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A HIGHLY ACCURATE ALGORITHM FOR COMPUTATION OF COMPLEX-VALUED BESSEL, NEUMANN AND HANKEL FUNCTIONS OF INTEGER ORDER

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Abstract. *In this paper, a highly accurate algorithm for computation of complex-valued Bessel, Neumann and Hankel functions of integer order is given. The algorithm enables the computation of these functions in the entire complex plane with quadruple precision, which can be reduced to double precision. The complex values of the Bessel and Neumann functions of the zeroth and first order can be computed in a special way for small, medium-sized and large arguments in the first quadrant of the complex plane. The mapping of functions from the first quadrant to the other quadrants is described by simple formulas. Bessel and Neumann functions of higher positive integer order can be computed using forward and backward recurrence relations. Two types of Hankel functions are linear combinations of the Bessel and Neumann functions. Bessel, Neumann and Hankel functions of negative integer order are equal to positive order functions up to the sign.*

Key words: *Bessel function, Neumann function, Hankel function, complex-valued function, Gauss-Legendre quadrature*

1. Introduction

Bessel functions of the first kind, Bessel functions of the second kind (Neumann functions) and Bessel functions of the third kind (Hankel functions) of the integer order occur frequently in mathematics, but also in various applied scientific fields, such as electromagnetic problems [1–10] and earthquake engineering problems [11]. Important formulas and various numerical methods for computation of these functions can be found in references [12–21].

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In this paper, a new highly accurate algorithm for the computation of complex-valued Bessel, Neumann and Hankel functions of integer order is developed. The computation of Bessel, Neumann and Hankel functions of integer order in all quadrants of the complex plane is based on the computation of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plane. For numerical reasons, it is advisable to use the scaling of Bessel, Neumann and Hankel functions to avoid the numerical instability that occurs with large magnitudes of complex variables.

Section 2 introduces scaled and unscaled Bessel, Neumann and Hankel functions. Section 3 contains equations for computation of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plane for small, medium-sized and large magnitudes of the complex argument. Equations for the translation of these functions from the other quadrants into the first quadrant are given in Sections 4–6. The computation of these functions of higher positive integer orders is presented in Section 7, and the validation of the proposed model is descriptively presented in Section 8.

2. Scaled and Unscaled Bessel, Neumann and Hankel functions

The unscaled functions of integer order $n = 0, \pm 1, \pm 2, \dots$ considered in this paper are Bessel functions $J_n(\bar{z})$, Neumann functions $Y_n(\bar{z})$ and Hankel functions, which are defined by:

$$H_n^{(1)}(\bar{z}) = J_n(\bar{z}) + \mathrm{j} \cdot Y_n(\bar{z}) \tag{1}$$

$$H_n^{(2)}(\bar{z}) = J_n(\bar{z}) - \mathrm{j} \cdot Y_n(\bar{z}) \tag{2}$$

where $\bar{z} = x + \mathrm{j} \cdot y$ is a complex independent variable, which consists of real part x and imaginary part y, whereas j is the imaginary unit. $H_n^{(1)}$ and $H_n^{(2)}$ denote Hankel functions of the first and second kind, respectively.

To avoid numerical stability problems, computation of scaled Bessel functions is introduced in the proposed model. In the first quadrant of the complex plane, as the imaginary part of the complex variable increases, the values of Bessel functions also increase.

Scaled and unscaled functions are related by:

$$B_n^s(\bar{z}) = \mathrm{e}^{-|y|} \cdot B_n(\bar{z}) \tag{3}$$

where:

$$B_n(\bar{z}) \in \left\{ J_n(\bar{z}), Y_n(\bar{z}), H_n^{(1)}(\bar{z}), H_n^{(2)}(\bar{z}) \right\} \tag{4}$$

$$B_n^s(\bar{z}) \in \left\{ J_n^s(\bar{z}), Y_n^s(\bar{z}), H_n^{(1)s}(\bar{z}), H_n^{(2)s}(\bar{z}) \right\} \tag{5}$$

The following equations are valid for negative integer order:

$$B_{-n}(\bar{z}) = (-1)^n \cdot B_n(\bar{z}) \tag{6}$$

$$B_{-n}^s(\bar{z}) = (-1)^n \cdot B_n^s(\bar{z}) \tag{7}$$

3. COMPUTATION OF BESSEL AND NEUMANN FUNCTIONS OF THE ZEROTH AND FIRST ORDER IN THE FIRST QUADRANT

3.1. Small magnitude of the complex argument

Computation of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plane $(x \geq 0,\ y \geq 0)$ for small magnitude of the complex argument:

$$|\overline{z}| \leq b_1 = \begin{cases} 15, & \text{for double precision} \\ 5, & \text{for quadruple precision} \end{cases} \tag{8}$$

is based on the computation of truncated power series [12, 14], which can be rewritten as:

