s) and discrete delta domain (γ) is expressed by (7).

3.2. Rational approximation of FOO using direct discretization in Delta domain

Fractional order operators need to be discretized to realize and implement the FOS. For the implementation in delta domain, it is required to develop the generating function and (7) is used for this purpose.

Revisiting (7), following relationship is obtained.

$$s^{\pm\alpha} = \left(\frac{1}{\Delta}\ln(1+\gamma\Delta)\right)^{\pm\alpha} \tag{8}$$

In order to expand (8), logarithmic function in the right hand side is to be approximated in closed form. A trapezoidal quadrature rule [33] is utilized to get the close form approximation of $\ln(1+\beta)$ as given below:

$$\ln(1+\beta) = \frac{6\beta+3\beta^2}{6+6\beta+\beta^2} \tag{9}$$

Combining (8) and (9), the fractional order operator in continuous time domain can be approximated in delta domain and expressed by (10).

$$s^{\pm\alpha} = \left(\frac{1}{\Delta}\frac{6\gamma\Delta+3\gamma^2\Delta^2}{6+6\gamma\Delta+\gamma^2\Delta^2}\right)^{\pm\alpha} = \left(\frac{6\gamma+3\gamma^2\Delta}{6+6\gamma\Delta+\gamma^2\Delta^2}\right)^{\pm\alpha} \tag{10}$$

A clear observation from (10) can be made that at limiting value of $\Delta \to 0$, $s \cong \gamma$ means at fast sampling interval ,the frequency variables in continuous time domain and discrete delta domain maps each other .Therefore , (10) can be treated as the direct relationship between variable 's' and 'γ' and right hand side of the (10) is called the generating function.

To get the rational approximation of FOO ($s^{\pm\alpha}$) in delta domain, direct discretization method proposed in [34] is considered in this work. The direct discretization method for discretizing the FOO in delta domain as explained in [34] is chosen in this work as the method proved to be superior to the other relevant methods in the literature. The generating function as shown in (10) is expanded using the continued fraction expansion method (CFE)[35].

The CFE formulation is mathematically expressed by (11).

$$(1+c)^d = 1+\cfrac{dc}{1+\cfrac{(1-d)c}{2+\cfrac{(1+d)c}{3+\cfrac{(2-d)c}{2+\cfrac{(2+d)c}{5+\cfrac{(3-d)c}{2+\ldots}}}}}} \tag{11}$$

The variable 'c' can be replaced by $\left(1-\frac{6\gamma+3\gamma^2\Delta}{6+6\gamma\Delta+\gamma^2\Delta^2}\right)$ with 'd' as 'α' to expand the generating function. So, the integer order rational transfer function in delta domain corresponding to the FOO can be expressed as

$$G_\delta(\gamma) = CFE\left(\frac{6\gamma+3\gamma^2\Delta}{6+6\gamma\Delta+\gamma^2\Delta^2}\right)^{\pm\alpha} \tag{12}$$

In this work, a third order approximation of s^{α} is considered through the direct discretization method in delta domain [34] and the coefficients of rational delta transfer function are enumerated in Table 1. The method is termed as CFE-2PGILOGDEL.

Table 1 Coefficients for third-order approximation of s^{α}

$Den_3 = (3/\Delta)\alpha / (\alpha+1) / (4096\alpha^6 + 26624\alpha^5 + 9472\alpha^4 - 201472\alpha^3 - 252944\alpha^2 + 331304\alpha + 506955)$

Coefficient	Numerator
A_0	$(30720\alpha^6 + 454416\alpha^3 - 36096\alpha^5 - 838259\alpha +$ $78360\alpha^2 - 4096\alpha^7 - 192000\alpha^4 + 506955)Den_3$
A_1	$(-938460\Delta\alpha + 1388142\Delta - 723408\Delta\alpha^2 +$ $608640\Delta\alpha^3 - 76800\Delta\alpha^5 + 12288\Delta\alpha^6 + 12288\Delta\alpha^4)Den_3$
A_2	$(-465120\alpha^2\Delta^2 - 195900\Delta^2\alpha + 128640\Delta^2\alpha^3$ $-15360\Delta^2\alpha^5 + 714105\Delta^2 + 57600\Delta^2\alpha^4)Den_3$
A_3	$+(-64320\Delta^3\alpha^2 + 7680\Delta^3\alpha^4 + 97950\Delta^3)Den_3$
Coefficient	**Denominator**
B_0	$(4096\alpha^7 + 30720\alpha^6 + 36096\alpha^5 - 192000\alpha^4$ $-454416\alpha^3 + 78360\alpha^2 + 838259\alpha + 506955) / Den_3$
B_1	$+(938460\Delta\alpha + 1388142\Delta - 723408\Delta\alpha^2 - 608640\Delta\alpha^3$ $+76800\Delta\alpha^5 + 12288\Delta\alpha^6 + 12288\Delta\alpha^4) / Den_3$
B_2	$+(-465120\alpha^2\Delta^2 + 195900\Delta^2\alpha - 128640\Delta^2\alpha^3$ $+15360\Delta^2\alpha^5 + 714105\Delta^2 + 57600\Delta^2\alpha^4) / Den_3$
B_3	$+(-64320\Delta^3\alpha^2 + 7680\Delta^3\alpha^4 + 97950\Delta^3) / Den_3$

