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EXACT ANALYTICAL SOLUTIONS OF CONTINUOUSLY GRADED MODELS OF FLAT LENSES BASED ON TRANSFORMATION OPTICS

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Abstract. *We present a study of exact analytic solutions for electric and magnetic fields in continuously graded flat lenses designed utilizing transformation optics. The lenses typically consist of a number of layers of graded index dielectrics in both the radial and longitudinal directions, where the central layer in the longitudinal direction primarily contributes to a bulk of the phase transformation, while other layers act as matching layers and reduce the reflections at the interfaces of the middle layer. Such lenses can be modeled as compact composites with continuous permittivity (and if needed) permeability functions which asymptotically approach unity at the boundaries of the composite cylinder. We illustrate the proposed procedures by obtaining the exact analytic solutions for the electric and magnetic fields for one simple special class of composite designs with radially graded parameters. To this purpose we utilize the equivalence between the Helmholtz equation of our graded flat lens and the quantum-mechanical radial Schrödinger equation with Coulomb potential, furnishing the results in the form of Kummer confluent hypergeometric functions. Our approach allows for a better physical insight into the operation of our transformation optics-based graded lenses and opens a path toward novel designs and approaches.*

Key words: *Flat lenses, Graded permittivity and permeability models, Transformation Optics, Exact analytical solutions*

1. INTRODUCTION

Flat lenses designs based on Transformation Optics (TO) and using left-handed (negative refractive index) metamaterials have been discussed in a number of recent publications ([1], [2]). Basically, using the electromagnetic design, one is able to design a lens with the full functionality of a conventional lens, but compressed in space and possibly having additional functionalities. It is possible to do this in a wide range of operating frequencies, including microwave, terahertz and optical. However, the metamaterial composites proposed for such designs may be difficult to manufacture,

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especially when the required values of relative magnetic permeability and relative dielectric permittivity are less than unity, as argued in [3]. In order to avoid problems with fabrication of metamaterials with suitable values of magnetic permeabilities, it is possible to set the value of $\mu_r = 1$ and to vary ε_r only to create the desired refractive index of $n = \sqrt{\mu_r \varepsilon_r}$, but at the cost of decreasing the efficiency ofthe composite lenses [4]. In [4] a plano-concave lens has been designed with metamaterials to obtain a gain above 13 dB in the frequency band between 10 and 12 GHz. Such a lens has a narrow bandwidth typical for a majority of designs using metamaterials. The conventional flat lens designs, using Ray Optics (RO) approach, avoid the abovementioned difficulties with TO designs, but they do not have the same flexibility to control the phase and amplitude of the fields within the lens structure. An approach to remedy the drawbacks of both TO and RO designs is the Field Manipulation (FM) method, described in [3].

The studies of the flat-lenses design approaches mentioned above, however, generally require a direct numerical approach in solving the field equations. In the present paper, we use an alternative approach and investigate the possibilities to identify and study some special designs that allow for the exact solutions of the field equations analogous to those obtained in studying various planar and cylindrical metamaterial structures [5] - [11]. The main motive for pursuing analytical solutions of the problems involving flat lenses is that the detailed knowledge of analytical structure of the field solutions may provide additional insights leading to improved or even entirely new designs. We apply our approach to a specific case of a gradient-index (GRIN) flat lens.

2. PROBLEM FORMULATION AND FIELD EQUATIONS

The Graded Index (GRIN) approach to the design of a flat lens is based on the concept of field transformation, similar to that proposed by Luneburg for the design of spherical lenses [12]. Similarly to Luneburg's approach, a desired field distribution in the output port (the exit aperture) is specified and the medium parameters of the intervening medium are determined such that the given field distribution in the input port (input aperture) is transformed to the desired field distribution in the exit plane. In many practical cases, this can be performed by tracing rays through a designed inhomogeneous medium. The design parameters of the lens include center frequency, focal length, thickness, and gain. The physical size (diameter D) of the lens will depend on the gain and the radial model function (e.g. radial dependence of the permittivity). One typical design layout is shown in Fig. 1. The design goal is to maximize the performance of the lens, and for that purpose we want to realize the desired phases on the face B of the lens while simultaneously maximizing the transmission coefficient over a broad frequency band. The problem is typically solved using a multi-layer structure, with the desired phase at the center frequency and a transmission coefficient as close to one as possible over abroad frequency band for each of the ten rings shown in Fig. 1.