$$J_0(\overline{z}) \approx \sum_{i=0}^{m} \left(a_i \cdot \left(\frac{\overline{z}}{20} \right)^{2i} \right) \tag{9}$$

$$J_1(\overline{z}) \approx \frac{\overline{z}}{2} \cdot \sum_{i=0}^{m} \left(\frac{a_i}{i+1} \cdot \left(\frac{\overline{z}}{20} \right)^{2i} \right) \tag{10}$$

$$Y_0(\overline{z}) \approx \frac{2}{\pi} \cdot \left(C + \ln \frac{\overline{z}}{2} \right) \cdot J_0(\overline{z}) - \frac{2}{\pi} \cdot \sum_{i=1}^{m} \left(a_i \cdot \Phi(i) \cdot \left(\frac{\overline{z}}{20} \right)^{2i} \right) \tag{11}$$

$$Y_1(\overline{z}) \approx -\frac{2}{\pi \cdot \overline{z}} + \frac{2}{\pi} \cdot \left(C + \ln \frac{\overline{z}}{2} \right) \cdot J_1(\overline{z}) - \frac{\overline{z}}{2 \cdot \pi} \cdot \sum_{i=0}^{m} \left(\frac{a_i \cdot \Psi(i)}{i+1} \cdot \left(\frac{\overline{z}}{20} \right)^{2i} \right) \tag{12}$$

where:

$$a_0 = 1 \quad ; \quad a_i = -\left(\frac{10}{i} \right)^2 \cdot a_{i-1} \quad \text{for} \quad i = 1, 2, \ldots, m \leq 40 \tag{13}$$

$$\Phi(i) = \sum_{k=1}^{i} \frac{1}{k} \quad \text{for} \quad i \geq 1 \tag{14}$$

$$\Psi(i) = \begin{cases} 2 \cdot \Phi(i) + \dfrac{1}{i+1}, & \text{for} \quad i \geq 1 \\ 1, & \text{for} \quad i = 0 \end{cases} \tag{15}$$

whereas C = 0.57721566490153286060651209008240243 is known as Euler constant.

The parameter b_1, introduced by Equation (8), as well as parameter m, which is equal to or less than 40, introduced by Equations (9) – (13), are determined based on our numerous numerical tests. The values of functions computed by proposed algorithm were compared with the values computed by the free online software package Wolfram Alpha.

For Bessel and Neumann functions of the zeroth and first order, the summation is terminated after $m = 40$ or after the following conditions are met:

$$\left| a_m \cdot \left(\frac{\overline{z}}{20} \right)^{2m} \right| \leq \left| \sum_{i=0}^{m} a_i \cdot \left(\frac{\overline{z}}{20} \right)^{2i} \right| \cdot 10^{-q} \quad \text{for } J_0(\overline{z}) \tag{16}$$

$$\left|\frac{a_m}{m+1}\cdot\left(\frac{\overline{z}}{20}\right)^{2m}\right| \le \left|\sum_{i=0}^{m}\frac{a_i}{i+1}\cdot\left(\frac{\overline{z}}{20}\right)^{2i}\right|\cdot 10^{-q} \quad \text{for } J_1(\overline{z}) \tag{17}$$

$$\left|a_m\cdot\Phi(m)\cdot\left(\frac{\overline{z}}{20}\right)^{2m}\right| \le \left|\sum_{i=1}^{m}\left(a_i\cdot\Phi(i)\cdot\left(\frac{\overline{z}}{20}\right)^{2i}\right)\right|\cdot 10^{-q} \quad \text{for } Y_0(\overline{z}) \tag{18}$$

$$\left|\frac{a_m\cdot\Psi(m)}{m+1}\cdot\left(\frac{\overline{z}}{20}\right)^{2m}\right| \le \left|\sum_{i=0}^{m}\left(\frac{a_i\cdot\Psi(i)}{i+1}\cdot\left(\frac{\overline{z}}{20}\right)^{2i}\right)\right|\cdot 10^{-q} \quad \text{for } Y_1(\overline{z}) \tag{19}$$

The parameter q in Equations (16) – (19), determined by our numerical tests depending on the desired accuracy, have the following values:

$$q = \begin{cases} 17, & \text{for double precision} \\ 34, & \text{for quadruple precision} \end{cases} \tag{20}$$