Therefore, 3rd order rational approximation of delta transfer function corresponding to FOO is given by (13).

$$s^{\alpha} = G_{\delta 3rd}(\gamma) = CFE\left(\frac{6\gamma + 3\Delta\gamma^2}{6 + 6\gamma\Delta + \Delta^2\gamma^2}\right)^{\alpha} = \frac{A_0 + A_1\gamma^{-1} + A_2\gamma^{-2} + A_3\gamma^{-3}}{B_0 + B_1\gamma^{-1} + B_2\gamma^{-2} + B_3\gamma^{-3}} \tag{13}$$

3rd order approximation of the FOO in continuous time domain transfer function, $G_{c3rd}(s)$ can be obtained using the Oustaloup's approximation method [1].

4. Finite Word Length Characteristics of Rational Delta Transfer Function

This paper deals with the effects of finite word length (FWL) representation of delta transfer function (G_{δ} (γ, α)) which is approximated using the direct discretization of fractional order operator. A shift operator parameterized transfer function ($G_q(z, \alpha)$) is also derived corresponding to the FOO using the trapezoidal rule (Tustin) [10] as a generating function and expressed by (14) to compare the effects of FWL representation.

$$(w(z^{-1}))^{\pm\alpha} = \left(\frac{2}{\Delta}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm\alpha} \tag{14}$$

Both the shift and delta operator based rational transfer functions are obtained using the generating function and CFE, the resultant transfer functions are taking the IIR filter form. If the range of fractional order (α) lies in the range between 0 and 1 means $\alpha \in (0,1)$,all the poles of $G_\delta(\gamma, \alpha)$ and $G_q(z, \alpha)$ are real and negative [19]. Therefore, in IIR form realization, both the transfer function takes the following forms.

$$G_q(z,\alpha) = r_{z0} + \sum_{k=1}^{j} r_{zk}\frac{1}{z-\bar{\sigma}_{zk}} \tag{15}$$

$$G_\delta(\gamma,\alpha) = r_{\gamma 0} + \sum_{k=1}^{j} r_{\gamma k}\frac{1}{\gamma-\bar{\sigma}_{\gamma k}} \tag{16}$$

where , r_{z0} , r_{zk} , $r_{\gamma 0}$, $r_{\gamma k}$ are the residues and $\bar{\sigma}_{zk}$, $\bar{\sigma}_{\gamma k}$ are the poles.

For the digital implementation of the fractional order controller or fractional order system, the corresponding irrational FOO is discretized and corresponding rational transfer functions are obtained in both the z-domain and δ-domain. Once the transfer functions are implemented digitally using finite number of bits or finite word length registers, round off of the coefficients like poles and residues are essential. In other word, the coefficients are to be quantized.

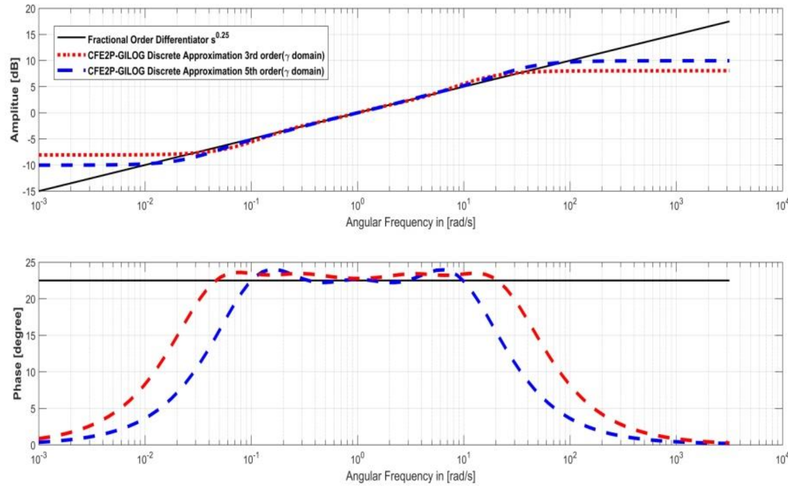
In this paper, frequency response analysis is done using both the desired coefficients and quantized coefficients for both discrete domain transfer functions. To study the finite word length characteristics of the delta domain transfer function corresponding to FOO, a 16b floating point representation (half precision) is used in this work. This is similar to the IEEE 754 32b floating point representation for single precision numbers. Half precision representation reduces the requirement of storage and increase the computational speed. In 16b representation format of coefficient quantization, 1 bit is reserved for sign, 6 bits are used to represent the exponent and rest of the 9 bits is used for normalized mantissa.