In Fig. 1 the following symbols are used:

$\mathbf{t}$ – thickness of the lens

$\mathbf{F}$ – focal length of the lens

$\boldsymbol{\phi_i}$ – phase of the plane wave incident from the left on the face A

$\mathbf{A, B}$ – the notation for the two faces (A and B) of the lens

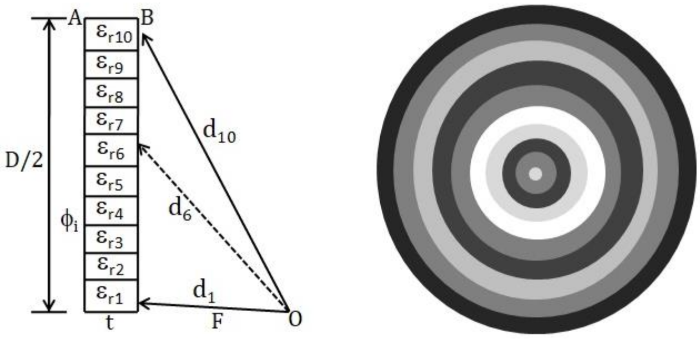

Fig. 1 Flat GRIN lens. Left: Cross section (side view) of the lens showing layers; right: top view of the lens

The middle layer perform a majority of the phase transformation, while the other layers act as matching layers to maximize the transmission of the waves incident from either side (graded antireflection structure). In the present approach we model the discrete structure shown in Fig. 1 by a cylindrical composite structure with the electric permittivity and permeability being continuous spatial functions

$$\varepsilon_{eff}(\omega) = \varepsilon_0 \varepsilon(r,z),\ \mu_{eff}(\omega) = \mu_0 \mu(r,z) \tag{1}$$

where (r, φ, z) is the set of cylindrical coordinates and the structure is centered around the z-axis. We consider a case of TE-wave propagation through the structure, so that the electric and magnetic field are

$$\boldsymbol{E} = E(r,z)\boldsymbol{e}_\varphi,\ \boldsymbol{H} = H_r(r,z)\boldsymbol{e}_r + H_z(r,z)\boldsymbol{e}_z \tag{2}$$

Here we note that the choice of TE-waves is by no means a restriction, and writing an analogous procedure for TM-waves is straightforward. In the case of TE-waves as described by (2), Maxwell equations for the scalar field components become

$$\frac{\partial E}{\partial z} = \mathrm{j}\omega\mu_0\mu(r,z)H_r,\ \frac{1}{r}\frac{\partial}{\partial r}(rE) = -\mathrm{j}\omega\mu_0\mu(r,z)H_z \tag{3}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \mathrm{j}\omega\varepsilon_0\varepsilon(r,z)E \tag{4}$$

Substituting equations (3) into (4), we obtain Helmholtz equation for the electric field

$$\frac{\partial^2}{\partial z^2}(rE) + \frac{\partial^2}{\partial r^2}(rE) - \frac{1}{\mu}\frac{\partial \mu}{\partial z}\frac{\partial}{\partial z}(rE) - \frac{1}{(\mu r)}\frac{\partial(\mu r)}{\partial r}\frac{\partial}{\partial r}(rE) + k^2\mu\varepsilon(rE) = 0 \tag{5}$$

or introducing a new function $W(r,z) = r\,E(r,z)$

$$\frac{\partial^2 W}{\partial z^2} + \frac{\partial^2 W}{\partial r^2} - \frac{1}{\mu}\frac{\partial \mu}{\partial z}\frac{\partial W}{\partial z} - \frac{1}{(\mu r)}\frac{\partial(\mu r)}{\partial r}\frac{\partial W}{\partial r} + k^2\mu\varepsilon\, W = 0 \tag{6}$$