3.2. Medium-sized magnitude of the complex argument

Computation of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plane $(x \ge 0,\ y \ge 0)$ for medium-sized magnitude of the complex argument, $b_1 < |\overline{z}| \le b_2$, where:

$$b_2 = \begin{cases} 40, & \text{for double precision} \\ 50, & \text{for quadruple precision} \end{cases} \tag{21}$$

is based on the application of Gauss-Legendre quadrature to integral representations of Bessel and Neumann functions:

$$J_0(\overline{z}) = \frac{2}{\pi}\cdot\int_0^{\pi/2}\cos(\overline{z}\cdot\sin\vartheta)\cdot d\vartheta \tag{22}$$

$$J_1(\overline{z}) = \frac{2}{\pi}\cdot\int_0^{\pi/2}\sin(\overline{z}\cdot\sin\vartheta)\cdot\sin\vartheta\cdot d\vartheta \tag{23}$$

$$Y_0(\overline{z}) = \frac{2}{\pi}\cdot\int_0^{\pi/2}\sin(\overline{z}\cdot\sin\vartheta)\cdot d\vartheta - \frac{2}{\pi}\cdot\int_0^{\infty}e^{-\overline{z}\cdot\sinh t}\cdot dt \tag{24}$$

$$Y_1(\overline{z}) = -\frac{2}{\pi}\cdot\int_0^{\pi/2}\cos(\overline{z}\cdot\sin\vartheta)\cdot\sin\vartheta\cdot d\vartheta - \frac{2}{\pi}\cdot\int_0^{\infty}e^{-\overline{z}\cdot\sinh t}\cdot\sinh t\cdot dt \tag{25}$$

where ϑ and t are independent variables.

The parameter b_2, introduced by Equation (21), is determined on the basis of our numerous numerical tests.

Due to the simpler numerical computation of the improper integral in Eq. (25), a new integral representation of $Y_1(\overline{z})$ can be obtained by substituting $\sinh t = \cosh t - e^{-t}$:

$$Y_1(\bar{z}) = -\frac{2}{\pi \cdot \bar{z}} - \frac{2}{\pi} \cdot \int_0^{\pi/2} \cos(\bar{z} \cdot \sin \vartheta) \cdot \sin \vartheta \cdot d\vartheta + \frac{2}{\pi} \cdot \int_0^{\infty} e^{-t-\bar{z} \cdot \sinh t} \cdot dt \qquad (26)$$

Computation of the scaled Bessel functions of the zeroth and first order for medium-sized magnitude of complex argument, $b_1 < |\bar{z}| \leq b_2$, using the Gauss-Legendre quadrature in the first quadrant of the complex plain $(x \geq 0;\ y \geq 0)$ is based on the following integral representations:

$$J_0^s(\bar{z}) = e^{-y} \cdot J_0(\bar{z}) = \frac{1}{\pi} \cdot \int_0^{\pi/2} (e^{-y+j \cdot \bar{z} \cdot \sin \vartheta} + e^{-y-j \cdot \bar{z} \cdot \sin \vartheta}) \cdot d\vartheta \qquad (27)$$

$$J_1^s(\bar{z}) = e^{-y} \cdot J_1(\bar{z}) = \frac{j}{\pi} \cdot \int_0^{\pi/2} (e^{-y-j \cdot \bar{z} \cdot \sin \vartheta} - e^{-y+j \cdot \bar{z} \cdot \sin \vartheta}) \cdot \sin \vartheta \cdot d\vartheta \qquad (28)$$

Computation of the scaled Neumann functions of the zeroth and first order for medium-sized magnitude of complex argument, $b_1 < |\bar{z}| \leq b_2$, using the Gauss-Legendre quadrature in the first octant of the complex plane, where $x \geq y \geq 0$, is based on the following integral representations:

$$Y_0^s(\bar{z}) = e^{-y} \cdot Y_0(\bar{z}) = \frac{j}{\pi} \cdot \int_0^{\pi/2} (e^{-y-j \cdot \bar{z} \cdot \sin \vartheta} - e^{-y+j \cdot \bar{z} \cdot \sin \vartheta}) \cdot d\vartheta - \frac{2}{\pi} \cdot \int_0^{\infty} e^{-y-\bar{z} \cdot \sinh t} \cdot dt \qquad (29)$$

$$Y_1^s(\bar{z}) = e^{-y} \cdot Y_1(\bar{z}) = -\frac{2 \cdot e^{-y}}{\pi \cdot \bar{z}} - \frac{1}{\pi} \cdot \int_0^{\pi/2} (e^{-y+j \cdot \bar{z} \cdot \sin \vartheta} + e^{-y-j \cdot \bar{z} \cdot \sin \vartheta}) \cdot \sin \vartheta \cdot d\vartheta + \frac{2}{\pi} \cdot \int_0^{\infty} e^{-t-y-\bar{z} \cdot \sinh t} \cdot dt \qquad (30)$$