5. RESULT ANALYSIS

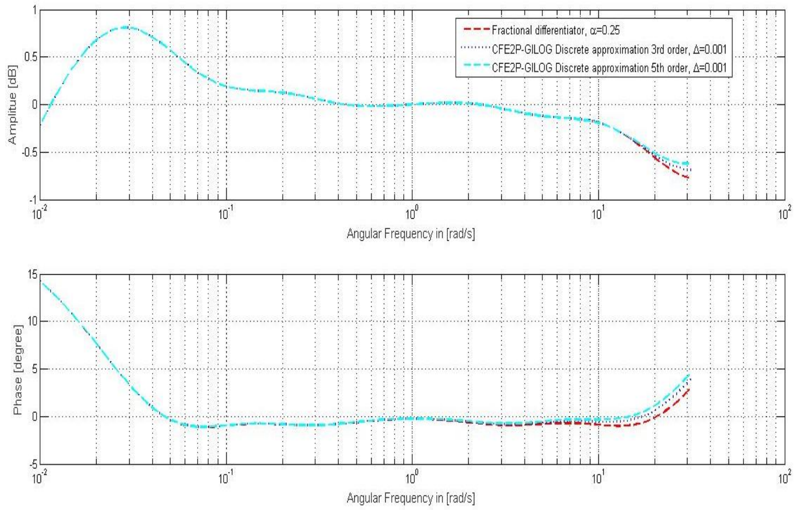
Pentium i7, 2.4 GHz processor with 32.0 GB RAM PC is used to perform the experimentation using MATLAB R2020a version.

A ¼ th order differentiator is considered as an example and 3rd order approximation ($G_{3rd}(\gamma,0.25)$) of it has been done using direct discretization in delta domain via CFE-2PGILOGDEL method [34] and is used as the backbone on which the finite word length effects has been studied in this work. Though fifth order approximation of the ¼ order differentiator in delta domain provides better frequency response but for the simplicity of operation, 3rd order approximation is considered in this paper. A comparison of frequency response for 3rd order ($G_{3rd}(\gamma,0.25)$) and 5th order approximations ($G_{5th}(\gamma,0.25)$) is graphically presented in Fig. 1. Frequency response analysis (Bode diagram) of the ¼ order differentiator ($G(s,0.25)$) and corresponding rational approximation of it in delta domain ($G_{3rd}(\gamma,0.25)$) are shown in Fig. 2a where rational approximations are obtained using three different sampling intervals such as $\Delta = 0.05$ sec, $\Delta = 0.01$ sec and $\Delta = 0.001$ sec.

It is clearly visible from Fig. 2b (error in approximation) that with faster sampling interval ($\Delta = 0.001$), frequency response of the approximate rational transfer function is more closely matching with that of the original continuous time fractional order differentiator over a large range of frequencies. This essence of delta operator property is making method unified when discretization of continuous time systems are done in delta domain.

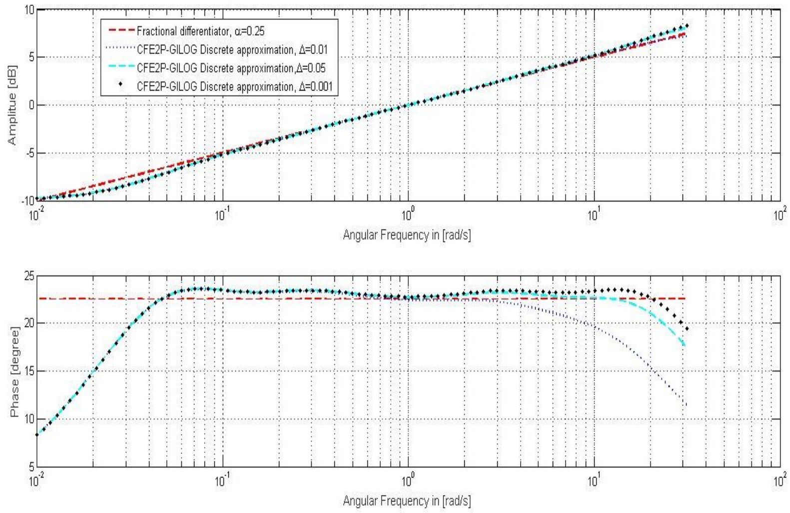


(a)