where $k^2 = \omega^2\varepsilon_0\mu_0$. The equation (5), or (6), is quite general. After choosing suitable model functions $\varepsilon(r,z) = \varepsilon_R(r)\varepsilon_Z(z)$and $\mu(r,z) = \mu_R(r)\mu_Z(z)$, if we can determine the analytic solution for the electric field $E = E(r,z)$, then using (3) we can readily obtain the magnetic field components $H_r = H_r(r,z)$ and $H_z = H_z(r,z)$ as well. The challenge is therefore to find suitable model functions $\varepsilon(r,z) = \varepsilon_R(r)\varepsilon_Z(z)$ and $\mu(r,z) = \mu_R(r)\mu_Z(z)$ that provide a reasonable resemblance of actual design structures like the one described in Table 1 and Fig. 2 of [3].

3. Analytics of a Simple Model of Composite Designs

At this stage, we need to restrict the form of the functions (1) to allow for a suitable analytical solution. Let us here consider a simple model where

$$\varepsilon(r,z) = \varepsilon(r) = 1 + \varepsilon_\Delta(r) \quad (\varepsilon_\Delta(r) \to 0,\ r \to \infty)\ , \quad \mu(r,z) = 1 \tag{7}$$

In (7) we require that at large distances ($r \to \infty$) the composite permittivity $\varepsilon(r)$ becomes unity, which describes the gradual transition to the free space outside the structure. This is simultaneously the condition for the antireflective behavior of the lens surface and thus the maximum input electromagnetic flux. Utilizing (7) and separating variables using the ansatz $(r,z) = r\,E(r,z) = Z(z)R(r)$, the equation (6) gives rise to two ordinary differential equations for the two functions, $Z(z)$ and $R(r)$, as follows

$$\frac{d^2Z}{dz^2} + k_z^2\,Z = 0 \qquad \Rightarrow \qquad Z(z) = e^{\pm \mathrm{j}k_z z} \tag{8}$$

$$\frac{d^2R}{dr^2} - \frac{1}{r}\frac{dR}{dr} + [k^2\varepsilon(r) - k_z^2]\,R = 0 \tag{9}$$

As indicated in (8), the solutions for $Z(z)$ are simple plane waves propagating in the z-direction, and we only need to solve equation (9). Introducing $(r) = \sqrt{r}\,\psi(r)$, the equation (9) becomes

$$\frac{d^2\psi}{dr^2} + \left[k^2\varepsilon(r) - k_z^2 - \frac{3}{4r^2}\right]\,\psi = 0 \tag{10}$$

Let us now introduce two constants $k_R^2 = k^2 - k_z^2$, $l = 1/2$, whereby the equation (10) becomes the well-known radial Schrödinger equation

$$\frac{d^2\psi}{dr^2} + \left[k_R^2 + k^2\varepsilon_\Delta(r) - \frac{l(l+1)}{r^2}\right]\,\psi = 0 \tag{11}$$

where we notice the following analogy between the parameters of the electromagnetic equation (11) and the parameters of the usual quantum-mechanical radial Schrödinger equation

$$k_R^2 \leftrightarrow \frac{2mE}{\hbar^2}, \qquad k^2\varepsilon_\Delta(r) \ \leftrightarrow\ -\frac{2mV(r)}{\hbar^2} \tag{12}$$