Our numerous numerical tests have shown that the infinite upper limits of the improper integrals in Eqs. (29) and (30) can be replaced by finite limits of the integrals without loss of accuracy:

$$\int_0^{\infty} e^{-y-\bar{z} \cdot \sinh t} \cdot dt \approx \int_0^{t_{m0}} e^{-j \cdot y \cdot \sinh t} \cdot e^{-(y+x \cdot \sinh t)} \cdot dt \qquad (31)$$

$$\int_0^{\infty} e^{-t-y-\bar{z} \cdot \sinh t} \cdot dt \approx \int_0^{t_{m1}} e^{-j \cdot y \cdot \sinh t} \cdot e^{-(t+y+x \cdot \sinh t)} \cdot dt \qquad (32)$$

where finite upper limits of the integrals can be defined by equations:

$$y + x \cdot \sinh t_{m0} = 75 \quad \Rightarrow \quad t_{m0} = \sinh^{-1}\left(\frac{75 - y}{x}\right) \qquad (33)$$

$$t_{m1} + y + x \cdot \sinh t_{m1} = 75 \qquad (34)$$

Upper integral limit t_{m1}, described by non-linear Equation (34), can be computed by Gauss-Newton iterative method using the following equation:

$$(t_{m1})_{k+1} = (t_{m1})_k - \frac{y + (t_{m1})_k + x \cdot \sinh((t_{m1})_k) - 75}{1 + x \cdot \cosh((t_{m1})_k)} \quad ; \quad k = 1, 2, \ldots \qquad (35)$$

with an initial value:

$$(t_{m1})_0 = \sinh^{-1}\left(\frac{75 - y - t_{m0}}{x}\right) \tag{36}$$

Computation of the scaled Neumann functions for medium-sized magnitude of complex argument, $b_1 < |\overline{z}| \leq b_2$, in the second octant of the complex plane, where $y > x \geq 0$, is also carried out using Gauss-Legendre quadrature. In this case, due to the greater attenuation of the integrands of improper integrals, the following expressions are used:

$$Y_0(\overline{z}) = \mathrm{j} \cdot J_0(\overline{z}) - \frac{2}{\pi} \cdot K_0(-\mathrm{j} \cdot \overline{z}) \tag{37}$$

$$Y_1(\overline{z}) = \mathrm{j} \cdot J_1(\overline{z}) + \mathrm{j} \cdot \frac{2}{\pi} \cdot K_1(-\mathrm{j} \cdot \overline{z}) \tag{38}$$

where K_0 and K_1 are modified Bessel functions of the second kind of the zeroth and first order, respectively.
From the integral representation of modified Bessel functions [12, 13]:

$$K_0(-\mathrm{j} \cdot \overline{z}) = \int_0^{\infty} \mathrm{e}^{\mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{39}$$

$$K_1(-\mathrm{j} \cdot \overline{z}) = \int_0^{\infty} \mathrm{e}^{\mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot \cosh t \cdot dt \tag{40}$$

where, due to numerical reasons, by substituting $\cosh t = \sinh t + \mathrm{e}^{-t}$, a new integral representation for K_1 can be obtained:

$$K_1(-\mathrm{j} \cdot \overline{z}) = \frac{\mathrm{j}}{\overline{z}} \cdot \mathrm{e}^{j \cdot \overline{z}} + \int_0^{\infty} \mathrm{e}^{-t + \mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{41}$$

After substituting Eqs. (22), (23), (39) and (41) into Eqs. (37) and (38), a new integral representation of unscaled and scaled Neumann functions in the second octant of the complex plane can be written as:

$$Y_0(\overline{z}) = \mathrm{j} \cdot J_0(\overline{z}) - \frac{2}{\pi} \cdot \int_0^{\infty} \mathrm{e}^{\mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{42}$$

$$Y_1(\overline{z}) = -\frac{2}{\pi \cdot \overline{z}} \cdot \mathrm{e}^{\mathrm{j} \cdot \overline{z}} + \mathrm{j} \cdot J_1(\overline{z}) + \mathrm{j} \cdot \frac{2}{\pi} \cdot \int_0^{\infty} \mathrm{e}^{-t + \mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{43}$$

$$Y_0^s(\overline{z}) = \mathrm{e}^{-y} \cdot Y_0(\overline{z}) = \mathrm{j} \cdot J_0^s(\overline{z}) - \frac{2}{\pi} \cdot \int_0^{\infty} \mathrm{e}^{-y + \mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{44}$$

$$Y_1^s(\overline{z}) = \mathrm{e}^{-y} \cdot Y_1(\overline{z}) = -\frac{2}{\pi \cdot \overline{z}} \cdot \mathrm{e}^{-y + \mathrm{j} \cdot \overline{z}} + \mathrm{j} \cdot J_1^s(\overline{z}) + \mathrm{j} \cdot \frac{2}{\pi} \cdot \int_0^{\infty} \mathrm{e}^{-t - y + \mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \tag{45}$$