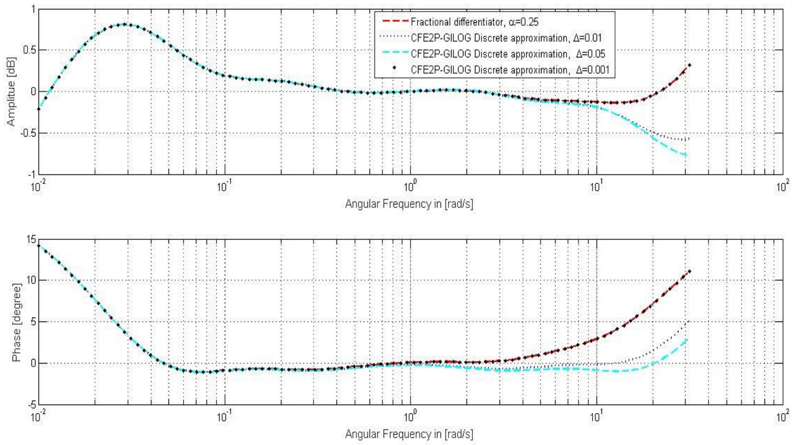


(b)

Fig. 1 a) Frequency responses of $G(s,0.25)$, $G_{3rd}(\gamma,0.25)$ and $G_{5th}(\gamma,0.25)$ at $\Delta = 0.001$ sec; b) Errors in approximations



(a)



(b)

Fig. 2 a) Frequency response of $G(s,0.25)$ and $G_{3rd}(\gamma,0.25)$ for sampling rates of $\Delta = 0.05$ sec, $\Delta = 0.01$ sec and $\Delta = 0.001$ sec; b) Errors in approximations

The residues and poles must be quantized since (15) and (16) are implemented using finite word length registers. To validate the results, an example of fractional order operator with degree $\alpha = 0.4$ is considered and the 3rd order approximation of the same in both the discrete z - domain and γ - domain when sampled at sampling interval $\Delta = 0.001$ sec is analysed.

The MATLAB computation to obtain the desired realization gives the following results. The poles and residues (desired) corresponding to

$G_{3rd}(z,0,4)$ provides: $\overline{\sigma}_{z1} = -0.8535$, $\overline{\sigma}_{z2} = 0.6869$, $\overline{\sigma}_{z3} = -0.1334$, $r_{z0} = -0.2530$, $r_{z1} = -0.1003$, $r_{z2} = -0.2467$ $r_{z2} = 1$.

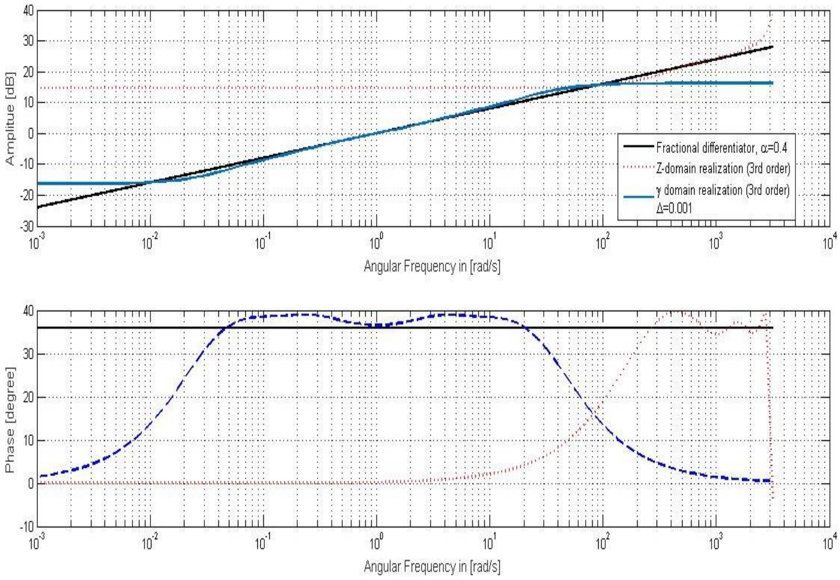
The poles and residues (desired) corresponding to $G_{3rd}(\gamma, 0.4)$ provides: $\overline{\sigma}_{\gamma 1} = 11.8987$, $\overline{\sigma}_{\gamma 2} = -1.2970$, $\overline{\sigma}_{\gamma 3} = -0.1858$, $r_{\gamma 0} = -20.8509$, $r_{\gamma 1} = -0.6461$, $r_{\gamma 2} = -0.0707$, $r_{\gamma 3} = 2.9565$.