Since we require that $\varepsilon_\Delta(r) \to 0$ when $r \to \infty$, the simplest model that we can adopt is the Coulomb potential

$$V(r) \sim -\frac{\alpha^2}{r} \qquad \Rightarrow \qquad \varepsilon_\Delta(r) = +\frac{\alpha^2}{r} \tag{13}$$

where α is a constant that must be chosen to provide the best fit to the presented graded model. Such a choice of $\varepsilon_\Delta(r)$ introduces an unphysical singularity of the permittivity function for $r = 0$, but with a proper choice of boundary conditions it can provide a sufficiently accurate model of the realistic graded permittivity structures. Substituting $\varepsilon_\Delta(r)$ from (13) into (11) we obtain

$$\frac{d^2\psi}{dr^2} + \left[k_R^2 + \frac{k^2\alpha^2}{r} - \frac{l(l+1)}{r^2}\right]\,\psi = 0 \tag{14}$$

The equation (14) has an exact analytical solution

$$\psi(r) = C_1 M\left(-\frac{\mathrm{j}k^2\alpha^2}{2k_R},\ l+\frac{1}{2}, 2\mathrm{j}k_R r\right) + C_2 W\left(-\frac{\mathrm{j}k^2\alpha^2}{2k_R}\ ,\ l+\frac{1}{2}, 2\mathrm{j}k_R r\right) \tag{15}$$

where $M(a,b,z) = M_{a,b}(z)$ and $W(a,b,z) = W(z)$ are Whittaker functions that can be expanded in terms ofKummer confluent hypergeometric functions F_1 and U. Based on the asymptotic behavior of the Whittaker functions for$r \to 0$ and $r \to \infty$, and the physical requirements on the behavior of the electric field functions $E(r,z)$, we see that we must choose $C_2 = 0$, such that for $l = 1/2$, we have

$$R(r) = \sqrt{r}\,\psi(r) = C_1\sqrt{r}\, M\left(-\frac{\mathrm{j}k^2\alpha^2}{2k_R},\ 1\ , 2\mathrm{j}k_R r\right) \tag{16}$$

and for waves propagating in the positive z-direction, we can write

$$E(r,z) = \frac{1}{r}Z(z)R(r) = C_1 e^{-\mathrm{j}k_z z} r^{-1/2}\, M\left(-\frac{\mathrm{j}k^2\alpha^2}{2k_R},\ 1, 2\mathrm{j}k_R r\right) \tag{17}$$

It is here convenient to express the result (17) in terms of Kummer confluent hypergeometric functions, in order to further clarify the mathematical properties of the electric field intensity function. Thus, we finally obtain

$$E(r,z) = C_1 r\, e^{-\mathrm{j}(k_z z + k_R r)} F_1\left(\frac{3}{2} - \frac{\mathrm{j}k^2\alpha^2}{2k_R},\ 3\ , 2\mathrm{j}k_R r\right) \tag{18}$$

The result (18) for the electric field intensity function $E(r,z)$ refers to the φ-component of the electric field due to the assumed TE-wave as defined in (2). It should however be noted that the assumption of the TE-wave is by no means limiting the generality of the results obtained in the present paper. The case of the TM-wave is fully analogous to the case of the TE-wave, and the only difference is that the result (18) is then valid for the magnetic field intensity function $H(r,z)$ which refers to the φ-component of the magnetic field. The electric field components $E_r = E_r(r,z)$ and $E_z = E_z(r,z)$ are then readily obtained using the TM-wave analogues of the equations (3). The choice of the TE-wave in the present paper was made for illustration purposes. Following the approach in [3], the relative permittivities are here assumed to be real functions and no dielectric losses are taken into account. It should however be noted that there is nothing in the present theory that limits the values of the relative permittivities to be real. It is fully feasible to use the present model with complex relative permittivities as well. This will be the subject of our future studies.

4. Study of a Specific Numerical Case

Let us now turn to the specific case of a GRIN lens studied in [3], where we have a structure with radially graded permittivities for the middle layer, as listed in Table 1.

Table 1 Radially graded permittivities of the middle layer of a GRIN lens.

Layer	1	2	3	4	5	6	7	8	9	10
$\bar{r}$ (mm)	1.5875	4.7625	7.9375	11.1125	14.2875	17.4625	20.6375	23.8125	26.9875	30.1625
$\varepsilon(\bar{r})$	25.5	24.5	22.3	18.5	14.55	10.5	7.65	5.5	3.5	1.65

Using the model function (7) with (13), we obtain the fitting graph as shown in Fig. 2, where we have chosen the parameter α to be equal to 0.36.