Infinite upper limits of improper integrals in Eqs. (44) and (45) can be replaced by finite limits of the integrals without loss of accuracy:

$$\int_0^{\infty} \mathrm{e}^{-y + \mathrm{j} \cdot \overline{z} \cdot \cosh t} \cdot dt \approx \int_0^{t_{m0}} \mathrm{e}^{\mathrm{j} \cdot x \cdot \cosh t} \cdot \mathrm{e}^{-y \cdot (1 + \cosh t)} \cdot dt \tag{46}$$

$$\int_0^{\infty} \mathrm{e}^{-t-y+\mathrm{j}\cdot\bar{z}\cdot\cosh t} \cdot dt \approx \int_0^{t_{m1}} \mathrm{e}^{\mathrm{j}\cdot x\cdot\cosh t} \cdot \mathrm{e}^{-t-y\cdot(1+\cosh t)} \cdot dt \tag{47}$$

where finite upper limits of the integrals can be defined for $y \le 37$ by equations:

$$y + y\cdot\cosh t_{m0} = 75 \quad \Rightarrow \quad t_{m0} = \cosh^{-1}\left(\frac{75-y}{y}\right) \tag{48}$$

$$t_{m1} + y + y\cdot\cosh t_{m1} = 75 \tag{49}$$

Upper integral limit t_{m1}, described by non-linear Equation (49), can be computed by Gauss-Newton iterative method using the following equation:

$$(t_{m1})_{k+1} = (t_{m1})_k - \frac{(t_{m1})_k + y + y\cdot\cosh\left((t_{m1})_k\right) - 75}{1 + y\cdot\sinh\left((t_{m1})_k\right)} \quad ; \quad k = 1, 2, \ldots \tag{50}$$

with an initial value:

$$(t_{m1})_0 = \cosh^{-1}\left(\frac{75-y-t_{m0}}{y}\right) \tag{51}$$

For $y > 37$ it can be taken that $t_{m0} = t_{m1} = 0$, and therefore the truncated improper integrals given by Eqs. (46) and (47) are equal to zero.

In the proposed algorithm, all finite integrals and truncated improper integrals are computed using Gauss-Legendre quadrature, where the total number of integration points is 100 for quadruple precision and 40 for double precision. In both cases, the algorithm is performed in quadruple precision.

3.3. Large magnitude of the complex argument

For large magnitude of the complex argument, $|\bar{z}| > b_2$, asymptotic approximations of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plain $(x \ge 0,\ y \ge 0)$ can be written as [12]:

$$J_0(\bar{z}) \sim \sqrt{\frac{2}{\pi\cdot\bar{z}}}\cdot\left[\bar{P}_0(\bar{z})\cdot\cos\left(\bar{z}-\frac{\pi}{4}\right) + \bar{Q}_0(\bar{z})\cdot\sin\left(\bar{z}-\frac{\pi}{4}\right)\right] \tag{52}$$

$$J_1(\bar{z}) \sim \sqrt{\frac{2}{\pi\cdot\bar{z}}}\cdot\left[\bar{Q}_1(\bar{z})\cdot\cos\left(\bar{z}-\frac{\pi}{4}\right) + \bar{P}_1(\bar{z})\cdot\sin\left(\bar{z}-\frac{\pi}{4}\right)\right] \tag{53}$$

$$Y_0(\bar{z}) \sim \sqrt{\frac{2}{\pi\cdot\bar{z}}}\cdot\left[\bar{P}_0(\bar{z})\cdot\sin\left(\bar{z}-\frac{\pi}{4}\right) - \bar{Q}_0(\bar{z})\cdot\cos\left(\bar{z}-\frac{\pi}{4}\right)\right] \tag{54}$$

$$Y_1(\bar{z}) \sim \sqrt{\frac{2}{\pi\cdot\bar{z}}}\cdot\left[\bar{Q}_1(\bar{z})\cdot\sin\left(\bar{z}-\frac{\pi}{4}\right) - \bar{P}_1(\bar{z})\cdot\cos\left(\bar{z}-\frac{\pi}{4}\right)\right] \tag{55}$$

where auxiliary functions $\bar{P}_0(\bar{z})$, $\bar{Q}_0(\bar{z})$, $\bar{P}_1(\bar{z})$ and $\bar{Q}_1(\bar{z})$ can be written as:

$$\bar{P}_0(\bar{z}) \approx 1+\sum_{k=1}^{m}\frac{p_k}{\bar{z}^{\,2k}}; \quad p_k = (-1)^k \cdot \frac{\prod_{i=1}^{2k}(2\cdot i-1)^2}{(2\cdot k)!\cdot 8^{2k}} \tag{56}$$

$$\bar{Q}_0(\bar{z}) \approx \frac{1}{8\cdot\bar{z}}\cdot\left(1+\sum_{k=1}^{m}\frac{q_k}{\bar{z}^{\,2k}}\right); \quad q_k = (-1)^k \cdot \frac{\prod_{i=1}^{2k}(2\cdot i+1)^2}{(2\cdot k+1)!\cdot 8^{2k}} \tag{57}$$