A 16b floating point representation is used to get approximated realization representation. The poles and residues (approximated) corresponding to:

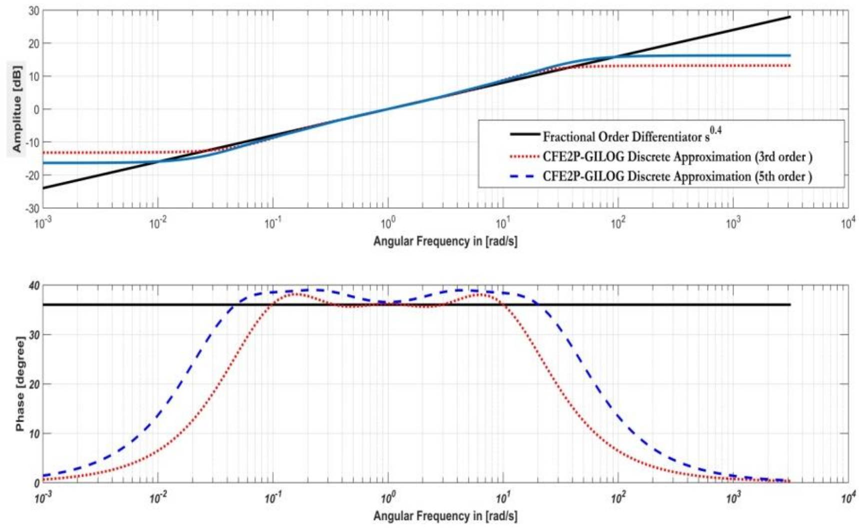
$G_{3rd}(z,0,4)$ provides: $\overline{\sigma}_{z1a} = -0.8535$, $\overline{\sigma}_{z2a} = 0.6869$, $\overline{\sigma}_{z3a} = -0.13339$, $r_{z0a} = -0.25299$, $r_{z1a} = -0.10029$, $r_{z2a} = -0.2467$.

The poles and residues (approximated) corresponding to $G_{3rd}(\gamma, 0.4)$ provides: $\overline{\sigma}_{\gamma 1a} = -11.8906$, $\overline{\sigma}_{\gamma 2a} = -1.2969$, $\overline{\sigma}_{\gamma 3a} = -0.1858$, $r_{\gamma 0a} = -20.8438$, $r_{\gamma 1a} = -0.6455$, $r_{\gamma 2a} = -0.0707$, $r_{\gamma 3a} = 2.9531$.

The frequency response obtained for the discrete time realization (*z*-domain and γ-domain) using the actual and quantized coefficients is shown in Fig. 3. In case of both the discrete time realizations, the absolute errors in magnitude and phase between the desired and approximated realization are shown in Fig. 4. From the response as shown in Figure 4, it may be noted that the error becomes much more in case of realization of the fractional order operator using shift operator parameterization (*z*-domain) where as in case of delta operator parameterization (*γ*-domain), almost zero error occurred using the quantized coefficients using 16 bit representation.



(a)



(b)

Fig. 3 a) Frequency response of the realization using *z*-domain and γ-domain for $\alpha = 0.4$ with $\Delta = 0.001$ sec; b) Fréquency responses of $G(s,0.4)$, $G_{3rd}(\gamma,0.4)$ and $G_{5rd}(\gamma,0.4)$ at $\Delta = 0.001$ sec

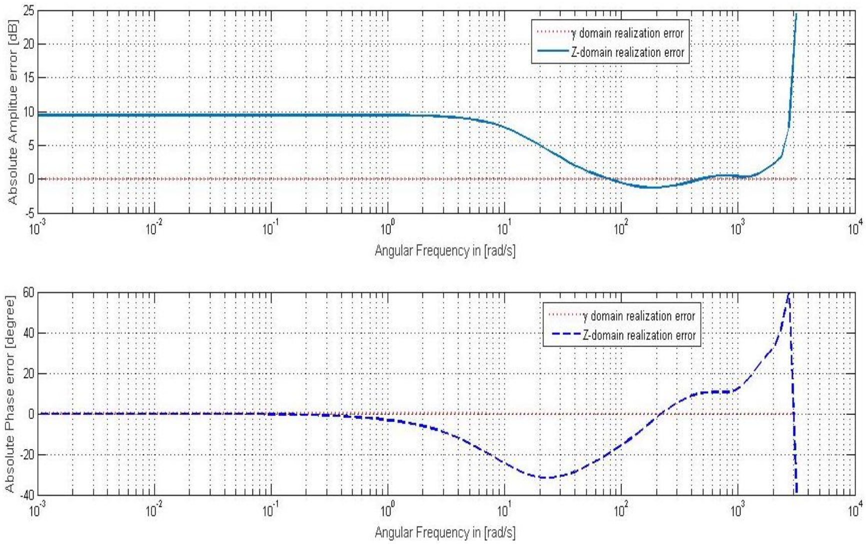


Fig. 4 Absolute errors between actual and approximated realization in *z*-domain and γ-domain

The same method can be used to get the discrete time realization of different fractional order operators. The Table 1 enumerates the desired and quantized coefficients for different fractional orders such as $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$.