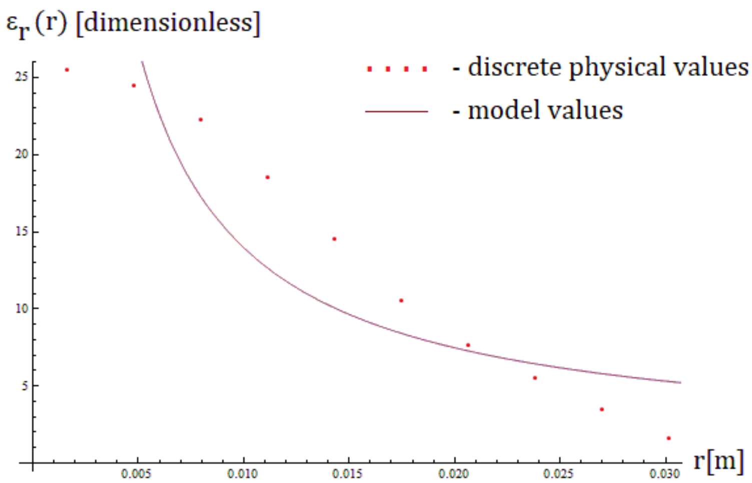

Fig. 2 Fitting of GRIN lens relative permittivity data using Coulomb function with $\alpha^2 = 0.36^2\text{m}$.

The cross section of the solution (18) for a constant z is shown in Fig. 3.

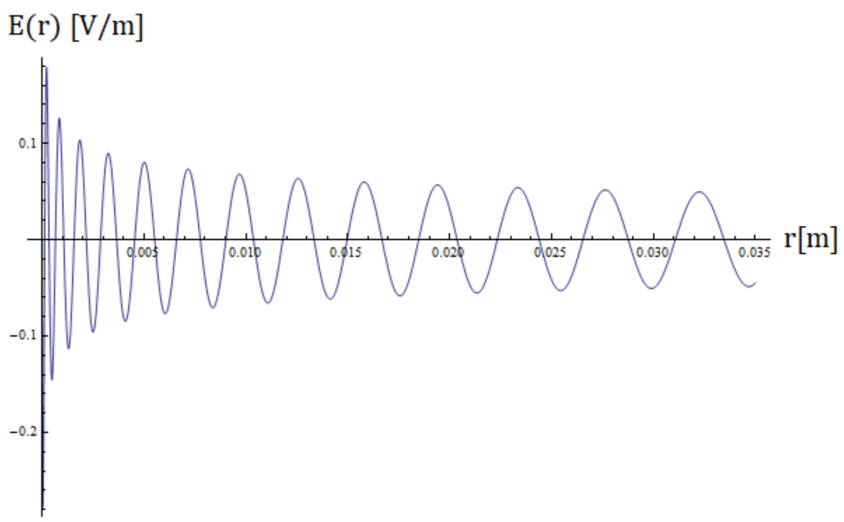

Fig. 3 Cross section of the electric field function E(r, z) for given constant z (z = 0), with $C_1 = 1$, $f = 30$ GHz, $k = 2\pi f/c$ and $k_z = 0.8\,k$.

Finally, a three dimensional plot of the solution (18) is shown in Fig. 4.