$$\bar{P}_1(\bar{z}) \approx 1+\sum_{k=1}^{m}\frac{f_k}{\bar{z}^{\,2k}}; \quad f_k = (-1)^k \cdot \frac{\prod_{i=1}^{2k}\left[4-(2\cdot i-1)^2\right]}{(2\cdot k)!\cdot 8^{2k}} \tag{58}$$

$$\bar{Q}_1(\bar{z}) \approx \frac{3}{8\cdot\bar{z}}\cdot\left(1+\sum_{k=1}^{m}\frac{g_k}{\bar{z}^{\,2k}}\right); \quad g_k = (-1)^k \cdot \frac{\prod_{i=1}^{2k}\left[4-(2\cdot i+1)^2\right]}{(2\cdot k+1)!\cdot 8^{2k}} \tag{59}$$

The parameter m in Equations (56) – (59) is based on our numerous numerical tests equal to or less than 25. The convergence is controlled in the same way as in the computation of Bessel and Neumann functions of the zeroth and first order given by Eqs. (16) – (19). Moreover, the unique convergence condition can be written as follows:

$$\left|\frac{c_m}{\bar{z}^{\,2m}}\right| \le \left|1+\sum_{k=1}^{m}\frac{c_k}{\bar{z}^{\,2k}}\right|\cdot 10^{-q}; \quad c_k \in \{p_k,\ q_k,\ f_k,\ g_k\} \tag{60}$$

Using the following mathematical identities:

$$\sin\left(\bar{z}-\frac{\pi}{4}\right)=\frac{\sin\bar{z}-\cos\bar{z}}{\sqrt{2}} \tag{61}$$

$$\cos\left(\bar{z}-\frac{\pi}{4}\right)=\frac{\sin\bar{z}+\cos\bar{z}}{\sqrt{2}} \tag{62}$$

$$\bar{F}(x,y)=\mathrm{e}^{-y}\cdot\sin\bar{z}=\frac{(\mathrm{e}^{-2y}+1)\cdot\sin x-\mathrm{j}\cdot(\mathrm{e}^{-2y}-1)\cdot\cos x}{2} \tag{63}$$

$$\bar{G}(x,y)=\mathrm{e}^{-y}\cdot\cos\bar{z}=\frac{(\mathrm{e}^{-2y}-1)\cdot\cos x+\mathrm{j}\cdot(\mathrm{e}^{-2y}-1)\cdot\sin x}{2} \tag{64}$$

asymptotic approximations of the scaled Bessel functions can be expressed by the following equations:

$$J_0^s(\bar{z}) \sim \frac{\bar{F}(x,y)\cdot(\bar{P}_0(\bar{z})+\bar{Q}_0(\bar{z}))+\bar{G}(x,y)\cdot(\bar{P}_0(\bar{z})-\bar{Q}_0(\bar{z}))}{\sqrt{\pi\cdot\bar{z}}} \tag{65}$$

$$J_1^s(\bar{z}) \sim \frac{\bar{F}(x,y)\cdot(\bar{P}_1(\bar{z})+\bar{Q}_1(\bar{z}))-\bar{G}(x,y)\cdot(\bar{P}_1(\bar{z})-\bar{Q}_1(\bar{z}))}{\sqrt{\pi\cdot\bar{z}}} \tag{66}$$

$$Y_0^s(\bar{z}) \sim \frac{\bar{F}(x,y)\cdot(\bar{P}_0(\bar{z})-\bar{Q}_0(\bar{z}))-\bar{G}(x,y)\cdot(\bar{P}_0(\bar{z})+\bar{Q}_0(\bar{z}))}{\sqrt{\pi\cdot\bar{z}}} \tag{67}$$

$$Y_1^s(\bar{z}) \sim \frac{-\bar{F}(x,y)\cdot(\bar{P}_1(\bar{z})-\bar{Q}_1(\bar{z}))-\bar{G}(x,y)\cdot(\bar{P}_1(\bar{z})+\bar{Q}_1(\bar{z}))}{\sqrt{\pi\cdot\bar{z}}} \tag{68}$$

3.4. Computation of Neumann functions by Wronskians

In the case when all four Bessel and Neumann functions of the zeroth and first order are computed, as well as if:

$$\left|J_0^s(\bar{z})\right| > 0 \tag{69}$$

the following Wronskians can be used for computation of the unscaled and scaled Neumann function of the first order in the first quadrant of the complex plane:

$$Y_1(\bar{z}) = \frac{J_1(\bar{z})\cdot Y_0(\bar{z})\cdot\pi\cdot\bar{z}-2}{\pi\cdot\bar{z}\cdot J_0(\bar{z})} \tag{70}$$

$$Y_1^S(\bar{z}) = \frac{J_1^S(\bar{z})\cdot Y_0^S(\bar{z})\cdot\pi\cdot\bar{z}-2\cdot e^{-2y}}{\pi\cdot\bar{z}\cdot J_0^S(\bar{z})} \tag{71}$$