The frequency response analysis using the approximate realization of the discrete *z*-domain and *γ*-domain transfer functions for $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ is illustrated in Fig. 5 using the quantized coefficients as listed in Table 1.

Table 2 Desired and approximated values of coefficients for *z*-domain and *γ*-domain transfer functions

α	Transfer Function	Desired residue	Approximate residue	Desired poles	Approximate. Poles
0.5	$z-TF$	0.849 + 0.00*i*	0.849 + 0.00*i*	-0.849 + 0.00*i*	-0.849 + 0.00*i*
		0.146 + 0.42*i*	0.146 + 0.42*i*	-0.146 + 0.42*i*	-0.146+ 0.42*i*
		0.146- 0.42*i*	0.146 - 0.42*i*	-0.1469 - 0.42*i*	-0.146- 0.42*i*
		-0.321 + 0.23*i*	-0.321+ 0.23*i*	0.321 + 0.23*i*	0.321 + 0.23*i*
		-0.321- 0.23*i*	-0.321- 0.23*i*	0.321- 0.23*i*	0.321 - 0.23*i*
				141.421	141.421
	$\gamma-TF$	-6207.750+ 0.0*i*	-6207.86+ 0.0*i*	-628.943+ 0.0*i*	-628.943 + 0.00*i*
		1036.50 + 11*i*	1036.43 + 11*i*	104.17 + 127.73*i*	104.17 + 127.73*i*
		1036.50- 11*i*	1036.43- 11*i*	104.17 - 127.77*i*	104.17 - 127.77*i*
		-3.87+ 0.00*i*	-4.30+ 0.00*i*	-19.116 + 0.000*i*	-19.116 + 0.000*i*
		-0.57+ 0.00*i*	-0.63+ 0.00*i*	-1.572+ 0.000*i*	-1.57 + 0.000*i*
		-0.03+ 0.00*i*	-0.05 + 0.00*i*	-0.23 + 0.00*i*	-0.23+ 0.0*i*
				7	7
0.6	$z-TF$	0.889 + 0.00*i*	0.88 + 0.00*i*	-0.889 + 0.000*i*	-0.889 + 0.000*i*
		0.15 + 0.30*i*	0.155 + 0.390*i*	-0.1545 + 0.390*i*	-0.155 + 0.3990*i*
		0.155 - 0.390*i*	0.155 - 0.3990*i*	-0.1545 - 0.390*i*	-0.155 - 0.390*i*
		-0.29 + 0.222*i*	-0.293 + 0.222*i*	0.2993 + 0.222*i*	0.293 + 0.2232*i*
		-0.293 - 0.222*i*	-0.293 - 0.222*i*	0.293 - 0.22*i*	0.293 - 0.222*i*
				380.736	380.733
	$\gamma-TF$	-627.50 + 0.00*i*	-627.88 + 0.000*i*	-623.76+ 0.000*i*	-624.43+ 0.000*i*
		1036.58 + 109.8*i*	1036.4 + 110.37*i*	133.39 + 134.237*i*	134.14 + 133.77*i*
		1036.58 - 109.8*i*	1036.46 - 110.37*i*	133.39 - 134.27*i*	134.174 - 133.77*i*
		-3.8 + 0.00*i*	-4.3075 + 0.000*i*	-24.889 + 0.00*i*	-19.119 + 0.000*i*
		-0.56 + 0.00*i*	-0.63594 + 0.00*i*	-1.721 + 0.00*i*	-1.720 + 0.00*i*
		-0.030 + 0.00*i*	-0.0522 + 0.000*i*	-0.258 + 0.00*i*	-0.239 + 0.00*i*
				11.1421	11.1421
0.7	$z-TF$	0.93 + 0.000*i*	0.9233 + 0.0*i*	-0.924+ 0.00i	-0.92 + 0.000*i*
		0.168 + 0.362*i*	0.1608 + 0.3642*i*	- 0.168+ 0.362i	-0.168 + 0.362*i*
		0.168 - 0.36402*i*	0.160 - 0.362*i*	- 0.168- 0.362i	-0.168 - 0.3642*i*
		-0.270 + 0.209*i*	-0.270 + 0.209*i*	0.2721+ 0.200i	0.271 + 0.20*i*
		-0.270 - 0.209*i*	-0.270 - 0.209*i*	0.2721- 0.200i	0.271 - 0.200*i*
				1024.9	1024.9