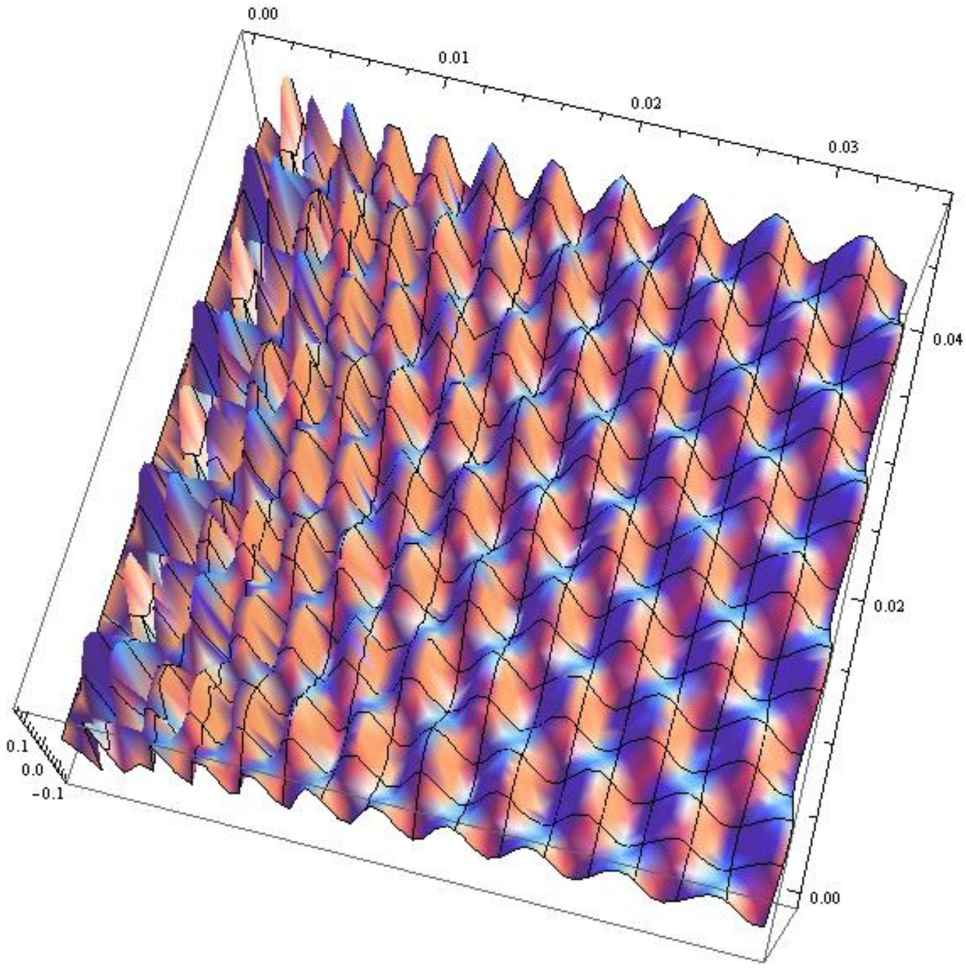

Fig. 4 Electric field function $E(r, z)$.

From Fig. 4 we readily see how the wave is radially focused while moving along the z-direction, as expected. The size of the wave amplitudes is not normalized with respect to any starting position, and does not reflect any specific initial electric field strength. Even though the Coulomb function is far from the optimum fit for the GRIN lens data, the obtained results can be used to describe simply and sufficiently accurately the chosen lens.

It should be noted here that our choice of the model function (Coulomb function) has been made based on the well known analytical solutions for that function. There is a number of other functions that also allow the exact analytic solutions of the problem at hand, in particular if the model is extended to allow the graded permeability of the lens layers. The studies of other models involving such more accurate model functions will be the subject of our coming papers.

5. CONCLUSIONS

The possibility to find exact analytic solutions for the electric and magnetic fields in continuously graded flat lenses has been studied. The flat lenses are modeled as compact composites with continuous permittivity and permeability functions which asymptotically approach unity at the boundaries of the composite cylinder. In order to illustrate the present approach, we obtain an exact analytic solution for the electric field intensity for an FM composite lens with constant magnetic permeability $\mu(r, z) = 1$ and radially dependent dielectric permittivity. In our coming research efforts, we see the need to look for the possible models with exact analytical (or at least perturbational and/or WKB) solutions for graded profiles of some more complex flat-lens designs studied in literature.

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