4. Computation of Functions of the Zeroth and First Order in the Second Quadrant

In the second quadrant of the complex plain $(x<0,\ y\geq 0)$ computation of the unscaled and scaled Bessel, Neumann and Hankel functions of integer order n can be translated in the first quadrant using the following expressions:

$$J_n(\bar{z}) = (-1)^n\cdot(J_n(-\bar{z}^*))^* \tag{72}$$

$$Y_n(\bar{z}) = (-1)^n\cdot\left[(Y_n(-\bar{z}^*))^*+\mathrm{j}\cdot 2\cdot(J_n(-\bar{z}^*))^*\right] \tag{73}$$

$$H_n^{(1)}(\bar{z}) = (-1)^n\cdot\left[\mathrm{j}\cdot(Y_n(-\bar{z}^*))^*-(J_n(-\bar{z}^*))^*\right] \tag{74}$$

$$H_n^{(2)}(\bar{z}) = (-1)^n\cdot\left[3\cdot(J_n(-\bar{z}^*))^*-\mathrm{j}\cdot(Y_n(-\bar{z}^*))^*\right] \tag{75}$$

$$J_n^s(\bar{z}) = (-1)^n\cdot(J_n^s(-\bar{z}^*))^* \tag{76}$$

$$Y_n^s(\bar{z}) = (-1)^n\cdot\left[(Y_n^s(-\bar{z}^*))^*+\mathrm{j}\cdot 2\cdot(J_n^s(-\bar{z}^*))^*\right] \tag{77}$$

$$H_n^{(1)s}(\bar{z}) = (-1)^n\cdot\left[\mathrm{j}\cdot(Y_n^s(-\bar{z}^*))^*-(J_n^s(-\bar{z}^*))^*\right] \tag{78}$$

$$H_n^{(2)s}(\bar{z}) = (-1)^n\cdot\left[3\cdot(J_n^s(-\bar{z}^*))^*-\mathrm{j}\cdot(Y_n^s(-\bar{z}^*))^*\right] \tag{79}$$

where asterisk * denotes the complex conjugation.

5. COMPUTATION OF FUNCTIONS OF THE ZEROTH AND FIRST ORDER IN THE THIRD QUADRANT

In the third quadrant of the complex plain $(x < 0,\ y < 0)$ computation of the unscaled and scaled Bessel and Neumann functions of integer order n can be translated in the first quadrant using the following expressions:

$$J_n(\overline{z}) = (-1)^n \cdot J_n(-\overline{z}) \tag{80}$$

$$Y_n(\overline{z}) = (-1)^n \cdot \left[Y_n(-\overline{z}) - \mathrm{j} \cdot 2 \cdot J_n(-\overline{z}) \right] \tag{81}$$

$$H_n^{(1)}(\overline{z}) = (-1)^n \cdot \left[3 \cdot J_n(-\overline{z}) + \mathrm{j} \cdot Y_n(-\overline{z}) \right] \tag{82}$$

$$H_n^{(2)}(\overline{z}) = (-1)^{n+1} \cdot \left[J_n(-\overline{z}) + \mathrm{j} \cdot Y_n(-\overline{z}) \right] \tag{83}$$

$$J_n^s(\overline{z}) = (-1)^n \cdot J_n^s(-\overline{z}) \tag{84}$$

$$Y_n^s(\overline{z}) = (-1)^n \cdot \left(Y_n^s(-\overline{z}) - \mathrm{j} \cdot 2 \cdot J_n^s(-\overline{z}) \right) \tag{85}$$

$$H_n^{(1)s}(\overline{z}) = (-1)^n \cdot \left[3 \cdot J_n^s(-\overline{z}) + \mathrm{j} \cdot Y_n^s(-\overline{z}) \right] \tag{86}$$

$$H_n^{(2)s}(\overline{z}) = (-1)^{n+1} \cdot \left[J_n^s(-\overline{z}) + \mathrm{j} \cdot Y_n^s(-\overline{z}) \right] \tag{87}$$

6. COMPUTATION OF FUNCTIONS OF THE ZEROTH AND FIRST ORDER IN THE FOURTH QUADRANT

In the fourth quadrant of the complex plain $(x \ge 0,\ y < 0)$ computation of the unscaled and scaled Bessel and Neumann functions of integer order n can be translated in the first quadrant using the following expressions:

$$J_n(\overline{z}) = (J_n(\overline{z}^*))^* \tag{88}$$

$$Y_n(\overline{z}) = (Y_n(\overline{z}^*))^* \tag{89}$$

$$H_n^{(1)}(\overline{z}) = (J_n(\overline{z}^*))^* + \mathrm{j} \cdot (Y_n(\overline{z}^*))^* = (H_n^{(2)}(\overline{z}^*))^* \tag{90}$$