	$\gamma - TF$	-6277.17 + 0.00*i*	-6277.18 + 0.00*i*	-621017 + 0.0*i*	-621017 + 0.0*i*
		103654 + 119.3*i*	103656 + 18*i*	131.8 + 15.03*i*	131.87 + 15.03*i*
		10365 - 119.3*i*	10365 - 119.378*i*	131.871 - 15.03*i*	131.871 - 15.03*i*
		-3.500 + 0.00*i*	-3.5020 + 0.000*i*	-34.554 + 0.00*i*	-34.54 + 0.0*i*
		-0.5243 + 0.00*i*	-0.5243 + 0.000*i*	-1.9 + 0.0*i*	-1.9 + 0.0*i*
		-0.0287 + 0.00*i*	-0.0287 + 0.00*i*	-0.282 + 0.0*i*	-0.2852 + 0.0*i*
				18.930	18.930
0.8	$z - TF$	0.954 + 0.00*i*	0.953 + 0.000*i*	-0.953 + 0.00*i*	-0.953 + 0.00*i*
		0.16 + 0.31*i*	0.1618 + 0.3181*i*	-0.1617 + 0.3181*i*	-0.1617 + 0.31*i*
		0.168 - 0.31*i*	0.1618 - 0.3181*i*	-0.1617 - 0.311*i*	-0.1617 - 0.311*i*
		-0.29 + 0.18*i*	-0.239 + 0.1898*i*	0.29 + 0.188*i*	0.239 + 0.188*i*
		-0.239 - 0.18*i*	-0.239 - 0.1898*i*	0.239 - 0.188*i*	0.23 - 0.18*i*
				2759.4	2759.4
	$\gamma - TF$	-620769 + 0.00*i*	-620769 + 0.00*i*	-615.49 + 0.00*i*	-615.4 + 0.00*i*
		103661 + 118.9*i*	1036611 + 118.9*i*	108.9 + 113.64*i*	108.39 + 113.64*i*
		103661 - 118.9*i*	1036611 - 118.9*i*	108.9 - 113.64*i*	108.39 - 113.64*i*
		-3.15 + 0.00*i*	-3.1725 + 0.00*i*	-53.90 + 0.000*i*	-53.960 + 0.00*i*
		-0.4736 + 0.00*i*	-0.476 + 0.00*i*	-2.104 + 0.00*i*	-2.1094 + 0.00*i*
		-0.016 + 0.0*i*	-0.0186 + 0.0*i*	-0.315 + 0.00*i*	-0.315 + 0.00*i*
				36.27	36.27
0.9	$z - TF$	0.978 + 0.000*i*	0.9783 + 0.000*i*	-0.9780 + 0.000*i*	-0.970 + 0.00*i*
		0.153 + 0.2525*i*	0.159 + 0.2525*i*	-0.1539 + 0.255*i*	-0.159 + 0.255*i*
		0.153 - 0.2525*i*	0.159 - 0.255*i*	-0.1539 - 0.2525*i*	-0.159 - 0.2525*i*
		-0.194 + 0.1593*i*	-0.19 + 0.15983*i*	0.1924 + 0.1592*i*	0.1924 + 0.159*i*
		-0.194 - 0.1593*i*	-0.194 - 0.1593*i*	0.1924 - 0.1592*i*	0.194 - 0.1592*i*
				742894	742894
	$\gamma - TF$	-676.65 + 0.0*i*	-676.65 + 0.0*i*	-628.92 + 0.00*i*	-628.92 + 0.00*i*
		10666 + 118.56*i*	10666 + 118.567*i*	109.66 + 130.09*i*	109.66 + 130.09*i*
		103666 - 118.56*i*	10366 - 118.56*i*	109.66 - 130.09*i*	109.66 - 130.07*i*
		-2.8 + 0.00*i*	-2.87 + 0.00*i*	-111.71 + 0.00*i*	-111.71 + 0.00*i*
		-0.426 + 0.00*i*	-0.426 + 0.00*i*	-2.343 + 0.00*i*	-2.34 + 0.00*i*
		-0.008 + 0.00*i*	-0.0087 + 0.00*i*	-0.34 + 0.00*i*	-0.34 + 0.00*i*
				93.02	93.018

The maximum percentage of relative error in magnitude and phase for realization of different fractional order operators are described in Table 2. It can be visualized that the maximum percentage relative error in magnitude is occurring in case of *z*-domain realization where as in case of γ-domain realization, the response characteristics are very much aligned with that of the continuous time domain results for fractional order differentiator of different orders (α).

This paper deals with the unified method for the digital realization of FOO in delta domain. The order of the factional order operator is considered from 0.5 to 0.9 to validate the method as compared to the order of FOO considered for digital realization in [19]. Moreover, the sampling interval is here considered as Δ = 0.001 sec which is much less than the sampling considered in [19]. As the sampling interval is reduced much, the notion of using delta operator is justified.