$$H_n^{(2)}(\overline{z}) = (J_n(\overline{z}^*))^* - \mathrm{j} \cdot (Y_n(\overline{z}^*))^* = (H_n^{(1)}(\overline{z}^*))^* \tag{91}$$

$$J_n^s(\overline{z}) = (J_n^s(\overline{z}^*))^* \tag{92}$$

$$Y_n^s(\overline{z}) = (Y_n^s(\overline{z}^*))^* \tag{93}$$

$$H_n^{(1)s}(\overline{z}) = (J_n^s(\overline{z}^*))^* + \mathrm{j} \cdot (Y_n^s(\overline{z}^*))^* = (H_n^{(2)s}(\overline{z}^*))^* \tag{94}$$

$$H_n^{(2)s}(\overline{z}) = (J_n^s(\overline{z}^*))^* - \mathrm{j} \cdot (Y_n^s(\overline{z}^*))^* = (H_n^{(1)s}(\overline{z}^*))^* \tag{95}$$

7. Computation of Functions of Higher Positive Integer Order

For computation of Bessel, Neumann and Hankel functions of higher positive integer orders, forward and backward recurrence relations can be used [14], [19–24]. Forward recurrence relations can be written as:

$$B_n(\bar{z}) = \frac{2\cdot(n-1)}{\bar{z}}\cdot B_{n-1}(\bar{z}) - B_{n-2}(\bar{z}); \quad n = 2, 3, ... \tag{96}$$

$$B_n^s(\bar{z}) = \frac{2\cdot(n-1)}{\bar{z}}\cdot B_{n-1}^s(\bar{z}) - B_{n-2}^s(\bar{z}); \quad n = 2, 3, ... \tag{97}$$

The Miller backward recurrence algorithm [14], [19–24] for the unscaled and scaled functions can be based on the assumptions that:

$$B_n(\bar{z}) = \bar{S}\cdot\bar{F}_n(\bar{z}) \tag{98}$$

$$B_n^s(\bar{z}) = \bar{S}\cdot\bar{F}_n(\bar{z}) \tag{99}$$

where:

$$\bar{S} = \frac{B_0(\bar{z})}{\bar{F}_0(\bar{z})} \quad \text{or} \quad \bar{S} = \frac{B_1(\bar{z})}{\bar{F}_1(\bar{z})} \tag{100}$$

$$\bar{S} = \frac{B_0^s(\bar{z})}{\bar{F}_0(\bar{z})} \quad \text{or} \quad \bar{S} = \frac{B_1^s(\bar{z})}{\bar{F}_1(\bar{z})} \tag{101}$$

where expressions with a larger denominator are chosen.
The Miller backward recurrence relation can be written as:

$$\bar{F}_n(\bar{z}) = \frac{2\cdot(n+1)}{\bar{z}}\cdot\bar{F}_{n+1}(\bar{z}) - \bar{F}_{n+2}(\bar{z}); \quad n = N, N-1, ..., 0 \tag{102}$$

where:

$$\bar{F}_{N+1}(\bar{z}) = 1 \quad ; \quad \bar{F}_{N+2}(\bar{z}) = 0 \tag{103}$$

Algorithms for estimating the integer parameter N can be found in [14].

8. Model Validation

The proposed model for highly accurate computation of Bessel, Neumann and Hankel functions of integer order in the entire complex plane for complex variables of arbitrary magnitude was implemented in a FORTRAN program. To determine the accuracy of the obtained results and the numerical stability of the model, a comparison was made with the publicly available program package Wolfram Alpha. While the developed FORTRAN program uses double-double precision computing, Wolfram Alpha can employ numbers with an arbitrary number of decimal places.

The minimum number of matching digits, when comparing the results obtained in the program written based on the proposed model with the results obtained in the Wolfram Alpha program package, for complex variables of arbitrary magnitude and order of functions, is 30, which is a satisfactory level of accuracy of this model.

9. Conclusion

The proposed algorithm for computation of complex-valued Bessel, Neumann and Hankel functions of integer order provides highly accurate results in the entire complex plane with quadruple precision, which can be reduced to double precision. The computation of these functions in all quadrants of the complex plane is based on the computation of Bessel and Neumann functions of the zeroth and first order in the first quadrant of the complex plane, using truncated power series for small arguments, Gauss-Legendre quadrature for medium-sized arguments and asymptotic approximations of the Bessel functions for large arguments. Bessel, Neumann and Hankel functions of higher positive integer order can be computed using forward and backward recurrence relations, whereas these functions of negative integer order are equal to positive order functions up to the sign. For numerical reasons, the functions can be scaled. The obtained results were compared with the results provided by the Wolfram Alpha software package. The methodology presented in this paper is also applicable to the computation of modified Bessel functions, which is an area of future research.

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