With the present approach, the maximum percentage of relative error in magnitude and phase are less as compared to the method adopted in [19].For example, in [19] , the maximum percentage of relative error in magnitude and phase are 13.33 % and 7.83% respectively in delta domain for $\alpha = 0.7$, where as in this work , the maximum percentage of relative error in magnitude and phase are 0.35 % and 4.52% respectively in delta domain for $\alpha = 0.7$. This proves the superiority of this proposed method.

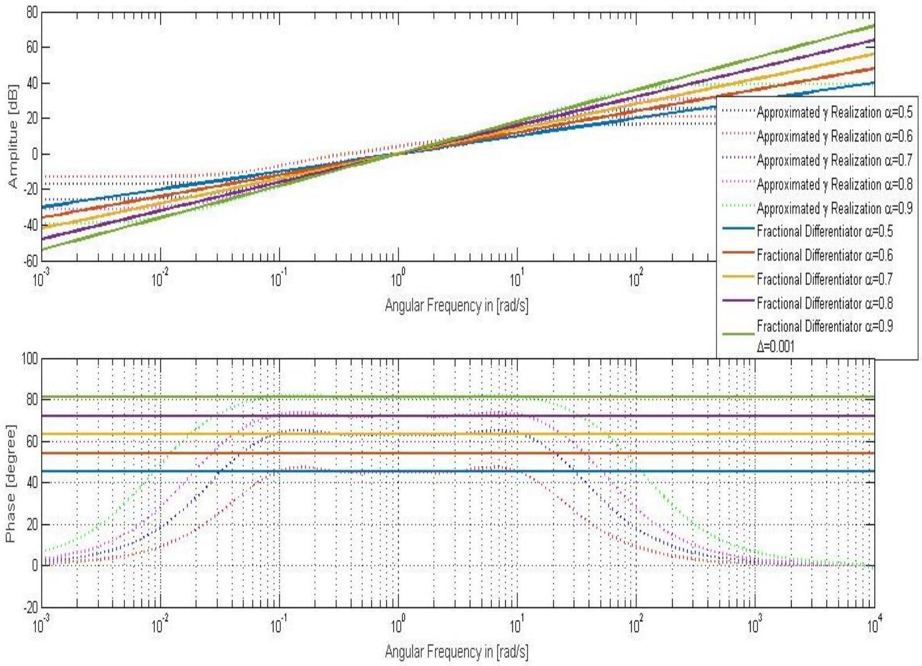


Fig. 5 Frequency response of $G_{3rd}(\gamma,\alpha)$ for $\alpha = 0.5,0.6,0.7,0.8,0.9$ at $\Delta = 0.001$sec

Table 3 Maximum percentage relative errors in magnitude and phase responses

α	Amplitude Max.% Relative Error $(z-TF)$	Amplitude Max.% Relative Error $(\gamma-TF)$	Phase Max.% Relative Error $(z-TF)$	Phase Max.% Relative Error $(\gamma-TF)$
0.5	72.2063	0.1414	8.054600	6.6341
0.6	73.0387	0.2316	52.33260	7.9628
0.7	89.4212	0.3516	132.3962	4.5271
0.8	89.6875	0.4701	133.5642	3.2106
0.9	93.4510	0.4682	145.4541	1.7190

6. CONCLUSIONS

This paper deals with the digital realization of fractional order operator using delta operator parameterization. Whenever any fractional order operator is represented using the corresponding rational approximated transfer function in discrete z -domain, the transfer function approximations become sensitive to coefficient variation resulting in a poor finite word length effect. Instead, delta operator parameterized rational transfer function is considerably less sensitive to parameter variation. In this paper a 3rd order approximation of the fractional order operator of different orders are considered and the operators are directly discretized to get the corresponding delta domain transfer functions. Through the direct discretization method used in this work for the discretization of the FOO in delta domain, the rational transfer function corresponding to the FOO becomes stable for fast sampling rate ($\Delta \to 0$) which may not be possible in all case of indirect discretization method at fast sampling interval. The frequency response analysis of the different transfer functions in delta domain using the desired coefficients and approximated coefficients using half precision quantization logic reveals that delta operator parameterized transfer function provides much more robust with respect to coefficient variation. The maximum percentage relative error for magnitude and phase for different fractional order operator are tabulated from Fig. 5 and the results as shown in Table 2 ensure that rational approximation using delta operator gives very small relative error as compared to the rational approximation using the shift operator parameterization (z-domain). Therefore, the digital transfer function rationalized using delta operator is a robust one for finite word length implementation. At fast sampling interval (Δ=0.001), the rational approximation using δ-domain resembles to the result obtained using continuous time domain representation making the realization a unified one.

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